On the structure of spectra of periodic elliptic operators

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Abstract

The paper discusses the problem of absolute continuity of spectra of periodic elliptic operators. A new result on absolute continuity for a matrix operator of Schrödinger type is obtained. It is shown that all types of operators for which the absolute continuity has previously been established can be reduced to this one. It is also discovered that some natural generalizations stumble upon an obstacle in the form of non-triviality of a certain analytic bundle on the two-dimensional torus.

1 Introduction

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Elliptic differential operators with periodic coefficients arise naturally in many areas of mathematical physics. One can mention among the most prominent examples quantum solid state theory, where the main operator of interest is the stationary Schrödinger operator

$$-\Delta + V(x) \tag{1}$$

with a potential $V(x)$ periodic with respect to a lattice in $\mathbb{R}^d$ (see for instance [4], [54]).

Another example that has gained importance due to recent advances in the theory of photonic crystals (e.g., [32], [38]) is the periodic Maxwell operator, one of realizations of which (in fact, of its square) can be written as

$$\frac{1}{\mu(x)}\nabla \times \frac{1}{\varepsilon(x)}\nabla\times \tag{2}$$

defined on zero-divergence vector fields in $\mathbb{R}^3$. Here the periodic functions $\varepsilon(x)$ and $\mu(x)$ represent the electric permittivity and magnetic permeability of the medium.

Scalar counterparts of this operator are

$$-\nabla \cdot \frac{1}{\varepsilon(x)}\nabla \tag{3}$$

and

$$-\frac{1}{\mu(x)}\Delta \tag{4}$$

(in the latter case the operator must be considered in a weighted L_2-space in order to be self-adjoint). These scalar operators also arise in studying periodic acoustic media.

Among other operators of interest one can mention the divergence type operators

$$\sum \partial_i a_{ij}(x)\partial_j \tag{5}$$

and magnetic Schrödinger operators

$$(\frac{1}{i}\nabla - A(x))^2 + V(x), \tag{6}$$

where the scalar functions $a_{ij}(x)$ and $V(x)$ and the vector magnetic potential $A(x)$ are assumed to be periodic with respect to a lattice.

Another extensively studied case is the one of a periodic Dirac operator

$$\frac{1}{i}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\alpha_j + V(x),$$

where α_j are self-adjoint matrices satisfying the anti-commutation relations

$$\alpha_j\alpha_i + \alpha_i\alpha_j = 2\delta_{ij}I$$

(Dirac or Pauli matrices) and the periodic matrix function V is the potential.

In all these cases the structure of the spectrum of the corresponding operator is of major interest. Consider for instance the periodic Schrödinger operator (1) in $\mathbb{R}^d$. Assuming that the potential is real and satisfies some mild conditions one can prove self-adjointness of this operator (see [54]). It is well known that the spectrum of such an operator has the so called band-gap structure, i.e. it consists of the union of finite closed segments $[a_j, b_j]$ such that $a_j \to \infty$ (e.g. [4], [28], [32], [34], [37], [54], [61]). It has been known to physicists for a long time (see for instance [4]) that the spectrum of this operator in $L_2(\mathbb{R}^d)$ does not contain any eigenvalues. Put in different terms, this means that none of the spectral bands $[a_j, b_j]$ collapses into a point. One can easily prove that there are no eigenvalues of finite multiplicity [28], however the proof of the general statement had to wait for a long time until the celebrated Thomas' theorem was proven in 1973 [68]. In fact, the paper [68] contained the proof of a more general statement: the spectrum of a self-adjoint periodic operator (1) in $3D$ is absolutely continuous. One should notice that the statement on absence of eigenvalues holds also in the non-self-adjoint case. In the self-adjoint situation the so called Floquet theory enables one to easily deduce absolute continuity from the absence of eigenvalues, since one can easily prove absence of the singular continuous spectrum (see for instance [37], [54], and [60]). Although the proofs of this implication in [37], [54], and [60] are formulated for the Schrödinger case only, the derivation is very general and holds for any self-adjoint periodic elliptic operator with sufficiently regular coefficients (see for instance the proof of Theorem 4.5.9 in [37]). In fact, the statement on absence of the singular continuous spectrum holds **with the same proof** for operators in periodic electro-magnetic fields in the case of a non-zero **rational** magnetic flux through the unit cell (i.e., when the magnetic potential is not periodic). Then one has an abelian group of the so called "magnetic translations" (or Zak group, e.g. [26], [27], [50], and [71]), which enables one to carry through the proof. In this situation, however, the point spectrum can arise (Landau levels).

The question arises on whether absence of eigenvalues (absolute continuity of spectrum in the self-adjoint case) is a property that all periodic elliptic differential operators enjoy. It is known, however (see p. 135–136 in [37]) that this is in general not true for elliptic operators of order four. On the other hand, the common belief is that periodic elliptic operators of second order do not have eigenvalues (or, in the self-adjoint case, have absolute continuous spectra), although it is rarely discussed why this must be true. Let us describe why the authors believe in this. Assuming that a periodic elliptic operator has an eigenvalue, one can derive from this existence of an eigenfunction of a super-exponential decay (see Theorems 4.1.5 and 4.1.6 in [37] and Theorem 12 in [36]). Existence of such fast decaying solutions is possible for fourth order equations, where even compactly supported solutions can exist (see p. 135–136 in [37] and the original paper [53]). In the case of equations of second order, such fast decay should probably violate uniqueness of continuation at infinity and hence be impossible. Unfortunately, the appropriate uniqueness theorems that prohibit super fast decay at infinity for equations of the type we consider are not known yet in full generality. The only applicable results of [30] and [46] deal with the Schrödinger operator with an electric potential only. Thus, due to absence of appropriate theorems of possible decay rate of solutions, the absolute continuity problem since the Thomas' paper [68] has been approached in a different way. It is reminded to the reader in Section 2. Let us list the known results. Thomas' theorem [68] was extended to all dimensions and a broader class of periodic Schrödinger operators in [54]. Papers [16]–[20] contained the proof of the absolute continuity of the spectrum for the Dirac operator with a periodic scalar electric potential. Thomas' result for the Schrödinger operator implies the similar statement for the operator

$$H_{\rho,V} = -\nabla \cdot \rho^2(x)\nabla + V(x) \tag{7}$$

with periodic functions $\rho \neq 0$ and V if ρ is smooth enough, due to the well known transformation

$$\rho^{-1} H_{\rho,V} \rho^{-1}, \tag{8}$$

which reduces the operator to the form (1). The disadvantage of this transform is that it requires smoothness of ρ, the condition which is probably not needed for the result to hold and which is violated in many applications (for instance, to photonic crystals). Thomas' theorem also handles the operator (4). However, the case of the magnetic Schrödinger operator (6) had resisted attempts to prove absolute continuity of the spectrum until the paper [31] where this was done for the case of small magnetic potentials. Finally, in the remarkable papers [8] and [65] the full strength statement about the magnetic Schrödinger operator was proven first in dimension two [8] and then in arbitrary dimension [65]. The elegant algebraic approach of the paper [8] works only in dimension two. This paper lead to a proof in arbitrary dimension [65]. Papers [8] and [65] triggered an avalanche of publications that significantly advanced the known results (see [9]–[14], [21]–[24], [41], [47], [48], [55]–[58], [66], and [63]). Some major questions, however (for instance, how to handle the operators with variable coefficients in the higher order terms) remain unanswered. One can find a rather comprehensive survey in [12] and [63].

The goal of this paper is to provide a new absolute continuity result for a matrix Schrödinger operator such that all operators treated so far can be transformed into this form. The proof uses standard microlocal analysis techniques and employs the idea of [65] in a simplified form. We were also influenced by the papers [49] and [63], although this might not be obvious from the text. We believe that this approach clarifies and unifies the know results. At the same time it singles out a problem of triviality of a certain analytic vector bundle on the torus, an obstruction which has not been noticed before. We would also want to notice that our main results do not require self-adjointness. This happens to be useful even for some self-adjoint problems, which can sometimes be transformed to simpler non-self-adjoint ones, as it is explained in Section 7. A limited version of the main result was announced in [41].

We want to make it clear that we did not even attempt to achieve optimal conditions on the coefficients of the equations, but rather to encompass a wider set of types of equations. Finding the best possible conditions on the coefficients is an important task (especially taking into account that in many applications, for instance to photonic crystals [32], [38], the coefficients in the leading terms are piecewise constant). This was, however, not among our goals, and so our conditions on coefficients are in most cases not optimal. One can find optimal or close to optimal conditions for some cases in the recent works [12], [55], [56].

Let us describe the structure of the paper. The next section provides a preliminary discussion of the problem, in particular an exposition of the original Thomas' approach [68]. Section 3 contains the formulation of the main result of the paper. The next section was written for convenience of the reader and provides a brief sketch of the proof. Section 5 delivers the proof of the main result in the absence of electric potential. In the next section, the known technique of including the electrical potential is indicated with appropriate references. Section 7 describes the algebraic reduction of all studied so far types of operators to the form contained in our main result. The paper ends with the sections containing concluding remarks and acknowledgments.

2 Formulation of the problem and preliminary discussion

Let $P(x, D)$ be an elliptic second order scalar or matrix differential operator in $\mathbb{R}^d$ with coefficients periodic with respect to a lattice Γ in $\mathbb{R}^d$. Here $D = -i\nabla$. For simplicity of presentation we will consider in the most of the text the case of the integer lattice $\mathbb{Z}^d$. It is easy, however, to modify the proofs in order to cover an arbitrary lattice. We will not yet specify any conditions on smoothness of the coefficients of P. The reader can assume so far the coefficients to be as smooth as needed. The natural domain of definition of the operator will then be the Sobolev space $H^2(\mathbb{R}^d)$ (or its vector analog in the case of systems, although we will not distinguish these cases in the notations for the spaces). Our goal is to prove nonexistence of non-zero L_2-solutions of the equation $Pu = \lambda u$. As it was mentioned in the introduction, in the self-adjoint case standard considerations show that this implies absolute continuity of the spectrum of P in $L_2(\mathbb{R}^d)$. We also note that the spectral parameter λ can be included in the operator, and hence we will assume from now on that $\lambda = 0$.

We will describe now the approach of paper [68] to the proof of absolute continuity of the spectrum. The reader should be aware that we will adhere to the language of multi-dimensional analyticity, although the original paper [68] and most of other publications on the subject use one-dimensional directional analyticity. The language of several complex variables is not only natural, but often unavoidable in theory of periodic PDEs, as one can conclude for instance from [35], [36], [42] [43], [44], and [52]. This is in parallel with the theory of PDEs with constant coefficients, where complex analysis in several variables is often the language of choice (see for instance [29] and [51]). Using the technique of analytic Fredholm operator functions in several complex variables (see [70], [37]) makes the methods more flexible, allowing for a wide class of equations, including non-self-adjoint ones.

The main technique of the so called Floquet theory (see e.g. [28], [36], [37], [54], [59], [60], and [61]) is the decomposition of the operator P in the space $L_2(\mathbb{R}^d)$ into the direct integral of operators on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$P = \int_B^{\oplus} P(k)dk,$$

where

$$B = \left\{k \in \mathbb{R}^d \,|\, k_j \in [-\pi, \pi], j = 1, ..., d\right\}$$

is the so called *Brillouin zone*, and the operator

$$P(k) = P(x, D + k)$$

acts on $\mathbb{Z}^d$-periodic functions, i.e. on functions on the torus

$$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d.$$

The vector k is usually called *quasimomentum*. This direct integral decomposition provides an isometry between $L_2(\mathbb{R}^d)$ and

$$\int_B^{\oplus} L_2(\mathbb{T}^d)dk.$$

Let us observe now that one can define the operator family $P(k)$ for all values $k \in \mathbb{C}^d$. Then $P(k)$ becomes an analytic (in fact, polynomial) function on $\mathbb{C}^d$ with values in the space of bounded Fredholm operators acting from $H^2(\mathbb{T}^d)$ into $L_2(\mathbb{T}^d)$. Consider now the following set that we will call the *Fermi surface*:

$$F = \left\{k \in \mathbb{C}^d \,|\, \exists u \neq 0, P(k)u = 0\right\} \subset \mathbb{C}^d.$$

It follows from the general theory of Fredholm operator functions (see for instance [37], [70] and references therein) that the Fermi surface is an analytic set in $\mathbb{C}^d$ (see details and references in [35] and [37]). In fact, it is the set of all zeros of an entire function $f(k)$ of a finite order in $\mathbb{C}^d$. Assume now that there exists a non-zero solution $u \in L_2(\mathbb{R}^d)$ of the equation $Pu = 0$. After expanding into the direct integral, we conclude immediately that the set of quasimomenta $k \in B$ for which the operator $P(k)$ has a nontrivial kernel must have positive measure. On the other hand, this set is the intersection $F \cap B$. We come to the conclusion that the function $f(k)$ is equal to zero on a set of a positive measure in the real subspace $\mathbb{R}^d \subset \mathbb{C}^d$. Due to standard uniqueness theorems for analytic functions, this implies that $f(k)$ is identically equal to zero, and hence $F = \mathbb{C}^d$. Looking at the definition of the Fermi surface we see that this leads to existence of a nontrivial solution of the equation

$$P(k)u = 0$$

on $\mathbb{T}^d$ for arbitrary $k \in \mathbb{C}^d$. So, we come to the following conclusion:

Theorem 1 *(see Theorem 4.1.5 in [37]) If there exists a vector $k \in \mathbb{C}^d$ such that the elliptic operator $P(k)$ on torus $\mathbb{T}^d$ has zero kernel, then the equation $Pu = 0$ in $\mathbb{R}^d$ has no non-trivial L_2-solutions.*

Remark 1

1. *For the case of a periodic Schrödinger operator this statement (in a little bit different form) was established by L. Thomas [68]. In the current form and even for hypoelliptic operators it is proven in [36], [37].*
2. *If in the self-adjoint case one applies this theorem to the operators $P - \lambda$ with arbitrary real λ, one concludes the absolute continuity of the spectrum of the operator P in $\mathbb{R}^d$ (e.g., Theorem 4.5.9 in [37]).*

Now the task, according to Theorem 1, becomes to find a quasimomentum $k \in \mathbb{C}^d$ such that the operator $P(k)$ on torus $\mathbb{T}^d$ is invertible. We will remind the standard procedure of choosing an appropriate vector k in the case of the periodic Schrödinger operator (1) (see [12], [37], [54], [68]). It can be chosen as $k = (\pi + i\rho, 0, ..., 0)$ with a large real ρ. Indeed, then the operator $P(k)$ is

$$(D + k)^2 + V(x).$$

Assume for simplicity that $V \in L_\infty(\mathbb{R}^d)$ (this assumption can be significantly weakened, as shown for instance in [6], [12], [36], [54], [55], [56], [68]). We will show that

$$\left|\left|(D + k)^2 u\right|\right|_{L_2(\mathbb{T}^d)} \geq \pi\rho \left|\left|u\right|\right|_{L_2(\mathbb{T}^d)} \tag{9}$$

for any $u \in H^2(\mathbb{T}^d)$. In order to derive the inequality (9), we use Fourier expansion

$$u(x) = \sum_{m \in \mathbb{Z}^d} \widehat{u}_m e^{2\pi i m \cdot x}.$$

Then

$$(D+k)^2 u = \sum_{m\in\mathbb{Z}^d} (2\pi m + k)^2 \widehat{u}_m e^{2\pi i m\cdot x}.$$

Due to the choice of k, we get

$$\left|(2\pi m + k)^2\right| \geq \left|Im(2\pi m + k)^2\right| = \left|(2\pi m_1 + \pi)\rho\right| \geq \pi\left|\rho\right|,$$

which proves (9). Now, choosing ρ such that $\pi\rho > ||V||_{L_\infty}$, one concludes that the equality

$$(D+k)^2 u + V(x)u = 0$$

is impossible for $u \neq 0$. Then according to the Theorem 1 we obtain that the equation

$$-\Delta u + Vu = 0$$

in $\mathbb{R}^d$ has no non-zero L_2-solutions. This proves the following result:

Theorem 2 *[68] The periodic Schrödinger operator H in $\mathbb{R}^d$ has no eigenvalues, and hence in the self-adjoint case its spectrum is absolutely continuous.*

Remark 2 *The statement on absence of eigenvalues does not require self-adjointness, so it holds even in the case of a complex periodic potential.*

3 The main result

The main object of study in this paper is the following matrix generalization of a periodic magnetic Schrödinger operator (6) in $\mathbb{R}^d$, $d \geq 2$:

$$H(\mathbf{a}, V) = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} g_{jk} \frac{\partial}{\partial x_k} - 2i\sum_{j=1}^{d} a_j(x)\frac{\partial}{\partial x_j} + V(x), \tag{10}$$

where $G = \{g_{jk}\}$ is a constant positive definite $d\times d$ matrix, the $n\times n$ matrix function $V(x)$ is the "electric potential" (in fact, it corresponds to a combination of the magnetic and electric potentials in (6)) and the $n\times n$ matrix functions $a_j(x)$ form an analog $\mathbf{a} = \{a_j(x)\}$ of a magnetic potential. Both potentials $\mathbf{a}$ and V are assumed to be periodic with respect to a lattice Γ in $\mathbb{R}^d$. A linear change of variables transforms Γ into the integer lattice $\mathbb{Z}^d$ without changing the form of the operator, so we will assume from now on that $\Gamma = \mathbb{Z}^d$. Although the matrix G cannot be diagonalized without destroying the equality $\Gamma = \mathbb{Z}^d$, the proofs will be provided for simplicity for the case when $G = I$ only, and then can be rather automatically adjusted to arbitrary positive definite G. So, we will concentrate for the sake of simplicity on the operator

$$H(\mathbf{a}, V) = -\Delta - 2i \sum_{j=1}^{d} a_j(x) \frac{\partial}{\partial x_j} + V(x) \tag{11}$$

with $\mathbb{Z}^d$-periodic potentials only, while the results will hold for any operator (10) with arbitrary lattice of periods.

In order to simplify the notations, we will use the standard functional spaces notations like $H^s(\mathbb{T}^d)$ or $L_2(\mathbb{T}^d)$ for vector and matrix valued functions, meaning that each of their entries belongs to the corresponding space.

According to the Theorem 1, if one desires to show absence of eigenvalues (or absolute continuity of the spectrum in the self-adjoint case), then the task should be to show existence of a quasimomentum $k \in \mathbb{C}^d$ such that the operator

$$H(k, \mathbf{a}, V) = (D + k)^2 + 2\mathbf{a}(x) \cdot (D + k) + V$$

has zero kernel on the torus $\mathbb{T}^d$. We will prove the following

Theorem 3 *Let the magnetic potential* $\mathbf{a}$ *belong to* $H^s(\mathbb{T}^d)$ *for some* $s > 3d/2 - 1$ *and satisfy the following conditions:*

1. *The average values of the components of* $\mathbf{a}$ *are equal to zero:*

$$\int_{\mathbb{T}^d} a_j(x) dx = 0, \; j = 1, ..., d. \tag{12}$$

2. *For any* $j, k = 1, ..., d$ *and* $x, y \in \mathbb{R}^d$ *the values of the matrix functions* a_j *satisfy the commutation relation*

$$[a_j(x), a_k(y)] + [a_k(x), a_j(y)] = 0. \tag{13}$$

Then there exist a constant $C > 0$*, a vector* $e \in \mathbb{R}^d$*, and a positive number* $\beta \in \mathbb{R}$*, such that*

$$||H(k, \mathbf{a}, 0)u||_{L_2} \geq C(|\rho| \, ||u||_{L_2} + ||u||_{H^1}) \tag{14}$$

for any $k = (\beta + i\rho)e \in \mathbb{C}^d$ *with sufficiently large* $\rho \in \mathbb{R}$ *and for any* $u \in H^2(\mathbb{T}^d)$.

Remark 3 *1. Notice that self-adjointness of the matrices a_j is not required.*

2. The commutation relation (13) implies in particular commutativity of the values of each of the functions $a_j(x)$, but not necessarily commutation between $a_j(x)$ and $a_k(y)$. Although commutativity of all the matrices $a_j(x)$ for different values of j and x is sufficient for (13), it is not necessary for the proof, and in fact the example of the Dirac operators leads to the case when such commutativity does not hold, while (13) still does.

3. We presume that the condition (13) is probably not necessary for the Theorem to hold. However, the attempts to prove the statement without it stumble upon an analytic obstacle in the form of non-triviality of a vector bundle. So, one cannot rule out possibility of surprises. See further discussion in Section 8 of the paper.

Theorem 3 leads to the immediate conclusion that the equation $H(k, \mathbf{a}, 0)u = 0$ has no non-zero solutions on $\mathbb{T}^d$ for large values of ρ. Then, as in the previous section, Thomas' Theorem 1 implies that the operator $H(\mathbf{a}, 0)$ has no eigenvalues (and hence has absolutely continuous spectrum in the self-adjoint case). Including an electric potential V that satisfies appropriate conditions is now a matter of technique (see [12] and [65]). For instance, using techniques described in [54], [37], and [65] this easily leads to the following result.

Theorem 4 *Assume that the conditions of Theorem 3 are satisfied and that $V \in L_{p,loc}(\mathbb{R}^d)$ with $p = 2$ for $d = 2$ and $p > d - 1$ for $d \geq 3$. Then the periodic operator $H(\mathbf{a}, V)$ has no eigenvalues in $L_2(\mathbb{R}^d)$.*

Notice that self-adjointness of any of the potentials is not assumed. In the self-adjoint case this implies

Theorem 5 *If the conditions of Theorem 4 are satisfied and the matrices $a_j(x)$ and $V(x)$ are self-adjoint for all j and x, then the spectrum of the operator $H(\mathbf{a}, V)$ is absolutely continuous.*

In fact, the conditions imposed on the magnetic and electric potentials are too restrictive. Using techniques of Section 5 in [12] one can extend the above results to the case when $V \in L_{p,loc}$ with $p > 1$ when $d = 2$, $p = d/2$ for

$d = 3$ and 4, and $p = d - 2$ when $d \geq 5$. In fact, for $d \geq 3$ the Lorentz spaces $L^0_{p,\infty}$ suffice [12]. We will not provide the proof of this extension here. One can refer to Section 5 in [12] for the method of its deduction from Theorem 3. One can find the optimal conditions on the electric potential V in terms of different functional classes in [12] and [55]–[58]. Although it is not clear yet how to optimize the conditions on the magnetic potential $\mathbf{a}$ for $d > 2$, for $d = 2$ the optimal conditions in L_p-classes are known, see the corresponding results and conjectures in [12], some new advances in [24], and discussion at the end of the paper.

4 Sketch of the proof

The approach we use is rather standard in the microlocal analysis: microlocal reduction of the problem to a standard one, treating this model case, and construction of a parametrix using a partition of unity. This finally leads to the invertibility of the operator for large values of ρ. Due to periodicity, the dual space will be discrete.

Fourier series expansion transfers the consideration to functions defined on the dual lattice $2\pi\mathbb{Z}^d$. The Fourier coefficients of a function f on the torus $\mathbb{T}^d$ will be denoted by $\widehat{f}(m)$. We will need the function

$$\Lambda(m, \rho) = (m^2 + \rho^2)^{1/2}$$

and the corresponding (pseudo-differential of first order) operator $\Lambda_\rho(D)$ on the torus $\mathbb{T}^d$ acting as

$$(\Lambda_\rho(D)f)\hat{}\,(m) = \Lambda(m, \rho)\widehat{f}(m).$$

Notice that

$$||\Lambda_\rho(D)f||_{L_2} \sim |\rho|\, ||u||_{L_2} + ||u||_{H^1}.$$

Consider the case of zero electric potential first. It is known (see [12], [14], [65], and Section 6) how to include then the electric potential.

Let us introduce some notations:

$$H = -\Delta + 2\mathbf{a}(x) \cdot D,$$

$$H_0 = -\Delta,$$

$$H(k) = (D + k)^2 + 2\mathbf{a}(x) \cdot (D + k),$$

and

$$H_0(k) = (D+k)^2.$$

We consider $H(k)$ and $H_0(k)$ as operators on the torus $\mathbb{T}^d$. Let $k = 2\pi(i\rho+\beta)e \in \mathbb{C}^d$, where $\beta \in \mathbb{R}$ is fixed, $\rho \in \mathbb{R}$ is arbitrarily large, and $e \in \mathbb{R}^d$. The idea behind the proof of the estimate (14) is the following. We will construct a family of operators R_ρ such that

$$R_\rho H(k)\Lambda_\rho^{-1} = I + T_\rho,$$

where

$$||T_\rho||_{L_2\to L_2} \to 0,\ \rho \to \infty$$

and

$$||R_\rho||_{L_2\to L_2} \leq const,\ \rho \to \infty.$$

This for large ρ would imply (14):

$$|\rho|\,||u||_{L_2} + ||u||_{H^1} \sim ||\Lambda_\rho(D)f||_{L_2} = \left|\left|(I+T_\rho)^{-1}R_\rho H(k)\Lambda_\rho^{-1}\Lambda_\rho(D)f\right|\right|_{L_2}$$

$$\leq C\,||H(k)f||_{L_2}.$$

Construction of the approximate inversion operators R_ρ is done locally and then pasted together using an appropriate partition of unity. Let us add some details. The symbol of the operator $H_0(k)$ is

$$H_0(k,m) = (2\pi m + k)^2 = 4\pi^2\left[(m+\beta e)^2 - \rho^2 + 2i\rho e\cdot(m+\beta e)\right].$$

We consider this as the principal symbol of the full operator $H(k)$. Notice that we include some lower order differential terms with parameter k into the principal symbol, which is rather standard when working with operators with parameters (see for instance [1]). The main difficulty is that the resulting problem with a parameter is not elliptic in the sense of [1]. The zero set of this symbol is

$$Z_\rho = \left\{m|\ (m+\beta e)^2 = \rho^2,\ e\cdot(m+\beta e) = 0\right\} \subset \mathbb{R}^d.$$

Now the situation looks differently away from the set Z_ρ and close to it. Namely, sufficiently far away from this set the principal part will dominate the magnetic one, hence an approximate inverse can be constructed by neglecting the magnetic part. One ends up using the estimates of Section 2. However, close to Z_ρ the magnetic potential part becomes of comparable strength with the principal one. Then one chooses a suitable finite covering of Z_ρ by sets of diameter ρ^{δ} with an appropriately chosen $\delta \in (0,1)$. In each of the open sets of this covering the operator $H(k)\Lambda_\rho^{-1}$ is split into two parts by linearizing its symbol at a point. The linear term is a model first order differential operator. It happens to be the generalized Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} + F(z) \tag{15}$$

on the complex plane, where the plane arises as a rational plane in $\mathbb{R}^d$, and the matrix function $F(z)$ is periodic and commutative. The remainder can be estimated away in the sense that approximate inversion of the model operator immediately leads to an approximate inverse for the whole sum. So, locally the problem is reduced to inverting the operator (15) on the torus. The nice trick that does exactly that was invented by A. Sobolev [65]. The idea is that under an appropriate choice of the vector e used above for defining the quasimomentum k, one can gauge away most of the magnetic potential a in (15). Hence the problem is reduced to the Cauchy-Riemann operator on the torus, which is an easily treatable case. As the result of this procedure, we have a set of local approximate inverse operators $R_{k,j}$ with suitable estimates. Then an appropriate partition of unity $\sum \phi_j = 1$ in the dual space finishes the job. Namely, let

$$\psi_j = \sum \phi_k,$$

where the sum runs over all values of k such that the supports of ϕ_j and ϕ_k intersect, and define the operator

$$R(k) = \sum_j \phi_j(k) R_{k,j} \psi_j(k).$$

It is constructed in such a way that

$$R(k)H(k)\Lambda_\rho^{-1} = I + T(k),$$

where

$$\lim_{\rho\to\infty} ||T(k)|| = 0. \tag{16}$$

This proves (14) and all the theorems stated in the previous section.

We claim that all types of periodic operators for which absolute continuity of the spectrum has been proven so far (Schrödinger, Dirac, Maxwell, etc.)

can be reduced to the case treated in Theorem 4. This will be shown in the Section 7. However, the case of a magnetic potential $\mathbf{a}$ not satisfying (13) is also of importance and still remains a challenge. We expect that the absolute continuity must also hold without assuming (13), see however discussion in Section 8.

5 Proof of the main estimate

We will now present the proof of Theorem 3.

Let us denote by H_0 the principal part $-\Delta$ of the operator $H = -\Delta + 2\mathbf{a}(x) \cdot D$ and by $H_0(m)$ its symbol $4\pi^2 m^2$. As before, the Fourier coefficients of a function f on the torus will be denoted by $\widehat{f}(m)$. We use the function $\Lambda(m, \rho) = (m^2 + \rho^2)^{1/2}$ introduced above and the corresponding (pseudo-differential of first order) operator $\Lambda_\rho(D)$ on the torus.

Let $n \in \mathbb{Z}^d$ be a primitive integer vector, i.e. such that $\lambda n \notin \mathbb{Z}^d$ for $\lambda \in (0,1)$. The particular choice of n will be specified later. We denote

$$e = \frac{n}{|n|} \in \mathbb{R}^d$$

and

$$\beta = \frac{1}{2\,|n|}.$$

Consider the quasimomentum $k = 2\pi(\beta + i\rho)e \in \mathbb{C}^d$, where $\rho > 0$. This choice of the quasimomentum guarantees invertibility of the operator $H_0(D+k)$ on the torus. This follows from the direct calculation of the symbol:

$$H_0(2\pi m + k) = (2\pi m + k)^2 = 4\pi^2 \left[(m + \beta e)^2 - \rho^2 + 2i\rho e \cdot (m + \beta e)\right].$$

Due to the specific choice of e, we now get

$$\begin{aligned} |H_0(2\pi m + k)| &\geq |Im H_0(2\pi m + k)| \\ &= 8\pi^2 \rho\, |n|^{-1} \,|n \cdot m + 0.5| \geq 4\pi^2 \rho\, |n|^{-1}. \end{aligned}$$

This proves invertibility. Besides, the norm in $L_2(\mathbb{T}^2)$ of the inverse operator is bounded from above by $|n|\,(4\pi^2\rho)^{-1}$.

The characteristic set, or the set of degeneration of the symbol is introduced as in the previous section:

$$Z_\rho = \left\{ m |\ (m + \beta e)^2 = \rho^2,\, e \cdot (m + \beta e) = 0 \right\}.$$

It can be represented as the intersection of the sphere

$$S_\rho = \left\{ m | \ (m + \beta e)^2 = \rho^2 \right\}$$

with the hyperplane

$$P = \{ m | \ e \cdot (m + \beta e) = 0 \} .$$

5.1 Inverse operator away from the characteristic set

Let us fix a number $\delta \in (0, 1/2)$. Consider the "good" set

$$U_{\rho,0} = \left\{ m | \ dist(m, Z_\rho) \geq \rho^\delta \right\} . \tag{17}$$

We call this set "good", since one can show that on this set the principal part dominates the magnetic terms. Namely, the following simple lemma holds.

Lemma 6 *There exist operators $R_{\rho,0}$ and $T_{\rho,0}$ such that for any function $\psi \in L_2(\mathbb{T}^d)$ with support of $\widehat{\psi}$ in $U_{\rho,0}$ one has*

$$R_{\rho,0}(k) H(k) \Lambda_\rho^{-1} \psi = \psi + T_{\rho,0} \psi,$$

where for $\rho \to \infty$ one has

$$||R_{\rho,0}||_{L_2 \to L_2} \leq C, \ \ ||T_{\rho,0}||_{L_2 \to L_2} \to 0. \tag{18}$$

Proof of the lemma. Consider the following representation:

$$H(k) \Lambda_\rho(D)^{-1} = H_0(k) \Lambda_\rho(D)^{-1} + A(k),$$

where

$$A(k) = 2\mathbf{a} \cdot (D + k) \Lambda_\rho(D)^{-1}.$$

Let us notice that the magnetic potential is bounded, so there is an uniform with respect to all parameters involved estimate

$$||A(k)||_{L_2 \to L_2} \leq const. \tag{19}$$

Let us now look at the constant coefficients part $H_0(D + k) \Lambda_\rho(D)^{-1}$. Its symbol is

$$4\pi^2 \left[(m + \beta e)^2 - \rho^2 + 2i\rho e \cdot (m + \beta e) \right] (m^2 + \rho^2)^{-1/2}.$$

We need to estimate this symbol from below on the set $U_{\rho,0}$. Pulling ρ out of the symbol, we reduce the problem to estimating from below the function

$$\left[q^2 - 1 + 2ie \cdot q\right] (q^2+1)^{-1/2}$$

when $q = \rho^{-1}(m + \beta e)$ is at a distance at least $\rho^{\delta-1}$ from the set Z_1. We claim that in this case

$$\left| \left[q^2 - 1 + 2ie \cdot q\right] (q^2+1)^{-1/2} \right| \geq C\rho^{\delta-1}. \tag{20}$$

Indeed, either q is at a distance at least $\rho^{\delta-1}$ from the unit sphere centered at the origin, or it is at a distance at least $\rho^{\delta-1}$ from the hyperplane $e \cdot q = 0$ being at the same time at a distance no more than $\rho^{\delta-1}$ from the unit sphere. In the first case the real part of our expression can be estimated from below by $\rho^{\delta-1}$. In order to show this, it is sufficient to rewrite the real part as

$$(|q| - 1)(|q| + 1)(q^2+1)^{-1/2},$$

which gives a lower bound of the kind we need. In the second case we will estimate the imaginary part. Namely, we know that $|q| \in (1 - \rho^{\delta-1}, 1 + \rho^{\delta-1})$ and $|e \cdot q| \geq \rho^{\delta-1}$. Then in the expression

$$2\, |e \cdot q|\, (q^2+1)^{-1/2}$$

the term $(q^2+1)^{-1/2}$ behaves as 1, and the rest is estimated from below by $\rho^{\delta-1}$. The estimate (20) implies that one can choose an operator $R_{\rho,0}$ as follows:

$$R_{\rho,0} = \left(H_0(k)\Lambda_\rho^{-1}\right)^{-1} \chi(D),$$

where χ is a function which is equal to 1 on $U_{\rho,0}$ and is supported in a slightly larger set of the same type. Then on functions ψ supported in $U_{\rho,0}$ one has

$$R_{\rho,0} H(k) \Lambda_\rho^{-1} \psi = \psi + T_{\rho,0}\psi,$$

where

$$T_{\rho,0} = \left(H_0(k)\Lambda_\rho^{-1}\right)^{-1} \chi A(k).$$

The estimates (18) now follow from (19) and (20). This proves the lemma.

5.2 Local behavior on the characteristic set

Now we have to deal with the "bad" set, i.e. the complement to $U_{\rho,0}$. Let us choose a point ξ on the characteristic set Z_ρ:

$$(\xi + \beta e)^2 = \rho^2, \; e \cdot (\xi + \beta e) = 0. \tag{21}$$

The conditions (21) are equivalent to the representation of the vector ξ in the form

$$\xi = -\beta e + \rho l, \tag{22}$$

where $l \in \mathbb{R}^d$ is a unit vector orthogonal to e. We will restrict consideration only to the rational points l, so the line tl contains an integer point. This reduces the choice of ξ to a dense subset in Z_ρ, which will be sufficient for our purpose. Let us also choose a neighborhood $V_{l,\rho}$ of the point ξ such that its diameter is bounded by $C\rho^\delta$ with an absolute constant C and a finite set of such neighborhoods covers the complement of $U_{\rho,0}$ with a finite (independent on ρ) multiplicity. The exact choice of these sets will be specified later. Consider the operator $H(k)\Lambda_\rho(D)^{-1}$. Its symbol is

$$h_k(m,x) = (m+k)^2(m^2+\rho^2)^{-1/2} + 2\mathbf{a}(x) \cdot (m+k)(m^2+\rho^2)^{-1/2}.$$

This operator acts as $Op(h_k)$. This means that in order to apply this operator to a periodic function one expands the function into the Fourier series (with m as the dual variable), then multiplies its terms by $h_k(m,x)$, and sums the series.

We are now going to linearize the symbol at the point $m = \xi$ in the neighborhood $V_{l,\rho}$.The linearization can be done by applying the Taylor expansion, or alternatively by a simple algebraic calculation that we are going to perform. In $V_{l,\rho}$ the function $(m^2+\rho^2)^{-1/2}$ behaves as $C\rho^{-1}$. Any point m in this neighborhood can be represented as $m = -\beta e + \rho l + \nu$, where $|\nu| \leq C\rho^\delta$. Using this representation and equality $k = (\beta + i\rho)e$, we get

$$(m+k)^2 + 2\mathbf{a}(x) \cdot (m+k)$$

$$= (\rho(ie+l)+\nu)^2 + 2(\rho(ie+l)+\nu) \cdot \mathbf{a}$$

$$= 2\rho(ie+l) \cdot (\nu + \mathbf{a}(x)) + \Phi = 2\rho(ie+l) \cdot (m + \beta e - \rho l + \mathbf{a}(x)) + \Phi,$$

where the symbol

$$\Phi(m,x) = \nu^2 + 2\nu \cdot \mathbf{a}$$

satisfies in $V_{l,\rho}$ the estimate

$$|\Phi(m,x)| \le C\rho^{2\delta}.$$

Hence, in the same neighborhood

$$\left|\Phi(m,x)(m^2+\rho^2)^{-1/2}\right| \le C\rho^{2\delta-1}.$$

Now the symbol $h_k(m,x)$ can be represented as

$$h_k(m,x) = \sqrt{2}(ie+l)\cdot(m+\beta e-\rho l+\mathbf{a}(x)) + \Psi(m,x),$$

where the symbol

$$\Psi = 2\rho(ie+l)\cdot(m+\beta e-\rho l+\mathbf{a}(x))\left((m^2+\rho^2)^{-1/2} - 1/\left(\sqrt{2}\rho\right)\right)+\Phi(m,x)(m^2+\rho^2)^{-1/2}$$

satisfies for large values of ρ the estimate

$$|\Psi(m,x)| \le C\rho^{2\delta-1}.$$

The constants in the last two estimates are absolute.

As the result, we can conclude that the operator $H(k)\Lambda_\delta(D)^{-1}$ can be represented on functions ψ whose Fourier coefficients are supported in $V_{l,\rho}$ as

$$H(k)\Lambda_\delta(D)^{-1} = \sqrt{2}H_\xi + \Psi, \tag{23}$$

where $H_\xi = Op(h_\xi)$ has the symbol

$$h_\xi(m) = (ie+l)\cdot(m+\beta e-\rho l+\mathbf{a}(x)) \tag{24}$$

and $\Psi = Op(\Psi(m,x))$ obeys the estimate

$$|\Psi(m,x)| \le C\rho^{2\delta-1} \tag{25}$$

on $V_{l,\rho}$. We remind the reader that here $\xi = -\beta e + \rho l$.

Let us notice that the symbol (24) corresponds to the first order differential operator on the torus of the form $\partial/\partial\overline{z} + F(z)$, if we identify the l-e plane with the complex plane $\mathbb{C}$. Here $F(z) = (ie+l)\cdot\mathbf{a}(x)$ is a periodic matrix function with zero average. This is the model operator that we will have to deal with.

Lemma 7 *The values of the matrix function $F = (ie + l) \cdot \mathbf{a}(x)$ commute.*

Proof of the lemma. Let us denote $(ie + l) = (s_1, ..., s_d)$. Then

$$(ie + l) \cdot \mathbf{a}(x) = \sum_j s_j a_j(x).$$

Now direct calculation shows that the condition (13) implies commutativity of these matrices for different values of x. Indeed,

$$[F(x), F(y)] = [\sum_j s_j a_j(x), \sum_k s_k a_k(y)] = \sum_{1 \le j,k \le d} s_j s_k [a_j(x), a_k(y)]$$

$$= \sum_{1 \le j \le d} s_j^2 [a_j(x), a_j(y)] + \sum_{1 \le j < k \le d} s_j s_k \left([a_j(x), a_k(y)] + [a_k(x), a_j(y)]\right).$$

Now all the terms in the last two sums are equal to zero due to (13).

The following lemma enables us to neglect the remainder term Ψ in (23) if δ is small enough.

Lemma 8 *There exists a constant C such that for any function v which has its Fourier series supported in $V_{l,\rho}$ the following estimate holds:*

$$||\Psi v||_{L_2} \le C \rho^{(d+4)\delta/2-1} ||v||_{L_2}. \tag{26}$$

In particular, if

$$\delta < 2/(d + 4), \tag{27}$$

then the operator norm of Ψ on the subspace of such functions tends to zero when $\rho \to \infty$.

The **proof of the lemma** is straightforward. Indeed,

$$|(\Psi v)(x)|^2 = \left| \sum_{m \in V_{l,\rho}} \Psi(m, x) \widehat{v}(m) e^{2\pi i m \cdot x} \right|^2.$$

Now the Cauchy-Schwarz inequality together with (25) and an estimate of the size of $V_{l.\rho}$ by $\rho^{d\delta}$ finish the job.

This lemma and expansion (23) show that it is sufficient for our purpose to invert the model operator H_ξ with uniform estimates on the norm of $(H_\xi)^{-1}$ on functions Fourier supported in $V_{l,\rho}$.

5.3 Inversion of the model operator

In this section our goal is to construct the inverse of the model inhomogeneous $\overline{\partial}$-operator described above and to get estimates for it. This will be used to construct a local approximate inverse operator in $V_{l,\rho}$ and then a global approximate inverse. In order to make the consideration better understandable, we will start with the simplest case and move toward the most general one. In each of these cases we will have to make an appropriate choice of vector e, which so far has been arbitrary.

5.3.1 Zero magnetic potential

In this case $\mathbf{a} = 0$ (which is the situation considered by L. Thomas [68]). Then the model operator (which we will denote by H_ξ^0) has constant coefficients. The imaginary part of its symbol

$$(ie + l) \cdot (m + \beta e - \rho l)$$

is equal to

$$e \cdot (m + \beta e)$$

(recall that vectors e and l are orthogonal). Let us now choose any nonzero vector $n \in \mathbb{Z}^d$ and denote $e = n/|n|$. We assume, as before that $\beta = 1/(2|n|)$. Then we have an estimate of the symbol from below:

$$\begin{aligned} |(ie + l) \cdot (m + \beta e - \rho l)| &\geq |e \cdot (m + \beta e)| \\ &= \frac{1}{|n|}\left| m \cdot n + \frac{1}{2}\right| \geq \frac{1}{2|n|}, \end{aligned}$$

and hence

$$\left|\left|H_\xi^0 \psi\right|\right|_{L_2} \geq \frac{1}{2|n|} \left|\left|\psi\right|\right|_{L_2}. \tag{28}$$

This means that we can use as an approximate inverse the operator $R_{l,\rho} = \frac{1}{\sqrt{2}}\left(H_\xi^0\right)^{-1}$ on functions with the Fourier support inside $V_{l,\rho}$. Indeed, we see that $||R_{l,\rho}|| \leq const$ and

$$\frac{1}{\sqrt{2}}\left(H_\xi^0\right)^{-1} H(D+k)\Lambda_\delta(D)^{-1} = \left(H_\xi^0\right)^{-1}\left(H_\xi^0 + \Psi\right) = I + \left(H_\xi^0\right)^{-1}\Psi.$$

Here the operator $T = R\Psi = \frac{1}{\sqrt{2}}\left(H_\xi^0\right)^{-1}\Psi$ obviously has the norm tending to zero for large values of ρ.

5.3.2 Small magnetic potential

Let us denote $||\mathbf{a}|| = \max_{x\in\mathbb{T}^d, j=1,...,d} ||a_j(x)||$, where $||a_j(x)||$ is the operator norm of the matrix $a_j(x)$ in the Euclidean space $\mathbb{C}^n$. We will assume that $||\mathbf{a}||$ is sufficiently small (it will be clear from what follows below how small it should be). This is the situation considered in [31]. Let us choose $n = e = (1, 0, ..., 0)$ and $\beta = 1/2$. We can consider our model operator $H_\xi + \Psi$ as a perturbation of the free one:

$$H_\xi + \Psi = H_\xi^0 + Op(2\pi(ie + l)\cdot \mathbf{a}) + \Psi.$$

If $||\mathbf{a}||$ is small enough and ρ is large enough, we conclude that

$$||T|| = \left|\left| \left(H_\xi^0\right)^{-1} \left(Op(2\pi(ie + l)\cdot \mathbf{a}) + \Psi\right)\right|\right| < 1,$$

and hence we can define the inverse operator

$$R_{l,\rho} = (I + T)^{-1} \left(H_\xi^0\right)^{-1}.$$

It is clear that uniform norm estimates from above hold for $R_{l,\rho}$.

5.3.3 2D case

The two dimensional case was settled in [8]. We will show here how it fits into the general scheme. Let us assume that $d = 2$. In this case the only vector $m \in \mathbb{R}^d$ such that $(ie + l)\cdot m = 0$ is $m = 0$. Since the magnetic potential has zero average and the values of $(ie + l)\cdot \mathbf{a}$ commute, one can gauge the magnetic potential away completely. Namely, let $\phi(x)$ be a commutative matrix function of $x \in \mathbb{R}^2$ periodic with respect to the lattice $\mathbb{Z}^2$ that also commutes with all values of $(ie+l)\cdot\mathbf{a}$. We can commute the model operator $H_\xi = (ie + l)\cdot(D + \beta e - \rho l) + (ie + l)\cdot \mathbf{a}$ with the exponent $e^{i\phi(x)}$:

$$e^{-i\phi(x)} \circ H_\xi \circ e^{i\phi(x)}.$$

This, due to commutativity of the values of all matrix functions involved produces the operator

$$(ie + l)\cdot(D + \beta e - \rho l) + (ie + l)\cdot(\mathbf{a} + \nabla\phi).$$

Assume now that we choose a function ϕ such that

$$(ie + l)\cdot\nabla\phi = -(ie + l)\cdot\mathbf{a}.$$

(One can easily identify this equality with an inhomogeneous $\overline{\partial}$-equation with respect to ϕ in the $e-l$ plane considered as the complex plane $\mathbb{C}$.) This is possible if $\mathbf{a}$ has zero average (which we do assume). Namely, the Fourier coefficients of the function ϕ can be found as

$$\widehat{\phi}(m) = -\frac{(ie+l)\cdot\widehat{\mathbf{a}}(m)}{(ie+l)\cdot m} \tag{29}$$

(we use the fact that $\widehat{\mathbf{a}}(0) = 0$). It is clear from the construction of $\phi(x)$ and commutativity of matrices $(ie+l)\cdot\widehat{\mathbf{a}}(m)$ that the matrix function ϕ is commutative and commutes with all matrices $(ie+l)\cdot\mathbf{a}(x)$. This enables us to apply the gauge transform as above, after which the magnetic potential goes away and we are again in the situation of the zero magnetic potential. In other words,

$$(H_\xi)^{-1} = e^{i\phi(x)}\left(H_\xi^0\right)^{-1} e^{-i\phi(x)}.$$

The formula (29) also provides an uniform (with respect to ρ and l) estimate

$$||\phi||_{L_\infty(\mathbb{T}^2)} \leq C_\varepsilon\, ||\mathbf{a}||_{H^\varepsilon(\mathbb{T}^2)}$$

for any positive ε. This gives uniform estimates on the L_2-norm of the gauge transform. Using now the considerations of the case when $\mathbf{a} = \mathbf{0}$, we end up with an uniform estimate from above for $R_{l,\rho} = (H_\xi)^{-1}$.

5.3.4 General case

The situation in dimensions higher than two (first considered in the scalar self-adjoint case in [65]) is definitely much trickier. One can see where the difficulty lies in comparison with $2D$. Namely, when $d > 2$, the division problem (29) cannot be resolved for arbitrary potential $\mathbf{a}$ with zero average. The reason is that the denominator in (29) has many zeros besides $m = 0$. This means that the magnetic potential cannot be gauged away completely. The clever trick invented by A. V. Sobolev is to use a gauge transform for "almost" eliminating the magnetic potential, rather than for its complete removal like it was done for $d = 2$. Let us describe this method in detail.

One needs to find an integer vector n (which will determine the vector $e = n/|n|$ and the number $\beta = 1/2\,|n|$ as before) and a large positive number L such that if we keep in the Fourier expansion of the magnetic potential $\mathbf{a}$ only the finite sum up to L, then the resulting approximation to the potential

$$\mathbf{a}_L(x) = \sum_{|m|\leq L} \widehat{\mathbf{a}}(m) e^{2\pi i m\cdot x} \tag{30}$$

can be gauged away according to (29). If additionally the norm of the remainder $\mathbf{a} - \mathbf{a}_L$ were sufficiently small, then we would have reduced the problem to the case of a small magnetic potential. So, now we need to find a vector $e = n/|n|$ and a number $L > 0$ such that

i) $(ie + l) \cdot m \neq 0$ for any rational vector l orthogonal to e and $m \neq 0$, $|m| < L$;

ii) $||\mathbf{a} - \mathbf{a}_L|| < \theta/|n|$, where θ is a given small constant. We would also like to have uniform control over the norm $||\mathbf{a}_L||_{H^\varepsilon}$.

If this is done, then we can find according to (29) a gauge transform with the phase $\phi(x)$ such that it eliminates the $\mathbf{a}_L$ part of the magnetic potential. Besides, $||\phi||_\infty \leq C(L, \mathbf{a})$, where $C(L, \mathbf{a})$ depends on L and $\mathbf{a}$ only (and in particular does not depend on ρ and l). This reduces the problem to the case of a small potential $\mathbf{a} - \mathbf{a}_L$, which was already treated before. Let us notice that "small" means less than $C\beta = \theta/|n|$ with some absolute constants C and θ. In other words, choosing a longer integer vector n we impose much stronger requirement on the part of the potential that must be gauged away.

Let us implement this program. The idea is that for an appropriately chosen integer direction n all integer vectors m that are orthogonal to n have a large norm. This means that the plane $e \cdot m = 0$ (and hence $(ie + l) \cdot m = 0$) hits the points of the dual lattice very far from the origin (i.e., for $|m| > L$ with a large L). If this is true, then the "tail" $\mathbf{a} - \mathbf{a}_L$ will be small. This requires, however, some estimates, which we are going to present. The first step is to discuss the relation between $|n|$ and L.

Lemma 9 *[65] For any integer $L > 0$ the equation*

$$m_1 + m_2 L + ... + m_d L^{d-1} = 0$$

has no non-zero integer solutions $m = (m_1, ..., m_d)$ with $|m_j| < L/2$. Moreover, if $m \neq 0$ and $|m_j| < L/2$, then the following estimate holds:

$$\left| m_1 + m_2 L + ... + m_d L^{d-1} \right| \geq 1. \tag{31}$$

Proof of the lemma. Let $m \neq 0$ and $|m_j| < L/2$. Let m_k $(k \geq 1)$ be the first non-zero component among $m_1, ..., m_d$. Then

$$\left| m_1 + m_2 L + ... + m_d L^{d-1} \right| = \left| m_k L^{k-1} + ... + m_d L^{d-1} \right|$$

$$= \left| L^k \left(\frac{m_k}{L} + \left(m_{k+1} + ... + m_d L^{d-k} \right) \right) \right| \geq L^k \frac{1}{L} = L^{k-1} \geq 1.$$

This proves both statements of the lemma.

We choose now the integer vector $n = (1, L, L^2, ..., L^{d-1})$, where L is a large integer L (note that $|n| \sim L^{d-1}$). Then the lemma above shows that all non-zero integer solutions of the equation $m \cdot n = 0$ lie outside the ball of radius $L/2 \sim C\,|n|^{1/(d-1)}$.

Consider the Fourier expansion of the magnetic potential:

$$\mathbf{a}(x) = \sum \widehat{\mathbf{a}}(m) e^{2\pi i m \cdot x}$$

and assume that $\sum ||\widehat{\mathbf{a}}(m)||\,|m|^{\mu} < \infty$ for some μ. Then

$$\sum_{|m|>L} ||\widehat{\mathbf{a}}(m)|| \leq L^{-\mu} \sum_{|m|>L} |m|^{\mu}\,||\widehat{\mathbf{a}}(m)|| = o(L^{-\mu}),\ L \to \infty.$$

If now $|n| = O(L^{\mu})$, we see that

$$|n| \sum_{|m|>L} ||\widehat{\mathbf{a}}(m)|| \to 0,\ L \to \infty.$$

Now the last lemma shows that if $\sum ||\widehat{\mathbf{a}}(m)||\,|m|^{d-1} < \infty$, in particular if $\mathbf{a} \in H^s(\mathbb{T}^d)$ for some $s > 3d/2 - 1$, our requirement ii) on smallness of the residual $\mathbf{a}_r = \mathbf{a} - \mathbf{a}_L$ is satisfied. Here $\mathbf{a}_L$ is defined as in (30).

Our goal now is to gauge away the part $(ie+l)\cdot\mathbf{a}_L$ of the potential in the model operator. This is possible if the vector $e = n/\,|n|$ is chosen as above. Namely, let us again look for a periodic commutative (and commuting with the matrices $(ie+l)\cdot\mathbf{a}$) matrix function $\phi(x)$ such that

$$(ie + l) \cdot \nabla\phi = -(ie + l) \cdot \mathbf{a}_L.$$

This is possible according to the Lemma 7 and formula (29), since we assume that $\widehat{\mathbf{a}}_L(0) = \widehat{\mathbf{a}}(0) = 0$ and that $\widehat{\mathbf{a}}_L(m) = 0$ whenever $(ie+l)\cdot m = 0$. Besides, the estimate (31) together with the definition of the vector e imply that

$$||\phi||_{L_\infty} \leq C_\varepsilon\,||\mathbf{a}||_{H^{d/2+\varepsilon}}$$

for any positive ε, where the constant C_ε does not depend on l and ρ (although it does depend on n).

We get now the representation

$$e^{-i\phi} H_\xi e^{i\phi} = H_\xi^0 + (ie + l) \cdot \mathbf{a}_r.$$

Choosing L sufficiently large, we achieve smallness of $\mathbf{a}_r$ and hence, as it was explained in the case of a small potential, invertibility of $H_\xi^0 + (ie + l) \cdot \mathbf{a}_r$ with a uniform estimate of the inverse. Then the operator

$$R_{l,\rho} = e^{i\phi} \left(H_\xi^0 + (ie + l) \cdot \mathbf{a}_r \right)^{-1} e^{-i\phi}$$

inverts the operator H_ξ and satisfies an uniform (with respect to ρ and l) estimate from above.

Let us summarize here that the maximal smoothness of $\mathbf{a}$ required so far in the case $d > 2$ is $\mathbf{a} \in H^s(\mathbb{T}^d)$ for some $s > 3d/2 - 1$.

5.4 Global inverse operator and completion of the proof

The last step of the proof requires to paste the local approximate inverse operators together, which is a rather standard procedure. Namely, let $\psi_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $\psi_0 \geq 0$, $\psi_0(\xi) = 1$ for $\xi \in [-0.5, 0.5]^d$, and $\psi_0(\xi) = 0$ for ξ outside $[-1,1]^d$. Now set $\psi_j(\xi) = \psi_0(\xi - j)$ for $j \in \mathbb{Z}^d$ and $\chi_j = \psi_j \left(\sum \psi_i\right)^{-1}$. Then $\{\chi_j\}$ forms a unity partition, diameter of the support of χ_j does not exceed $2\sqrt{d}$, and the multiplicity of the covering of the whole space by the supports of these functions is finite.

Let us now fix $\delta \in (0, 2/(d+4))$ and define the stretched functions

$$\chi_{\rho j}^1(\xi) = \chi_j(\rho^{-\delta}\xi).$$

This new set of functions still forms a unity partition and besides

$$\left| D^\alpha \chi_{\rho j}^1 \right| \leq C_\alpha \rho^{-\delta|\alpha|}. \tag{32}$$

The support of $\chi_{\rho j}^1$ belongs to the cube

$$Q_{\rho j} = \{\xi |\ \rho^{-\delta}\xi - j \in [-1,1]^d\}$$

Now we need a set of cut-off functions with a larger support. Namely, we define

$$\chi_{\rho j}^2 = \sum_s \chi_{\rho s}^1,$$

where the sum is taken over all values of s for which the supports of functions $\chi_{\rho s}^1$ and $\chi_{\rho j}^1$ intersect. We obviously have estimates for $\chi_{\rho j}^2$ similar to those for $\chi_{\rho j}^1$. Besides,

$$\chi_{\rho j}^1 \chi_{\rho j}^2 = \chi_{\rho j}^1.$$

Consider the set of all values of j such that the support of $\chi^2_{\rho j}$ is disjoint with the characteristic set Z_ρ. Then obviously the supports of the corresponding $\chi^1_{\rho j}$of belong to the set $U_{\rho,0}$ as defined in Section 5.1. Hence, according to that section, in each of the corresponding cubes $Q_{\rho j}$ there exists an operator $R_{\rho j}$ such that $||R_{\rho j}|| < C$ uniformly with respect to ρ,

$$R_{\rho j} H(k) \Lambda_\rho^{-1} \psi = \psi + T_{\rho j} \psi$$

on functions ψ whose Fourier series is supported in $Q_{\rho j}$, and

$$||T_{\rho j}|| \to 0, \ \ \rho \to \infty$$

uniformly in ρ.

Consider now the values of j for which the support of $\chi^2_{\rho j}$ does touch the characteristic set. Then, according to the considerations of the previous section, on each of the corresponding cubes $Q_{\rho j}$ there also exist operators $R_{\rho j}$ and $T_{\rho j}$ with the same properties.

We are ready to construct an approximate inverse for the whole operator $H(k)\Lambda_\rho^{-1}$. We define it as follows:

$$R_\rho = \sum_j \chi^1_{\rho j} R_{\rho j} \chi^2_{\rho j},$$

where we denote the multiplier of the Fourier series by the function $\chi^1_{\rho j}$ by the same expression $\chi^1_{\rho j}$. Let us show that for large values of ρ this is indeed an approximate inverse for $H(k)$.

Lemma 10 *For sufficiently large values of ρ the following relation holds:*

$$R_\rho H(k) \Lambda_\rho^{-1} = I + T_\rho,$$

where $||T_\rho||_{L_2 \to L_2} < 1$.

As it was explained before, this would prove the Theorem 3 and the consequent results.

Proof of the lemma. Let us compute the product $R_\rho H(k) \Lambda_\delta^{-1}$:

$$R_\rho H(k) \Lambda_\rho^{-1} = \sum_j \chi^1_{\rho j} R_{\rho j} \chi^2_{\rho j} H(k) \Lambda_\rho^{-1}$$

$$= \sum_j \chi^1_{\rho j} R_{\rho j} H(k) \Lambda_\rho^{-1} \chi^2_{\rho j} + \sum_j \chi^1_{\rho j} R_{\rho j} \left[\chi^2_{\rho j}, H(k) \Lambda_\rho^{-1} \right].$$

Here $[A, B]$ denotes the commutator of the operators A and B. The first sum in this expression is

$$\left(\sum_j \chi^1_{\rho j} \right) I + \sum_j \chi^1_{\rho j} T_{\rho j} \chi^2_{\rho j} = I + T_\rho.$$

Due to finiteness of the multiplicity of the covering by the supports of $\chi^2_{\rho j}$, we conclude that

$$||T_\rho||_{L_2 \to L_2} \leq C \max_j ||T_{\rho j}||_{L_2 \to L_2} \to 0$$

when $\rho \to \infty$. So, the only remaining task is to show that

$$\left|\left| \left[\chi^2_{\rho j}, H(k) \Lambda_\rho^{-1} \right] \right|\right|_{L_2 \to L_2} \to 0$$

uniformly with respect to j when $\rho \to \infty$. Let us compute the commutator explicitly, taking into account that

$$H(k) = (D + k)^2 + 2\mathbf{a} \cdot (D + k).$$

The second order term $(D + k)^2$ has constant coefficients, and so commutes with $\chi^2_{\rho j}$, which is itself a constant coefficient operator. Hence the commutator can be expressed as follows:

$$2[\chi^2_{\rho j}, \mathbf{a}] \cdot (D + k) \Lambda_\rho^{-1}.$$

The operator $(D + k)\Lambda_\rho^{-1}$ is obviously bounded in L_2 uniformly with respect to ρ. So, the question is whether

$$\left|\left| [\chi^2_{\rho j}, \mathbf{a}] \right|\right|_{L_2 \to L_2} \to 0$$

uniformly with respect to j when $\rho \to \infty$. Consider the block matrix of this operator in the basis of harmonics $e^{2\pi i m \cdot x}$. The matrix of the operator of multiplication by $\mathbf{a}$ has entries $\{\widehat{\mathbf{a}}_{m-m'}\}$, where as before $\widehat{\mathbf{a}}_m$ are the Fourier coefficients of $\mathbf{a}$. Then the matrix of $B = [\chi^2_{\rho j}, \mathbf{a}]$ is

$$b_{m,m'} = \left(\chi^2_{\rho j}(m) - \chi^2_{\rho j}(m') \right) \widehat{\mathbf{a}}_{m-m'} = \nabla \chi^2_{\rho j}(m' + \theta(m - m')) \cdot (m - m') \widehat{\mathbf{a}}_{m-m'}.$$

Using the estimate (32), we conclude that

$$||b_{m,m'}|| \leq C\rho^{-\delta} \, |m - m'| \, ||\widehat{\mathbf{a}}_{m-m'}|| \, .$$

If now we assume that

$$\sum_m |m| \, ||\widehat{\mathbf{a}}_m|| < \infty, \tag{33}$$

this would lead to the estimate of l_1- and l_∞- operator norms of the matrix $B = \{b_{m,m'}\}$ from above by $C\rho^{-\delta}$. This, by interpolation, implies that its l_2-norm is estimated by the same number, and hence uniformly with respect to j tends to zero when $\rho \to \infty$. The estimate (33), in turn, follows for instance from the assumption that $\mathbf{a} \in H^{d/2+1+\varepsilon}$.

6 Inclusion of an electric potential

In this section we will explain how to derive Theorem 4 from Theorem 3. Assume that V belongs to the space $L_{p,loc}(\mathbb{R}^d)$ with $p = 2$ for $d = 2$ and $p > d - 1$ for $d \geq 3$. The way one can include the electric potential under these conditions is rather standard (see the proofs of Theorem XIII.10 in [54], Theorem 4.1.8 in [37], and Sections 3.5 and 7.2 in [65]). Namely, Young and Hölder inequalities easily lead to the following estimate between the electric potential and free terms:

$$||Vu||_{L^t(\mathbb{T}^d)} \leq C\rho^{-1+(d-1)/p} \, ||H(k,0,0)u||_{L^t(\mathbf{T}^d)} \tag{34}$$

for any $u \in H^2(\mathbb{T}^d)$ (see the simple details in the proofs of Theorem 4.1.8 in [37] and Lemma 7.2 in [65]). Besides, one has an obvious estimate

$$||(H(k,0,0) - H(k,\mathbf{a},0)) \, u||_{L^t(\mathbf{T}^d)} \leq C \, ||\Lambda_\rho(D)u||_{L^t(\mathbf{T}^d)} \, .$$

This inequality, (34), (14), and conditions on p imply that

$$\begin{aligned}
||Vu||_{L^t(\mathbb{T}^d)} &\leq \varepsilon \, ||H(k,0,0)u||_{L^t(\mathbf{T}^d)} \\
&\leq \varepsilon \left(||H(k,\mathbf{a},0)u||_{L^t(\mathbf{T}^d)} + ||(H(k,0,0) - H(k,\mathbf{a},0)) \, u||_{L^t(\mathbf{T}^d)} \right) \\
&\leq \varepsilon \left(||H(k,\mathbf{a},0)u||_{L^t(\mathbf{T}^d)} + C \, ||\Lambda_\rho(D)u||_{L^t(\mathbf{T}^d)} \right) \\
&\leq C\varepsilon \, ||H(k,\mathbf{a},0)u||_{L^t(\mathbf{T}^d)} \, ,
\end{aligned}$$

where $\varepsilon = C\rho^{-1+(d-1)/p}$ tends to zero when ρ goes to infinity. This implies that for large ρ the equation

$$H(k, \mathbf{a}, V)u = H(k, \mathbf{a}, 0)u + Vu = 0$$

has no non-zero solutions on the torus, which concludes the proof of the theorem.

As we already remarked before, it is possible to weaken the conditions on the electric potential V to those as in [12]. This, however, requires a more delicate analytic technique rather than usage of (34). This method is described in [12], and we will not present it here.

7 Other types of operators

In this section we will show that all the types of periodic operators for which absolute continuity has been proven, in spite of their seemingly different appearance, can be reduced to the form (11). Thus, Theorem 4 applies. Speaking of the **types** of operators, we neglect the question about the optimal conditions on the coefficients. In some of the results that we quote these conditions are more relaxed than ours. We did not pursue the goal of obtaining optimal classes of coefficients for which our methods and results would hold.

The following remark is due. With the exception of the trivial case of an isotropic operator (7), the only situations when the absolute continuity has been proven for an operator with variable coefficients in the leading terms, where the ones considered in [47] and [48]. There the result was established for the Schrödinger operator with variable periodic anisotropic metric in $2D$ and for the Maxwell operator in a periodic isotropic medium. Albeit these operators appear to have truly variable coefficients in the leading terms, it will be shown below, that in fact they also boil down to (11). In other words, there has not been any true success achieved yet for the variable coefficient case. This is not to diminish the role of the remarkable papers [47] and [48], which not only contained important new results, but also introduced some innovative analytic tricks that might be useful in more general situations.

7.1 Schrödinger operator

The standard scalar periodic Schrödinger operator

$$(D-A)^2+V=\sum\left(\frac{1}{i}\frac{\partial}{\partial x_j}-a_j(x)\right)^2+V(x)$$

is obviously a particular case of (11) for $n=1$. Hence, the results of Theorems 4 and 5 apply.

7.2 Dirac operator

After the absolute continuity of the spectra of the Schrödinger operator with a periodic electric potential was established in [68], the similar result was obtained for the periodic Dirac operator in [16]–[19]. This study of the Dirac operator was continued in [9], [13], and [20]–[24].

Let us recall what the periodic Dirac operator in $\mathbb{R}^d$ is. Let α_j $(j=1,2,...,d)$ be Hermitian $m\times m$ matrices for some $m\in\mathbb{N}$ satisfying the relations

$$\alpha_j\alpha_l+\alpha_l\alpha_j=2\delta_{jl}I, \tag{35}$$

where δ_{jl} is the Kronecker delta, and I is the unit $m\times m$ matrix. Consider the operator

$$\mathcal{D}=\frac{1}{i}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\alpha_j+V(x)$$

acting on vector-functions from $\mathbb{R}^d$ to $\mathbb{C}^m$. Here $V(x)$ is a function with values in $m\times m$ matrices periodic with respect to a lattice Γ in $\mathbb{R}^d$. Let us assume that $V(x)$ belongs to $L_{r,loc}$ with $r>2$ when $d=2$ and V is continuous in higher dimensions. If additionally V is self-adjoint, then the operator $\mathcal{D}$ is self-adjoint in $L_2(\mathbb{R}^d)$ with the domain $H^1(\mathbb{R}^d)$.

Conjecture 11 *The periodic self-adjoint Dirac operator $\mathcal{D}$ has absolutely continuous spectrum.*

This conjecture has not been proven in full generality even for smooth matrix potentials V. As it was explained before, in order to establish absolute continuity, it is sufficient to prove absence of eigenvalues. This must hold also in the non-selfadjoint case.

Conjecture 12 *Without the assumption of self-adjointness of the potential V the periodic Dirac operator still has no L_2-eigenfunctions.*

We will now explain under what condition one can reduce study of this problem to the one for an operator (11) and hence prove absolute continuity. We will show then that all cases when the conjecture has been proven fall into this category. We will do this first without paying attention to the conditions on the potential, At this stage the reader may think that V is of the class C^1. The requirements on V will be specified a little bit later.

Assume that there exists an L_2-eigenfunction u of the Dirac operator, i.e.

$$\mathcal{D}u = \frac{1}{i}\sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\alpha_j + Vu = \lambda u. \tag{36}$$

Ellipticity enables is to conclude that then $u \in H^2$. This leads to the equality

$$\mathcal{D}^2 u = \lambda^2 u,$$

or

$$-\Delta u - i\sum_j \left(V(x)\alpha_j + \alpha_j V(x)\right)\frac{\partial u}{\partial x_j} + \left(V^2 - i\sum_j \alpha_j \frac{\partial V}{\partial x_j}\right) u = \lambda^2 u.$$

It is clear that we are in the situation of an operator (11). However, Theorems 3 and 4 require the "magnetic potential" matrices $a_j(x) = V(x)\alpha_j + \alpha_j V(x)$ to satisfy (13). This leads, according to Theorems 4 and 5 to the following statement:

Theorem 13 *Assume that*

1. *The average values of the matrix functions $a_j(x) = V(x)\alpha_j + \alpha_j V(x)$, $j = 1, ..., 2$ are equal to zero.*

2. *The matrices $V(x)\alpha_j + \alpha_j V(x)$ for different values of x and j satisfy the commutation relation (13).*

3. *$(V(x)\alpha_j + \alpha_j V(x)) \in H^s_{loc}(\mathbb{R}^d)$ with $s > \frac{3}{2}d - 1$.*

4. *$\left(V^2 - \sum_j \alpha_j \frac{\partial V}{\partial x_j}\right) \in L_{p,loc}(\mathbb{R}^d)$ with $p = 2$ for $d = 2$ and $p > d - 1$ for $d > 2$.*

Then the operator (36) has no L_2-eigenfunctions. In particular, in the self-adjoint case (i.e., when the matrix $V(x)$ is self-adjoint) the spectrum is absolutely continuous.

The main (and probably unnecessary) restriction is the commutation condition (13) on the matrices $(V(x)\alpha_j + \alpha_j V(x))$. We want to point out, however, that in all cases known to the authors when absolute continuity of the periodic Dirac operator has been proven so far, this condition is satisfied. Indeed, all papers [9], [13], and [16]-[24] assume a very specific types of the potentials. Namely, it is assumed that the operator has the following form:

$$\sum_{j=1}^{d} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - c_j \right) \alpha_j + V_0 \alpha_{d+1} + V_1 I,$$

where c_j, V_0, and V_1 are scalar functions, and the family $\{\alpha_1, ..., \alpha_{d+1}\}$ satisfies the same anti-commutativity condition (35) that has been assumed so far for $\{\alpha_1, ..., \alpha_d\}$. In this case

$$V(x) = -\sum_{j=1}^{d} c_j \alpha_j + V_0 \alpha_{d+1} + V_1 I \tag{37}$$

and hence for $l \le d$

$$2a_l(x) = (V(x)\alpha_l + \alpha_l V(x))$$
$$= -\sum_{j=1}^{d} c_j(x)\alpha_j\alpha_l + V_0(x)\alpha_{d+1}\alpha_l + V_1(x)\alpha_l - \sum_{j=1}^{d} c_j\alpha_l\alpha_j + V_0\alpha_l\alpha_{d+1} + V_1\alpha_l$$
$$= -2c_l(x)I + 2V_1(x)\alpha_l.$$

Then

$$[a_l(x), a_j(y)] + [a_j(x), a_l(y)] = V_1(x)V_1(y)\left([\alpha_l, \alpha_j] + [\alpha_j, \alpha_l]\right) = 0,$$

so the condition (13) is satisfied.

In some cases [24] potentials of a little bit more general type than (37) are considered:

$$V(x) = -\sum_{j=1}^{d} c_j \alpha_j + V_0 + V_1,$$

where the matrix function V_0 commutes and V_1 anti-commutes with the matrices $\alpha_1, ...\alpha_d$. One can check as in the calculation above that (13) still holds.

7.3 2D operators with variable metric and global isothermal coordinates

Most of the known results on absolute continuity of periodic elliptic operators deal with the operators with constant coefficients in the principal part (modulo the isotropic operator (7), where the metric conformal to the flat one is eliminated by the substitution (8)). The only exceptions were the innovative papers [47] and [48] by A. Morame. We will consider the operators treated in these papers in this and the next sub-sections in order to show that they in fact boil down to a Schrödinger operator (11), and hence belong to the realm of the operators with the constant coefficient leading terms.

The paper [47] contains the proof of absolute continuity of the spectrum of the periodic Schrödinger operator with variable metric in $2D$:

$$\sum_{1\leq j,k\leq 2} |g|^{-1/2}\left(D_{x_j}-a_j(x)\right) g^{j,k}(x)\,|g|^{1/2}\left(D_{x_k}-a_k(x)\right)+V(x),\ x\in\mathbb{R}^2.$$

Here the metric $g(x)$ (a positive definite matrix function) is assumed to be periodic with respect to a lattice Γ and of the class C^∞. The magnetic potential a is Γ-periodic and belongs to C^∞, and the electric potential V is Γ-periodic and belongs to L_∞. The operator, in order to be self-adjoint, needs to be considered in the appropriate L_2-space, namely in $L_2(\mathbb{R}^2;|g|^{1/2}\,dx)$.

We will show here that even under much milder conditions on the coefficients this operator can be reduced to a scalar periodic Schrödinger operator. Existence of such a reduction does not diminish the relevance of the result and its proof provided in [47]. Besides being the first paper where the absolute continuity problem in presence of variable coefficient second order terms was successfully treated, [47] also contains new ingenious analytic tricks that might be useful in other cases.

Consider the following more general second order periodic operator in $2D$:

$$\begin{aligned} H &= D^*g(x)D+a(x)\cdot\nabla+V(x)\\ &= -\sum_{1\leq j,k\leq 2}\frac{\partial}{\partial x_j}g_{j,k}(x)\frac{\partial}{\partial x_k}+\sum_{1\leq j\leq 2}a_j(x)\frac{\partial}{\partial x_j}+V(x), \end{aligned} \tag{38}$$

where we do not yet specify conditions on the coefficients g, a, and V except that g is a positive matrix function, and all the coefficients are periodic with respect to a lattice Γ in $\mathbb{R}^2$. Let first $g(x)$ be a scalar, i.e. $g(x)=b(x)I_2$

where $b(x)$ is a sufficiently smooth positive scalar function and I_2 is the 2×2 unit matrix. We call such an operator isotropic. Then, as it was explained in the introduction, the problem of absolute continuity of the spectrum of such an operator can be transformed to the same problem for the Schrödinger operator $b^{-1/2} H \circ b^{-1/2}$. Our goal in this section is to show that in $2D$ one can always transform the operator H to an isotropic one by a periodic change of coordinates, and hence reduce the absolute continuity problem to the well studied Schrödinger case.

First of all, the reader is probably aware of existence of local coordinate changes that do this. Such coordinates are called isothermal or conformal and are used for local study of surfaces in $\mathbb{R}^3$ (e.g., Section 13 of Ch. 2 in [25] and p. 376 in [67]), for reduction of second order elliptic PDEs in two variables to the canonical form ([15]), and in complex analysis ([2], [69]). While [25] and [67] assume for simplicity significant smoothness (and even analyticity) of the metric g, one can find in [2], [15], and [69] mild conditions under which such coordinates exist.

Lemma 14 *($8 in Ch. IV of [15] and [69]). Let m be a non-negative integer and $\alpha \in (0,1)$. Assume that the entries of the matrix function $g(x)$ in (38) belong to $C^{m+\alpha}(\mathbb{R}^2)$. Then in a neighborhood of any point $x_0 \in \mathbb{R}^2$ there exists a change of variables $y(x) \in C^{m+1+\alpha}(\mathbb{R}^2)$ that transforms the operator (38) to a one with the leading term*

$$-\nabla \cdot \sigma(y) \nabla$$

with a scalar function σ that belongs to $C^{m+\alpha}$ in a neighborhood of $y(x_0)$.

So, locally the anisotropy of the operator can be eliminated by switching to the isothermal variables y. The interesting observation is that such variables can be chosen globally for a periodic operator. Before going into the details, we want to mention that it was pointed to the first author by L. Friedlander and S. Novikov that the uniformization theorem for Riemann surfaces (see for instance Section 11G in Ch. III of [3]) implies existence of global periodic isothermal coordinates that do the reduction. Namely, the following statement holds:

Proposition 15 *Under the same conditions on the matrix $g(x)$ as in Lemma 14, there exists a global change of variables $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the* *operator into an isotropic one and such that $\Psi(0) = 0$ and for any integer vector $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$*

$$\Psi(x + \mathbf{n}) = \Psi(x) + n_1\Psi((1, 0)) + n_2\Psi((0, 1)). \tag{39}$$

The smoothness of the global change Ψ is the same as of the local one.

The equality (39) means that Ψ transforms the original $\mathbb{Z}^2$-periodic operator (38) into an isotropic one which is periodic with respect to the lattice generated by the basis vectors $\Psi((1, 0))$ and $\Psi((0, 1))$. In particular, the operator is transformed into the one for which the absolute continuity problem can be resolved by reduction to the Schrödinger case. This implies the result of [47] under much milder conditions on the coefficients than the ones in [47] (see for instance discussion in [14]).

Proof of Proposition 15. The local isothermal coordinates that exist according to Lemma 14 can be chosen orientation preserving. They provide an atlas of local coordinate charts on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We can think that they map local coordinate patches onto the unit disk in $\mathbb{C}$ (identified with $\mathbb{R}^2$). Then transforms between the intersecting local charts are orientation preserving and conformal, and hence holomorphic. This introduces a structure of a Riemannian surface $\mathcal{R}$ on $\mathbb{T}^2$. Since $\mathcal{R}$ is a compact Riemannian surface of genus $g = 1$, there exists unique up to a constant factor holomorphic differential (differential 1-form) ω, since the dimension of the vector space of such differentials is equal to the genus (e.g., Theorem 10-3 in [62] and Section 24A in Ch. V of [3]). Holomorphy means that in the local isothermal coordinates x, y the differential looks like $f(z)dz$ with a holomorphic function $f(z)$, where $z = x + iy$. This differential has no zeros (i.e., $f(z)$ has no zeros in each local chart), since otherwise we could create by shifts zeros at an arbitrary point, which would contradict to the one-dimensionality of the space of such differentials. Let us consider the covering mapping

$$\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathcal{R}$$

and pull back this differential ω to a differential Ω on the whole $\mathbb{R}^2$. Let us also fix a point (for instance, the origin 0) in $\mathbb{R}^2$. Then integrating Ω from this point to a variable point x in the plane, we get the mapping

$$\Psi(x) = \int_0^x \Omega$$

from $\mathbb{R}^2$ into the complex plane. It is easy to see that this mapping is a homeomorphism of $\mathbb{R}^2$. It maps the lattice of periods isomorphically onto another lattice and gives the global periodic isothermal coordinates. Notice that due to analyticity of the differential there was no loss of smoothness in the process of moving from local to global isothermal coordinates. This finishes the proof.

7.4 Maxwell operator

Consider a $3D$ medium described by positive electric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$. Both functions are scalar (thus the medium is isotropic), periodic with respect to a lattice Γ in $\mathbb{R}^3$, measurable, and bounded away from zero and infinity:

$$0 < c < \varepsilon(x), \mu(x) < C < \infty.$$

Later on in this section it will be assumed that these functions are of the class C^1. We will denote by $\mathbf{E}(x)$ and $\mathbf{H}(x)$ the electric and magnetic fields. Consider the Hilbert space $\mathcal{L} = L_2(\mathbb{R}^3; \mathbb{C}^3, \varepsilon) \oplus L_2(\mathbb{R}^3; \mathbb{C}^3, \mu)$ of pairs $(\mathbf{E}, \mathbf{H})$. Here $L_2(\mathbb{R}^3; \mathbb{C}^3, s)$ denotes the L_2-space with respect to the measure $s\,dx$ of vector functions on $\mathbb{R}^3$ with values in $\mathbb{C}^3$. The subspace $L \subset \mathcal{L}$ consists of pairs $(\mathbf{E}, \mathbf{H})$ that satisfy the zero divergence conditions

$$\nabla \cdot \mu\mathbf{H} = \mathbf{0},\ \nabla \cdot \varepsilon\mathbf{E} = \mathbf{0}. \tag{40}$$

The Maxwell operator M acts in L as follows:

$$M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & i\frac{1}{\varepsilon}\nabla \times \mathbf{H} \\ -i\frac{1}{\mu}\nabla \times \mathbf{E} & 0 \end{pmatrix}, \tag{41}$$

with the domain consisting of fields $(\mathbf{E}, \mathbf{H})$ such $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ belong to $L_2(\mathbb{R}^3; \mathbb{C}^3)$. This operator is self-adjoint [7]. The reader can find the spectral theory of Maxwell operator developed in high generality in [7].

It is believed that the spectrum of the Maxwell operator M in a periodic medium is absolutely continuous. This question is relevant, for instance for the theory of photonic crystals [38]. This belief was justified for the case of C^∞ functions ε and μ in [48]. The goal of this section is to show that the essence of what was done in [48] boils down to a reduction of the problem to a non-self-adjoint matrix Schrödinger operator of the type (11), so then Theorem 4 applies. This was noticed by T. Suslina [63].

The crucial observation of the paper [48] was the following statement that can be verified by a direct calculation:

Lemma 16 *Let the functions ε and μ be of the class C^2 and $\mathbf{u} \in H^2_{loc}(\mathbb{R}^3, \mathbb{C}^3)$ be such that $\nabla \cdot \varepsilon \mathbf{u} = \mathbf{0}$. Then*

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{u} = -\frac{1}{\mu\sqrt{\varepsilon}} \Delta\left(\sqrt{\varepsilon}\mathbf{u}\right) + \left(\varepsilon \nabla \left(\frac{1}{\mu\varepsilon}\right)\right) \times (\nabla \times \mathbf{u}) + W\mathbf{u}, \qquad (42)$$

where W is a 3×3 matrix, whose entries are expressible in terms of ε and μ, and the equality holds in the distributional sense.

Suppose now that we have an L_2-eigenfunction $(\mathbf{E}, \mathbf{H})$ of the Maxwell operator. This means that

$$\begin{cases} i\frac{1}{\varepsilon}\nabla \times \mathbf{H} & = & \lambda \mathbf{E} \\ -i\frac{1}{\mu}\nabla \times \mathbf{E} & = & \lambda \mathbf{H} \end{cases}. \qquad (43)$$

In this case further application of the *curl* operator is possible, which leads to

$$\begin{cases} \nabla \times \frac{1}{\mu}\nabla \times \mathbf{E} = & i\lambda \nabla \times \mathbf{H} = & \lambda\varepsilon\mathbf{E} \\ \nabla \times \frac{1}{\varepsilon}\nabla \times \mathbf{H} = & -i\lambda\nabla \times \mathbf{E} = & -\lambda\mu\mathbf{H} \\ \nabla \cdot \varepsilon\mathbf{E} & = \mathbf{0} & \\ \nabla \cdot \mu\mathbf{H} & = \mathbf{0} & \end{cases}. \qquad (44)$$

Using now the Lemma for the first two equations, we conclude that

$$\begin{cases} -\frac{1}{\mu\sqrt{\varepsilon}}\Delta\left(\sqrt{\varepsilon}\mathbf{E}\right) + \left(\varepsilon\nabla\left(\frac{1}{\mu\varepsilon}\right)\right) \times (\nabla \times \mathbf{E}) + W_1\mathbf{E} - \lambda\varepsilon\mathbf{E} & = & 0 \\ -\frac{1}{\varepsilon\sqrt{\mu}}\Delta\left(\sqrt{\mu}\mathbf{H}\right) + \left(\mu\nabla\left(\frac{1}{\mu\varepsilon}\right)\right) \times (\nabla \times \mathbf{H}) + W_2\mathbf{H} + \lambda\mu\mathbf{H} & = & 0 \end{cases},$$

where W_j are 3×3 matrices. Using the equations (43) again to eliminate the *curl* terms, we obtain the system

$$\begin{cases} -\frac{1}{\mu\sqrt{\varepsilon}}\Delta\left(\sqrt{\varepsilon}\mathbf{E}\right) + i\lambda\mu\left(\varepsilon\nabla\left(\frac{1}{\mu\varepsilon}\right)\right) \times \mathbf{H} + W_1\mathbf{E} - \lambda\varepsilon\mathbf{E} & = & 0 \\ -\frac{1}{\varepsilon\sqrt{\mu}}\Delta\left(\sqrt{\mu}\mathbf{H}\right) - i\lambda\varepsilon\left(\mu\nabla\left(\frac{1}{\mu\varepsilon}\right)\right) \times \mathbf{E} + W_2\mathbf{H} + \lambda\mu\mathbf{H} & = & 0 \end{cases}. \qquad (45)$$

Let us introduce a new six-dimensional vector-function $\mathbf{v} = (\sqrt{\varepsilon}\mathbf{E}, \sqrt{\mu}\mathbf{H})$. It belongs to L_2, due to the assumptions that $(\mathbf{E}, \mathbf{H})$ is in L_2 and the functions ε and μ are bounded. For this vector-function we rewrite (45) as the equation

$$-\Delta\mathbf{v} + V\mathbf{v} = 0, \qquad (46)$$

where V is a 6×6 periodic matrix-function with bounded entries obtained from ε, μ, and λ according to the above procedure. This equation is an eigenvalue problem for the matrix operator $-\Delta + V$, which is obviously of the type (11). This completes our reduction. Notice that the matrix V is in general not self-adjoint, in spite the fact that we have started with a self-adjoint operator $\dot{M}$. According to Theorem 4, this is irrelevant, and (46) has no non-zero L_2-solutions.

Theorem 17 *Under the conditions stated in this section, the Maxwell operator M has absolutely continuous spectrum.*

8 Concluding remarks

1. It remains a challenge to prove the absolute continuity of the spectrum of the general second order periodic elliptic operator

$$-\sum_{1\leq j,k\leq 2} \frac{\partial}{\partial x_j} g_{j,k}(x) \frac{\partial}{\partial x_k} + \sum_{1\leq j\leq 2} a_j(x) \frac{\partial}{\partial x_j} + V(x)$$

   in dimensions higher than two even for smooth metric $\{g_{j,k}(x)\}$. The trick with isothermal coordinates works only in $2D$. Similarly, absolute continuity has not been proven for the periodic Maxwell operator in anisotropic media, i.e. when the material tensors $\varepsilon(x)$ and $\mu(x)$ are matrices rather than scalars. One can assume arbitrary smoothness of ε and μ. It looks like new approaches are needed in order to treat these problems. One of them could be to obtain appropriate theorems on allowed rate of decay of solutions that generalize those of [30] and [46]. If one could prohibit super-exponential decay of eigenfunctions, then results of [36], [37] would imply the absolute continuity (see the corresponding discussion in the Introduction). The other obvious possibility is to try to push through the Thomas' approach. This, however, might happen to be too difficult in the presence of variable coefficients in the leading terms. As this paper makes clear, all known results essentially deal with the constant coefficient principal part only.

2. In the previous remark we listed the problems that remain unresolved in the case of smooth coefficients. For some other operators, the sufficiently smooth coefficients do not pose a challenge anymore, but re-

ducing conditions on the coefficients is an important task. The optimal conditions on the electric potential in terms of standard functional spaces have already been approached in [12], [63], and [55]–[58] (see also [14] and [33], where the potential is allowed to contain parts supported on a graph, or delta-type terms). This is not so for the magnetic potential. Conditions imposed in [65] and in the current paper are definitely unnecessarily strong. One can find in [12] a reasonable conjecture about the optimal conditions on a, which has been justified in $d = 2$, and some recent advances in [24]. The situation is even worse with the coefficients in the leading terms. Namely, the reduction (8) of the isotropic operator (7) to the Schrödinger case works only when the metric $\rho(x)$ is sufficiently smooth (for instance, belongs to H^2_{loc}). This, however, is far from what the applied problems request, since many applications work with composite media, where the coefficient $\rho(x)$ is piecewise constant. The same remark applies to the discussion in the Section 7.3, where isothermal coordinates are used. Analogously, applications to photonic crystals ([32], [38]) require considering piecewise constant dielectric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$. There is little doubt that the spectra must be absolute continuous in these cases, but proving this might require new techniques.

3. We are tempted to say that the commutation condition (13) looks irrelevant and should not be necessary. There are only two observations that stop us short of doing so. First of all, as it is shown above, all known results relate to the "commutative" situation only. Secondly, the simplest "non-commutative" case of the Dirac operator with arbitrary smooth **matrix** potential V has not been covered, in spite of significant attention the Dirac operator has attracted. So, one cannot exclude the possibility of surprises. Let us now take a closer look at the condition (13). In its absence the proof stalls at the inversion of the model operator $\frac{\partial}{\partial \bar{z}} + F(z)$ on the torus, when the matrix function $F(z)$ is not commutative. In order for the process of gauging away most of the magnetic potential to go through, one needs to find an invertible matrix function $B(z)$ (the replacement of the exponent $\exp i\phi(z)$ that was used before) on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that
$$\frac{\partial B}{\partial \bar{z}} = BF.$$
This problem is known in complex analysis in several variables. It is

known (see, for instance [45]) that its solution always exists locally. It is straightforward to check that the ratios of local solutions are holomorphic. This means that the local solutions determine an analytic vector bundle on the torus $\mathbb{T}^2$. Thus, existence of the needed global solution is equivalent to analytic triviality of this bundle. One wanders whether a version of Sobolev's trick ([65] and Section 5.3.4 of the current paper) could let one to eliminate this analytic obstruction and to carry through gauging away most of the magnetic potential. In the commutative case this is not hard to do, since one can describe the obstacle as non-vanishing of some Fourier coefficients of F, which is probably impossible in general.

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