Three-Dimensional Stability of Burgers Vortices: the Low Reynolds Number Case.

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Abstract

In this paper we establish rigorously that the family of Burgers vortices of the three-dimensional Navier-Stokes equation is stable for small Reynolds numbers. More precisely, we prove that any solution whose initial condition is a small perturbation of a Burgers vortex will converge toward another Burgers vortex as time goes to infinity, and we give an explicit formula for computing the change in the circulation number (which characterizes the limiting vortex completely.) We also give a rigorous proof of the existence and stability of non-axisymmetric Burgers vortices provided the Reynolds number is sufficiently small, depending on the asymmetry parameter.

1 Introduction

Numerical simulations of turbulent flows have lead to the general conclusion that vortex tubes serve as important organizing structures for such flows – in the memorable phrase of [9] they form the "sinews of turbulence". After the discovery by Burgers [1] of the explicit vortex solutions of the three-dimensional Navier-Stokes equation which now bear his name, these solutions have been used to model various aspects of turbulent flows [17]. It was also observed in numerical computations of fluid flows that the vortex tubes present in these simulations usually did not exhibit the axial symmetry of the explicit Burgers solution, but rather an elliptical core region. This lead to a search for non-axisymmetric vortices [13], [9], [6]. While no rigorous proof of their existence was available until recently, perturbative calculations and extensive numerical simulations have lead to the expectation that stationary vortical solutions of the three-dimensional Navier-Stokes equation do exist for any Reynolds number and all values of the asymmetry parameter (which we define below) between zero and one.

When addressing the stability of Burgers vortices, it is very important to specify the class of allowed perturbations. If we consider just two-dimensional perturbations (i.e., perturbations which do not depend on the axial variable), then fairly complete answers are known. Robinson and Saffman [13] computed perturbatively the eigenvalues of the linearized operator at the Burgers vortex and proved its stability for sufficiently small Reynolds numbers. Numerical computations of these eigenvalues were performed by Prochazka and Pullin [10], and no instability was found up to $Re = 10^4$. A similar conclusion was drawn for non-symmetric vortices [11]. The first mathematical work is [5], where we proved that the axisymmetric Burgers vortex is *globally stable* with respect to integrable, two-dimensional perturbations, for any value of the Reynolds number. Decay rates in time of spatially localized perturbations were also computed, explaining partially the numerical results of [10]. Building on this work the existence and local stability of slightly asymmetric vortices with respect to two dimensional perturbations was proved in [4] for arbitrary Reynolds numbers.

The stability issue is much more difficult if we allow for perturbations which depend on the axial variable too, and very few results have been obtained so far in this truly three-dimensional case. One early study by Leibovich and Holmes [7] concluded that one could not prove global stability for any Reynolds number solely by means of energy methods. Using a kind of Fourier expansion in the axial variable, Rossi and Le Dizès [14] showed that the point spectrum of the linearized operator is associated with purely two-dimensional perturbations. Crowdy [2] obtained a formal asymptotic expansion of the eigenfunctions in the axial variable. In an important recent work, Schmid and Rossi [16] rewrote the linearized equations in a form which allowed them to compute numerically the evolution of various Fourier modes, from which they concluded that eventually all perturbative modes will be damped out.

In this paper we address rigorously the existence of non-axisymmetric vortices and the stability with respect to three-dimensional perturbations of both the symmetric and non-symmetric vortex solutions. More precisely we will prove that, for all values of the asymmetry parameter between zero and one, non-axisymmetric vortices exist at least for small Reynolds numbers. In addition, we show that this family of vortex solutions is, in the language of dynamical systems theory, *asymptotically stable with shift*. That is to say, if we take initial conditions that are small perturbations of a vortex solution, the resulting solution of the Navier-Stokes equation will converge toward a vortex solution, but not, in general, the one which we initially perturbed. We also give a formula for computing the limiting vortex toward which the solution converges.

We now state our results more precisely. The three-dimensional Navier-Stokes equations for an incompressible fluid with constant density $\bar{\rho}$ and kinematic viscosity ν are the partial differential equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} \,=\, \nu\Delta\mathbf{u} - \frac{1}{\bar{\rho}}\nabla p\;, \quad \nabla\cdot\mathbf{u} \,=\, 0\;. \tag{1}$$

Here $\mathbf{u}(x,t)$ is the velocity of the fluid and $p(x,t)$ its pressure. Equation (1) will be considered in the whole space $\mathbf{R}^3$. Burgers vortices are particular solutions of (1) which are perturbations of the background straining flow

$$\mathbf{u}^s(x) = \begin{pmatrix} \gamma_1 x_1 \\ \gamma_2 x_2 \\ \gamma_3 x_3 \end{pmatrix} , \quad p^s(x) = -\frac{1}{2}\bar{\rho}(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + \gamma_3^2 x_3^2) , \tag{2}$$

where $\gamma_1, \gamma_2, \gamma_3$ are real constants satisfying $\gamma_1 + \gamma_2 + \gamma_3 = 0$. We restrict ourselves to the case of an *axial strain* aligned with the vertical axis, namely we assume $\gamma_1, \gamma_2 < 0$ and $\gamma_3 > 0$. Setting $\mathbf{u} = \mathbf{u}^s + \mathbf{U}$, we obtain the following evolution equation for the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{U}$:

$$\partial_t \mathbf{\Omega} + (\mathbf{U}\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{U} + (\mathbf{u}^s\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{u}^s = \nu\Delta\mathbf{\Omega} , \quad \nabla\cdot\mathbf{\Omega} = 0 . \tag{3}$$

Under reasonable assumptions which will be satisfied for the solutions we consider, the rotational part $\mathbf{U}$ of the velocity can be recovered from the vorticity $\mathbf{\Omega}$ by means of the Biot-Savart law:

$$\mathbf{U}(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(\mathbf{x}-\mathbf{y})\times\mathbf{\Omega}(y)}{|x-y|^3}\,\mathrm{d}y , \quad x \in \mathbf{R}^3 . \tag{4}$$

For notational simplicity we begin by discussing the *axisymmetric* case where $\gamma_1 = \gamma_2 = -\gamma_3/2$. In this situation, it is well-known [1] that (3) has a family of explicit stationary solutions of the form $\mathbf{\Omega} = \Gamma\hat{\mathbf{\Omega}}^B$, where $\Gamma \in \mathbf{R}$ is a parameter and

$$\hat{\mathbf{\Omega}}^B(x_\perp) = \begin{pmatrix} 0 \\ 0 \\ \hat{\Omega}^B(x_\perp) \end{pmatrix} , \quad \hat{\Omega}^B(x_\perp) = \frac{\gamma}{4\pi\nu}\, e^{-\gamma|x_\perp|^2/(4\nu)} . \tag{5}$$

Here $x_\perp = (x_1, x_2)$, $|x_\perp|^2 = x_1^2 + x_2^2$, and $\gamma \equiv \gamma_3 > 0$. The velocity field corresponding to $\Gamma\hat{\mathbf{\Omega}}^B$ is $\Gamma\hat{\mathbf{U}}^B$, where

$$\hat{\mathbf{U}}^B(x_\perp) = \frac{1}{2\pi}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \frac{1}{|x_\perp|^2}\Big(1 - e^{-\gamma|x_\perp|^2/(4\nu)}\Big) . \tag{6}$$

These solutions are called the *axisymmetric Burgers vortices.* Observe that $\int_{\mathbf{R}^2} \hat{\Omega}^B \,\mathrm{d}x_\perp = 1$, so that the parameter Γ represents the circulation of the velocity field $\Gamma\hat{\mathbf{U}}^B$ at infinity (in the horizontal plane $x_3 = 0$). Following [9], we define the Reynolds number associated with the Burgers vortex $\Gamma\hat{\mathbf{U}}^B$ as

$$R = \frac{|\Gamma|}{\nu} . \tag{7}$$

Our principal result concerns the evolution of solutions of (3) with initial conditions that are close to a Burgers vortex. Unlike in much previous work the perturbations we consider do not merely depend on the transverse variables $x_\perp$, but also on x_3. We prove that any solution of (3) starting sufficiently close to the Burgers vortex with circulation Γ converges as $t \to +\infty$ toward a Burgers vortex with circulation Γ' close to Γ, and we give an explicit formula for computing the difference $\Gamma' - \Gamma$ in terms of the initial perturbation.

We now introduce some function spaces to measure the size of our perturbations. Roughly speaking, we require the perturbations to decay as inverse powers of $x_\perp$ as

$|x_\perp| \to \infty$, but need only boundedness in x_3. To be specific, we use the following weight function

$$b(x_\perp) \;=\; (1+\gamma x_1^2/\nu)^{1/2}(1+\gamma x_2^2/\nu)^{1/2}\;, \quad x_\perp = (x_1,x_2) \in \mathbf{R}^2\;. \tag{8}$$

Given any $m \ge 0$, we define $L^2(m) \;=\; \{\omega : \mathbf{R}^2 \to \mathbf{R}\,|\, \|\omega\|_{L^2(m)} < \infty\}$, where

$$\|\omega\|^2_{L^2(m)} \;=\; \frac{1}{\gamma\nu}\int_{\mathbf{R}^2} b(x_\perp)^{2m}|\omega(x_\perp)|^2 \,\mathrm{d}x_\perp\;. \tag{9}$$

In other words, a function ω belongs to $L^2(m)$ if and only if ω, $|x_1|^m\omega$, $|x_2|^m\omega$, and $|x_1x_2|^m\omega$ are square integrable over $\mathbf{R}^2$. For later use, we observe that $L^2(m)$ is continuously embedded into $L^1(\mathbf{R}^2)$ if $m > 1/2$, i.e. there exists $C > 0$ such that $\|\omega\|_{L^1} \le C\|\omega\|_{L^2(m)}$ for all $\omega \in L^2(m)$.

Our main space $X^2(m)$ will be the set of all $\omega : \mathbf{R}^3 \to \mathbf{R}$ such that $x_\perp \mapsto \omega(x_\perp, x_3) \in L^2(m)$ for all $x_3 \in \mathbf{R}$, and such that the map $x_3 \mapsto \omega(\cdot, x_3)$ is bounded and continuous from $\mathbf{R}$ into $L^2(m)$. As is easily verified, $X^2(m) \simeq C^0_b(\mathbf{R}, L^2(m))$ is a Banach space equipped with the norm

$$\|\omega\|_{X^2(m)} \;=\; \sup_{x_3\in\mathbf{R}} \|\omega(\cdot, x_3)\|_{L^2(m)}\;. \tag{10}$$

Remark 1.1 *If $\boldsymbol{\omega} \;=\; (\omega_1,\omega_2,\omega_3)$ is a vector field whose components are elements of $X^2(m)$, we shall often write $\boldsymbol{\omega} \in X^2(m)$ instead of $\boldsymbol{\omega} \in X^2(m)^3$, and $\|\boldsymbol{\omega}\|_{X^2(m)}$ instead of $\|(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}\|_{X^2(m)}$. A similar abuse of notation will occur for other function spaces too.*

Consider initial conditions for the vorticity equation (3) which are a perturbation of the Burgers vortex:

$$\boldsymbol{\Omega}^0(x) \;=\; \Gamma\hat{\boldsymbol{\Omega}}^B(x_\perp) + \boldsymbol{\omega}^0(x)\;, \tag{11}$$

with $\Gamma \in \mathbf{R}$ and $\boldsymbol{\omega}^0 \in X^2(m)^3$. Define

$$\delta\Gamma \;=\; \Big(\frac{\gamma}{2\pi\nu}\Big)^{1/2}\int_{\mathbf{R}}\int_{\mathbf{R}^2} e^{-\gamma x_3^2/(2\nu)}\omega_3^0(x_\perp,x_3)\,\mathrm{d}x_\perp\,\mathrm{d}x_3\;. \tag{12}$$

Just to make sure the notation is clear, in the integrand ω_3^0 refers to the third component of the initial perturbation $\boldsymbol{\omega}^0$.

Theorem 1.2 *Fix $m > 3/2$, and assume that $(\gamma_1,\gamma_2,\gamma_3) = \gamma(-\frac{1}{2},-\frac{1}{2},1)$. For any $\mu \in (0,1/2)$, there exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that if $|\Gamma| \le R_0\nu$ and $\|\boldsymbol{\omega}_0\|_{X^2(m)} \le \varepsilon_0$, then the solution $\boldsymbol{\Omega}(x,t)$ of (3) with initial condition (11) converges as t tends to infinity to the Burgers vortex with circulation number $\Gamma' = \Gamma + \delta\Gamma$, where $\delta\Gamma$ is given by (12). Convergence is with respect to the $L^2(m)$ norm in $x_\perp$ and uniformly on compact sets in x_3. More explicitly, if $I \subset \mathbf{R}$ is any compact interval we have*

$$\sup_{x_3\in I}\|\boldsymbol{\Omega}(\cdot,x_3,t) - \Gamma'\hat{\boldsymbol{\Omega}}^B(\cdot)\|_{L^2(m)} \;=\; \mathcal{O}(e^{-\mu\gamma t})\;, \quad t \to +\infty\;. \tag{13}$$

Remark 1.3 *Here and in the sequel, all constants are independent of the strain γ and the viscosity ν. In fact, both parameters will shortly be eliminated by a rescaling.*

The proof of Theorem 1.2 uses ideas from our analysis of the stability of the two-dimensional Oseen vortices in [5]. The main observation is that, if we linearize equation (3) at the Burgers vortex $\Gamma\hat{\mathbf{\Omega}}^B$ for small Γ, we obtain a small perturbation of a non-constant coefficient differential operator for which we can explicitly compute an integral representation of the associated semigroup. This semigroup decays exponentially when acting on functions $\boldsymbol{\omega} \in X^2(m)^3$ *provided* $\omega_3 \in X_0^2(m)$, where

$$X_0^2(m) = \left\{\omega \in X^2(m) \;\middle|\; \int_{\mathbf{R}^2} \omega(x_\perp, x_3)\,\mathrm{d}x_\perp = 0 \quad \text{for all } x_3 \in \mathbf{R}\right\} . \tag{14}$$

Thus an important step in the proof consists in decomposing the perturbation as $\boldsymbol{\omega}(x,t) = \varphi(x_3,t)\hat{\mathbf{\Omega}}^B(x_\perp) + \tilde{\boldsymbol{\omega}}(x,t)$, where

$$\varphi(x_3,t) = \int_{\mathbf{R}^2} \omega_3(x_\perp, x_3, t)\,\mathrm{d}x_\perp\ , \quad x_3 \in \mathbf{R}\ , \quad t \ge 0\ .$$

By construction $\tilde{\omega}_3 \in X_0^2(m)$, hence $\tilde{\boldsymbol{\omega}}(x,t)$ will decay exponentially to zero by the remark above. Now, the crucial point is that $\varphi(x_3,t)$ satisfies the amazingly simple equation

$$\partial_t\varphi + \gamma x_3 \partial_3 \varphi = \nu \partial_3^2 \varphi\ ,$$

which can be solved explicitly, see (51) below. From the solution formula we see that $\varphi(x_3,t)$ converges uniformly on compact sets to the value $\delta\Gamma$ as $t \to \infty$, and (13) follows. In other words, φ is a "zero mode" which is responsible for the fact that the family of Burgers vortices is only asymptotically stable *with shift*.

While the axisymmetric vortex solution has been extensively studied because of the explicit formulas for its velocity and vorticity fields, numerical experiments on turbulent flows seem to indicate that the vortex tubes that are prominent in these flows are not symmetric, but rather elliptical in cross section. A natural way to obtain such vortices is to assume that the straining flow is not axisymmetric, namely

$$\gamma_1 = -\frac{\gamma}{2}(1+\lambda)\ , \quad \gamma_2 = -\frac{\gamma}{2}(1-\lambda)\ , \quad \gamma_3 = \gamma\ , \tag{15}$$

where $\gamma > 0$ and $\lambda \in [0,1)$ is an additional parameter which measures the asymmetry of the strain. While no explicit formulas for the vortex are known when $\lambda > 0$, extensive perturbative and numerical investigations indicate that there do exist stationary solutions of (3) for $0 \le \lambda < 1$, which for λ close to zero are small perturbations of the vorticity field of the axisymmetric Burgers vortex [13], [9], [11]. As in the symmetric case, there is in fact a family of vortices for each value of $\lambda \in [0,1)$ parametrized by the total circulation Γ, but when $\lambda > 0$ these solutions are *not* just multiples of one another.

In Section 2 we give a simple but rigorous proof of the existence of these non-axisymmetric vortex solutions for all values of $\lambda \in [0,1)$, provided the circulation Reynolds number of the flow is sufficiently small (depending on λ). A complementary result is obtained in [4] where we prove that, if $\lambda > 0$ is sufficiently small, non-axisymmetric vortex solutions exist for *all* values of the Reynolds number.

The construction of these non-axisymmetric vortices requires some work, so as a first step we rewrite the Navier-Stokes and associated vorticity equation in non-dimensional form. This simplifies the expressions for the solutions and also the equations themselves. Fix $\lambda \in [0,1)$ and assume that the straining flow is given by (2), (15) for some $\gamma > 0$. We replace the variables x, t and the functions $\mathbf{u}, p$ with the dimensionless quantities

$$\tilde{x} = \Bigl(\frac{\gamma}{\nu}\Bigr)^{1/2} x \;, \quad \tilde{t} = \gamma t \;, \quad \tilde{\mathbf{u}} = \frac{\mathbf{u}}{(\gamma\nu)^{1/2}} \;, \quad \tilde{p} = \frac{p}{\bar{\rho}\gamma\nu} \;, \tag{16}$$

where $\nu > 0$ is the kinematic viscosity. Dropping the tildes for simplicity, we see that the new functions $\mathbf{u}, p$ satisfy the Navier-Stokes equation (1) with $\nu = \bar{\rho} = 1$. Similarly the new straining flow $\mathbf{u}^s$ is given by (2), (15) with $\gamma = 1$. Setting $\mathbf{u} = \mathbf{u}^s + \mathbf{U}$, we obtain that $\boldsymbol{\Omega} = \nabla \times \mathbf{U}$ satisfies (3) with $\nu = 1$, namely

$$\partial_t \boldsymbol{\Omega} + (\mathbf{U}\cdot\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{U} + (\mathbf{u}^s\cdot\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}^s \;=\; \Delta\boldsymbol{\Omega} \;, \quad \nabla\cdot\boldsymbol{\Omega} \;=\; 0 \;. \tag{17}$$

Thus the main effect of the change of variables (16) is to set $\gamma = \nu = 1$ everywhere. In particular, in the dimensionless variables the weight function (8) becomes $b(x_\perp) = (1+x_1^2)^{1/2}(1+x_2^2)^{1/2}$, and the norm (9) reduces to $\|\omega\|_{L^2(m)} = \|b^m\omega\|_{L^2}$.

To formulate our result, we define

$$\mathcal{G}_\lambda(x_\perp) \;=\; \frac{\sqrt{1-\lambda^2}}{4\pi}\, e^{-\frac{1}{4}((1+\lambda)x_1^2+(1-\lambda)x_2^2)} \;, \quad x_\perp = (x_1,x_2) \in \mathbf{R}^2 \;. \tag{18}$$

If $\lambda = 0$, $\mathcal{G}_\lambda(x_\perp)$ is just the vorticity field (5) of the symmetric Burgers vortex written in the new coordinates, and the family of these vortices is indexed by the non-dimensionalized circulation number $\rho = \Gamma/\nu$. As we show below, for any $\lambda \in (0,1)$, $\mathcal{G}_\lambda(x_\perp)$ is still the leading order approximation to the vorticity of the non-axisymmetric Burgers vortex, for small Reynolds number $|\rho|$. Our precise result is:

Theorem 1.4 *Fix $m > 3/2$, $\lambda \in [0,1)$, and assume that $(\gamma_1,\gamma_2,\gamma_3)$ is given by (15) with $\gamma = 1$. There exist $R_1(\lambda) > 0$ and $K_1(\lambda) > 0$ such that, for $|\rho| \le R_1$, the vorticity equation (17) has a stationary solution $\boldsymbol{\Omega}^B(x_\perp;\rho,\lambda)$ which satisfies*

$$\boldsymbol{\Omega}^B(x_\perp;\rho,\lambda) \;=\; \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp;\rho,\lambda) \end{pmatrix} \;, \quad \int_{\mathbf{R}^2} \Omega^B(x_\perp;\rho,\lambda)\,\mathrm{d}x_\perp \;=\; \rho \;, \tag{19}$$

and

$$\|\Omega^B(\cdot;\rho,\lambda) - \rho\mathcal{G}_\lambda(\cdot)\|_{L^2(m)} \;\le\; K_1\rho^2 \;. \tag{20}$$

Furthermore, $\Omega^B(\cdot;\rho,\lambda)$ is a smooth function of ρ and λ, and there is no other stationary solution of (3) of the form (19) satisfying $\|\Omega^B - \rho\mathcal{G}_\lambda\|_{L^2(m)} \le 2R_1$.

Remark 1.5 *The proof shows that $R_1(\lambda) \to 0$ and $K_1(\lambda) \to \infty$ as $\lambda \to 1$. On the other hand, $R_1(0) > 0$ and $K_1(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to 0$. In particular, setting $\lambda = 0$ in (20), we recover that $\Omega^B(\cdot;\rho,0) = \rho\mathcal{G}_0$.*

Remark 1.6 *Theorem 2 shows that the asymmetric Burgers vortex $\boldsymbol{\Omega}^B(x_\perp;\rho,\lambda)$ decays rapidly as $|x_\perp| \to \infty$, since the parameter $m > 3/2$ is arbitrary (note, however, that the constants R_1, K_1 depend on m). In fact, proceeding as in [4], it is possible to show that $\boldsymbol{\Omega}^B$ has a Gaussian decay as $|x_\perp| \to \infty$. Moreover, $\boldsymbol{\Omega}^B$ is also a smooth function of $x_\perp$, see Remark 2.3 below.*

Finally, we prove that these families of non-symmetric vortices are stable in the same sense as the symmetric Burgers vortices are.

Theorem 1.7 *Fix $m > 3/2$, $\lambda \in [0, 1)$, and assume that $(\gamma_1, \gamma_2, \gamma_3)$ is given by (15) with $\gamma = 1$. For any $\mu \in (0, \frac{1}{2}(1-\lambda))$, there exist $R_2(\lambda) > 0$ and $\varepsilon_2(\lambda) > 0$ such that, if $|\rho| \le R_2$ and if $\mathbf{\Omega}^0(x) = \mathbf{\Omega}^B(x_\perp; \rho, \lambda) + \boldsymbol{\omega}^0(x)$ with $\boldsymbol{\omega}^0 \in X^2(m)^3$ satisfying*

$$\|\boldsymbol{\omega}^0\|_{X^2(m)} + \lambda\|\partial_3\varphi^0\|_{L^\infty}^2 \le \varepsilon_2 , \quad \textit{where} \quad \varphi^0(x_3) = \int_{\mathbf{R}^2} \omega_3^0(x_\perp, x_3)\,\mathrm{d}x_\perp , \tag{21}$$

then the solution $\mathbf{\Omega}(x, t)$ of (17) with initial data $\mathbf{\Omega}^0$ converges as $t \to +\infty$ to the vortex solution $\mathbf{\Omega}^B(x_\perp; \rho', \lambda)$, where $\rho' = \rho + \delta\rho$ and $\delta\rho = (2\pi)^{-1/2} \int_{\mathbf{R}} e^{-x_3^2/2}\varphi^0(x_3)\,\mathrm{d}x_3$, see (12). More precisely, for any compact interval $I \subset \mathbf{R}$, we have

$$\sup_{x_3\in I} \|\mathbf{\Omega}(\cdot, x_3, t) - \mathbf{\Omega}^B(\cdot; \rho', \lambda)\|_{L^2(m)} = \mathcal{O}(e^{-\mu t}) , \quad t \to +\infty . \tag{22}$$

The symmetric case $\lambda = 0$ is included in Theorem 1.7, which therefore subsumes Theorem 1.2. Note however that the assumptions on the initial data are more restrictive when $\lambda > 0$, because we then need a condition on $\partial_3\varphi^0$. This is due to the fact that non-axisymmetric Burgers vortices with different circulation numbers are not multiples of one another.

The rest of the text is organized as follows. In Section 2, we prove the existence of non-axisymmetric Burgers vortices for small Reynolds numbers. The core of the paper is Section 3, where we show that these families of vortices are asymptotically stable with shift. Section 4 is an appendix where we collect various estimates on the semigroup associated to the linearized vorticity equation, together with a few remarks concerning the Biot-Savart law.

2 Existence of non-axisymmetric Burgers vortices

The properties of non-axisymmetric Burgers vortices seem first to have been studied by Robinson and Saffman [13] who used perturbative methods to investigate their existence for small values of the Reynolds number. There were many further investigations in the intervening years - we mention particularly the perturbative study of the large Reynolds number limit of these vortices by Moffatt, Kida and Ohkitani [9], and the numerical work of Prochazka and Pullin [11]. However, as far as we know there has been no rigorous proof of the existence of these types of solutions and so in this section we present a simple argument which proves the existence of non-symmetric vortices in the case of small Reynolds number.

Fix $\lambda \in [0, 1)$ and assume that $\gamma_1, \gamma_2, \gamma_3$ are given by (15) with $\gamma = 1$. Motivated by the perturbative calculations of [13] we look for stationary solutions of (17) of the form

$$\mathbf{\Omega}^B(x_\perp) = \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp) \end{pmatrix} , \qquad \int_{\mathbf{R}^2} \Omega^B(x_\perp)\,\mathrm{d}x_\perp = \rho ,$$

for some $\rho \in \mathbf{R}$ (recall that $|\rho|$ is the Reynolds number). Since $\boldsymbol{\Omega}^B$ depends only on the horizontal variable $x_\perp = (x_1, x_2)$ and has only the third component nonzero, the associated velocity field $\mathbf{U}^B$ depends only on $x_\perp$ and has only the first two components nonzero. Thus $\mathbf{U}^B$ is naturally identified with a two-dimensional velocity field $\bar{\mathbf{U}}^B$ which can be computed using the two-dimensional version of the Biot-Savart law:

$$\bar{\mathbf{U}}^B(x_\perp) \;=\; \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{|x_\perp - y_\perp|^2}\begin{pmatrix} y_2 - x_2 \\ x_1 - y_1 \end{pmatrix} \Omega^B(y_\perp)\,\mathrm{d}y_\perp\ . \tag{23}$$

Inserting these expressions into (17), we see that Ω^B satisfies the scalar equation

$$\bar{\mathbf{U}}^B \cdot \nabla_\perp \Omega^B \;=\; (\mathcal{L}_\perp + \lambda\mathcal{M})\Omega^B\ , \tag{24}$$

where $\mathcal{L}_\perp$ and $\mathcal{M}$ are the differential operators

$$\mathcal{L}_\perp \;=\; \Delta_\perp + \frac{1}{2}(x_\perp \cdot \nabla_\perp) + 1\ , \quad \mathcal{M} \;=\; \frac{1}{2}(x_1\partial_1 - x_2\partial_2)\ . \tag{25}$$

Here we have used the natural notations $\nabla_\perp = (\partial_1, \partial_2)$ and $\Delta_\perp = \partial_1^2 + \partial_2^2$.

We shall solve (24) in the weighted space $L^2(m)$ defined by (9) (with $\gamma = \nu = 1$). Our approach rests on the fact that the spectrum of the linear operator $\mathcal{L}_\perp + \lambda\mathcal{M}$ in $L^2(m)$ can be explicitly computed, see Section 4.2. If $m > 1/2$, this operator turns out to be invertible on the invariant subspace $L^2_0(m)$ defined by

$$L^2_0(m) \;=\; \left\{\omega \in L^2(m) \,\middle|\, \int_{\mathbf{R}^2} \omega(x_\perp)\,\mathrm{d}x_\perp = 0\right\}\ . \tag{26}$$

This allows to rewrite (24) as a fixed point problem which is easily solved by a contraction argument.

As a preliminary step, let $\bar{\mathbf{V}}_\lambda(x_\perp)$ be the two-dimensional velocity field obtained from $\mathcal{G}_\lambda(x_\perp)$ by the Biot-Savart law (23). Using (18) and (25) one can easily verify that

$$(\mathcal{L}_\perp + \lambda\mathcal{M})\mathcal{G}_\lambda = 0\ , \quad \text{and} \quad \int_{\mathbf{R}^2} \mathcal{G}_\lambda(x_\perp)\,\mathrm{d}x_\perp = 1\ .$$

If we are given $\Omega^B \in L^2(m)$ with $m > 1/2$ and if $\rho = \int_{\mathbf{R}^2} \Omega^B\,\mathrm{d}x_\perp$, we can decompose

$$\Omega^B \;=\; \rho\mathcal{G}_\lambda + \omega\ , \quad \bar{\mathbf{U}}^B \;=\; \rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}\ , \tag{27}$$

where $\omega \in L^2_0(m)$ and $\bar{\mathbf{u}}$ is the velocity obtained from ω by the Biot-Savart law (23). With these notations, finding a solution to (24) is equivalent to solving

$$(\mathcal{L}_\perp + \lambda\mathcal{M})\omega \;=\; (\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}) \cdot \nabla_\perp(\rho\mathcal{G}_\lambda + \omega)\ , \quad \omega \in L^2_0(m)\ . \tag{28}$$

Note that $(\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}) \cdot \nabla_\perp(\rho\mathcal{G}_\lambda + \omega) = \nabla_\perp \cdot ((\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}})(\rho\mathcal{G}_\lambda + \omega))$ since $\bar{\mathbf{V}}_\lambda$ and $\bar{\mathbf{u}}$ are divergence-free. Thus the right-hand side of the (28) has zero mean as expected.

The next proposition ensures that the operator $\mathcal{L}_\perp + \lambda\mathcal{M}$ is invertible on $L^2_0(m)$ and that $(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp$ defines a bounded operator from $L^p(m)$ into $L^2_0(m)$, where $L^p(m)$ is the weighted space defined in (72).

Proposition 2.1 *Fix $m > 3/2$ and $\lambda \in [0,1)$. There exists $C(m,\lambda) > 0$ such that, for all $f \in L^2_0(m)$,*

$$\|(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} f\|_{L^2(m)} \le C\|f\|_{L^2(m)} \ . \tag{29}$$

Moreover, if $p \in (1,2]$, there exists $C(m,\lambda,p) > 0$ such that, for all $g \in L^p(m)$,

$$\|(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} \partial_i g\|_{L^2(m)} \le C\|g\|_{L^p(m)} \ , \quad i = 1,2 \ . \tag{30}$$

Proof: Let $\mathcal{T}_\lambda(t)$ denote the strongly continuous semigroup generated by $\mathcal{L}_\perp + \lambda\mathcal{M}$, the properties of which are studied in the Section 4.2. If $f \in L^2_0(m)$, we know from (73) that

$$\|\mathcal{T}_\lambda(t) f\|_{L^2(m)} \ \le \ C\, e^{-\frac{1}{2}(1-\lambda)t}\|f\|_{L^2(m)} \ , \quad t \ge 0 \ . \tag{31}$$

Thus $\mathcal{L}_\perp + \lambda\mathcal{M}$ is invertible on $L^2_0(m)$ and we have the Laplace formula

$$(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} f \ = \ -\int_0^\infty \mathcal{T}_\lambda(t) f \,\mathrm{d}t \ , \quad f \in L^2_0(m) \ . \tag{32}$$

Combining (31), (32), we easily obtain (29). Assume now that $f = \partial_i g$ for some $i \in \{1,2\}$ and some $g \in L^p(m)$. Using (32) and (74) or (75), we obtain an estimate of the form

$$\|(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} \partial_i g\|_{L^2(m)} \ \le \ C\int_0^\infty a(t)^{-\frac{1}{p}}\, e^{-\frac{1}{2}(1-\lambda)t}\|g\|_{L^p(m)} \,\mathrm{d}t \ ,$$

where $a(t) = 1 - e^{-t}$. Since $p > 1$, the singularity in the integral at $t = 0$ is integrable and (30) follows. □

We can now rewrite (28) as $\omega = F_{\lambda,\rho}(\omega)$, where $F_{\lambda,\rho} : L^2_0(m) \to L^2_0(m)$ is defined by

$$F_{\lambda,\rho}(\omega) \ = \ (\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp \cdot ((\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}) \cdot (\rho\mathcal{G}_\lambda + \omega)) \ . \tag{33}$$

For any $r > 0$, let $B_m(0,r)$ denote the closed ball of radius r centered at the origin in $L^2_0(m)$. The main result of this section is:

Proposition 2.2 *Fix $m > 3/2$ and $\lambda \in [0,1)$. There exist $R_1(\lambda) > 0$ and $K_1(\lambda) > 0$ such that, if $|\rho| \le R_1$, then $F_{\lambda,\rho}$ has a unique fixed point $\omega_{\lambda,\rho}$ in $B_m(0,2R_1)$. Moreover $\omega_{\lambda,\rho}$ is contained in $B_m(0,K_1\rho^2)$ and $\omega_{\lambda,\rho}$ is a smooth function of both λ and ρ.*

Proof: Let $\mathcal{B}_\lambda : L^2(m) \times L^2(m) \to L^2_0(m)$ be the bilinear map defined by

$$\mathcal{B}_\lambda(\Omega_1,\Omega_2) \ = \ (\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp \cdot (\bar{\mathbf{U}}_1\Omega_2) \ ,$$

where $\bar{\mathbf{U}}_1$ is the velocity field obtained from Ω_1 by the Biot-Savart law (23). If $1 < p < 2$, then $\|\bar{\mathbf{U}}_1\Omega_2\|_{L^p(m)} \le C\|\Omega_1\|_{L^2(m)}\|\Omega_2\|_{L^2(m)}$ by Corollary 4.5. Using in addition (30), we see that there exists $C_1(\lambda) > 0$ such that

$$\|\mathcal{B}_\lambda(\Omega_1,\Omega_2)\|_{L^2(m)} \ \le \ C_1(\lambda)\|\Omega_1\|_{L^2(m)}\|\Omega_2\|_{L^2(m)} \ , \quad \Omega_1,\Omega_2 \in L^2(m) \ .$$

Since $F_{\lambda,\rho}(\omega) = \mathcal{B}_\lambda(\rho\mathcal{G}_\lambda + \omega, \rho\mathcal{G}_\lambda + \omega)$, we obtain, for all $\omega \in L^2_0(m)$,

$$\|F_{\lambda,\rho}(\omega)\|_{L^2(m)} \ \le \ C_2(\lambda)\rho^2 + C_3(\lambda)(2|\rho|\|\omega\|_{L^2(m)} + \|\omega\|^2_{L^2(m)}) \ , \tag{34}$$

where $C_2(\lambda) = \|\mathcal{B}_\lambda(\mathcal{G}_\lambda, \mathcal{G}_\lambda)\|_{L^2(m)}$ and $C_3(\lambda) = C_1(\lambda)\max(1, \|\mathcal{G}_\lambda\|_{L^2(m)})$. Similarly, for all $\omega_1, \omega_2 \in L^2_0(m)$,

$$\|F_{\lambda,\rho}(\omega_1) - F_{\lambda,\rho}(\omega_2)\|_{L^2(m)} \;\le\; C_3(\lambda)\|\omega_1 - \omega_2\|_{L^2(m)}(2|\rho| + \|\omega_1\|_{L^2(m)} + \|\omega_2\|_{L^2(m)}) \;. \quad (35)$$

When $\lambda = 0$, $\mathcal{G}_\lambda$ is radially symmetric and $\bar{\mathbf{V}}_\lambda$ is azimuthal, hence $\bar{\mathbf{V}}_\lambda \cdot \nabla_\perp \mathcal{G}_\lambda = 0$. Thus $C_2(0) = 0$, so that $C_2(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to 0$.

Now, choose $R_1 > 0$ sufficiently small so that

$$C_2 R_1 \le 1 \;, \quad \text{and} \quad 8C_3R_1 \;\le\; 1 \;.$$

If $|\rho| \le R_1$ and $2C_2\rho^2 \le r \le 2R_1$, estimates (34) and (35) imply that $F_{\lambda,\rho}$ maps the ball $B_m(0, r)$ into itself and is a strict contraction there. More precisely, if $\omega_1, \omega_2 \in B_m(0, r)$, then

$$\|F_{\lambda,\rho}(\omega_1)\|_{L^2(m)} \;\le\; r \;, \quad \text{and} \quad \|F_{\lambda,\rho}(\omega_1) - F_{\lambda,\rho}(\omega_2)\|_{L^2(m)} \;\le\; \frac{3}{4}\|\omega_1 - \omega_2\|_{L^2(m)} \;.$$

By the contraction mapping theorem, $F_{\lambda,\rho}$ has a unique fixed point $\omega_{\lambda,\rho}$ in $B_m(0, r)$. Choosing $r = 2R_1$, we obtain the existence and uniqueness claim in Proposition 2.2. Then setting $r = K_1\rho^2$ with $K_1 = 2C_2$, we see that $\omega_{\lambda,\rho} \in B_m(0, K_1\rho^2)$. Finally, the smoothness property is a immediate consequence of the implicit function theorem. Indeed, the map $(\omega, \lambda, \rho) \mapsto F_{\lambda,\rho}(\omega)$ is obviously C^∞ from $L^2_0(m) \times [0, 1) \times \mathbf{R}$ into $L^2_0(m)$, and the partial differential

$$D_\omega F_{\lambda,\rho}(\omega) \;=\; \tilde{\omega} \mapsto \mathcal{B}_\lambda(\tilde{\omega}, \rho\mathcal{G}_\lambda + \omega) + \mathcal{B}_\lambda(\rho\mathcal{G}_\lambda + \omega, \tilde{\omega})$$

satisfies $\|D_\omega F_{\lambda,\rho}(\omega)\| \le 3/4$ whenever $|\rho| \le R_1$ and $\omega \in B_m(0, 2R_1)$. Thus $\mathbf{1} - D_\omega F_{\lambda,\rho}(\omega)$ is invertible at $\omega = \omega_{\lambda,\rho}$, and the implicit function theorem implies that $\omega_{\lambda,\rho}$ is a smooth function of both λ and ρ. $\square$

Theorem 1.4 is a direct consequence of Proposition 2.2. Indeed, if $|\rho| \le R_1(\lambda)$, we set $\Omega^B(x_\perp; \rho, \lambda) = \rho\mathcal{G}_\lambda(x_\perp) + \omega_{\lambda,\rho}(x_\perp)$, where $\omega_{\lambda,\rho}$ is as in Proposition 2.2, and we denote by $\bar{\mathbf{U}}^B(x_\perp; \rho, \lambda)$ the two-dimensional velocity field obtained from Ω^B by the Biot-Savart law (23). Then

$$\mathbf{\Omega}^B(x_\perp; \rho, \lambda) \;=\; \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp; \rho, \lambda) \end{pmatrix} \;, \quad \mathbf{U}^B(x_\perp; \rho, \lambda) \;=\; \begin{pmatrix} \bar{U}_1^B(x_\perp; \rho, \lambda) \\ \bar{U}_2^B(x_\perp; \rho, \lambda) \\ 0 \end{pmatrix} \quad (36)$$

is a stationary solution of (17) which has all the desired properties. In particular, since $\omega_{\lambda,\rho} \in L^2_0(m)$, we have

$$\int_{\mathbf{R}^2} \Omega^B(x_\perp; \rho, \lambda)\,\mathrm{d}x_\perp \;=\; \rho \;, \quad (37)$$

while the fact that $\omega_{\lambda,\rho} \in B_m(0, K_1\rho^2)$ implies that (20) holds. For later use, we observe that there exists $C(\lambda, m) > 0$ such that, for $|\rho| \le R_1$,

$$\|\Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C|\rho| \;, \quad \text{and} \quad \|\partial_\rho \Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C \;. \quad (38)$$

Moreover $\|\partial_\rho^2 \Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C\lambda$, because in the symmetric case $\Omega^B(\cdot; \rho, 0) = \rho\mathcal{G}_0$ so that $\partial_\rho^2 \Omega^B(\cdot; \rho, 0) = 0$.

Remark 2.3 *We chose to solve (24) in $L^2(m)$ because this is basically the space we shall use in Section 3 to study the stability of the vortices. But it is clear from the proof of Proposition 2.2 that nothing important changes if we replace $L^2(m)$ with the corresponding Sobolev space*

$$H^k(m) = \left\{ f \in L^2(m) \,\middle|\, \partial_1^i \partial_2^j f \in L^2(m) \text{ for all } i,j \in \mathbf{N} \text{ with } i+j \le k \right\} ,$$

for any $k \in \mathbf{N}$. This shows that the asymmetric Burgers vortex $\Omega^B(x_\perp;\rho,\lambda)$ is a smooth function of $x_\perp$ too. In particular, by Sobolev embedding, $b^m\Omega^B(\cdot;\rho,\lambda) \in C_b^0(\mathbf{R}^2)$ (the space of all continuous and bounded functions on $\mathbf{R}^2$) and we have the analogue of (38):

$$\sup_{x_\perp \in \mathbf{R}^2} b(x_\perp)^m |\Omega^B(x_\perp;\rho,\lambda)| \le C|\rho| \ , \quad \sup_{x_\perp \in \mathbf{R}^2} b(x_\perp)^m |\partial_\rho \Omega^B(x_\perp;\rho,\lambda)| \le C \ . \tag{39}$$

Moreover, since $\Omega^B(\cdot;\rho,\lambda) \in L^p(\mathbf{R}^2)$ for all $p \in [1,+\infty]$, Proposition 4.4 implies that $\mathbf{U}(\cdot;\rho,\lambda) \in L^q(\mathbf{R}^2) \cap C_b^0(\mathbf{R}^2)$ for all $q \in (2,\infty]$, and there exists $C(q,m,\lambda) > 0$ such that

$$\|\mathbf{U}^B(\cdot;\rho,\lambda)\|_{L^q(\mathbf{R}^2)} \le C|\rho| \ , \quad \text{and} \quad \|\partial_\rho \mathbf{U}^B(\cdot;\rho,\lambda)\|_{L^q(\mathbf{R}^2)} \le C \ . \tag{40}$$

3 Stability with respect to three-dimensional perturbations

We now prove that the family of vortices constructed in the previous section is asymptotically stable with shift, provided the circulation Reynolds number is sufficiently small, depending on the asymmetry parameter $\lambda \in [0,1)$. In particular, our result applies to the classical family of symmetric Burgers vortices ($\lambda = 0$).

Throughout this section we fix some $\lambda \in [0,1)$. For $|\rho|$ sufficiently small we denote by $\mathbf{\Omega}^B(x_\perp;\rho)$, $\mathbf{U}^B(x_\perp;\rho)$ the asymmetric vortex (36) with total circulation ρ (to simplify the notation, we omit the dependence on λ). As we mentioned in the introduction, if we slightly perturb the vortex $\mathbf{\Omega}^B(\cdot;\rho)$ the solution of the vorticity equation will converge toward another vortex with a possibly different circulation. This means that we must allow the parameter ρ to depend on time. Also, since the perturbations we consider may depend on the axial variable x_3, it turns out to be convenient to approximate the solutions by vortices with different circulation numbers in different x_3 sections. In other words, we will consider solutions of (17) of the form

$$\mathbf{\Omega}(x,t) = \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp;\rho+\varphi(x_3,t)) \end{pmatrix} + \begin{pmatrix} \omega_1(x,t) \\ \omega_2(x,t) \\ \omega_3(x,t) \end{pmatrix} , \tag{41}$$

where $\varphi(x_3,t)$ is determined so that $\int_{\mathbf{R}^2} \omega_3(x_\perp,x_3,t)\,\mathrm{d}x_\perp = 0$ for all x_3 and t. In view of (37), it is obvious that any pertubation of $\mathbf{\Omega}^B(\cdot;\rho)$ that is integrable with respect to the transverse variables $x_\perp$ can be decomposed in a unique way as in (41). Similarly, we write the rotational part of the velocity field as

$$\mathbf{U}(x,t) = \begin{pmatrix} \tilde{U}_1^B(x,t;\rho,\varphi) \\ \tilde{U}_2^B(x,t;\rho,\varphi) \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} , \tag{42}$$

where $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$ is the velocity field obtained from the vorticity $\mathbf{\Omega}^B(x_\perp;\rho+\varphi(x_3,t))$ by the Biot-Savart law (4). It will be shown in Proposition 4.11 that $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$ is a small perturbation of $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))$ if φ varies slowly in the x_3 direction, namely there exists $C(\lambda)>0$ such that

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x,t;\rho,\varphi) - \mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))| \,\le\, C\|\partial_3\varphi(\cdot,t)\|_{L^\infty}\,. \tag{43}$$

Let $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)^T$ and $\mathbf{u} = (u_1,u_2,u_3)^T$ denote the remainder terms in (41) and (42) respectively. By construction, $\mathbf{u}$ is the velocity field obtained from $\boldsymbol{\omega}$ by the Biot-Savart law (4). Remark that $\nabla\cdot\mathbf{u}=0$, but $\nabla\cdot\boldsymbol{\omega} = -(\partial_\rho\Omega^B)\partial_3\varphi\neq 0$, hence $\boldsymbol{\omega}\neq\nabla\times\mathbf{u}$. In broadest terms, our strategy is to show that $\boldsymbol{\omega}(x,t)$ and $\partial_3\varphi(x,t)$ converge to zero as time tends to infinity, so that the vorticity $\mathbf{\Omega}(x,t)$ approaches one of the vortices $\mathbf{\Omega}^B(\cdot;\rho')$ constructed in Section 2. With that in mind, we now write out the evolution equations for $\boldsymbol{\omega}$ and φ.

Inserting (41), (42) into (17) and using the identity $(\mathbf{U}\cdot\nabla)\mathbf{\Omega}-(\mathbf{\Omega}\cdot\nabla)\mathbf{U} = \nabla\times(\mathbf{\Omega}\times\mathbf{U})$, we obtain after straightforward calculations:

$$\partial_t\boldsymbol{\omega} \,=\, \mathbb{L}\boldsymbol{\omega} + \mathbb{P}_\varphi\boldsymbol{\omega} + \mathbb{N}(\boldsymbol{\omega}) + \mathbb{H}(\varphi)\,, \tag{44}$$

where the various terms in the right-hand side are defined as follows.

• The linear operator $\mathbb{L}$ is the leading order part of the equation, which takes into account the diffusion and the effects of the background strain:

$$\mathbb{L}\boldsymbol{\omega} \,=\, \Delta\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}^s - (\mathbf{u}^s\cdot\nabla)\boldsymbol{\omega} \,=\, \begin{pmatrix} (\mathcal{L}+\gamma_1)\omega_1 \\ (\mathcal{L}+\gamma_2)\omega_2 \\ (\mathcal{L}+\gamma_3)\omega_3 \end{pmatrix}\,.$$

Here $\gamma_1,\gamma_2,\gamma_3$ are given by (15) with $\gamma=1$, and

$$\mathcal{L} \,=\, \Delta - (\mathbf{u}^s\cdot\nabla) \,=\, \Delta + \frac{1}{2}(x_\perp\cdot\nabla_\perp) + \frac{\lambda}{2}(x_1\partial_1 - x_2\partial_2) - x_3\partial_3\,. \tag{45}$$

• The term $\mathbb{P}_\varphi\boldsymbol{\omega} = \nabla\times(\tilde{\mathbf{U}}^B\times\boldsymbol{\omega} + \mathbf{u}\times\mathbf{\Omega}^B)$ describes the linear interaction between the perturbation and the modulated vortex, namely:

$$\mathbb{P}_\varphi\boldsymbol{\omega} \,=\, \begin{pmatrix} \partial_2(\tilde{U}_1^B\omega_2 - \tilde{U}_2^B\omega_1) + \partial_3(\tilde{U}_1^B\omega_3 + u_1\Omega^B) \\ \partial_1(\tilde{U}_2^B\omega_1 - \tilde{U}_1^B\omega_2) + \partial_3(\tilde{U}_2^B\omega_3 + u_2\Omega^B) \\ -\partial_1(\tilde{U}_1^B\omega_3 + u_1\Omega^B) - \partial_2(\tilde{U}_2^B\omega_3 + u_2\Omega^B) \end{pmatrix}\,. \tag{46}$$

Here and in the sequel, to simplify the notation, we write $\mathbf{\Omega}^B$ instead of $\mathbf{\Omega}^B(\cdot;\rho+\varphi)$ and $\tilde{\mathbf{U}}^B$ instead of $\tilde{\mathbf{U}}^B(\cdot;\rho,\varphi)$.

• The term $\mathbb{N}(\boldsymbol{\omega}) = \nabla\times(\mathbf{u}\times\boldsymbol{\omega})$ collects all the nonlinear contributions in $\boldsymbol{\omega}$, specifically:

$$\mathbb{N}(\boldsymbol{\omega}) \,=\, \begin{pmatrix} \partial_2(u_1\omega_2 - u_2\omega_1) + \partial_3(u_1\omega_3 - u_3\omega_1) \\ \partial_1(u_2\omega_1 - u_1\omega_2) + \partial_3(u_2\omega_3 - u_3\omega_2) \\ \partial_1(u_3\omega_1 - u_1\omega_3) + \partial_2(u_3\omega_2 - u_2\omega_3) \end{pmatrix}\,. \tag{47}$$

• Finally, $\mathbb{H}(\varphi) = \mathbb{L}\mathbf{\Omega}^B + \nabla\times(\tilde{\mathbf{U}}^B\times\mathbf{\Omega}^B) - \partial_t\mathbf{\Omega}^B$ is an inhomogeneous term which is due to the fact that $\mathbf{\Omega}^B(\cdot;\rho+\varphi)$ fails to be a solution of (17) if φ is not identically constant.

A simple calculation shows that $\mathbb{H}_i(\varphi) = \partial_3(\tilde{U}_i^B \Omega^B)$ for $i = 1, 2$. The third component of $\mathbb{H}(\varphi)$ has a more complicated expression:

$$\begin{aligned} \mathbb{H}_3(\varphi) \;&=\; (\mathcal{L}_\perp + \lambda \mathcal{M})\Omega^B - \nabla_\perp \cdot (\tilde{\mathbf{U}}^B \Omega^B) + (\partial_\rho^2 \Omega^B)(\partial_3 \varphi)^2 \\ &\quad -(\partial_\rho \Omega^B)(\partial_t \varphi + x_3 \partial_3 \varphi - \partial_3^2 \varphi)\ , \end{aligned} \tag{48}$$

where $\mathcal{L}_\perp$ and $\mathcal{M}$ are defined in (25).

Equation (44) governs the evolution of both φ and $\boldsymbol{\omega}$. To separate out the evolution equation for φ, we recall that $\boldsymbol{\omega}$ satisfies the constraint $\int_{\mathbf{R}^2} \omega_3(x_\perp, x_3, t)\,\mathrm{d}x_\perp = 0$. If we integrate the third component of the vectorial equation (44) with respect to the transverse variables $x_\perp$, the first three terms in the right-hand side give no contribution, as can be seen from the formulas (45), (46), (47). So we must impose

$$\int_{\mathbf{R}^2} \mathbb{H}_3(\varphi)\,\mathrm{d}x_\perp \;=\; 0\ , \quad \text{for all } x_3 \text{ and } t\ . \tag{49}$$

As is clear from (24), the first term in the right-hand side of (48) has zero mean with respect to $x_\perp$, and so does the second term because it is explicitly in divergence form. On the other hand, differentiating (37) with respect to ρ, we obtain the identities $\int_{\mathbf{R}^2} \partial_\rho \Omega^B\,\mathrm{d}x_\perp = 1$ and $\int_{\mathbf{R}^2} \partial_\rho^2 \Omega^B\,\mathrm{d}x_\perp = 0$. Thus (49) gives the evolution equation for φ:

$$\partial_t \varphi + x_3 \partial_3 \varphi \;=\; \partial_3^2 \varphi\ . \tag{50}$$

Remarkably, this equation is *linear* and completely *decoupled* from the rest of the system. As is easily verified, the solution of (50) with initial data $\varphi(x_3, 0) = \varphi^0(x_3)$ is given by the explicit formula

$$\varphi(x_3, t) \;=\; (G_t * \varphi^0)(x_3 e^{-t})\ , \quad x_3 \in \mathbf{R}\ , \quad t > 0\ , \tag{51}$$

where

$$G_t(x_3) \;=\; \sqrt{\frac{1}{2\pi(1-e^{-2t})}}\, \exp\Bigl(-\frac{x_3^2}{2(1-e^{-2t})}\Bigr)\ , \quad x_3 \in \mathbf{R}\ , \quad t > 0\ . \tag{52}$$

The following simple estimates will be useful:

Proposition 3.1 *If $\varphi^0 \in C_b^0(\mathbf{R})$, the solution of (50) with initial data φ^0 satisfies*

$$\|\varphi(\cdot, t)\|_{L^\infty} \;\le\; \|\varphi^0\|_{L^\infty}\ , \quad \|\partial_3 \varphi(\cdot, t)\|_{L^\infty} \;\le\; \frac{e^{-t}}{\sqrt{1 - e^{-2t}}}\|\varphi^0\|_{L^\infty}\ , \quad t > 0\ . \tag{53}$$

If moreover $\partial_3 \varphi^0 \in L^\infty(\mathbf{R})$, we also have

$$\|\partial_3 \varphi(\cdot, t)\|_{L^\infty} \;\le\; e^{-t}\|\partial_3 \varphi^0\|_{L^\infty}\ , \quad t \ge 0\ . \tag{54}$$

Proof: Since $\|G_t\|_{L^1} = 1$, it follows immediately from (51) that $\|\varphi(\cdot, t)\|_{L^\infty} \le \|\varphi^0\|_{L^\infty}$. If $\partial_3 \varphi^0 \in L^\infty(\mathbf{R})$, the same argument gives (54), because

$$\partial_3 \varphi(x_3, t) \;=\; e^{-t}(G_t * \partial_3 \varphi^0)(x_3 e^{-t}) \;=\; e^{-t}(\partial_3 G_t * \varphi^0)(x_3 e^{-t})\ , \quad t > 0\ . \tag{55}$$

To prove the second estimate in (53), we use the last expression in (55) and observe that $\|\partial_3 G_t\|_{L^1} = C/\sqrt{1 - e^{-2t}}$, where $C = \sqrt{2/\pi} < 1$. □

Remark 3.2 *Proposition 3.1 shows in particular that (51) defines a semigroup of bounded linear operators on $C_b^0(\mathbf{R})$, the space of all bounded and continuous functions on $\mathbf{R}$ equipped with the L^∞ norm. It is easy to verify that this semigroup is* not strongly continuous *in time, due to the dilation factor e^{-t} in (51) which in turn originates in the unbounded advection term $x_3\partial_3$ in (50). However, if we equip $C_b^0(\mathbf{R})$ with the (weaker) topology of uniform convergence on compact sets, then (51) defines a continuous function of time.*

We now return to the evolution equation for $\boldsymbol{\omega}$. Using (48), equation (50) for φ, and equation (24) satisfied by the asymmetric vortex Ω^B, we obtain for the inhomogeneous term $\mathbb{H}(\varphi)$ the simpler expression

$$\mathbb{H}(\varphi) \,=\, \begin{pmatrix} \partial_3(\tilde{U}_1^B\Omega^B) \\ \partial_3(\tilde{U}_2^B\Omega^B) \\ \nabla_\perp\cdot((\mathbf{U}^B-\tilde{\mathbf{U}}^B)\Omega^B) + (\partial_\rho^2\Omega^B)(\partial_3\varphi)^2 \end{pmatrix}\,, \tag{56}$$

where as usual $\mathbf{U}^B = \mathbf{U}^B(\cdot;\rho+\varphi)$. Before starting the rigorous analysis, let us briefly comment here on why we expect solutions of (44) to go to zero as t goes to infinity. Given $m > 3/2$, we assume that $\omega_i \in X^2(m)$ for $i = 1, 2, 3$, where $X^2(m)$ is the space defined in (10). By construction, ω_3 then belongs to the subspace $X_0^2(m)$ given by (14). As we show in Section 4.3, the linear operator $\mathbb{L}$ has spectrum that lies in the half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}\, z \le -\frac{1}{2}(1-\lambda)\}$ when acting on $X^2(m)\times X^2(m)\times X_0^2(m)$. Thus, the semigroup generated by this operator can be expected to decay like $\exp(-\frac{1}{2}(1-\lambda)t)$. The remaining linear terms in the equation, namely $\mathbb{P}_\varphi(\omega)$, contain a factor of the vortex solution which is proportional to $\rho+\varphi$ (see (39) and (100)) and hence, for small Reynolds number, they will be a small perturbation of $\mathbb{L}$ and will not destroy the exponential decay. The same is true for the nonlinear terms $\mathbb{N}(\omega)$ provided we restrict ourselves to sufficiently small perturbations. Finally, the inhomogeneous term $\mathbb{H}(\varphi)$ decays at least like e^{-t} by (43) and Proposition 3.1, so we expect the solution $\boldsymbol{\omega}(x,t)$ of (44) to converge exponentially to zero if the initial data are sufficiently small.

We now put these heuristic arguments into a rigorous form. Let $\mathbb{X}(m)$ be the Banach space $X^2(m)\times X^2(m)\times X_0^2(m)$ equipped with the norm $\|\boldsymbol{\omega}\|_{\mathbb{X}(m)} = \|\omega_1\|_{X^2(m)}+\|\omega_2\|_{X^2(m)}+\|\omega_3\|_{X^2(m)}$. As is shown in Proposition 4.6, the linear operator $\mathcal{L}$ is the generator of a semigroup $e^{t\mathcal{L}}$ of bounded operators on $X^2(m)$, hence the same is true for the operator $\mathbb{L}$ acting on $\mathbb{X}(m)$. A natural idea is then to use Duhamel's formula to rewrite (44) as an integral equation:

$$\boldsymbol{\omega}(t) \,=\, e^{t\mathbb{L}}\boldsymbol{\omega}^0 + \int_0^t e^{(t-s)\mathbb{L}}\Big(\mathbb{P}_\varphi\boldsymbol{\omega}(s) + \mathbb{N}(\boldsymbol{\omega}(s)) + \mathbb{H}(\varphi(s))\Big)\,\mathrm{d}s\,, \quad t \ge 0\,, \tag{57}$$

which can then be solved by a fixed point argument. A problem with this approach is that the semigroup $e^{t\mathcal{L}}$ fails to be strongly continuous on $X^2(m)$, essentially for the reason mentioned in Remark 3.2. To restore continuity, it is thus necessary to equip $X^2(m)$ with a weaker topology. For any $n \in \mathbf{N}^*$ we define the seminorm

$$|\omega|_{X_n^2(m)} \,=\, \sup_{|x_3|\le n} \|\omega(\cdot,x_3)\|_{L^2(m)}\,, \tag{58}$$

and we denote by $X^2_{\rm loc}(m)$ the space $X^2(m)$ equipped with the topology defined by the family of seminorms (58) for $n \in \mathbf{N}^*$, i.e. the topology of the Fréchet space $C^0(\mathbf{R}, L^2(m))$. In other words, a sequence ω_k converges to zero in $X^2_{\rm loc}(m)$ if and only if $|\omega_k|_{X^2_n(m)} \to 0$ as $k \to \infty$ for all $n \in \mathbf{N}^*$, namely if $\omega_k(x_3)$ converges to zero in $L^2(m)$ *uniformly on compact sets* in x_3. We define the product space $\mathbb{X}_{\rm loc}(m)$ in a similar way. Then Proposition 4.6 shows that the semigroup $e^{t\mathbb{L}}$ is strongly continuous on $\mathbb{X}_{\rm loc}(m)$, and the integrals in (57) can be defined as $\mathbb{X}_{\rm loc}(m)$-valued Riemann integrals, see Corollary 4.7 and Remark 4.8.

Since we expect $\boldsymbol{\omega}(t)$ to converge exponentially to zero as $t \to \infty$, we shall solve (57) in the Banach space

$$\mathbb{Y}_\mu(m) \;=\; \{\boldsymbol{\omega} \in C^0([0,+\infty), \mathbb{X}_{\rm loc}(m)) \,|\, \|\boldsymbol{\omega}\|_{\mathbb{Y}_\mu(m)} < \infty\}~,$$

for some $\mu > 0$, where

$$\|\boldsymbol{\omega}\|_{\mathbb{Y}_\mu(m)} \;=\; \sup_{t\ge 0} e^{\mu t} \|\boldsymbol{\omega}(t)\|_{\mathbb{X}(m)}~.$$

Given initial data $\varphi^0 \in C^0_b(\mathbf{R})$ and $\boldsymbol{\omega}^0 \in \mathbb{X}(m)$, we first define $\varphi(x_3,t)$ by (51), and then use the integral equation (57) to determine $\boldsymbol{\omega}(t)$ for all $t \ge 0$. Our main result is:

Proposition 3.3 *Fix $\lambda \in [0,1)$, $m > 3/2$, and $0 < \mu < \frac{1}{2}(1-\lambda)$. There exist positive constants $\rho_2 > 0$, $\varepsilon_2 > 0$, and $K_2 > 0$ such that, if $|\rho| \le \rho_2$, $\varepsilon \le \varepsilon_2$, and if $\varphi^0 \in C^0_b(\mathbf{R})$ satisfies $\|\varphi^0\|_{L^\infty} + \lambda\|\partial_3\varphi^0\|^2_{L^\infty} \le \varepsilon$, then for all $\boldsymbol{\omega}^0 \in \mathbb{X}(m)$ with $\|\boldsymbol{\omega}^0\|_{\mathbb{X}(m)} \le \varepsilon$ equation (57) has a unique solution $\boldsymbol{\omega} \in \mathbb{Y}_\mu(m)$ with $\|\boldsymbol{\omega}\|_{Y_\mu(m)} \le K_2\varepsilon$.*

Proof: Fix $\lambda \in [0,1)$, $m > 3/2$, and $0 < \mu < \frac{1}{2}(1-\lambda)$. To simplify the notations, we shall write $\mathbb{X}$ instead of $\mathbb{X}(m)$ and $\mathbb{Y}$ instead of $\mathbb{Y}_\mu(m)$. For any $r > 0$, we denote by $B_X(0,r)$ (respectively, $B_Y(0,r)$) the closed ball of radius $r > 0$ centered at the origin in $\mathbb{X}$ (respectively, $\mathbb{Y}$). Let $\varphi^0 \in C^0_b(\mathbf{R})$ and denote by $\varphi(x_3,t)$ the solution of (50) with initial data φ^0. Given $\rho \in \mathbf{R}$, $\boldsymbol{\omega}^0 \in \mathbb{X}$, and $\boldsymbol{\omega} \in \mathbb{Y}$, we estimate the various terms in the right-hand side of (57).

We begin with the linear term $e^{t\mathbb{L}}\boldsymbol{\omega}^0$. From Proposition 4.6 we know that the linear operator $\mathcal{L}+1 \equiv \hat{\mathcal{L}}_{1+\lambda,1-\lambda}$ generates a semigroup $S_t = e^{t(\mathcal{L}+1)}$ which is strongly continuous on $X^2_{\rm loc}(m)$, uniformly bounded on $X^2(m)$, and which decays like $e^{-\frac{1}{2}(1-\lambda)t}$ on $X^2_0(m)$. Since

$$e^{t\mathbb{L}}\boldsymbol{\omega}^0 \;=\; \Big(e^{t(\mathcal{L}+\gamma_1)}\omega^0_1\,,\, e^{t(\mathcal{L}+\gamma_2)}\omega^0_2\,,\, e^{t(\mathcal{L}+\gamma_3)}\omega^0_3\Big)^T~,$$

where $\gamma_1, \gamma_2, \gamma_3$ are given by (15) with $\gamma = 1$, we deduce that $t \mapsto e^{t\mathbb{L}}\boldsymbol{\omega}^0$ is continuous in $\mathbb{X}_{\rm loc}$ and satisfies

$$\begin{aligned}\|e^{t\mathbb{L}}\boldsymbol{\omega}^0\|_{\mathbb{X}} \;&\le\; C\Big(e^{-\frac{3+\lambda}{2}t}\|\omega^0_1\|_{X^2(m)} + e^{-\frac{3-\lambda}{2}t}\|\omega^0_2\|_{X^2(m)} + e^{-\frac{1-\lambda}{2}t}\|\omega^0_3\|_{X^2(m)}\Big)\\ &\le\; C_1 e^{-\frac{1-\lambda}{2}t}\|\boldsymbol{\omega}^0\|_{\mathbb{X}}~. \qquad (59)\end{aligned}$$

Note that it is crucial here that $\omega^0_3 \in X^2_0(m)$, otherwise we do not get any decay at all.

We next consider the linear term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_\varphi\boldsymbol{\omega}(s)\,{\rm d}s$. For $s \ge 0$ and $i \in \{1,2\}$, we know from Proposition 4.11 that $\tilde{U}^B_i(s) \equiv \tilde{U}^B_i(\cdot,s;\rho,\varphi(s)) \in C^0_b(\mathbf{R}^3)$ and $\|\tilde{U}^B_i(s)\|_{L^\infty} \le C(|\rho| + \|\varphi(s)\|_{L^\infty}) \le C(|\rho| + \|\varphi^0\|_{L^\infty})$ by Proposition 3.1. Since $\omega_j(s) \in X^2(m)$ for $j \in \{1,2,3\}$, it follows that $\tilde{U}^B_i(s)\omega_j(s) \in X^2(m)$ and

$$\|\tilde{U}^B_i(s)\omega_j(s)\|_{X^2(m)} \;\le\; \|\tilde{U}^B_i(s)\|_{L^\infty(\mathbf{R}^3)}\|\omega_j(s)\|_{X^2(m)} \;\le\; C\rho'\|\omega_j(s)\|_{X^2(m)}~,$$

where $\rho' = |\rho| + \|\varphi^0\|_{L^\infty}$. Moreover, it is not difficult to verify that $s \mapsto \tilde{U}_i^B(s)\omega_j(s)$ is continuous in $X^2_{\rm loc}(m)$. Similarly, $u_i(s) \in X^q(0)$ for $q \in (2, +\infty)$ by Proposition 4.9 and $\Omega^B(s) \equiv \Omega^B(\cdot; \rho + \varphi(s)) \in X^p(m)$ for $p \in [1, +\infty]$ by Remark 2.3, hence $u_i(s)\Omega^B(s) \in X^2(m)$ by Hölder's inequality and

$$\|u_i(s)\Omega^B(s)\|_{X^2(m)} \le \|u_i(s)\|_{X^4(0)}\|\Omega^B(s)\|_{X^4(m)} \le C\rho'\|\omega_i(s)\|_{X^2(m)} .$$

Again $s \mapsto u_i(s)\Omega^B(s)$ is continuous in $X^2_{\rm loc}(m)$. Thus Corollary 4.7 and Remark 4.8 show that the three components of the vector $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_\varphi\boldsymbol{\omega}(s)\,{\rm d}s$ are well defined and continuous in $X^2_{\rm loc}(m)$ for $t \ge 0$. Using Proposition 4.6, we can estimate the first component as follows:

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\Big(\partial_2(\tilde{U}_1^B\omega_2 - \tilde{U}_2^B\omega_1) + \partial_3(\tilde{U}_1^B\omega_3 + u_1\Omega^B)\Big)(s)\,{\rm d}s\Big\|_{X^2(m)} \\
&\quad\le C\int_0^t \frac{e^{-2(t-s)}}{a(t-s)^{1/2}}\Big(\|\tilde{U}_1^B(s)\omega_2(s)\|_{X^2(m)} + \|\tilde{U}_2^B(s)\omega_1(s)\|_{X^2(m)}\Big)\,{\rm d}s \\
&\quad+ C\int_0^t \frac{e^{-\frac{3+\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\Big(\|\tilde{U}_1^B(s)\omega_3(s)\|_{X^2(m)} + \|u_1(s)\Omega^B(s)\|_{X^2(m)}\Big)\,{\rm d}s ,
\end{aligned}$$

where $a(t) = 1 - e^{-t}$. (This estimate could be sharpened somewhat by using the functions $a_1(t)$, $a_2(t)$, and $1 - e^{-2t}$ which appear in Proposition 4.6, but they would lead to no qualitative improvement in the final result and so we use this somewhat simpler form.) The other two components can be estimated in exactly the same way except for a slower exponential decay of the linear semigroup, see (59). Summarizing, we obtain:

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_\varphi\boldsymbol{\omega}(s)\,{\rm d}s\Big\|_{\mathbb{X}} \le C\rho'\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\,\|\boldsymbol{\omega}(s)\|_{\mathbb{X}}\,{\rm d}s \le C_2\rho' e^{-\mu t}\|\boldsymbol{\omega}\|_{\mathbb{Y}} . \qquad (60)$$

We now consider the nonlinear term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,{\rm d}s$. Let $1 < p < 2$. For $s \ge 0$ and $i, j \in \{1, 2, 3\}$, we know from Corollary 4.10 that $u_i(s)\omega_j(s) \in X^p(m)$ with

$$\|u_i(s)\omega_j(s)\|_{X^p(m)} \le C\|\omega_i(s)\|_{X^2(m)}\|\omega_j(s)\|_{X^2(m)} .$$

Moreover $s \mapsto u_i(s)\omega_j(s)$ is continuous in $X^p_{\rm loc}(m)$. Thus, by Remark 4.8, the integral $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,{\rm d}s$ is well defined and continuous in $\mathbb{X}_{\rm loc}$ for $t \ge 0$. Proceeding as above we can estimate the first component as follows:

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\Big(\partial_2(u_1\omega_2 - u_2\omega_1) + \partial_3(u_1\omega_3 - u_3\omega_1)\Big)(s)\,{\rm d}s\Big\|_{X^2(m)} \\
&\quad\le C\int_0^t \frac{e^{-2(t-s)}}{a(t-s)^{1/p}}\Big(\|u_1(s)\omega_2(s)\|_{X^p(m)} + \|u_2(s)\omega_1(s)\|_{X^p(m)}\Big)\,{\rm d}s \\
&\quad+ C\int_0^t \frac{e^{-\frac{3+\lambda}{2}(t-s)}}{a(t-s)^{1/p}}\Big(\|u_1(s)\omega_3(s)\|_{X^p(m)} + \|u_3(s)\omega_1(s)\|_{X^p(m)}\Big)\,{\rm d}s .
\end{aligned}$$

Repeating the same arguments for the other two components, we thus find

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,{\rm d}s\Big\|_{\mathbb{X}} \le C\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/p}}\,\|\boldsymbol{\omega}(s)\|^2_{\mathbb{X}}\,{\rm d}s \le C_3 e^{-\mu t}\|\boldsymbol{\omega}\|^2_{\mathbb{Y}} . \qquad (61)$$

Finally, we turn our attention to the inhomogeneous term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s$. To bound the first two components, we observe that $\partial_3(\tilde U_i^B(s)\Omega^B(s)) \in X^2(m)$ with

$$\|\partial_3(\tilde U_i^B(s)\Omega^B(s))\|_{X^2(m)} \le C\rho'\|\partial_3\varphi(s)\|_{L^\infty}\ , \quad i = 1,2. \tag{62}$$

Indeed, $\partial_3(\tilde U_i^B\Omega^B) = \tilde U_i^B\partial_3\Omega^B + \Omega^B\partial_3\tilde U_i^B$. Since $\Omega^B = \Omega^B(x_\perp;\rho+\varphi(x_3,t))$, one has $\partial_3\Omega^B = (\partial_\rho\Omega^B)(\partial_3\varphi)$, hence by (38), (100)

$$\|\tilde U_i^B\partial_3\Omega^B\|_{X^2(m)} \le \|\tilde U_i^B\|_{L^\infty}\|\partial_3\Omega^B\|_{X^2(m)} \le C\rho'\|\partial_3\varphi\|_{L^\infty}\ .$$

Moreover, as $\partial_3\tilde{\mathbf{U}}^B$ is the velocity field obtained from $\partial_3\mathbf{\Omega}^B = (\partial_\rho\mathbf{\Omega}^B)(\partial_3\varphi)$ by the Biot-Savart law, the proof of Proposition 4.11 shows that $\|\partial_3\tilde U_i^B\|_{L^\infty} \le C\|\partial_3\varphi\|_{L^\infty}$, hence

$$\|\Omega^B\partial_3\tilde U_i^B\|_{X^2(m)} \le \|\Omega^B\|_{X^2(m)}\|\partial_3\tilde U_i^B\|_{L^\infty} \le C\rho'\|\partial_3\varphi\|_{L^\infty}\ ,$$

which proves (62). As usual, one checks that $s \mapsto \partial_3(\tilde U_i^B(s)\Omega^B(s))$ is continuous in $X^2_{\mathrm{loc}}(m)$ for $s > 0$. Using (53), (62), the first component of the inhomogeneous term can be estimated as follows:

$$\begin{aligned}\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\partial_3(\tilde U_1^B\Omega^B)(s)\,\mathrm{d}s\Big\|_{X^2(m)} &\le C\rho'\int_0^t e^{-\frac{3+\lambda}{2}(t-s)}\frac{e^{-s}}{a(s)^{1/2}}\|\varphi^0\|_{L^\infty}\,\mathrm{d}s\\ &\le Ce^{-\mu t}\rho'\|\varphi^0\|_{L^\infty}\ ,\end{aligned}$$

and the second one is bounded in exactly the same way. To bound the third component, we first remark that $(U_i^B(s) - \tilde U_i^B(s))\Omega^B(s)$ belongs to $X^2(m)$ for $i = 1,2$ and depends continuously on $s > 0$ in $X^2_{\mathrm{loc}}(m)$. By (38), (43), (53),

$$\|(U_i^B - \tilde U_i^B)\Omega^B\|_{X^2(m)} \le \|U_i^B - \tilde U_i^B\|_{L^\infty}\|\Omega^B\|_{X^2(m)} \le C\rho'\|\partial_3\varphi\|_{L^\infty}\ ,$$

hence using (53) we find

$$\begin{aligned}&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_3)}\nabla_\perp\cdot(\mathbf{U}^B(s) - \tilde{\mathbf{U}}^B(s))\Omega^B(s)\,\mathrm{d}s\Big\|_{X^2(m)}\\ &\qquad\le C\rho'\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\frac{e^{-s}}{a(s)^{1/2}}\|\varphi^0\|_{L^\infty}\,\mathrm{d}s \le Ce^{-\mu t}\rho'\|\varphi^0\|_{L^\infty}\ .\end{aligned}$$

On the other hand, $\partial_\rho^2\Omega^B(s)(\partial_3\varphi(s))^2$ lies in $X^2(m)$ and depends continuously on $s > 0$ in $X^2_{\mathrm{loc}}(m)$. As was mentioned before Remark 2.3, $\|\partial_\rho^2\Omega^B(s)\|_{X^2(m)} \le C\lambda$. If $\lambda > 0$, we assume that $\partial_3\varphi^0 \in L^\infty(\mathbf{R})$ and using (54) we obtain

$$\begin{aligned}\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_3)}(\partial_\rho^2\Omega^B(s))(\partial_3\varphi(s))^2\,\mathrm{d}s\Big\|_{X^2(m)} &\le C\lambda\int_0^t e^{-\frac{1-\lambda}{2}(t-s)}e^{-2s}\|\partial_3\varphi^0\|_{L^\infty}^2\,\mathrm{d}s\\ &\le C\lambda e^{-\mu t}\|\partial_3\varphi^0\|_{L^\infty}^2\ .\end{aligned}$$

Thus we have shown that $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s$ is well defined and continuous in $\mathbb{X}_{\mathrm{loc}}$ for $t \ge 0$. Moreover,

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s\Big\|_{\mathbb{X}} \le C_4 e^{-\mu t}\rho'\|\varphi^0\|_{L^\infty} + C_5\lambda e^{-\mu t}\|\partial_3\varphi^0\|_{L^\infty}^2\ . \tag{63}$$

Given $\boldsymbol{\omega} \in \mathbb{Y}$, we denote by $(\mathbb{F}\boldsymbol{\omega})(t)$ the right-hand side of (57). Estimates (59), (60), (61), (63) show that $t \mapsto (\mathbb{F}\boldsymbol{\omega})(t) \in \mathbb{Y}$. Thus $\mathbb{F}$ maps $\mathbb{Y}$ into itself and

$$\|\mathbb{F}\boldsymbol{\omega}\|_{\mathbb{Y}} \le C_1\|\boldsymbol{\omega}^0\|_{\mathbb{X}} + C_2\rho'\|\boldsymbol{\omega}\|_{\mathbb{Y}} + C_3\|\boldsymbol{\omega}\|_{\mathbb{Y}}^2 + C_4\rho'\|\varphi^0\|_{L^\infty} + C_5\lambda\|\partial_3\varphi^0\|_{L^\infty}^2 , \qquad (64)$$

where $\rho' = |\rho| + \|\varphi^0\|_{L^\infty}$. Moreover, if $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{Y}$, the same estimates show that

$$\|\mathbb{F}\boldsymbol{\omega}_1 - \mathbb{F}\boldsymbol{\omega}_2\|_{\mathbb{Y}} \le \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{\mathbb{Y}}\Big(C_2\rho' + C_3(\|\boldsymbol{\omega}_1\|_{\mathbb{Y}} + \|\boldsymbol{\omega}_2\|_{\mathbb{Y}})\Big) , \qquad (65)$$

because the linear term $e^{t\mathbb{L}}\boldsymbol{\omega}^0$ and the inhomogeneous term depending on $\mathbb{H}(\varphi)$ drop out when we consider the difference $\mathbb{F}\boldsymbol{\omega}_1 - \mathbb{F}\boldsymbol{\omega}_2$. Now, choose $\rho_2 > 0$ and $\varepsilon_2 > 0$ small enough so that

$$\rho_2 + \varepsilon_2 \le \min\Big(1, R_1, \frac{1}{2C_2}\Big) , \quad \text{and} \quad \varepsilon_2 \le \frac{1}{32C_3(C_1 + C_4 + C_5)} ,$$

where R_1 is as in Proposition 2.2. Assume that $|\rho| \le \rho_2$, $\varepsilon \le \varepsilon_2$, $\|\varphi^0\|_{L^\infty} + \lambda\|\partial_3\varphi^0\|_{L^\infty}^2 \le \varepsilon$, and $\|\boldsymbol{\omega}^0\|_{\mathbb{X}} \le \varepsilon$. If $4(C_1 + C_4 + C_5)\varepsilon \le r \le 1/(8C_3)$, then (64) shows that $\mathbb{F}$ maps the ball $B_Y(0,r)$ into itself. Indeed, under the assumptions above we have $C_2\rho' \le 1/2$ and $C_3 r \le 1/4$, hence if $\boldsymbol{\omega} \in B_Y(0,r)$ then (64) implies

$$\|\mathbb{F}\boldsymbol{\omega}\|_{\mathbb{Y}} \le C_1\varepsilon + \frac{r}{2} + \frac{r}{4} + C_4\varepsilon + C_5\varepsilon = (C_1 + C_4 + C_5)\varepsilon + \frac{3r}{4} \le r .$$

Similarly, $\|\mathbb{F}\boldsymbol{\omega}_1 - \mathbb{F}\boldsymbol{\omega}_2\|_{\mathbb{Y}} \le \frac{3}{4}\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{\mathbb{Y}}$ if $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in B_Y(0,r)$. By the contraction mapping theorem, $\mathbb{F}$ has thus a unique fixed point $\boldsymbol{\omega}$ in $B_Y(0,r)$. Choosing $r = K_2\varepsilon$ with $K_2 = 4(C_1 + C_4 + C_5)$, we see that $\boldsymbol{\omega}$ is the unique solution of (57) such that $\|\boldsymbol{\omega}\|_{\mathbb{Y}} \le K_2\varepsilon$. □

Theorem 1.7 is a direct consequence of Proposition 3.3. Indeed, suppose that the initial condition for the vorticity is $\boldsymbol{\Omega}^0(x) = \boldsymbol{\Omega}^B(x_\perp;\rho) + \boldsymbol{\omega}^0(x)$, where $\boldsymbol{\omega}^0 \in X^2(m)^3$ satisfies (21). Then we can decompose $\boldsymbol{\Omega}^0(x) = \boldsymbol{\Omega}^B(x_\perp;\rho + \varphi^0(x_3)) + \tilde{\boldsymbol{\omega}}^0(x)$, where φ^0 is as in (21) and $\tilde{\boldsymbol{\omega}}^0$ belongs to $\mathbb{X}(m)$, namely $\tilde{\omega}_3^0 \in X_0^2(m)$. Moreover, there exists $C(m,\lambda) > 0$ such that

$$\|\tilde{\boldsymbol{\omega}}^0\|_{\mathbb{X}} + \|\varphi^0\|_{L^\infty} + \lambda\|\partial_3\varphi^0\|_{L^\infty}^2 \le C\varepsilon_2 ,$$

and so the smallness conditions on the perturbation in Theorem 1.7 imply those in Proposition 3.3. We deduce that the solution of (17) with initial data $\boldsymbol{\Omega}^0$ satisfies $\boldsymbol{\Omega}(x,t) = \boldsymbol{\Omega}^B(x_\perp;\rho + \varphi(x_3,t)) + \tilde{\boldsymbol{\omega}}(x,t)$ for some $\tilde{\boldsymbol{\omega}} \in \mathbb{Y}_\mu(m)$, hence $\boldsymbol{\Omega}(x,t)$ converges exponentially in $X^2(m)^3$ toward the modulated vortex $\boldsymbol{\Omega}^B(x_\perp;\rho + \varphi(x_3,t))$. On the other hand, from (51), (52), we see that, for any $x_3 \in \mathbf{R}$, $\varphi(x_3,t)$ converges toward the limiting value

$$\lim_{t\to\infty}\varphi(x_3,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-\frac{1}{2}y^2}\varphi^0(y)\,\mathrm{d}y \equiv \delta\rho ,$$

and that $\sup_{x_3\in I}|\varphi(x_3,t) - \delta\rho| = \mathcal{O}(e^{-t})$ for any compact interval $I \subset \mathbf{R}$. Thus the difference $\|\boldsymbol{\Omega}^B(\cdot;\rho + \varphi(x_3,t)) - \boldsymbol{\Omega}^B(\cdot;\rho + \delta\rho)\|_{L^2(m)}$ will converge exponentially to zero as $t \to \infty$, uniformly for x_3 in any compact interval. Combining these estimates, we obtain (22).

4 Appendix

In this appendix we collect a number of technical estimates used in the main body of the paper. They relate mostly to the behavior of the semigroup generated by the linearization of the vorticity equation around the Burgers vortex. We also prove some estimates relating the vorticity field to the corresponding velocity field defined by the Biot-Savart law.

4.1 The one-dimensional Fokker-Planck operator

Fix $\alpha > 0$, and consider the one-dimensional linear equation

$$\partial_t \omega \;=\; \mathcal{L}_\alpha \omega \;\equiv\; \partial_x^2 \omega + \frac{\alpha}{2} x \partial_x \omega + \frac{\alpha}{2} \omega \;, \tag{66}$$

where $x \in \mathbf{R}$ and $t \geq 0$. If $\omega(x,t) = \tilde{\omega}(\sqrt{\alpha} x, \alpha t)$, then $\partial_t \tilde{\omega} = \mathcal{L}_1 \tilde{\omega}$ hence we could assume without loss of generality that $\alpha = 1$. However for our purposes it is more convenient to keep $\alpha > 0$ arbitrary.

The linear operator $\mathcal{L}_\alpha$ is formally conjugated to the Hamiltonian of the harmonic operator in quantum mechanics:

$$e^{\frac{\alpha x^2}{8}} \mathcal{L}_\alpha \, e^{-\frac{\alpha x^2}{8}} \;=\; L_\alpha \;\equiv\; \partial_x^2 - \frac{\alpha^2 x^2}{16} + \frac{\alpha}{4} \;.$$

As is well-known, the spectrum of L_α in $L^2(\mathbf{R})$ is a sequence of simple eigenvalues:

$$\sigma(L_\alpha) \;=\; \Big\{ -\frac{n\alpha}{2} \,\Big|\, n = 0, 1, 2, \dots \Big\} \;,$$

and the associated eigenfunctions are the Hermite functions $h_n(x) = e^{\alpha x^2/8} \partial_x^n e^{-\alpha x^2/4}$. This observation, however, is not sufficient to determine the whole spectrum of $\mathcal{L}_\alpha$ because we want to consider this operator acting on a space of functions with algebraic (rather than Gaussian) decay at infinity.

For any $m \geq 0$ and $p \geq 1$ we define the space $\mathrm{L}^p(m) = \{ f \in L^p(\mathbf{R}) \,|\, w^m f \in L^p(\mathbf{R}) \}$, where $w(x) = (1 + x^2)^{1/2}$. This Banach space is equipped with the natural norm

$$\|f\|_{\mathrm{L}^p(m)} \;=\; \|w^m f\|_{L^p} \;=\; \Big(\int_{\mathbf{R}} |w(x)^m f(x)|^p \, \mathrm{d}x \Big)^{1/p} \;.$$

The parameter m determines the decay rate at infinity of functions in $\mathrm{L}^p(m)$. For instance, it is easy to verify that $\mathrm{L}^2(m) \hookrightarrow L^1(\mathbf{R})$ if (and only if) $m > 1/2$, because in that case $w^{-m} \in L^2(\mathbf{R})$ so that any $f \in \mathrm{L}^2(m)$ satisfies

$$\int_{\mathbf{R}} |f(x)| \, \mathrm{d}x \;=\; \int_{\mathbf{R}} w(x)^m |f(x)| w(x)^{-m} \, \mathrm{d}x \;\leq\; \|w^m f\|_{L^2} \|w^{-m}\|_{L^2} \;=\; C \|f\|_{\mathrm{L}^2(m)} \;, \tag{67}$$

by Hölder's inequality. If $m > 1/2$, we thus define

$$\mathrm{L}^2_0(m) \;=\; \Big\{ f \in \mathrm{L}^2(m) \,\Big|\, \int_{\mathbf{R}} f(x) \, \mathrm{d}x = 0 \Big\} \;.$$

This closed subspace of $\mathrm{L}^2(m)$ is clearly invariant under the evolution defined by (66).

In ([3], Appendix A) it is shown that the spectrum of $\mathcal{L}_\alpha$ in $\mathrm{L}^2(m)$ is

$$\sigma_m(\mathcal{L}_\alpha) \;=\; \Bigl\{-\frac{n\alpha}{2}\,\Bigl|\, n = 0,1,2,\dots\Bigr\} \cup \Bigl\{z \in \mathbf{C}\,\Bigl|\,\mathrm{Re}\,(z) \le \alpha\Bigl(\frac{1}{4} - \frac{m}{2}\Bigr)\Bigr\}\;. \tag{68}$$

Thus, in addition to the discrete spectrum of the harmonic oscillator, the operator $\mathcal{L}_\alpha$ also has essential spectrum due to the slow spatial decay of functions in $\mathrm{L}^2(m)$. Note however that this essential spectrum can be pushed far away from the imaginary axis by taking $m \ge 0$ sufficiently large. Therefore, if m is large, the relevant part of the spectrum of $\mathcal{L}_\alpha$ is still given by the first few eigenvalues of L_α. In particular zero is an isolated eigenvalue of $\mathcal{L}_\alpha$ if $m > 1/2$, and the rest of the spectrum is strictly contained in the left-half plane. If we restrict ourselves to the invariant subspace $\mathrm{L}^2_0(m)$, the spectrum of $\mathcal{L}_\alpha$ is unchanged except for the zero eigenvalue (which is absent).

Equation (66) can be explicitly solved as $\omega(t) = e^{t\mathcal{L}_\alpha}\omega(0)$, where

$$\Bigl(e^{t\mathcal{L}_\alpha} f\Bigr)(x) \;=\; \frac{e^{\alpha t/2}}{(4\pi a(t))^{1/2}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4a(t)}} f(ye^{\alpha t/2})\,\mathrm{d}y\;, \quad x \in \mathbf{R}\;, \quad t > 0\;,$$

and $a(t) = (1 - e^{-\alpha t})/\alpha$. Using this expression, it is straightforward to verify that $e^{t\mathcal{L}_\alpha}$ defines a strongly continuous semigroup in $\mathrm{L}^2(m)$ for any $m \ge 0$. Moreover, $e^{t\mathcal{L}_\alpha}$ maps $\mathrm{L}^2_0(m)$ into $\mathrm{L}^2_0(m)$ if $m > 1/2$, and the following estimates hold (see [3], Appendix A):

Proposition 4.1 *If $m > 1/2$, the semigroup $e^{t\mathcal{L}_\alpha}$ is uniformly bounded in $\mathrm{L}^2(m)$ for all $t \ge 0$. Moreover, if $m > 3/2$, there exists $C(m,\alpha) > 0$ such that, for all $f \in \mathrm{L}^2_0(m)$,*

$$\|e^{t\mathcal{L}_\alpha} f\|_{\mathrm{L}^2(m)} \;\le\; C\,e^{-\alpha t/2}\|f\|_{\mathrm{L}^2(m)}\;, \quad t \ge 0\;. \tag{69}$$

Finally, if $1 \le p \le 2$ and $m > 3/2$, then $e^{t\mathcal{L}_\alpha}\partial_x$ defines a bounded operator from $\mathrm{L}^p(m)$ into $\mathrm{L}^2_0(m)$ and there exists $C(m,\alpha,p) > 0$ such that, for all $f \in \mathrm{L}^p(m)$,

$$\|e^{t\mathcal{L}_\alpha}\partial_x f\|_{\mathrm{L}^2(m)} \;\le\; C\frac{e^{-\alpha t/2}}{a(t)^{\frac{1}{2p}+\frac{1}{4}}}\|f\|_{\mathrm{L}^p(m)}\;, \quad t > 0\;, \tag{70}$$

where $a(t) = (1 - e^{-\alpha t})/\alpha$.

4.2 Two-dimensional estimates

We next consider the two-dimensional equation

$$\partial_t\omega \;=\; \mathcal{L}_{\alpha_1,\alpha_2}\omega \;\equiv\; \Delta\omega + \frac{\alpha_1}{2}x_1\partial_1\omega + \frac{\alpha_2}{2}x_2\partial_2\omega + \frac{\alpha_1+\alpha_2}{2}\omega\;, \tag{71}$$

where $x \in \mathbf{R}^2$, $t \ge 0$, and $\alpha_1 \ge \alpha_2 > 0$. In the particular case where $\alpha_1 = 1 + \lambda$ and $\alpha_2 = 1 - \lambda$ for some $\lambda \in [0,1)$, we see that $\mathcal{L}_{\alpha_1,\alpha_2} = \mathcal{L}_\perp + \lambda\mathcal{M}$, where $\mathcal{L}_\perp, \mathcal{M}$ are defined in (25). Note that the parameters α_1, α_2 cannot be eliminated by a rescaling, unless $\alpha_1 = \alpha_2$.

We study the operator $\mathcal{L}_{\alpha_1,\alpha_2}$ in the weighted space

$$L^p(m) \;=\; \{f \in L^p(\mathbf{R}^2)\,|\,b^m f \in L^p(\mathbf{R}^2)\}\;, \quad \|f\|_{L^p(m)} = \|b^m f\|_{L^p}\;, \tag{72}$$

where $p \geq 1$, $m \geq 0$, and $b(x_1,x_2) = w(x_1)w(x_2) = (1+x_1^2)^{1/2}(1+x_2^2)^{1/2}$. It is clear that $L^2(m) = \mathrm{L}^2(m)\otimes\mathrm{L}^2(m)$, where $\mathrm{L}^2(m)$ is the one-dimensional space defined in the previous paragraph and $\otimes$ denotes the tensor product of Hilbert spaces, see [12]. Comparing the definitions (66), (71), we see that our operator can be decomposed as $\mathcal{L}_{\alpha_1,\alpha_2} = \mathcal{L}_{\alpha_1}\otimes\mathbf{1} + \mathbf{1}\otimes\mathcal{L}_{\alpha_2}$, where $\mathbf{1}$ denotes the identity operator. It follows that the spectrum of $\mathcal{L}_{\alpha_1,\alpha_2}$ in $L^2(m)$ is just the sum

$$\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2}) \;=\; \sigma_m(\mathcal{L}_{\alpha_1}) + \sigma_m(\mathcal{L}_{\alpha_2}) \;,$$

where $\sigma_m(\mathcal{L}_{\alpha_i})$ is given by (68) for $i = 1,2$. In particular, zero is an isolated eigenvalue of $\mathcal{L}_{\alpha_1,\alpha_2}$ if $m > 1/2$, and if $m > 3/2$ there exists $\mu > \alpha_2/2$ such that

$$\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2}) \;\subset\; \Bigl\{0\,,\,-\frac{\alpha_2}{2}\Bigr\} \,\cup\, \Bigl\{z\in\mathbf{C}\,\Big|\,\mathrm{Re}\,(z) \leq -\mu\Bigr\}\;.$$

(Recall that we assumed $\alpha_1 \geq \alpha_2$.) Moreover, if $m > 1/2$, the subspace $L^2_0(m)$ defined by (26) is invariant under the action of $\mathcal{L}_{\alpha_1,\alpha_2}$, and the restriction of $\mathcal{L}_{\alpha_1,\alpha_2}$ to $L^2_0(m)$ has spectrum $\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2})\setminus\{0\}$. Thus $\mathcal{L}_{\alpha_1,\alpha_2}$ is invertible in $L^2_0(m)$ if $m > 1/2$, with bounded inverse.

The semigroup generated by $\mathcal{L}_{\alpha_1,\alpha_2}$ satisfies $e^{t\mathcal{L}_{\alpha_1,\alpha_2}} = e^{t\mathcal{L}_{\alpha_1}}\otimes e^{t\mathcal{L}_{\alpha_2}}$. Thus, using Proposition 4.1, we immediately obtain the following estimates:

Proposition 4.2 *If $m > 1/2$, the semigroup $\mathrm{e}^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is uniformly bounded in $L^2(m)$ for all $t \geq 0$. Moreover, if $m > 3/2$, there exists $C(m,\alpha_1,\alpha_2) > 0$ such that, for all $f \in L^2_0(m)$,*

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}f\|_{L^2(m)} \;\leq\; C\,e^{-\alpha_2 t/2}\|f\|_{L^2(m)}\;,\quad t\geq 0\;. \tag{73}$$

Finally, if $1 \leq p \leq 2$ and $m > 3/2$, then $\mathrm{e}^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k$ defines a bounded operator from $L^p(m)$ into $L^2_0(m)$ for $k = 1,2$, and there exists $C(m,\alpha_1,\alpha_2,p) > 0$ such that, for all $f \in L^p(m)$,

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_1 f\|_{L^2(m)} \;\leq\; C\frac{e^{-\alpha_1 t/2}}{a_1(t)^{\frac{1}{2p}+\frac{1}{4}}a_2(t)^{\frac{1}{2p}-\frac{1}{4}}}\|f\|_{L^p(m)}\;,\quad t>0\;, \tag{74}$$

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_2 f\|_{L^2(m)} \;\leq\; C\frac{e^{-\alpha_2 t/2}}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}}a_2(t)^{\frac{1}{2p}+\frac{1}{4}}}\|f\|_{L^p(m)}\;,\quad t>0\;, \tag{75}$$

where

$$a_i(t) \;=\; \frac{1-e^{-\alpha_i t}}{\alpha_i} \;=\; \int_0^t e^{-\alpha_i s}\,\mathrm{d}s\;,\quad i=1,2\;.$$

Remark 4.3 *For $p \in [1,2]$ and $m > 1/2$, we also have the following bound:*

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}f\|_{L^2(m)} \;\leq\; \frac{C}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}}a_2(t)^{\frac{1}{2p}-\frac{1}{4}}}\|f\|_{L^p(m)}\;,\quad t>0\;.$$

We conclude this paragraph with a short discussion of the two-dimensional Biot-Savart law:

$$\mathbf{u}(x) \;=\; \frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{1}{|x-y|^2}\begin{pmatrix} y_2-x_2\\ x_1-y_1\end{pmatrix}\omega(y)\,\mathrm{d}y\;,\quad x\in\mathbf{R}^2\;. \tag{76}$$

Proposition 4.4 *Let $\mathbf{u}$ be the velocity field defined from ω via the Biot-Savart law (76).*
i) If $\omega \in L^p(\mathbf{R}^2)$ for some $p \in (1,2)$, then $\mathbf{u} \in L^q(\mathbf{R}^2)$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, and there exists $C(p) > 0$ such that $\|\mathbf{u}\|_{L^q} \le C\|\omega\|_{L^p}$.
ii) If $\omega \in L^p(\mathbf{R}^2) \cap L^q(\mathbf{R}^2)$ for some $p \in [1,2)$ and some $q \in (2,+\infty]$ then $\mathbf{u} \in C_b^0(\mathbf{R}^2)$ and there exists $C(p,q) > 0$ such that

$$\|\mathbf{u}\|_{L^\infty} \;\le\; C\|\omega\|_{L^p}^{\alpha}\|\omega\|_{L^q}^{1-\alpha}\;, \quad \textit{where} \quad \frac{1}{2} \;=\; \frac{\alpha}{p} + \frac{1-\alpha}{q}\;.$$

Proof: Assertion i) is a direct consequence of the Hardy-Littlewood-Sobolev inequality [8]. For a proof of ii), see for instance ([3], Lemma 2.1). □

We deduce from Proposition 4.4 the following useful bound on the product $\mathbf{u}\omega$:

Corollary 4.5 *Assume that $\omega_1, \omega_2 \in L^2(m)$ for some $m > 1/2$, and let $\mathbf{u}_1$ be the velocity field obtained from ω_1 via the Biot-Savart law (76). Then $\mathbf{u}_1\omega_2 \in L^p(m)$ for all $p \in (1,2)$, and there exists $C(m,p) > 0$ such that*

$$\|\mathbf{u}_1\omega_2\|_{L^p(m)} \le C\|\omega_1\|_{L^2(m)}\|\omega_2\|_{L^2(m)}\;.$$

Proof: Assume that $1 < p < 2$. By Hölder's inequality

$$\|\mathbf{u}_1\omega_2\|_{L^p(m)} \;=\; \|b^m\mathbf{u}_1\omega_2\|_{L^p} \;\le\; \|\mathbf{u}_1\|_{L^q}\|b^m\omega_2\|_{L^2}\;, \quad \text{where} \quad \frac{1}{q} \;=\; \frac{1}{p} - \frac{1}{2}\;.$$

Now $\|\mathbf{u}_1\|_{L^q} \le C\|\omega_1\|_{L^p}$ by Proposition 4.4, and $\|\omega_1\|_{L^p} \le C\|\omega_1\|_{L^2(m)}$ because $L^2(m) \hookrightarrow L^p(\mathbf{R}^2)$ for $p \in [1,2]$ if $m > 1/2$. This gives the desired result. □

4.3 The three-dimensional semigroup

This section is devoted to the three-dimensional equation

$$\partial_t\omega \;=\; \hat{\mathcal{L}}_{\alpha_1,\alpha_2}\omega \;\equiv\; \Delta\omega + \frac{\alpha_1}{2}x_1\partial_1\omega + \frac{\alpha_2}{2}x_2\partial_2\omega - x_3\partial_3\omega + \frac{\alpha_1+\alpha_2}{2}\omega\;, \tag{77}$$

where $x \in \mathbf{R}^3$, $t \ge 0$, and $\alpha_1 \ge \alpha_2 > 0$. In the particular case where $\alpha_1 = 1+\lambda$ and $\alpha_2 = 1-\lambda$, we have $\hat{\mathcal{L}}_{\alpha_1,\alpha_2} = \mathcal{L}+1$ where $\mathcal{L}$ is defined in (45).

It is important to realize that the evolution defined by (77) is essentially contracting in the transverse variables $x_\perp = (x_1, x_2)$ and expanding in the axial variable x_3. This is due to the signs of the advection terms, which in turn originate in our choice of the straining flow (2). For this reason we can assume that the solutions of (77) decay to zero as $|x_\perp| \to \infty$, but we cannot impose any decay in the x_3 variable (otherwise the solutions will not stay uniformly bounded for all times in the corresponding norm). This motivates the following choice of our function space. For $p \ge 1$ and $m \ge 0$, we introduce the Banach space

$$X^p(m) \;\equiv\; C_b^0(\mathbf{R}, L^p(m)) \;=\; \{\omega : \mathbf{R} \to L^p(m) \,|\, \omega \text{ is bounded and continuous}\} \tag{78}$$

equipped with the norm

$$\|\omega\|_{X^p(m)} \;=\; \sup_{x_3\in\mathbf{R}} \|\omega(x_3)\|_{L^p(m)}\;.$$

For any $n \in \mathbf{N}^*$ we also define the seminorm

$$|\omega|_{X^p_n(m)} \;=\; \sup_{|x_3|\le n} \|\omega(x_3)\|_{L^p(m)}~, \tag{79}$$

and we denote by $X^p_{\rm loc}(m)$ the space $X^p(m)$ equipped with the topology defined by the family of seminorms (79) for $n \in \mathbf{N}^*$. For later use, we observe that the ball $\{\omega \in X^p(m)\,|\,\|\omega\|_{X^p(m)} \le R\}$ is closed in $X^p_{\rm loc}(m)$ for any $R > 0$.

At least formally, the space $X^2(m)$ can be thought of as the tensor product $C^0_b(\mathbf{R}) \otimes L^2(m)$, i.e. the space generated by linear combinations of elements of the form $\omega(x_\perp, x_3) = f(x_3)g(x_\perp)$, with $f \in C^0_b(\mathbf{R})$ and $g \in L^2(m)$. In this picture, the linear operator defined by (77) can be decomposed as $\hat{\mathcal{L}}_{\alpha_1,\alpha_2} = \hat{\mathcal{L}}_3 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_{\alpha_1,\alpha_2}$, where $\mathcal{L}_{\alpha_1,\alpha_2}$ is defined in (71) and $\hat{\mathcal{L}}_3$ is the one-dimensional operator $\hat{\mathcal{L}}_3 f = \partial_3^2 f - x_3\partial_3 f$. It is readily verified that $\hat{\mathcal{L}}_3$ generates a semigroup in $C^0_b(\mathbf{R})$ given by the explicit formula

$$(e^{t\hat{\mathcal{L}}_3} f)(x_3) \;=\; (G_t * f)(x_3 e^{-t})~, \quad x_3 \in \mathbf{R}~, \quad t > 0~, \tag{80}$$

where G_t is defined in (52), and we know from Section 4.2 that $\mathcal{L}_{\alpha_1,\alpha_2}$ generates a strongly continuous semigroup in $L^2(m)$. Thus we expect that $\hat{\mathcal{L}}_{\alpha_1,\alpha_2}$ will generate a semigroup $\{S_t\}_{t\ge0}$ in $X^2(m)$ given by $S_t = e^{t\hat{\mathcal{L}}_3} \otimes e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$, or explicitly

$$(S_t\omega)(x_3) \;=\; \int_{\mathbf{R}} G_t(x_3e^{-t} - y_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(y_3)\Big)\,{\rm d}y_3~, \quad x_3 \in \mathbf{R}~, \quad t > 0~. \tag{81}$$

We shall prove that these heuristic considerations are indeed correct in the sense that (81) defines a semigroup of bounded operators in $X^2(m)$ with the property that $\omega(t) = S_t\omega$ is the solution of (77) with initial data $\omega \in X^2(m)$. However, the map $t \mapsto S_t\omega$ is not continuous in the topology of $X^2(m)$, but only in the (weaker) topology of $X^2_{\rm loc}(m)$. This is due to the fact that equation (77) has "infinite speed of propagation" in the sense that the advection term in the vertical variable is unbounded, see Remark 3.2.

Proposition 4.6 *For any $m \ge 0$, the family $\{S_t\}_{t\ge0}$ defined by (81) and $S_0 = \mathbf{1}$ is a semigroup of bounded linear operators on $X^2(m)$. If $\omega_0 \in X^2(m)$ and $\omega(t) = S_t\omega_0$, then $\omega : [0,+\infty) \to X^2_{\rm loc}(m)$ is continuous, and $\omega(t)$ solves (77) for $t > 0$. For any $R > 0$, if $B_R = \{f \in X^2(m)\,|\,\|f\|_{X^2(m)} \le R\}$ is equipped with the topology of $X^2_{\rm loc}(m)$, then $S_t : B_R \to X^2_{\rm loc}(m)$ is continuous, uniformly in time on compact intervals. Moreover:*

i) If $m > 1/2$ then S_t is uniformly bounded on $X^2(m)$ for all $t \ge 0$.

ii) If $m > 3/2$ there exists $C(m,\alpha_1,\alpha_2) > 0$ such that, for all ω in the subspace $X^2_0(m)$ defined in (14),

$$\|S_t\omega\|_{X^2(m)} \;\le\; C\,e^{-\frac{\alpha_2}{2}t}\|\omega\|_{X^2(m)}~, \quad t \ge 0~. \tag{82}$$

iii) If $p \in [1,2]$ and $m > 3/2$, $S_t\partial_k$ defines a bounded operator from $X^p(m)$ into $X^2(m)$ for $t > 0$ and $k = 1,2,3$, and there exists $C(m,\alpha_1,\alpha_2,p) > 0$ such that

$$\|S_t\partial_1\omega\|_{X^2(m)} \;\le\; C\,\frac{e^{-\alpha_1 t/2}}{a_1(t)^{\frac{1}{2p}+\frac14}a_2(t)^{\frac{1}{2p}-\frac14}}\,\|\omega\|_{X^p(m)}~, \tag{83}$$

$$\|S_t\partial_2\omega\|_{X^2(m)} \;\le\; C\,\frac{e^{-\alpha_2 t/2}}{a_1(t)^{\frac{1}{2p}-\frac14}a_2(t)^{\frac{1}{2p}+\frac14}}\,\|\omega\|_{X^p(m)}~, \tag{84}$$

$$\|S_t\partial_3\omega\|_{X^2(m)} \;\le\; \frac{C}{\sqrt{1-e^{-2t}}}\,\frac{1}{a_1(t)^{\frac{1}{2p}-\frac14}a_2(t)^{\frac{1}{2p}-\frac14}}\,\|\omega\|_{X^p(m)}~, \tag{85}$$

where $a_1(t), a_2(t)$ are as in Proposition 4.2.

Proof: We first rewrite (81) in a slightly more convenient form. By (52) we have $G_t(y) = c(t)^{-1/2}G(c(t)^{-1/2}y)$, where $c(t) = 1 - e^{-2t}$ and $G(z) = (2\pi)^{-1/2}e^{-z^2/2}$. Thus setting $y_3 = x_3e^{-t} + c(t)^{1/2}z_3$ in (81), we obtain the equivalent formula

$$(S_t\omega)(x_3) \;=\; \int_{\mathbf{R}} G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3e^{-t} + c(t)^{1/2}z_3)\Big)\,\mathrm{d}z_3\;, \quad x_3 \in \mathbf{R}\;, \quad t \ge 0\;. \tag{86}$$

Fix $m \ge 0$. If $\omega \in X^2(m)$, then for any $t \ge 0$ the map $x_3 \mapsto e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)$ also belongs to $X^2(m)$, because $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a bounded operator on $L^2(m)$ by Proposition 4.2. Thus it follows immediately from (86) that $S_t\omega \in X^2(m)$ and

$$\|S_t\omega\|_{X^2(m)} \;\le\; \sup_{x_3\in\mathbf{R}} \|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)\|_{L^2(m)} \;\le\; N_m(t)\|\omega\|_{X^2(m)}\;, \tag{87}$$

where $N_m(t) = \|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\|_{L^2(m)\to L^2(m)}$. The semigroup formula $S_{t_1+t_2} = S_{t_1}S_{t_2}$ is easily verified using (81), Fubini's theorem, and the identity

$$\int_{\mathbf{R}} G_{t_1}(xe^{-t_1} - y)G_{t_2}(ye^{-t_2} - z)\,\mathrm{d}y \;=\; G_{t_1+t_2}(xe^{-(t_1+t_2)} - z)\;.$$

Thus $\{S_t\}_{t\ge 0}$ is a semigroup of bounded operators in $X^2(m)$.

On the other hand, by (86), we have for all $k \in \mathbf{N}$:

$$\|S_t\omega(x_3)\|_{L^2(m)} \;\le\; \int_{-k}^{k} G(z_3)N_m(t)\|\omega(x_3e^{-t} + c(t)^{1/2}z_3)\|_{L^2(m)}\,\mathrm{d}z_3 + N_m(t)\|\omega\|_{X^2(m)}\varepsilon_k\;,$$

where $\varepsilon_k = \int_{|z|\ge k} G(z)\,\mathrm{d}z \to 0$ as $k \to \infty$. Since $|x_3e^{-t} + c(t)^{1/2}z_3| \le n + k$ whenever $|x_3| \le n$ and $|z_3| \le k$, we deduce that for all $n \in \mathbf{N}^*$:

$$|S_t\omega|_{X^2_n(m)} \;\le\; N_m(t)\Big(|\omega|_{X^2_{n+k}(m)} + \varepsilon_k\|\omega\|_{X^2(m)}\Big)\;. \tag{88}$$

This bound implies that $S_t : B_R \to X^2_{\mathrm{loc}}(m)$ is continuous, uniformly in time on compact intervals.

Furthermore, if $\omega \in X^2(m)$ and $t > 0$, we have

$$\begin{aligned}(S_t\omega - \omega)(x_3) \;&=\; \int_{\mathbf{R}} G(z_3)e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\Big(\omega(x_3e^{-t} + c(t)^{1/2}z_3) - \omega(x_3)\Big)\,\mathrm{d}z_3 \\ &\quad+\; e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3) - \omega(x_3)\;,\end{aligned}$$

hence proceeding as above we find for all $n, k \in \mathbf{N}^*$:

$$\begin{aligned}|S_t\omega - \omega|_{X^2_n(m)} \;&\le\; \int_{-k}^{k} G(z_3)N_m(t)\sup_{|x_3|\le n}\|\omega(x_3e^{-t} + c(t)^{1/2}z_3) - \omega(x_3)\|_{L^2(m)}\,\mathrm{d}z_3 \\ &\quad+\; 2N_m(t)\varepsilon_k\|\omega\|_{X^2(m)} + \sup_{|x_3|\le n}\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3) - \omega(x_3)\|_{L^2(m)}\;.\end{aligned}$$

The last term goes to zero as $t \to 0+$ because $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a strongly continuous semigroup on $L^2(m)$ and $\omega : [-n, n] \to L^2(m)$ is continuous (hence has compact range). Similarly, for each $k \in \mathbf{N}$, the integral goes to zero as $t \to 0+$ by Lebesgue's dominated convergence theorem, because $\omega : [-n-k, n+k] \to L^2(m)$ is uniformly continuous. Since $\varepsilon_k \to 0$ as $k \to \infty$, we conclude that $S_t\omega \to \omega$ in $X^2_{\mathrm{loc}}(m)$ as $t \to 0+$. Then, using the semigroup property, we deduce that the map $t \mapsto S_t\omega$ is continuous to the right at any $t \geq 0$. Finally, if $t > \varepsilon > 0$, we have $S_t\omega - S_{t-\varepsilon}\omega = S_{t-\varepsilon}(S_\varepsilon\omega - \omega)$, hence by (88)

$$|S_t\omega - S_{t-\varepsilon}\omega|_{X^2_n(m)} \;\leq\; N_m(t-\varepsilon)\Big(|S_\varepsilon\omega - \omega|_{X^2_{n+k}(m)} + \varepsilon_k(1 + N_m(\varepsilon))\|\omega\|_{X^2(m)}\Big) \;.$$

Since $|S_\varepsilon\omega - \omega|_{X^2_{n+k}(m)} \to 0$ as $\varepsilon \to 0+$ for all $k, n \in \mathbf{N}^*$, this shows that $t \mapsto S_t\omega$ is also continuous to the left at any $t > 0$.

Next, using (86), (80), and the explicit formula for $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$, it is rather straightforward to verify that, for any $\omega \in X^2(m)$, the map $(x_\perp, x_3, t) \mapsto \omega(x_\perp, x_3, t) = ((S_t\omega)(x_3))(x_\perp)$ is smooth and satisfies (77) for $t > 0$. Thus $S_t\omega$ is indeed the solution of (77) with initial data ω.

It remains to establish the decay properties of S_t:

i) If $m > 1/2$, we know from Proposition 4.2 that $N_m(t) \leq C$ for all $t \geq 0$, hence $\{S_t\}_{t\geq 0}$ is uniformly bounded on $X^2(m)$ by (87).

ii) If $m > 3/2$ and $\omega \in X^2_0(m)$, then $\omega(x_3) \in L^2_0(m)$ for all $x_3 \in \mathbf{R}$ and (82) follows immediately from (87) and (73).

iii) If $k = 1, 2$, we define $S_t\partial_k$ by

$$(S_t\partial_k\omega)(x_3) \;=\; \int_{\mathbf{R}} G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k\omega(x_3e^{-t} + c(t)^{1/2}z_3)\Big)\,\mathrm{d}z_3 \;. \qquad (89)$$

If $m > 3/2$ and $p \in [1, 2]$, we know from Proposition 4.2 that $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k$ is a bounded operator from $L^p(m)$ into $L^2_0(m)$ satisfying (74) or (75). Thus the formula (89) defines a bounded operator from $X^p(m)$ into $X^2_0(m)$ and

$$\|S_t\partial_k\omega\|_{X^2(m)} \;\leq\; \sup_{x_3\in\mathbf{R}} \|\mathrm{e}^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k\omega(x_3)\|_{L^2(m)} \;.$$

Thus (83), (84) follow immediately from (74), (75). Finally we define $S_t\partial_3$ by

$$(S_t\partial_3\omega)(x_3) \;=\; -\frac{1}{c(t)^{1/2}}\int_{\mathbf{R}} \partial_3G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3e^{-t} + c(t)^{1/2}z_3)\Big)\,\mathrm{d}z_3 \;. \qquad (90)$$

We know that $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a bounded operator from $L^p(m)$ into $L^2(m)$, see Remark 4.3. Since $\partial_3G \in L^1(\mathbf{R}^2)$, we thus find

$$\begin{aligned}\|S_t\partial_3\omega\|_{X^2(m)} \;&\leq\; \frac{C}{c(t)^{1/2}}\sup_{x_3\in\mathbf{R}}\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)\|_{L^2(m)} \\ &\leq\; \frac{C}{c(t)^{1/2}}\frac{1}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}}a_2(t)^{\frac{1}{2p}-\frac{1}{4}}}\|\omega\|_{X^p(m)} \;,\end{aligned}$$

which is (85). This concludes the proof. $\square$

Corollary 4.7 *Let $T > 0$ and let $f : [0,T] \to X^2(m)$ be a bounded function satisfying $f \in C^0([0,T], X^2_{\rm loc}(m))$. Then the map $F : [0,T] \to X^2(m)$ defined by*

$$F(t) = \int_0^t S_{t-s} f(s)\,\mathrm{d}s\ , \quad 0 \le t \le T\ ,$$

satisfies $F \in C^0([0,T], X^2_{\rm loc}(m))$, and $\|F(t)\|_{X^2(m)} \le \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s$, where N_m is as in (87).

Proof: For any $t \in [0,T]$, we define $\psi_t : [0,t] \to X^2(m)$ by $\psi_t(s) = S_{t-s}f(s)$. Since $f \in C^0([0,T], X^2_{\rm loc}(m))$ and since the semigroup S_t is continuous on $X^2_{\rm loc}(m)$ as described in Proposition 4.6, it is easy to verify that the map $\psi_t : [0,t] \to X^2_{\rm loc}(m)$ is also continuous. As $X^2_{\rm loc}(m)$ is a subspace of the Fréchet space $C^0(\mathbf{R}, L^2(m))$, the integral $\int_0^t \psi_t(s)\,\mathrm{d}s$ can be defined as in ([15], Theorem 3.17). However, in the present case, we can also use the following “pedestrian” construction (which agrees with the general one). For any $n \in \mathbf{N}^*$, we define

$$F_n(t) = \int_0^t \chi_n \psi_t(s)\,\mathrm{d}s = \int_0^t \chi_n S_{t-s} f(s)\,\mathrm{d}s\ ,$$

where χ_n denotes the map $x_3 \mapsto \mathbf{1}_{[-n,n]}(x_3)$. Clearly $\chi_n\psi_t(s)$ is a continuous function of $s \in [0,t]$ with values in the Banach space $C^0([-n,n], L^2(m))$, hence $F_n(t) \in C^0([-n,n], L^2(m))$ can be defined for any $t \in [0,T]$ as a Banach-valued Riemann integral. Using again the continuity of the semigroup S_t one finds that $F_n : [0,T] \to C^0([-n,n], L^2(m))$ is continuous and satisfies

$$|F_n(t)|_{X^2_n(m)} \le \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s \le C(T)\int_0^t \|f(s)\|_{X^2(m)}\,\mathrm{d}s\ .$$

(Note that $t \mapsto N_m(t)$ and $t \mapsto \|f(t)\|_{X^2(m)}$ are lower semicontinuous, hence measurable.) Now, for each $t \in [0,T]$, it is clear that $(F_m(t))(x_3) = (F_n(t))(x_3)$ if $|x_3| \le n \le m$, hence there is a unique $F(t) \in C^0(\mathbf{R}, L^2(m))$ such that $(F(t))(x_3) = (F_n(t))(x_3)$ whenever $|x_3| \le n$. By construction, $\|F(t)\|_{X^2(m)} \le \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s$ for all $t \in [0,T]$, and $F \in C^0([0,T], X^2_{\rm loc}(m))$. $\square$

Remark 4.8 *Similarly, if $p \in (1,2]$ and $k \in \{1,2,3\}$, Proposition 4.6 implies that, if $f : [0,T] \to X^p(m)$ is bounded in $X^p(m)$ and continuous in $X^p_{\rm loc}(m)$, the map $F : [0,T] \to X^2(m)$ defined by*

$$F(t) = \int_0^t S_{t-s}\partial_k f(s)\,\mathrm{d}s\ , \quad 0 \le t \le T\ ,$$

is bounded in $X^2(m)$ and continuous in $X^2_{\rm loc}(m)$. In that case, for each $n \in \mathbf{N}^*$ and each $t \in (0,T]$, $F_n(t) \equiv \chi_n F(t)$ is defined by a “generalized” Riemann integral, because the integrand has a singularity at $s = t$.

4.4 The three-dimensional Biot-Savart law

In this final section, we discuss the three-dimensional Biot-Savart law, namely

$$\mathbf{u}(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(\mathbf{x}-\mathbf{y}) \times \boldsymbol{\omega}(y)}{|x-y|^3}\,\mathrm{d}y\ , \quad x \in \mathbf{R}^3\ . \tag{91}$$

We first prove the analogue of Proposition 4.4 in the spaces $X^p(m)$ defined by (78).

Proposition 4.9 *Fix $m > 1/2$. If $\boldsymbol{\omega} \in X^2(m)$, the velocity field given by (91) satisfies $\mathbf{u} \in X^q(0)$ for all $q \in (2,\infty)$, and there exists $C(m,q) > 0$ such that $\|\mathbf{u}\|_{X^q(0)} \le C\|\boldsymbol{\omega}\|_{X^2(m)}$.*

Proof: Assume that $\omega \in X^2(m)$ for some $m > 1/2$. For all $x = (x_\perp, x_3) \in \mathbf{R}^3$, we have by Fubini's theorem

$$|\mathbf{u}(x_\perp, x_3)| \le C \int_{\mathbf{R}^3} \frac{|\boldsymbol{\omega}(y_\perp, y_3)|}{|x_\perp - y_\perp|^2 + (x_3 - y_3)^2} \,\mathrm{d}y_\perp \,\mathrm{d}y_3 = C \int_{\mathbf{R}} F(x_\perp; x_3, y_3) \,\mathrm{d}y_3 ,$$

where

$$F(x_\perp; x_3, y_3) = \int_{\mathbf{R}^2} \frac{|\boldsymbol{\omega}(y_\perp, y_3)|}{|x_\perp - y_\perp|^2 + (x_3 - y_3)^2} \,\mathrm{d}y_\perp .$$

By Minkowski's inequality, it follows that

$$\|\mathbf{u}(\cdot, x_3)\|_{L^q(\mathbf{R}^2)} \le C \int_{\mathbf{R}} \|F(\cdot; x_3, y_3)\|_{L^q(\mathbf{R}^2)} \,\mathrm{d}y_3 . \tag{92}$$

If $2 < q < \infty$, we shall show that there exists $H_q \in L^1(\mathbf{R})$ and $C > 0$ such that

$$\|F(\cdot; x_3, y_3)\|_{L^q(\mathbf{R}^2)} \le C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(m)} H_q(x_3 - y_3) , \quad x_3, y_3 \in \mathbf{R} . \tag{93}$$

Together with (92), this gives $\|\mathbf{u}(\cdot, x_3)\|_{L^q(\mathbf{R}^2)} \le C\|\boldsymbol{\omega}\|_{X^2(m)}$ for all $x_3 \in \mathbf{R}$, which is the desired bound. Since the Biot-Savart law is invariant under spatial translations, the same arguments show that, for all $x_3 \in \mathbf{R}$,

$$\|\mathbf{u}(\cdot, x_3 + \varepsilon) - \mathbf{u}(\cdot, x_3)\|_{L^q(\mathbf{R}^2)} \le C \int_{\mathbf{R}} \|\boldsymbol{\omega}(\cdot, y_3 + \varepsilon) - \boldsymbol{\omega}(\cdot, y_3)\|_{L^2(m)} H_q(x_3 - y_3) \,\mathrm{d}y_3 .$$

As $\varepsilon \to 0$ the right-hand side converges to zero by Lebesgue's dominated convergence theorem, thus $\mathbf{u} \in C_b^0(\mathbf{R}, L^q(\mathbf{R}^2)) \equiv X^q(0)$.

To prove (93), for any $a \in \mathbf{R}$ we define $f_a(y_\perp) = (|y_\perp|^2 + a^2)^{-1}$. If $a \ne 0$, then $f_a \in L^r(\mathbf{R}^2)$ for all $r > 1$, and there exists $C_r > 0$ such that

$$\|f_a\|_{L^r(\mathbf{R}^2)} \le \frac{C_r}{|a|^{2-\frac{2}{r}}} .$$

Since $F(\cdot; x_3, y_3) = |\boldsymbol{\omega}(\cdot, y_3)| * f_{x_3 - y_3}$, Young's inequality implies

$$\|F(\cdot; x_3, y_3)\|_{L^q(\mathbf{R}^2)} \le C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)} \|f_{x_3-y_3}\|_{L^p(\mathbf{R}^2)} \le \frac{C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)}}{|x_3 - y_3|^{2-\frac{2}{p}}} , \tag{94}$$

where $1 + \frac{1}{q} = \frac{1}{2} + \frac{1}{p}$, hence $2 - \frac{2}{p} = 1 - \frac{2}{q} < 1$. On the other hand, by Hölder's inequality,

$$\begin{aligned} F(x_\perp; x_3, y_3) &= \int_{\mathbf{R}^2} b^m(y_\perp) |\boldsymbol{\omega}(y_\perp, y_3)| \frac{1}{b^m(y_\perp)(|x_\perp - y_\perp|^2 + (x_3 - y_3)^2)} \,\mathrm{d}y_\perp \\ &\le \|b^m \boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)} \Big(\frac{1}{b^{2m}} * f_{x_3-y_3}^2\Big)^{1/2}(x_\perp) . \end{aligned}$$

Using Young's inequality again, we obtain

$$\begin{aligned}\|F(\cdot;x_3,y_3)\|_{L^q(\mathbf{R}^2)} &\le C\|\boldsymbol{\omega}(\cdot,y_3)\|_{L^2(m)}\|b^{-m}\|_{L^2(\mathbf{R}^2)}\|f_{x_3-y_3}\|_{L^q(\mathbf{R}^2)}\\ &\le \frac{C\|\boldsymbol{\omega}(\cdot,y_3)\|_{L^2(m)}}{|x_3-y_3|^{2-\frac{2}{q}}}\ ,\end{aligned} \tag{95}$$

where $2-\frac{2}{q}>1$. Combining (94), (95), we obtain (93). □

An easy consequence is the analogue of Corollary 4.5:

Corollary 4.10 *Assume that $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2 \in X^2(m)$ for some $m>1/2$, and let $\mathbf{u}_1$ be the velocity field obtained from $\boldsymbol{\omega}_1$ via the Biot-Savart law (91). Then $\mathbf{u}_1\boldsymbol{\omega}_2 \in X^p(m)$ for all $p\in(1,2)$, and there exists $C(m,p)>0$ such that*

$$\|\mathbf{u}_1\boldsymbol{\omega}_2\|_{X^p(m)} \le C\|\boldsymbol{\omega}_1\|_{X^2(m)}\|\boldsymbol{\omega}_2\|_{X^2(m)}\ . \tag{96}$$

Proof: Assume that $1<p<2$ and $\frac{1}{q}=\frac{1}{p}-\frac{1}{2}$. By Hölder's inequality, we have for all $x_3\in\mathbf{R}$

$$\|b(\cdot)^m\mathbf{u}_1(\cdot,x_3)\boldsymbol{\omega}_2(\cdot,x_3)\|_{L^p(\mathbf{R}^2)} \le \|\mathbf{u}_1(\cdot,x_3)\|_{L^q(\mathbf{R}^2)}\|b(\cdot)^m\boldsymbol{\omega}_2(\cdot,x_3)\|_{L^2(\mathbf{R}^2)}\ .$$

Taking the supremum over x_3 and using Proposition 4.9, we obtain (96). Moreover, since $\mathbf{u}_1\in X^q(0)$ and $\boldsymbol{\omega}_2\in X^2(m)$, it is clear that $x_3\mapsto \mathbf{u}_1(\cdot,x_3)\boldsymbol{\omega}_2(\cdot,x_3)$ is continuous from $\mathbf{R}$ into $L^p(m)$. □

In the rest of this section, we fix some $\lambda\in[0,1)$. Given $\rho\in\mathbf{R}$ and $\varphi\in C_b^1(\mathbf{R})$, our goal is to compare the velocity field $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3))$ defined by (36) with the velocity field $\tilde{\mathbf{U}}^B(x;\rho,\varphi)$ obtained from $\boldsymbol{\Omega}^B(x_\perp;\rho+\varphi(x_3))$ via the Biot-Savart law. (As in Section 3 we omit the dependence on λ for simplicity.) Since $\boldsymbol{\Omega}^B$ has only the third component nonzero, (91) implies that $\tilde{\mathbf{U}}^B$ has only the first two components nonzero:

$$\begin{pmatrix}\tilde{U}_1^B(x;\rho,\varphi)\\ \tilde{U}_2^B(x;\rho,\varphi)\end{pmatrix} = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{1}{|x-y|^3}\begin{pmatrix}y_2-x_2\\ x_1-y_1\end{pmatrix}\Omega^B(y_\perp;\rho+\varphi(y_3))\,\mathrm{d}y_\perp\,\mathrm{d}y_3\ . \tag{97}$$

On the other hand, for any $x_3\in\mathbf{R}$, $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3))$ is obtained from $\boldsymbol{\Omega}^B(x_\perp;\rho+\varphi(x_3))$ via the two-dimensional Biot-Savart law (76), which can be written in the form

$$\begin{pmatrix}U_1^B(x_\perp;\rho+\varphi(x_3))\\ U_2^B(x_\perp;\rho+\varphi(x_3))\end{pmatrix} = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{1}{|x-y|^3}\begin{pmatrix}y_2-x_2\\ x_1-y_1\end{pmatrix}\Omega^B(y_\perp;\rho+\varphi(x_3))\,\mathrm{d}y_\perp\,\mathrm{d}y_3\ , \tag{98}$$

because

$$\int_{\mathbf{R}}\frac{1}{|x-y|^3}\,\mathrm{d}y_3 \equiv \int_{\mathbf{R}}\frac{1}{(|x_\perp-y_\perp|^2+(x_3-y_3)^2)^{3/2}}\,\mathrm{d}y_3 = \frac{2}{|x_\perp-y_\perp|^2}\ . \tag{99}$$

Using these representation formulas, it is easy to show that the velocity fields $\tilde{\mathbf{U}}^B$, $\mathbf{U}^B$ are close if the function φ varies sufficiently slowly.

Proposition 4.11 *Fix $\lambda \in [0,1)$, and assume that $\rho \in \mathbf{R}$ and $\varphi \in C_b^1(\mathbf{R})$ satisfy $|\rho| + \|\varphi\|_{L^\infty} \le R_1(\lambda)$, where R_1 is defined in Proposition 2.2. Then $\tilde{\mathbf{U}}^B(\cdot;\rho,\varphi) \in C_b^0(\mathbf{R}^3)$, and there exists $C(\lambda) > 0$ such that*

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x;\rho,\varphi)| \;\le\; C(|\rho| + \|\varphi\|_{L^\infty}) \;, \tag{100}$$

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x;\rho,\varphi) - \mathbf{U}^B(x_\perp;\rho+\varphi(x_3))| \;\le\; C\|\varphi'\|_{L^\infty} \;. \tag{101}$$

Proof: Since $\Omega^B(x_\perp;\rho)$ is a continuous function of $x_\perp \in \mathbf{R}^2$ which decays rapidly as $|x_\perp| \to \infty$, uniformly in $\rho \in [-R_1, R_1]$, it is not difficult to verify that the velocity field $\tilde{\mathbf{U}}^B(x;\rho,\varphi)$ defined by (97) depends continuously on $x \in \mathbf{R}^3$. Next using (99) and (39) with $m = 1$, we find

$$|\tilde{\mathbf{U}}^B(x;\rho,\varphi)| \;\le\; \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{|x_\perp - y_\perp|}\frac{C(|\rho| + \|\varphi\|_{L^\infty})}{b(y_\perp)}\,\mathrm{d}y_\perp \;.$$

Since $b^{-1} \in L^p(\mathbf{R}^2)$ for all $p \in (1,\infty]$, the above integral is uniformly bounded for all $x_\perp \in \mathbf{R}^2$ (see Proposition 4.4), and we obtain (100).

Finally, taking the difference of (97) and (98), we see that $|\tilde{\mathbf{U}}^B - \mathbf{U}^B| \le C(D_1 + D_2)$, where

$$D_i(x) \;=\; \int_{\mathbf{R}^3} \frac{|x_i - y_i|}{|x-y|^3}\Big|\Omega^B(y_\perp;\rho+\varphi(y_3)) - \Omega^B(y_\perp;\rho+\varphi(x_3))\Big|\,\mathrm{d}y_\perp\,\mathrm{d}y_3 \;, \quad i = 1,2 \;.$$

But

$$\begin{aligned}\Big|\Omega^B(y_\perp;\rho+\varphi(y_3)) - \Omega^B(y_\perp;\rho+\varphi(x_3))\Big| \;&\le\; \Big|\int_{x_3}^{y_3} \partial_\rho\Omega^B(y_\perp;\rho+\varphi(z))\varphi'(z)\,\mathrm{d}z\Big| \\ &\le\; |x_3 - y_3|\|\varphi'\|_{L^\infty} \sup_{|\rho|\le R_1} |\partial_\rho\Omega^B(y_\perp;\rho)| \;.\end{aligned}$$

Since

$$\int_{\mathbf{R}} \frac{|x_3 - y_3|}{|x-y|^3}\,\mathrm{d}y_3 \;=\; \frac{2}{|x_\perp - y_\perp|} \;, \quad \text{and} \quad \frac{|x_i - y_i|}{|x_\perp - y_\perp|} \;\le\; 1 \;,$$

we thus find

$$\|D_i\|_{L^\infty} \;\le\; 2\|\varphi'\|_{L^\infty}\int_{\mathbf{R}^2} \sup_{|\rho|\le R_1} |\partial_\rho\Omega^B(y_\perp;\rho)|\,\mathrm{d}y_\perp \;, \quad i = 1,2 \;. \tag{102}$$

Using now (39) with $m = 2$, we see that the integrand is bounded by $C/b(y_\perp)^2$, hence the integral in (102) is finite. This gives (101). □

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