Mean Field Dynamics of Boson Stars

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Abstract

We consider a quantum mechanical system of N bosons with relativistic dispersion interacting through a mean field Coulomb potential (attractive or repulsive). We choose the initial wave function to describe a condensate, where the N bosons are all in the same one-particle state. Starting from the N-body Schrödinger equation, we prove that, in the limit $N \to \infty$, the time evolution of the one-particle density is governed by the relativistic nonlinear Hartree equation. This equation is used to describe the dynamics of boson stars (Chandrasekhar theory). The corresponding static problem was rigorously solved in [10].

1 Introduction

We consider a system of N identical bosons with relativistic dispersion relation and with a mean field Coulomb interaction. The system is described on the Hilbert space $L^2_s(\mathbb{R}^{3N}, \mathrm{d}\mathbf{x})$, the subspace of $L^2(\mathbb{R}^{3N}, \mathrm{d}\mathbf{x})$ containing all functions symmetric with respect to permutations. The Hamiltonian of the system is given by

$$H_N = \sum_{j=1}^{N} (1 - \Delta_j)^{1/2} + \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \, . \tag{1.1}$$

Here we use units with $\hbar = m = 1$ (where m denotes the mass of the bosons). The Hamiltonian H_N defines a mean field interaction among the bosons because the coupling constant is proportional to $1/N$: with this scaling the kinetic and potential part of the energy are typically of the same order. This condition is necessary in order to have a mean field description of the system in the limit of large N.

The constant λ can be positive or negative, corresponding to repulsive and attractive interaction. In the case of repulsive interaction we will have no restriction on the value of λ. The choice of a negative λ, which corresponds to an attractive Coulomb potential, leads to a Chandrasekhar theory of boson stars, where general relativity effects are neglected. In this case we need to impose the condition $\lambda > -4/\pi$. To understand why this assumption is necessary, note that, if $\lambda > -4/\pi$, the Hamiltonian H_N, with domain $D(H_N) = H^1(\mathbb{R}^{3N})$, is self adjoint and stable, in the sense that the ground state energy of H_N divided by the number of particle N, is bounded below, uniformly in N. On the other hand, for $\lambda < -4/\pi$, the Hamiltonian H_N is unstable: the ground state energy

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per particle diverges to $-\infty$ as $N \to \infty$. This phenomenon is usually referred to as the collapse of the system: the energy is minimized by letting the particles closer and closer, because the increase of the kinetic energy (due to the localization of the particles) is not enough, if $\lambda < -4/\pi$, to compensate for the decrease of the potential energy. These results were derived in [10], where the authors prove that the energy per particle of the ground state of the system is given, in the limit $N \to \infty$, by the minimization of a one-particle energy functional:

$$\begin{aligned}\lim_{N\to\infty} \frac{E_{\rm GS}(\lambda)}{N} &= \inf_{\varphi:\|\varphi\|=1} \mathcal{E}(\varphi,\overline{\varphi}) \\ &= \inf_{\varphi:\|\varphi\|=1} \left(\int \mathrm{d}x\, |(1-\Delta)^{1/4}\varphi(x)|^2 + \frac{\lambda}{2}\int \mathrm{d}x\, (V * |\varphi|^2)(x)\, |\varphi(x)|^2 \right).\end{aligned} \tag{1.2}$$

Here we defined $V(x) = |x|^{-1}$. The r.h.s. is negative infinity if $\lambda < -4/\pi$. The existence of a critical coupling $\lambda_{\rm crit} = -4/\pi$ is due to the fact that the kinetic energy, which behaves, for large momenta, as $|\nabla|$, and the potential $|x|^{-1}$ scale in the same way.

In this paper we are interested in the dynamics generated by H_N in the limit $N \to \infty$. From (1.2) one can expect that the macroscopic dynamics of the system are described, in the limit $N \to \infty$, by the one particle nonlinear relativistic Hartree equation

$$i\partial_t \varphi_t(x) = \frac{\delta \mathcal{E}}{\delta \overline{\varphi}}(\varphi_t(x), \overline{\varphi_t}(x)) = (1-\Delta)^{1/2}\varphi_t(x) + \lambda (V * |\varphi_t|^2)(x)\varphi_t(x)\,. \tag{1.3}$$

Some important properties of this equation, such as the global well-posedness for $\lambda > \lambda_{\rm crit}$, are proven in [9].

To formulate the convergence towards the nonlinear Hartree equation (1.3) more precisely, we define next the marginal distributions of an N-particle wave function ψ_N, and we investigate their time evolution. Given an N-particle wave function $\psi(\mathbf{x}) \in L^2_s(\mathbb{R}^{3N}, \mathrm{d}\mathbf{x})$ with $\|\psi\| = 1$, we define the corresponding density matrix $\gamma_N = |\psi_N\rangle\langle\psi_N|$ as the orthogonal projection onto ψ. The kernel of γ_N is

$$\gamma_N(\mathbf{x};\mathbf{x}') = \psi_N(\mathbf{x})\overline{\psi_N}(\mathbf{x}')\,.$$

More generally, a density matrix γ_N is a non-negative trace class operator on $L^2_s(\mathbb{R}^{3N})$ with $\mathrm{Tr}\,\gamma_N = 1$. For $k = 1, \dots N$, the k-particle marginal distribution of γ_N, denoted by $\gamma_N^{(k)}$, is defined by taking the partial trace over the last $N-k$ variables of γ_N, that is, the kernel of $\gamma_N^{(k)}$ is given by

$$\gamma_N^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\, \gamma_N(\mathbf{x}_k, \mathbf{x}_{N-k}; \mathbf{x}'_k, \mathbf{x}_{N-k})\,.$$

Here and henceforth we use the notation

$$\mathbf{x} = (x_1, \dots, x_N), \quad \mathbf{x}_k = (x_1, \dots, x_k), \quad \mathbf{x}_{N-k} = (x_{k+1}, x_{k+2}, \dots, x_N)$$

with $x_j \in \mathbb{R}^3$, for all $j = 1, \dots N$, and analogously for the primed variables. Since $\mathrm{Tr}\,\gamma_N = 1$, it follows immediately that $\mathrm{Tr}\,\gamma_N^{(k)} = 1$ for all $k = 1, \dots, N$.

The time evolution of the system is governed by the Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N\, \psi_{N,t} \tag{1.4}$$

or, equivalently, by the Heisenberg equation

$$i\partial_t \gamma_{N,t} = [H_N, \gamma_{N,t}]$$

for the dynamics of γ_N. From the Schrödinger equation we can also derive a hierarchy of N equations, usually called the BBGKY hierarchy, describing the evolution of the marginal distributions $\gamma^{(k)}_{N,t}$ for $k = 1, \dots, N$. Using the permutation symmetry we find, for $k = 1, \dots, N$,

$$\begin{aligned} i\partial_t \gamma^{(k)}_{N,t} = & \sum_{j=1}^k \left[(1-\Delta_{x_j})^{1/2}, \gamma^{(k)}_{N,t} \right] + \frac{\lambda}{N} \sum_{1 \le i < j \le k} \left[V(x_i - x_j), \gamma^{(k)}_{N,t} \right] \\ & + \lambda (1 - \frac{k}{N}) \sum_{j=1}^k \mathrm{Tr}_{k+1} \left[V(x_j - x_{k+1}), \gamma^{(k+1)}_{N,t} \right] . \end{aligned} \tag{1.5}$$

(Recall that $V(x) = |x|^{-1}$). Here we use the convention $\gamma^{(k)}_N = 0$, if $k = N+1$. Moreover we denote by Tr_{k+1} the partial trace over the variable x_{k+1}. In terms of the kernels $\gamma^{(k)}_{N,t}(\mathbf{x}_k, \mathbf{x}'_k)$ the last equation can be written as

$$\begin{aligned} i\partial_t \gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) = & \sum_{j=1}^k ((1-\Delta_{x_j})^{1/2} - (1-\Delta_{x'_j})^{1/2}) \gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) \\ & + \frac{\lambda}{N} \sum_{1 \le i < j \le k} \left(V(x_i - x_j) - V(x'_i - x'_j) \right) \gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) \\ & + \lambda (1 - \frac{k}{N}) \sum_{j=1}^k \int \mathrm{d}x_{k+1} \left(V(x_j - x_{k+1}) - V(x'_j - x_{k+1}) \right) \gamma^{(k+1)}_{N,t}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) . \end{aligned}$$

In the limit $N \to \infty$ the BBGKY hierarchy (1.5) formally converges to the infinite hierarchy of equations

$$i\partial_t \gamma^{(k)}_t = \sum_{j=1}^k \left[(1-\Delta_{x_j})^{1/2}, \gamma^{(k)}_t \right] + \lambda \sum_{j=1}^k \mathrm{Tr}_{k+1} \left[V(x_j - x_{k+1}), \gamma^{(k+1)}_t \right] \tag{1.6}$$

for all $k \ge 1$. It is easy to check that, if the initial data is factorized, that is if

$$\gamma^{(k)}_{t=0}(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi(x_j) \overline{\varphi}(x'_j)$$

for all $k \ge 1$, then the infinite hierarchy (1.6) has the solution

$$\gamma^{(k)}_t(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi_t(x_j) \overline{\varphi_t}(x'_j)$$

where $\varphi_t(x)$ is the solution of the non-linear one-particle Hartree equation (1.3) with initial data $\varphi_{t=0} = \varphi$ (for $\lambda > \lambda_{\text{crit}} = -4/\pi$, (1.3) is known to have a unique global solution in the space $H^{m/2}(\mathbb{R}^3)$, for every $m \ge 1$, see [9]). Hence, if we consider a factorized initial wave function $\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j)$, if we fix $k \ge 1$ and $t \in \mathbb{R}$, and if we denote by $\gamma^{(k)}_{N,t}$ the k-particle marginal distribution associated with the time evolution $\psi_{N,t}$ of ψ_N, then we can expect that, in a suitable weak topology,

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k; \mathbf{x}'_k) \to \gamma^{(k)}_t(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi_t(x_j) \overline{\varphi_t}(x'_j) \quad \text{for } N \to \infty \tag{1.7}$$

where φ_t is the solution of the Hartree equation (1.3). The aim of this paper is to give a rigorous proof of this statement.

The first result of this type was proven in [7] for non-relativistic dispersions and for smooth potentials. This work was generalized to bounded potentials in [13]. In [6] the authors show the convergence (1.7), for a non-relativistic dispersion, and for any integrable potential: they use the formalism of second quantization and they need the initial state to be coherent (and thus the number of particles cannot be fixed). In [4] the convergence (1.7) was proven for bosons with non-relativistic dispersion interacting through a Coulomb potential. Partial results for the non-relativistic Coulomb case were also established in [1]. In [2], a joint work with L. Erdős and H.-T. Yau, we consider a system of N non-relativistic bosons, interacting, in the mean-field scaling, through an N-dependent potential $V_N(x) = N^{3\beta}V(N^\beta x)$, with $0 < \beta < 3/5$, which converges to a delta-function in the limit $N \to \infty$: for this potential we prove the convergence of solutions of the finite hierarchy (1.5) (with $V(x)$ replaced by $V_N(x)$) to solutions of the infinite hierarchy (1.6) (with V replaced by $\delta(x)$). A recent overview of rigorous results and open problems concerning the nonlinear Hartree equation and its derivation as the mean field limit of large bosonic systems can be found in [5].

Note that the non-relativistic case considered in all these works is technically easier than the relativistic case we are considering in the present paper: the presence of a quadratic dispersion (the non-relativistic kinetic energy of a boson is given by the negative Laplacian, quadratic in the momentum) makes the control of the Coulomb singularity much simpler than in the relativistic case, where the kinetic energy is only linear in the momentum (for large momenta, $(1+p^2)^{1/2} \simeq |p|$). Hence, although our general strategy is the same as in [4], we need here more refined estimates.

Next we explain the methods we use to prove (1.7) in some details. The main technical tool in our analysis is an a-priori estimate (Theorem 4.4), which guarantees a certain smoothness of the N-body wave function (and thus of the corresponding marginal densities), uniformly in N and t. As in [4], we derive the a-priori estimate from energy estimates, which control powers of the energy in terms of the corresponding powers of the kinetic energy. In order to prove these energy estimates we need to introduce an N-dependent cutoff in the Hamiltonian. For $\varepsilon > 0$ we define the cutoff Hamiltonian

$$\widetilde{H}_N = \sum_{j=1}^N (1-\Delta_j)^{1/2} + \lambda \sum_{i<j}^N \frac{1}{|x_i - x_j| + \varepsilon N^{-1}} \,. \tag{1.8}$$

In $\widetilde{H}_N$ the Coulomb singularity has been regularized on the length scale $|x| \simeq \varepsilon N^{-1}$. In Proposition 4.1 we then prove the following operator bound (valid on the space $L^2_s(\mathbb{R}^{3N})$ of functions invariant w.r.t. permutations) for powers of the Hamiltonian $\widetilde{H}_N$:

$$\widetilde{H}_N^k \geq \; C^k N^k \; (1-\Delta_1)^{1/2}(1-\Delta_2)^{1/2} \dots (1-\Delta_k)^{1/2} \tag{1.9}$$

for all $k \geq 1$, and for all N large enough (depending on k and on the cutoff ε). Hence, if the initial state $\psi_N \in L^2_s(\mathbb{R}^{3N})$ is such that

$$(\psi_N, \widetilde{H}_N^k \psi_N) \leq C^k N^k \tag{1.10}$$

for all $k \geq 0$, then, using the conservation of the energy, from (1.9) we can derive bounds on higher derivatives of the solution $\psi^\varepsilon_{N,t}$ of the Schrödinger equation

$$i\partial_t \psi^\varepsilon_{N,t} = \widetilde{H}_N \psi^\varepsilon_{N,t} \tag{1.11}$$

with regularized interaction, and with initial data $\psi^\varepsilon_{N,t=0} = \psi_N$. Unfortunately, a typical factorized wave function $\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j)$ does not satisfies (1.10) (unless $\varphi \in C_0^\infty(\mathbb{R}^3)$, an assumption we want to avoid). Therefore, the introduction of an additional cutoff in the initial one-particle wave function φ is required. Given $\kappa > 0$ and a one particle wave function $\varphi \in H^1(\mathbb{R}^3)$, we define its regularized version by

$$\varphi^\kappa = \exp(-\kappa|p|/N)\varphi \qquad \text{with } p = -i\nabla\,.$$

Then the regularized N-particles wave function $\psi_N^\kappa(\mathbf{x}) = \prod_{j=1}^N \varphi^\kappa(x_j)$ can be proven to satisfy (1.10) for every $k \geq 1$ and for all N large enough (depending on k and κ). This follows from Proposition 4.2, where we derive an upper bound for the expectation of powers of the Hamiltonian. Hence if we denote by $\psi_{N,t}^{\varepsilon,\kappa}$ the evolution of the regularized initial data ψ_N^κ generated by $\widetilde{H}_N$, then (1.9) implies that

$$(\psi_{N,t}^{\varepsilon,\kappa}, (1-\Delta_1)^{1/2} \dots (1-\Delta_k)^{1/2} \psi_{N,t}^{\varepsilon,\kappa}) \leq C^k \tag{1.12}$$

for all $k \geq 1$ and for all N large enough (depending on k, ε and κ). For fixed $\varepsilon, \kappa > 0$ we denote by $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}_{N,t}^{(k)}\}_{k=1}^N$ the family of marginal distributions associated with the wave function $\psi_{N,t}^{\varepsilon,\kappa}$.

Using the bound (1.12) we can prove the compactness of the sequence $\widetilde{\Gamma}_{N,t}$ with respect to an appropriate weak topology. Moreover we can show that any limit point $\Gamma_{\infty,t} = \{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ of $\widetilde{\Gamma}_{N,t}$ satisfies the infinite hierarchy (1.6) with initial value $\gamma_{\infty,t=0}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \varphi(x_j)\overline{\varphi}(x'_j)$ (the cutoffs ε and κ, which regularize the interaction and the initial data, disappear in the limit $N \to \infty$) and that the solution of (1.6) is unique (in an appropriate space). This implies that

$$\gamma_{\infty,t}^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \varphi_t(x_j)\overline{\varphi_t}(x_j),$$

for every $k \geq 1$, where φ_t is the solution of the non-linear Hartree equation (1.3) with initial data $\varphi_{t=0} = \varphi$. Finally, we prove that the difference between the physical evolution (generated by the Hamiltonian H_N) of the initial wave function $\psi_N(\mathbf{x}) = \prod_{j=1}^k \varphi(x_j)$ and the modified evolution (generated by $\widetilde{H}_N$) of the approximate initial wave function ψ_N^κ converges to zero, for ε and κ converging to zero, uniformly in N (for N large enough). This concludes the proof of (1.7).

The paper is organized as follows. In Section 2 we introduce some Banach spaces of density matrices, and we equip them with weak topologies, useful to take the limit $N \to \infty$. In Section 3 we state our main result and give a sketch of its proof. In Section 4 we prove the energy estimates for the regularized Hamiltonian $\widetilde{H}_N$, and we use them to derive the a-priori bound (1.12). In Section 5 we show the compactness of the sequence of marginals $\widetilde{\gamma}_{N,t}^{(k)}$ associated with the wave function $\psi_{N,t}^{\varepsilon,\kappa}$ with respect to the weak topology. In Section 6 we demonstrate that any limit point of the marginal densities $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}_{N,t}^{(k)}\}_{k\geq 1}$ satisfies the infinite hierarchy (1.6). In Section 7 we show the uniqueness of the solution of (1.6). In Section 8 we remove the cutoffs, letting $\varepsilon, \kappa \to 0$. Finally, in Section 9, we collect some important technical lemmas, used throughout the paper.

Notations. Throughout the paper, we will use the notation $S_j = (1+p_j^2)^{1/4} = (1-\Delta_j)^{1/4}$ for integer $j \geq 1$. Moreover, with an abuse of notation, we will denote by $\|\varphi\|$ the L^2-norm of the function φ and by $\|A\|$ the operator norm of the the operator A. We will use the symbol Tr_k to denote the partial trace over the k'th variable. In general we will use the symbol Tr to denote the trace over all variables involved: sometimes, instead of Tr, we will use the symbol $\mathrm{Tr}^{(k)}$ to stress the total number of variables over which the trace is taken. In general, we will denote by C a universal constant, depending, possibly, only on the coupling constant λ and on the H^1-norm of the initial one-particle wave function φ. If C also depends on other quantities, we will usually stress them explicitly.

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2 Banach Spaces of Density Matrices

For $k \geq 1$, we denote by $\mathcal{L}_k^1$ and by $\mathcal{K}_k$ the space of trace class and, respectively, of compact operators on the k-particle Hilbert space $L^2(\mathbb{R}^{3k}, \mathrm{d}\mathbf{x}_k)$. It is a well known fact (see, for example, Theorem VI.26 in [11]) that $(\mathcal{L}_k^1, \|\,.\,\|_1) = (\mathcal{K}_k, \|\,.\,\|)^*$, where $\|\,.\,\|_1$ denotes the trace norm, and $\|\,.\,\|$ the operator norm. This induces a weak* topology on $\mathcal{L}_k^1$. We define $\mathcal{L}^1$ as the direct sum over $k \geq 1$ of the spaces $\mathcal{L}_k^1$, that is

$$\mathcal{L}^1 = \bigoplus_{k\geq 1} \mathcal{L}_k^1 = \{\Gamma = \{\gamma^{(k)}\}_{k\geq 1} : \gamma^{(k)} \in \mathcal{L}_k^1, \quad \forall\, k \geq 1\}\,.$$

We equip the space $\mathcal{L}^1$ with the product of the weak* topologies on $\mathcal{L}_k^1$ for all $k \geq 1$. A sequence $\Gamma_N = \{\gamma_N^{(k)}\}_{k\geq 1}$ converges to $\Gamma = \{\gamma^{(k)}\}_{k\geq 1}$ in $\mathcal{L}^1$ for $N \to \infty$ if and only if $\gamma_N^{(k)}$ converges to $\gamma^{(k)}$ with respect to the weak* topology of $\mathcal{L}_k^1$ for all $k \geq 1$.

To prove our main theorem (Theorem 3.1), we also need to define a different topology, which makes use of the smoothness of the marginal densities $\gamma_{N,t}^{(k)}$ and allows us to pass the smoothness to the limit $N \to \infty$. For $\gamma^{(k)} \in \mathcal{L}_k^1$, we define the norm

$$\|\gamma^{(k)}\|_{\mathcal{H}_k} = \mathrm{Tr}\,|S_1 \dots S_k\, \gamma^{(k)} S_k \dots S_1|$$

and the corresponding Banach space

$$\mathcal{H}_k = \{\gamma^{(k)} \in \mathcal{L}_k^1 : \|\gamma^{(k)}\|_{\mathcal{H}_k} < \infty\}.$$

Recall that we use the notation $S_j = (1 + p_j^2)^{1/4} = (1 - \Delta_j)^{1/4}$. Moreover we define the space of operators

$$\mathcal{A}_k = \{T^{(k)} = S_1 \dots S_k K^{(k)} S_k \dots S_1 : K^{(k)} \in \mathcal{K}_k\}$$

with the norm

$$\|T^{(k)}\|_{\mathcal{A}_k} = \|S_1^{-1} \dots S_k^{-1} T^{(k)} S_k^{-1} \dots S_1^{-1}\|$$

where $\|\,.\,\|$ denotes the operator norm. Then we have, for every $k \geq 1$,

$$(\mathcal{H}_k, \|\,.\,\|_{\mathcal{H}_k}) = (\mathcal{A}_k, \|\,.\,\|_{\mathcal{A}_k})^* \tag{2.1}$$

and thus we have a weak* topology on the space $\mathcal{H}_k$. The proof of (2.1) is identical to the proof of Lemma 3.1 in [4]. Note that the weak* topology on $\mathcal{H}_k$ is stronger than the weak* topology on the larger space $\mathcal{L}_k^1$: if a sequence $\gamma_N^{(k)}$ converges to $\gamma^{(k)}$ with respect to the weak* topology of $\mathcal{H}_k$, then it also converges with respect to the weak* topology of $\mathcal{L}_k^1$.

We will prove that the densities $\widetilde{\gamma}_{N,t}^{(k)}$ associated with the wave function $\psi_{N,t}^{\varepsilon,\kappa}$ (defined by the regularized Schrödinger equation (1.11)), are continuous functions of t with values in $\mathcal{H}_k$ (continuous w.r.t. the weak* topology of $\mathcal{H}_k$). In order to prove the continuity of limits of $\gamma_{N,t}^{(k)}$, we want to invoke the Arzela-Ascoli Theorem, which is only applicable to metric spaces. Fortunately, since the space $\mathcal{A}_k$ is separable (which follows from the fact that the space $\mathcal{K}_k$ of compact operators is separable), it turns out that the weak* topology of $\mathcal{H}_k$, when restricted to the unit ball of $\mathcal{H}_k$, is metrizable. In fact, because of the separability of $\mathcal{A}_k$, we can fix a dense countable subset of the unit ball of $\mathcal{A}_k$: we denote it by $\{J_i^{(k)}\}_{i\geq 1} \in \mathcal{A}_k$, with $\|J_i^{(k)}\|_{\mathcal{A}_k} = 1$ for all $i \geq 1$. Using the operators $J_i^{(k)}$ we define the following metric on $\mathcal{H}_k$: for $\gamma^{(k)}, \widetilde{\gamma}^{(k)} \in \mathcal{H}_k$ we set

$$\rho_k(\gamma^{(k)}, \widetilde{\gamma}^{(k)}) := \sum_{i=1}^{\infty} 2^{-i} \left| \text{Tr } J_i^{(k)} \left(\gamma^{(k)} - \widetilde{\gamma}^{(k)} \right) \right| . \tag{2.2}$$

Then the topology induced by the metric $\rho_k(.,.)$ and the weak* topology are equivalent on the unit ball of $\mathcal{H}_k$ (see [12], Theorem 3.16). In other words, a uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{H}_k$ converges to $\gamma^{(k)} \in \mathcal{H}_k$ with respect to the weak* topology, if and only if $\rho_k(\gamma_N^{(k)}, \gamma^{(k)}) \to 0$ for $N \to \infty$.

For a fixed $T \geq 0$ we consider the space $C([0,T], \mathcal{H}_k)$ of functions of $t \in [0,T]$ with values in $\mathcal{H}_k$, which are continuous with respect to the metric ρ_k (for uniformly bounded functions, the continuity with respect to ρ_k is equivalent to continuity with respect to the weak* topology of $\mathcal{H}_k$). On the space $C([0,T], \mathcal{H}_k)$ we define the metric $\widehat{\rho}_k$ by

$$\widehat{\rho}_k(\gamma^{(k)}(.), \widetilde{\gamma}^{(k)}(.)) = \sup_{t\in[0,T]} \rho_k(\gamma^{(k)}(t), \widetilde{\gamma}^{(k)}(t)) . \tag{2.3}$$

Finally, we define the space $\mathcal{H}$ as the direct sum over $k \geq 1$ of the spaces $\mathcal{H}_k$, that is

$$\mathcal{H} = \bigoplus_{k\geq 1} \mathcal{H}_k = \{\Gamma = \{\gamma^{(k)}\}_{k\geq 1} : \gamma^{(k)} \in \mathcal{H}_k, \quad \forall\, k \geq 1\} ,$$

and, for a fixed $T \geq 0$, we consider the space

$$C([0,T], \mathcal{H}) = \bigoplus_{k\geq 1} C([0,T], \mathcal{H}_k),$$

equipped with the product of the topologies induced by the metric $\widehat{\rho}^{(k)}$ on $C([0,T], \mathcal{H}_k)$. That is, for $\Gamma_{N,t} = \{\gamma_{N,t}^{(k)}\}_{k\geq 1}$ and $\Gamma_t = \{\gamma_t^{(k)}\}_{k\geq 1}$ in $C([0,T], \mathcal{H})$, we have $\Gamma_{N,t} \to \Gamma_t$ for $N \to \infty$ if and only if, for every fixed $k \geq 1$,

$$\widehat{\rho}_k(\gamma_{N,t}^{(k)}, \gamma_t^{(k)}) = \sup_{t\in[0,T]} \rho_k(\gamma_{N,t}^{(k)}, \gamma_t^{(k)}) \to 0$$

for $N \to \infty$.

3 Main Result

We are now ready to formulate our main theorem. We also give its proof, using results which will be established later on.

Theorem 3.1. *Let $\varphi \in H^1(\mathbb{R}^3)$, with $\|\varphi\| = 1$, and $\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j)$. Denote by $\psi_{N,t}$ the time evolution of ψ_N determined by the Schrödinger equation (1.4), and let $\gamma_{N,t}^{(k)}$, for $k = 1, \ldots, N$,* *be the marginal distributions corresponding to $\psi_{N,t}$. Assume that the coupling constant $\lambda > -4/\pi$. Then for all fixed $t \in \mathbb{R}$, we have, for $N \to \infty$,*

$$\Gamma_{N,t} = \{\gamma^{(k)}_{N,t}\}_{k\geq 1} \to \Gamma_t = \{\gamma^{(k)}_t\}_{k\geq 1} \qquad \text{on} \quad \mathcal{L}^1 = \bigoplus_{k\geq 1} \mathcal{L}^1_k \tag{3.1}$$

with respect to the product of the weak topologies of $\mathcal{L}^1_k$. Here*

$$\gamma^{(k)}_t(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \varphi_t(x_j)\overline{\varphi_t(x'_j)} \tag{3.2}$$

where φ_t is the solution of the non-linear Hartree equation

$$i\partial_t\varphi_t = (1-\Delta)^{1/2}\varphi_t + \lambda\left(\frac{1}{|\cdot|} * |\varphi_t|^2\right)\varphi_t \tag{3.3}$$

with initial data $\varphi_{t=0} = \varphi$.

Remarks.

i) The factorization assumption for the initial state ψ_N is only necessary in the limit $N \to \infty$. In other words, if $\gamma^{(k)}_N$ denotes the k-particle marginal associated with ψ_N, it is enough to assume that, for every fixed $k \geq 1$,

$$\mathrm{Tr}\left|\gamma^{(k)}_N - \gamma^{(k)}\right| \to 0 \quad \text{as } N \to \infty$$

where $\gamma^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \varphi(x_j)\overline{\varphi(x'_j)}$ for some $\varphi \in H^1(\mathbb{R}^3)$.

ii) The result can also be easily generalized to the case where the initial state is not in a pure state. If we assume the initial density matrix to be given by

$$\gamma_N(\mathbf{x};\mathbf{x}') = \prod_{j=1}^N \gamma^{(1)}(x_j;x'_j),$$

for some $\gamma^{(1)} \in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, we can prove the convergence (3.1); in this case the limit $\Gamma_t = \{\gamma^{(k)}_t\}_{k\geq 1}$ is given by $\gamma^{(k)}_t(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \gamma^{(1)}_t(x_j;x'_j)$ where $\gamma^{(1)}_t(x;x')$ is the solution of the nonlinear equation

$$i\partial_t\gamma^{(1)} = \left[(1-\Delta)^{1/2} + \lambda\left(\frac{1}{|.|} * \rho_t\right), \gamma^{(1)}\right] \quad \text{with} \quad \rho_t(x) = \gamma^{(1)}(x;x)$$

with initial data $\gamma^{(1)}_{t=0} = \gamma^{(1)}$. This equation is equivalent to the Hartree equation (3.3), if $\gamma^{(1)}(x;x') = \varphi(x)\overline{\varphi}(x')$.

iii) Although we state our theorem specifically for the Coulomb potential $V(x) = \lambda|x|^{-1}$, it is clear that our result and our proof still hold true for $V(x) = A(x)|x|^{-1} + B(x)$, if, for example $A, B \in \mathcal{S}(\mathbb{R}^3)$, the Schwarz class.

Proof. To simplify the notation we will assume $t \geq 0$. The proof for $t < 0$ is similar.

Step 1. Introduction of the cutoffs. For $\kappa > 0$ we define the regularized version of the initial one-particle wave function $\varphi \in H^1(\mathbb{R}^3)$ by

$$\varphi^\kappa = \exp(-\kappa|p|/N)\varphi$$

with $p = -i\nabla$. Note that the regularized wave function φ^κ depends on N. We use the notation

$$\psi_N^\kappa(\mathbf{x}) = \prod_{j=1}^N \varphi^\kappa(x_j)$$

for the N body wave function corresponding to the regularized φ^κ. We denote by $\psi_{N,t}^\kappa$ its time evolution with respect to the physical Hamiltonian H_N (1.1), and by $\psi_{N,t}^{\kappa,\varepsilon}$ its time evolution with respect to the modified Hamiltonian $\widetilde{H}_N$ defined in (1.8). We also use the notation $\gamma_{N,\kappa,t}^{(k)}$ and $\widetilde{\gamma}_{N,t}^{(k)} = \gamma_{N,\kappa,\varepsilon,t}^{(k)}$ for the family of marginal distributions corresponding to $\psi_{N,t}^\kappa$ and $\psi_{N,t}^{\kappa,\varepsilon}$, respectively.

Step 2. A priori bounds. In Section 4, Theorem 4.4, we prove that for any $k \geq 1$ and for all $\varepsilon, \kappa > 0$ there exists $N_0 = N_0(k, \varepsilon, \kappa)$ such that

$$\operatorname{Tr} |S_1 \dots S_k \, \widetilde{\gamma}_{N,t}^{(k)} S_k \dots S_1| \leq C^k \tag{3.4}$$

for all $N > N_0$ and for all $t \in \mathbb{R}$. The constant C is independent of $k, \varepsilon, \kappa, N, t$.

Step 3. Compactness of $\widetilde{\Gamma}_{N,t}$. Fix $T \geq 0$. From Theorem 5.1 it follows that, for every fixed $k \geq 1$, the sequence $\widetilde{\gamma}_{N,t}^{(k)} \in C([0,T], \mathcal{H}_k)$ is compact with respect to the metric $\widehat{\rho}_k$ (defined in (2.3)). This implies that the sequence $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}_{N,t}^{(k)}\}_{k=1}^N$ is compact in the space $C([0,T], \mathcal{H}) = \bigoplus_{k \geq 1} C([0,T], \mathcal{H}_k)$ with respect to the product of the topologies generated by the metrics $\widehat{\rho}_k$. Here we need the following standard argument (choice of the diagonal subsequence) to prove that, given a sequence N_j, with $N_j \to \infty$ for $j \to \infty$, there exists a subsequence M_j of N_j with $M_j \to \infty$ for $j \to \infty$, such that $\widetilde{\gamma}_{M_j,t}^{(k)}$ converges as $j \to \infty$, for all $k \geq 1$; by the compactness of $\widetilde{\gamma}_{N,t}^{(1)}$ there is a subsequence $\alpha_{1,j}$ of N_j, with $\alpha_{1,j} \to \infty$ for $j \to \infty$, such that $\widetilde{\gamma}_{\alpha_{1,j},t}^{(1)}$ converges. Next, by the compactness of $\widetilde{\gamma}_{N,t}^{(2)}$, there exists a subsequence $\alpha_{2,j}$ of $\alpha_{1,j}$ with $\alpha_{2,j} \to \infty$ and such that $\widetilde{\gamma}_{\alpha_{2,j},t}^{(2)}$ converges as $j \to \infty$ (clearly, since $\alpha_{2,j}$ is a subsequence of $\alpha_{1,j}$, $\widetilde{\gamma}_{\alpha_{2,j},t}^{(1)}$ converges as well). Inductively we can define subsequences $\alpha_{j,\ell}$ for all $ell \geq 1$, and we can set $M_j = \alpha_{j,j}$: then $M_j \to \infty$ for $j \to \infty$ and $\widetilde{\gamma}_{M_j,t}^{(k)}$ converges as $j \to \infty$, for all $k \geq 1$.

It also follows from Theorem 5.1 that, if $\Gamma_{\infty,t} = \{\gamma_{\infty,t}^{(k)}\}_{k \geq 1} \in C([0,T], \mathcal{H})$ denotes an arbitrary limit point of $\widetilde{\Gamma}_{N,t}$, then we have

$$\operatorname{Tr} |S_1 \dots S_k \gamma_{\infty,t}^{(k)} S_k \dots S_1| \leq C^k \tag{3.5}$$

for all $t \in [0,T]$ and for all $k \geq 1$ (with the same constant C as in (3.4)).

Step 4. Convergence to the infinite hierarchy. In Theorem 6.1, we prove that any limit point $\Gamma_{\infty,t} \in C([0,T], \mathcal{H})$ of the sequence $\widetilde{\Gamma}_{N,t}$ (with respect to the product of the topologies $\widehat{\rho}_k$) is a solution of the infinite hierarchy (1.6), with initial value $\Gamma_{\infty,t=0} = \{\gamma_0^{(k)}\}_{k \geq 1}$, where

$$\gamma_0^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi(x_j) \overline{\varphi(x'_j)} \tag{3.6}$$

(the κ-dependence of φ^κ disappears in the limit $N \to \infty$).

Step 5. Uniqueness of the infinite hierarchy. In Theorem 7.1 we demonstrate that for any given $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1} \in \mathcal{H}$, satisfying the estimate

$$\|\gamma_0^{(k)}\|_{\mathcal{H}_k} = \text{Tr}\, |S_1 \dots S_k \gamma_0^{(k)} S_k \dots S_1| \leq C^k$$

for some $C > 0$, there is at most one solution $\Gamma_{\infty,t} = \{\gamma_{\infty,t}^{(k)}\}_{k\geq 1} \in C([0,T],\mathcal{H})$ of the infinite hierarchy (1.6) which satisfies $\Gamma_{\infty,t=0} = \Gamma_0$ and

$$\|\gamma_{\infty,t}^{(k)}\|_{\mathcal{H}_k} \leq C^k \tag{3.7}$$

for all $t \in [0,T]$ and $k \geq 1$. This, together with (3.5), and with the fact that, by Step 4, every limit point of $\widetilde{\Gamma}_{N,t}$ satisfies the infinite hierarchy (1.6), immediately implies that the sequence $\widetilde{\Gamma}_{N,t}$ converges for $N \to \infty$ (a compact sequence with only one limit point is always convergent). Next, we note that the family of density $\Gamma_t = \{\gamma_t^{(k)}\}_{k\geq 1}$, with $\gamma_t^{(k)}$ defined by (3.2), satisfies (1.6) and the initial condition (3.6) (the existence and uniqueness of a global solution of the nonlinear Hartree equation (3.3), for $\lambda > \lambda_{\text{crit}}$, is proven in [9]). Moreover, it also satisfies the bound (3.7). In fact, by Lemma 9.3,

$$\|\gamma_t^{(k)}\|_{\mathcal{H}_k} = \text{Tr}\, |S_1 \dots S_k \gamma_t^{(k)} S_k \dots S_1| = (\varphi_t, (1-\Delta)^{1/2} \varphi_t)^k \leq C^k \,.$$

It follows that

$$\widetilde{\Gamma}_{N,t} \to \Gamma_t \qquad \text{as } N \to \infty$$

on $C([0,T],\mathcal{H})$, with respect to the product of the topologies induced by $\widehat{\rho}_k$, for $k \geq 1$. This, in particular, implies that, for every fixed $t \in [0,T]$, we have $\widetilde{\Gamma}_{N,t} \to \Gamma_t$ in $\mathcal{H}$, with respect to the product of the weak* topologies of $\mathcal{H}_k$ (which, since the sequence $\widetilde{\gamma}_{N,t}^{(k)}$ is uniformly bounded in N, are equivalent to the topologies generated by the metrics ρ_k). In turn this implies that, for every fixed $t \in [0,T]$, we have

$$\widetilde{\Gamma}_{N,t} \to \Gamma_t \qquad \text{on} \quad \mathcal{L}^1 = \bigoplus_{k\geq 1} \mathcal{L}^1_k, \tag{3.8}$$

with respect to the product of the weak* topologies on $\mathcal{L}^1_k$, for $k \geq 1$.

Step 6. Removal of the cutoffs. Recall that $\psi_{N,t}$ denotes the evolution of the initial function ψ_N generated by the Hamiltonian H_N, defined in (1.1). Moreover, $\psi_{N,t}^\kappa$ and $\psi_{N,t}^{\kappa,\varepsilon}$ denote the time evolution of the regularized initial wave function ψ_N^κ with respect to the dynamics generated by the original Hamiltonian H_N and, respectively, by the regularized Hamiltonian $\widetilde{H}_N$ (defined in (1.8)). We remind that $\gamma_{N,t}^{(k)}$, $\gamma_{N,\kappa,t}^{(k)}$ and $\widetilde{\gamma}_{N,t}^{(k)} = \gamma_{N,\kappa,\varepsilon,t}^{(k)}$ denote the marginal distributions associated with $\psi_{N,t}$, to $\psi_{N,t}^\kappa$ and, respectively, with $\psi_{N,t}^{\kappa,\varepsilon}$. In Proposition 8.1, we show that

$$\|\psi_{N,t}^\kappa - \psi_{N,t}^{\varepsilon,\kappa}\| \leq Ct\, \varepsilon^{1/4}$$

for every N large enough (depending on ε, κ). This clearly implies that

$$\text{Tr}\, |\gamma_{N,t,\kappa}^{(k)} - \gamma_{N,t,\kappa,\varepsilon}^{(k)}| \leq Ct\, \varepsilon^{1/4} \tag{3.9}$$

for all $k \geq 1$ and for all N large enough (depending on ε, κ). Moreover, in Proposition 8.2, we show that

$$\|\psi_{N,t}^\kappa - \psi_{N,t}\| = \|\psi_N^\kappa - \psi_N\| \leq C\kappa$$

for all N and $t \in \mathbb{R}$. Therefore

$$\text{Tr } |\gamma^{(k)}_{N,\kappa,t} - \gamma^{(k)}_{N,t}| \leq C\kappa \tag{3.10}$$

for all $k \geq 1$, $t \in \mathbb{R}$ and $N \geq 1$.

Step 7. Conclusion of the proof. For every fixed $k \geq 1$ and $t \in [0,T]$, and for every compact operator $J^{(k)} \in \mathcal{K}_k$, we have

$$\begin{aligned} \left| \text{Tr } J^{(k)} \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_t \right) \right| \leq \; & \|J^{(k)}\| \text{ Tr } \left| \gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,\kappa,t} \right| + \|J^{(k)}\| \text{ Tr } \left| \gamma^{(k)}_{N,\kappa,t} - \gamma^{(k)}_{N,\kappa,\varepsilon,t} \right| \\ & + \left| \text{Tr } J^{(k)} \left(\gamma^{(k)}_{N,\kappa,\varepsilon,t} - \gamma^{(k)}_t \right) \right| , \end{aligned}$$

where $\gamma^{(k)}_t$ is defined in (3.2). By (3.9) and (3.10), for any fixed $\delta > 0$, we can find κ and ε small enough such that both the first and the second term on the r.h.s. of the last equation are smaller than $\delta/3$, uniformly in N, if N is large enough. Finally, by (3.8), the last term can be made smaller than $\delta/3$ by choosing N sufficiently large (and keeping ε and κ fixed). Thus, for every fixed $t \in [0,T]$, $k \geq 1$, $J^{(k)} \in \mathcal{K}_k$, and $\delta > 0$, we have

$$\left| \text{Tr } J^{(k)} \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{\infty,t} \right) \right| \leq \delta$$

for all N large enough. In other words, $\Gamma_{N,t} \to \Gamma_{\infty,t}$ with respect to the product of the weak* topologies of $\mathcal{L}^1_k$, $k \geq 1$, for every fixed $t \in [0,T]$. Since $T < \infty$ is arbitrary, this completes the proof of the theorem. $\square$

4 A–Priori Estimates

The aim of this section is to prove an priori bound, of the form

$$\text{Tr } |S_1 \dots S_k \widetilde{\gamma}^{(k)}_{N,t} S_k \dots S_1| \leq C^k , \tag{4.1}$$

for the marginals $\widetilde{\gamma}^{(k)}_{N,t}$ associated with the solution $\psi^{\varepsilon,\kappa}_{N,t}$ of the modified Schrödinger equation (1.11) with regularized initial data $\psi^{\varepsilon,\kappa}_{N,t=0} = \psi^{\kappa}_N$ (recall that $S_j = (1-\Delta_j)^{1/4}$). To prove (4.1) we derive an upper and a lower bound for powers of $\widetilde{H}_N$ in terms of corresponding powers of the kinetic energy.

Proposition 4.1 (Lower Bound for $\widetilde{H}^k_N$). *We distinguish two cases.*

i) If $\lambda > 0$: For every $k \geq 0$, and $C < 1$, there exists $N_0 = N_0(k, C, \varepsilon)$ (depending also on the cutoff ε in the potential) such that

$$(\psi, \widetilde{H}^k_N \psi) \geq C^k \left\{ N^k (\psi, S_1^2 \dots S_k^2 \psi) + N^{k-1} (\psi, S_1^4 S_2^2 \dots S_{k-1}^2 \psi) \right\} \tag{4.2}$$

for all $N > N_0$, and for all ψ invariant with respect to permutations.

ii) If $-4/\pi < \lambda < 0$: For every $k \geq 0$, and $C < 1$, there exists $N_0 = N_0(k, C, \varepsilon)$ such that

$$(\psi, \widetilde{H}^k_N \psi) \geq C^k (1 + \frac{\pi}{4}\lambda)^{[k/2]} \left\{ N^k (\psi, S_1^2 \dots S_k^2 \psi) + N^{k-1} (\psi, S_1^4 S_2^2 \dots S_{k-1}^2 \psi) \right\} \tag{4.3}$$

for all $N > N_0$, and for all ψ symmetric with respect to permutations. Here $[k/2] = k/2$ if k is even, and $[k/2] = (k+1)/2$ if k is odd.

Proof. We use a two-step induction over k. For $k=0$ the statement is trivial. For $k=1$, and $\lambda>0$, the claim follows by the positivity of the potential. For $k=1$ and $\lambda<0$, it follows from the operator inequality

$$\frac{1}{|x_1-x_2|}\leq\frac{\pi}{2}S_1^2$$

(see Lemma 9.1, part (i)) and by the symmetry of ψ with respect to permutations. Next we assume that the claim holds true for $k=n$, and we prove it for $k=n+2$. We consider the case $\lambda<0$ (if $\lambda>0$ the proof is easier), and we assume that n is even (for odd n the proof is analogous). By the induction assumption, for any $C<1$, there exists $N_0(n,C,\varepsilon)$ such that

$$(\psi,\widetilde{H}_N^{n+2}\psi)=(\psi,\widetilde{H}_N\widetilde{H}_N^n\widetilde{H}_N\psi)\geq C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\,(\psi,\widetilde{H}_NS_1^2\dots S_n^2\widetilde{H}_N\psi)\tag{4.4}$$

for all $N>N_0(n,C,\varepsilon)$, and for all ψ symmetric with respect to permutations (note that also $\widetilde{H}_N\psi$ is symmetric). Here we are neglecting, because of its positivity, the contribution arising from the second term in the parenthesis on the r.h.s. of (4.3). Writing

$$\widetilde{H}_N=\sum_{j=n+1}^NS_j^2+h_N\qquad\text{with}\qquad h_N=\sum_{j=1}^nS_j^2+\frac{\lambda}{N}\sum_{i<j}^N\frac{1}{|x_i-x_j|+\varepsilon N^{-1}},$$

equation (4.4) implies that

$$\begin{aligned}(\psi,\widetilde{H}_N^{n+2}\psi)\geq{}&C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\sum_{i,j\geq n+1}(\psi,S_i^2S_1^2\dots S_n^2S_j^2\psi)\\&+C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\sum_{i\geq n+1}\left((\psi,S_i^2S_1^2\dots S_n^2h_N\psi)+\text{c.c.}\right),\end{aligned}$$

where *c.c.* denotes the complex conjugate. Using the permutation symmetry we find

$$\begin{aligned}(\psi,\widetilde{H}_N^{n+2}\psi)\geq{}&C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\\&\times\Big[(N-n)(N-n-1)(\psi,S_1^2\dots S_{n+2}^2\psi)+(N-n)(\psi,S_1^4S_2^2\dots S_{n+1}^2\psi)\\&+\lambda\frac{(N-n)(n+1)n}{2N}(\psi,V_{1,2}^\varepsilon S_1^2\dots S_{n+1}^2\psi)+\text{c.c.}\\&+\lambda\frac{(N-n)(n+1)(N-n-1)}{N}(\psi,V_{1,n+2}^\varepsilon S_1^2\dots S_{n+1}^2\psi)+\text{c.c.}\\&+\lambda\frac{(N-n)(N-n-1)(N-n-2)}{2N}(\psi,V_{n+2,n+3}^\varepsilon S_1^2\dots S_{n+1}^2\psi)+\text{c.c.}\Big]\end{aligned}\tag{4.5}$$

where we use the notation $V_{i,j}^\varepsilon=1/(|x_i-x_j|+\varepsilon N^{-1})$. We first estimate the last term (for positive λ, this term is positive, and thus can be neglected). Since $V_{n+2,n+3}^\varepsilon$ commutes with S_j, for all $j\leq n+1$, we have

$$(\psi,V_{n+2,n+3}^\varepsilon S_1^2\dots S_{n+1}^2\psi)=(\psi,S_1\dots S_{n+1}V_{n+2,n+3}^\varepsilon S_{n+1}\dots S_1\psi)\leq\frac{\pi}{2}\,(\psi,S_1^2\dots S_{n+2}^2\psi)\tag{4.6}$$

because $V_{n+2,n+3}^\varepsilon\leq|x_{n+2}-x_{n+3}|^{-1}\leq(\pi/2)\,S_{n+2}^2$ (Lemma 9.1, part (i)). Now we turn our attention to the fourth term in (4.5). We have

$$\begin{aligned}\left|(\psi,V_{1,n+2}^\varepsilon S_1^2\dots S_{n+1}^2\psi)\right|&=\left|(\psi,S_{n+1}\dots S_2V_{1,n+2}^\varepsilon S_1^2S_2\dots S_{n+1}\psi)\right|\\&=\left|(\psi,S_{n+2}S_{n+1}\dots S_1\{S_1^{-1}S_{n+2}^{-1}V_{1,n+2}^\varepsilon S_1S_{n+2}^{-1}\}S_1\dots S_{n+1}S_{n+2}\psi)\right|\\&\leq\text{const}\,(\psi,S_1^2\dots S_{n+2}^2\psi)\end{aligned}\tag{4.7}$$

where we used that, by Lemma 9.1, part (ii), the norm $\|S_1^{-1}S_{n+2}^{-1}V^\varepsilon_{1,n+2}S_1S_{n+2}^{-1}\|$ is finite, uniformly in ε. Finally we consider the third term in (4.5). To this end we note that

$$\begin{aligned}(\psi, V^\varepsilon_{1,2}S_1^2\dots S_{n+1}^2\psi) &= (\psi, S_{n+1}\dots S_3 V^\varepsilon_{1,2}S_1^2S_2^2S_3\dots S_{n+1}\psi)\\ &= (\psi, S_{n+1}\dots S_3S_2S_1^2\{S_1^{-2}S_2^{-1}V^\varepsilon_{1,2}S_2\}S_1^2S_2\dots S_{n+1}\psi)\,.\end{aligned}$$

For every $\delta > 0$ there exists, by Lemma 9.1, part (iii), a constant $C_{\varepsilon,\delta} > 0$ (depending also on ε) such that

$$\left|(\psi, V^\varepsilon_{1,2}S_1^2\dots S_{n+1}^2\psi)\right| \leq C_{\varepsilon,\delta}N^\delta(\psi, S_1^4S_2^2\dots S_{n+1}^2\psi)\,. \tag{4.8}$$

We fix $0 < \delta < 1$. Then, inserting (4.6), (4.7), and (4.8) into (4.5), we find

$$\begin{aligned}(\psi, \widetilde{H}_N^{n+2}\psi) \geq\; & C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\Big\{(N-n)(N-n-1) + \frac{\pi}{4}\lambda\frac{(N-n)(N-n-1)(N-n-2)}{N}\\ &\qquad - \text{const}\,\frac{(N-n)(n+1)(N-n-1)}{N}\Big\}(\psi, S_1^2\dots S_{n+2}^2\psi)\\ &+ C^nN^n(1+\frac{\pi}{4}\lambda)^{n/2}\Big\{(N-n) - C_{\varepsilon,\delta}N^\delta\Big\}(\psi, S_1^4S_2^2\dots S_{n+1}^2\psi)\\ =\; & C^nN^{n+2}(1+\frac{\pi}{4}\lambda)^{n/2}\frac{(N-n)(N-n-1)}{N^2}\\ &\qquad\times\left(1+\frac{\pi}{4}\lambda\frac{N-n-2}{N} - \text{const}\,\frac{n+1}{N}\right)(\psi, S_1^2\dots S_{n+2}^2\psi)\\ &+ C^nN^{n+1}(1+\frac{\pi}{4}\lambda)^{n/2}\frac{N-n}{N}\left(1 - C_{\varepsilon,\delta}\frac{N^\delta}{N-n}\right)(\psi, S_1^4S_2^2\dots S_{n+1}^2\psi)\,.\end{aligned}$$

Next, since $C < 1$, we can find $N_0(n+2, C, \varepsilon) > N_0(n, C, \varepsilon)$ such that the four inequalities

$$\frac{(N-n)(N-n-1)}{N^2} \geq C, \quad \left(1+\frac{\pi}{4}\lambda\frac{N-n-2}{N} - \text{const}\,\frac{n+1}{N}\right) \geq C\left(1+\frac{\pi}{4}\lambda\right) \quad \text{and}$$

$$\frac{N-n}{N} \geq C, \qquad \left(1 - C_{\varepsilon,\delta}\frac{N^\delta}{N-n}\right) \geq C$$

are satisfied for all $N > N_0(n+2, C, \varepsilon)$. This implies that

$$(\psi, \widetilde{H}_N^{n+2}\psi) \geq C^{n+2}\left(1+\frac{\pi}{4}\lambda\right)^{(n+2)/2}\left\{N^{n+2}(\psi, S_1^2\dots S_{n+2}^2\psi) + N^{n+1}(\psi, S_1^4S_2^2\dots S_{n+1}^2\psi)\right\}$$

for all $N > N_0(n+2, C)$, and completes the proof of the proposition. □

In the next proposition, we prove an upper bound for powers of $\widetilde{H}_N$.

Proposition 4.2 (Upper Bound for $\widetilde{H}_N^k$). *For $\ell \geq 1$, we use the notation $\boldsymbol{\alpha}_\ell = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{N}_+^\ell$, and $|\boldsymbol{\alpha}_\ell| = \alpha_1 + \dots + \alpha_\ell$. Assume C is large enough (depending only on λ). Then, for every fixed $k \geq 0$, $\varepsilon > 0$ and for every $a > 0$ there exists $0 \leq C(k, \varepsilon, a) < \infty$ (also depending on λ) such that*

$$(\psi, \widetilde{H}_N^k\psi) \leq C^k(\psi, \left(S_1^2 + \dots S_N^2\right)^k\psi) + C(k,\varepsilon,a)\,N^a\sum_{\ell=1}^{k-1}\sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = k} N^\ell\,(\psi, S_1^{2\alpha_1}\dots S_\ell^{2\alpha_\ell}\psi) \tag{4.9}$$

for all N and for all ψ symmetric with respect to permutations of the N particles.

Proof. We prove the proposition by a two step induction over k. The statement is clear for $k = 0$ and $k = 1$ (in this case one can choose $C(k = 1, \varepsilon, a) = 0$). We assume now that the statement holds for $k = n$, and we prove it for $k = n + 2$. From the induction assumption, with a replaced by $a(n-1)/n$, and since ψ and $\widetilde{H}_N \psi$ are symmetric with respect to permutations, we have

$$\begin{aligned}(\psi, \widetilde{H}_N^{n+2} \psi) &= (\psi, \widetilde{H}_N \widetilde{H}_N^n \widetilde{H}_N \psi) \\ &\leq C^n (\psi, \widetilde{H}_N (S_1^2 + \dots + S_N^2)^n \widetilde{H}_N \psi) \\ &\quad + C(n, \varepsilon, a(n-1)/n) \, N^{a(n-1)/n} \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = k} N^r \, (\psi, \widetilde{H}_N \, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \widetilde{H}_N \psi) \, . \end{aligned} \tag{4.10}$$

We start by considering the first term on the r.h.s. of the last equation. Applying a Schwarz inequality and using again the permutation symmetry, we find

$$\begin{aligned}&(\psi, \widetilde{H}_N (S_1^2 + \dots + S_N^2)^n \widetilde{H}_N \psi) \\ &\qquad \leq \; 2(\psi, (S_1^2 + \dots + S_N^2)^{n+2} \psi) + N(N-1) \lambda^2 (\psi, V_{1,2}^{\varepsilon} (S_1^2 + \dots + S_N^2)^n V_{1,2}^{\varepsilon} \psi) \, .\end{aligned}$$

The weighted Minkowski inequality

$$(S_1^2 + \dots + S_N^2)^n \leq 2(S_3^2 + \dots + S_N^2)^n + c(n)(S_1^{2n} + S_2^{2n})$$

(where the constant $c(n)$ can be chosen as $c^n n^n$, for some $c > 0$) implies that

$$\begin{aligned}(\psi, \widetilde{H}_N (S_1^2 + \dots + S_N^2)^n \widetilde{H}_N \psi) \leq 2 \, &(\psi, (S_1^2 + \dots + S_N^2)^{n+2} \psi) \\ &+ 2N(N-1) \lambda^2 (\psi, V_{1,2}^{\varepsilon} (S_3^2 + \dots + S_N^2)^n V_{1,2}^{\varepsilon} \psi) \\ &+ 4N^2 \lambda^2 c(n) (\psi, V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon} \psi) \, .\end{aligned} \tag{4.11}$$

The second term on the r.h.s. of the last equation can be bounded by

$$\begin{aligned}2N(N-1) \lambda^2 (\psi, V_{1,2}^{\varepsilon} (S_3^2 + \dots + S_N^2)^n V_{1,2}^{\varepsilon} \psi) &\leq C \, N(N-1) (\psi, (S_3^2 + \dots S_N^2)^n S_1^2 S_2^2 \psi) \\ &\leq C \, (\psi, (S_1^2 + \dots + S_N^2)^{n+2} \psi)\end{aligned} \tag{4.12}$$

because $V_{1,2}^{\varepsilon}$ commutes with $(S_3^2 + \dots + S_N^2)$, and $(V_{1,2}^{\varepsilon})^2 \leq |x_1 - x_2|^{-2} \leq C \, S_1^2 S_2^2$ (see Lemma 9.1) and because $N(N-1)(\psi, S_1^2 S_2^2 \psi) \leq (\psi, (S_1^2 + \dots + S_N^2)^2 \psi)$ for permutation invariant ψ. Next we consider the last term on the r.h.s. of (4.11). Using Lemma 9.2, for every $a > 0$, $n \geq 0$ and $\varepsilon > 0$, we find a constant $D(n, \varepsilon, a)$ such that

$$N^2 \, V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon} \leq D(n, \varepsilon, a) N^a \, N^2 \sum_{m=1}^{n} S_1^{2(n-m)} S_1^4 S_2^2 N^{m-1} \, .$$

In the sum over m, we can add more factors of S_j^2, so that the order of each monomial is $2(n+2)$ (and we can compare this contribution with the sum in the r.h.s. of (4.9)). More precisely we can bound

$$S_1^{2(n-m)} S_1^4 S_2^2 = S_1^{2(n-m+2)} S_2^2 \leq S_1^{2(n-m+2)} S_2^2 S_3^2 \dots S_m^2 S_{m+1}^2$$

because $S_j^2 \geq 1$. Hence, for every $a > 0$, there exists a constant $C_1(n, \varepsilon, a)$ (also depending on λ) such that

$$4c(n) \lambda^2 N^2 (\psi, V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon} \psi) \leq C_1(n, \varepsilon, a) N^a \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell \, (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) \, .$$

From the last equation, together with (4.11), (4.12), it follows that

$$
\begin{aligned}
(\psi, \widetilde{H}_N (S_1^2 + \cdots + S_N^2)^n \widetilde{H}_N \psi) \leq C^2 (\psi, (S_1^2 + \cdots + S_N^2)^{n+2} \psi) & \\
+ C_1(n,\varepsilon,a)\, N^a \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) N^\ell &
\end{aligned}
\tag{4.13}
$$

provided C is large enough (depending only on λ).

Next we consider the second term on the r.h.s. of (4.10). Applying the Schwarz inequality, we find

$$
\begin{aligned}
\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} & N^r \, (\psi, \widetilde{H}_N \, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \widetilde{H}_N \psi) \\
& \leq 2 \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \, (\psi, (S_1^2 + \cdots + S_N^2)^2 S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \psi) \\
& \quad + \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \sum_{i<j} (\psi, V_{i,j}^\varepsilon S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V_{i,j}^\varepsilon \psi) \, .
\end{aligned}
\tag{4.14}
$$

Using the permutation symmetry, the first term on the r.h.s. of last equation can be rewritten as

$$
\begin{aligned}
& \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \, (\psi, (S_1^2 + \cdots + S_N^2)^2 S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \psi) \\
& = \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \Big\{ r \, (\psi, S_1^{2(\alpha_1+2)} S_2^{2\alpha_2} \dots S_r^{2\alpha_r} \psi) + 2r(r-1) \, (\psi, S_1^{2(\alpha_1+1)} S_2^{2(\alpha_2+1)} S_3^{2\alpha_3} \dots S_r^{2\alpha_r} \psi) \\
& \qquad + 2r(N-r)(\psi, S_1^{2(\alpha_1+1)} \dots S_r^{2\alpha_r} S_{r+1}^2 \psi) + (N-r)(\psi, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} S_{r+1}^4 \psi) \\
& \qquad + 2(N-r)(N-r-1)(\psi, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} S_{r+1}^2 S_{r+2}^2 \psi) \Big\}
\end{aligned}
\tag{4.15}
$$

which clearly implies that

$$
\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \, (\psi, (S_1^2 + \cdots + S_N^2)^2 S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \psi) \leq \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) \, .
$$

The second term on the r.h.s. of (4.14), on the other hand, can be rewritten as

$$
\begin{aligned}
\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} & N^r \sum_{i<j} (\psi, V_{i,j}^\varepsilon S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V_{i,j}^\varepsilon \psi) \\
& = \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \Big\{ \frac{r(r-1)}{2} (\psi, V_{1,2}^\varepsilon S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V_{1,2}^\varepsilon \psi) \\
& \qquad + r(N-r)(\psi, V_{1,r+1}^\varepsilon \, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V_{1,r+1}^\varepsilon \psi) \\
& \qquad + \frac{1}{2}(N-r)(N-r-1)(\psi, V_{r+1,r+2}^\varepsilon \, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V_{r+1,r+2}^\varepsilon \psi) \Big\} \, .
\end{aligned}
\tag{4.16}
$$

To bound the last contribution we use that, since $V^\varepsilon_{r+1,r+2}$ commutes with $S_1,\dots,S_r$, and since $V^\varepsilon_{r+1,r+2}\leq C\,S^2_{r+1}S^2_{r+2}$,

$$(\psi, V^\varepsilon_{r+1,r+2}\, S_1^{2\alpha_1}\dots S_r^{2\alpha_r} V^\varepsilon_{r+1,r+2}\psi)\leq C\,(\psi, S_1^{2\alpha_1}\dots S_r^{2\alpha_r} S^2_{r+1}S^2_{r+2}\psi).$$

Hence

$$\begin{aligned}\sum_{r=1}^{n-1}\sum_{\boldsymbol{\alpha}_r:|\boldsymbol{\alpha}_r|=n} N^r(N-r)(N-r-1)(\psi, V^\varepsilon_{r+1,r+2} S_1^{2\alpha_1}\dots S_r^{2\alpha_r} V^\varepsilon_{r+1,r+2}\psi)&\\ \leq C\sum_{\ell=1}^{n+1}\sum_{\boldsymbol{\alpha}_\ell:|\boldsymbol{\alpha}_\ell|=n+2} N^\ell(\psi, S_1^{2\alpha_1}\dots S_\ell^{2\alpha_\ell}\psi)\,.&\end{aligned}\tag{4.17}$$

As for the second term in (4.16), we write

$$(\psi, V^\varepsilon_{1,r+1}\, S_1^{2\alpha_1}\dots S_r^{2\alpha_r} V^\varepsilon_{1,r+1}\psi)=(\psi, S_2^{\alpha_2}\dots S_r^{\alpha_r} V^\varepsilon_{1,r+1} S_1^{2\alpha_1} V^\varepsilon_{1,r+1} S_r^{\alpha_r}\dots S_2^{\alpha_2}\psi)$$

and then we observe that, by Lemma 9.2, for every $a>0$ we can find $D(n,\varepsilon,a)$ such that

$$V^\varepsilon_{1,r+1}S_1^{2\alpha_1}V_{1,r+1}\leq D(n,\varepsilon,a)N^{a/n}\sum_{m=1}^{\alpha_1} S_1^{2(\alpha_1-m+2)}S^2_{r+1}\,N^{m-1}\,.\tag{4.18}$$

This implies that, for a suitable constant $C_2(n,\varepsilon,a)$,

$$\begin{aligned}&\sum_{r=1}^{n-1}\sum_{\boldsymbol{\alpha}_r:|\boldsymbol{\alpha}_r|=n} N^r r(N-r)(\psi, V^\varepsilon_{1,r+1}\, S_1^{2\alpha_1}\dots S_r^{2\alpha_r} V^\varepsilon_{1,r+1}\psi)\\ &\quad\leq C_2(n,\varepsilon,a)\,N^{a/n}\sum_{r=1}^{n-1}\sum_{\boldsymbol{\alpha}_r:|\boldsymbol{\alpha}_r|=n}\sum_{m=1}^{\alpha_1} N^{r+m}(\psi, S_1^{2(\alpha_1-m+2)}S_2^{2\alpha_2}\dots S_r^{2\alpha_r}S^2_{r+1}\psi)\\ &\quad\leq C_2(n,\varepsilon,a)N^{a/n}\sum_{\ell=1}^{n+1}\sum_{\boldsymbol{\alpha}_\ell:|\boldsymbol{\alpha}_\ell|=n+2} N^\ell(\psi, S_1^{2\alpha_1}\dots S_\ell^{2\alpha_\ell}\psi)\,.\end{aligned}\tag{4.19}$$

In the last inequality we used that in the sum over m we can add more derivatives (until the total order of each summand is $2(n+2)$), according to the bound

$$N^{r+m}\,S_1^{2(\alpha_1-m+2)}S_2^{2\alpha_2}\dots S_r^{2\alpha_r}S^2_{r+1}\leq N^{r+m}\,S_1^{2(\alpha_1-m+2)}S_2^{2\alpha_2}\dots S_r^{2\alpha_r}S^2_{r+1}S^2_{r+2}\dots S^2_{r+m}.$$

Moreover, note that $r+m\leq r+\alpha_1=r+n-(\alpha_2+\dots+\alpha_r)\leq r+n-(r-1)\leq n+1$ (this explains why the sum over the index ℓ, in the last line of (4.19), only goes up to $n+1$).

Finally, we consider the first term in (4.16). We have, using the symmetry with respect to permutations,

$$\begin{aligned}(\psi, V^\varepsilon_{1,2}\, S_1^{2\alpha_1}\dots S_r^{2\alpha_r} V^\varepsilon_{1,2}\psi)&=(\psi, S_3^{\alpha_3}\dots S_r^{\alpha_r} V^\varepsilon_{1,2} S_1^{2\alpha_1}S_2^{2\alpha_2} V^\varepsilon_{1,2} S_r^{\alpha_r}\dots S_3^{\alpha_2}\psi)\\ &\leq(\psi, S_3^{\alpha_3}\dots S_r^{\alpha_r} V^\varepsilon_{1,2} S_1^{2(\alpha_1+\alpha_2)} V^\varepsilon_{1,2} S_r^{\alpha_r}\dots S_3^{\alpha_2}\psi).\end{aligned}$$

Applying again Lemma 9.2, we find, for every $a>0$, a constant $D(n,\varepsilon,a)$ such that

$$V^\varepsilon_{1,2}S_1^{2(\alpha_1+\alpha_2)}V^\varepsilon_{1,2}\leq D(n,\varepsilon,a)\,N^{a/n}\sum_{m=1}^{\alpha_1+\alpha_2} S_1^{2(\alpha_1+\alpha_2-m+2)}S_2^2N^{m-1}\,.$$

Hence

$$\begin{aligned}
&\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r r(r-1) (\psi, V^\varepsilon_{1,2}\, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V^\varepsilon_{1,2} \psi) \\
&\qquad \leq C_3(n,\varepsilon,a) N^{a/n} \sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} \sum_{m=1}^{\alpha_1+\alpha_2} N^{r+m-1} (\psi, S_1^{2(\alpha_1+\alpha_2-m+2)} S_2^2 S_3^{2\alpha_3} \dots S_r^{2\alpha_r} \psi) \\
&\qquad \leq C_3(n,\varepsilon,a) N^{a/n} \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) \,.
\end{aligned} \tag{4.20}$$

Similarly to (4.19), in order to control the terms in the sum over m, we used the trivial bound

$$N^{r+m-1} S_1^{2(\alpha_1+\alpha_2-m+2)} S_2^2 S_3^{2\alpha_3} \dots S_r^{2\alpha_r} \leq N^{r+m-1} S_1^{2(\alpha_1+\alpha_2-m+2)} S_2^2 S_3^{2\alpha_3} \dots S_r^{2\alpha_r} S_{r+1}^2 \dots S_{r+m-1}^2 \,,$$

and the inequality $r+m-1 \leq r-1+\alpha_1+\alpha_2 \leq r+k-1-(\alpha_3+\dots\alpha_r) \leq n+1$.

Inserting (4.17), (4.19), and (4.20) in (4.16) we find

$$\begin{aligned}
&\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \sum_{i<j} (\psi, V^\varepsilon_{i,j} S_1^{2\alpha_1} \dots S_r^{2\alpha_r} V^\varepsilon_{i,j} \psi) \\
&\qquad\qquad \leq \; C_4(n,\varepsilon,a)\, N^{a/n} \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) \,.
\end{aligned}$$

Combining the last bound and (4.15) with (4.14) we find

$$\begin{aligned}
&\sum_{r=1}^{n-1} \sum_{\boldsymbol{\alpha}_r : |\boldsymbol{\alpha}_r| = n} N^r \, (\psi, \widetilde{H}_N \, S_1^{2\alpha_1} \dots S_r^{2\alpha_r} \widetilde{H}_N \psi) \\
&\qquad\qquad \leq C_5(n,\varepsilon,a) N^{a/n} \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell \, (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi) \,.
\end{aligned}$$

Defining $C(n+2,\varepsilon,a) = C^n C_1(n,\varepsilon,a) + C(n,\varepsilon,a(n-1)/n) \cdot C_5(n,\varepsilon,a)$, the last equation, together with (4.13) and (4.10), implies that

$$\begin{aligned}
(\psi, \widetilde{H}_N^{n+2} \psi) &\leq C^{n+2} (\psi, (S_1^2 + \dots + S_N^2)^{n+2} \psi) \\
&\quad + C(n+2,a,\varepsilon) N^a \sum_{\ell=1}^{n+1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = n+2} N^\ell \, (\psi, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell} \psi)
\end{aligned}$$

and completes the proof of the proposition. □

In order to apply the last proposition to our initial state ψ_N^κ we need the following lemma, which shows, together with Proposition 4.2, that

$$(\psi_N^\kappa, \widetilde{H}_N^k \psi_N^\kappa) \leq C^k \tag{4.21}$$

for N large enough (depending on ε, κ and k).

Lemma 4.3. *Fix $\varphi \in H^1(\mathbb{R}^3)$ with $\|\varphi\| = 1$. For $\kappa > 0$ define $\varphi^\kappa = \exp(-\kappa|p|/N)\varphi$. Let $\psi_N^\kappa(\mathbf{x}) = \prod_{j=1}^N \varphi^\kappa(x_j)$.*

i) We have $\|\varphi^\kappa\| \leq 1$ and

$$(\varphi^\kappa, (1+p^2)^{k/2}\varphi^\kappa) \leq C^k(1 + k!\,(N/\kappa)^{k-2})\|\varphi\|_{H^1}^2 .$$

ii) For every $k \geq 1$ and $\kappa > 0$ there exists a constant $C(k,\kappa)$ such that

$$\sum_{\ell=1}^{k-1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = k} N^\ell\,(\psi_N^\kappa, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell}\psi_N^\kappa) \leq C(k,\kappa) N^{k-1} .$$

iii) For fixed k, and κ, there exists $N_0(k,\kappa)$ such that

$$(\psi_N^\kappa, (S_1^2 + \dots S_N^2)^k \psi_N^\kappa) \leq C^k N^k$$

for all $N > N_0$ (the constant C is independent of k, κ and N).

Proof. i) The inequality $\|\varphi^\kappa\| \leq 1$ follows from $e^{-\kappa|p|/N} \leq 1$. Next we compute

$$\begin{aligned}(\varphi^\kappa, (1+p^2)^{k/2}\varphi^\kappa) &\leq 2^{k/2} \int \mathrm{d}p\,(1+|p|^k)e^{-2\kappa|p|/N}|\widehat{\varphi}(p)|^2 \\ &\leq 2^{k/2}\left(1 + \sup_p \left(|p|^{k-2}e^{-2\kappa|p|/N}\right)\right) \int \mathrm{d}p\,(1+|p|^2)|\widehat{\varphi}(p)|^2 \\ &\leq C^k(1 + k!\,(N/\kappa)^{k-2})\,\|\varphi\|_{H^1}^2 .\end{aligned}$$

ii) For any $\ell \leq k-1$ and $\boldsymbol{\alpha}_\ell = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{N}_+^\ell$, with $|\boldsymbol{\alpha}_\ell| = k$, we have, using part i),

$$\begin{aligned}N^\ell(\psi_N^\kappa, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell}\psi_N^\kappa) &\leq N^\ell \prod_{j=1}^{\ell} (\varphi^\kappa, (1+p^2)^{\alpha_j/2}\varphi^\kappa) \\ &\leq N^k N^{\ell-k} \prod_{j=1}^{\ell} C^{\alpha_j}(1 + \alpha_j!\,\kappa^{2-\alpha_j}N^{\alpha_j-2})\|\varphi\|_{H^1}^2 \\ &\leq C^k N^k \|\varphi\|_{H^1}^{2k} \prod_{j=1}^{\ell}(N^{1-\alpha_j} + \alpha_j!\,\kappa^{2-\alpha_j}N^{-1}) .\end{aligned}$$

Since $\alpha_j \geq 1$ for all $j = 1, \dots, \ell$, and since $\ell \leq k-1$ there is at least one $j \in \{1, \dots, \ell\}$ such that $\alpha_j \geq 2$. Thus,

$$\sum_{\ell=1}^{k-1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = k} N^\ell(\psi_N^\kappa, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell}\psi_N^\kappa) \leq C^k \|\varphi\|_{H^1}^{2k} N^{k-1} \prod_{j=1}^{k}(1 + k!\kappa^{2-k})^k = N^{k-1}C(k,\kappa) .$$

iii) By the permutation symmetry of ψ_N^κ, and by part ii), we have

$$\begin{aligned}(\psi_N^\kappa, (S_1^2 + \dots + S_N^2)^k\psi_N^\kappa) &\leq N^k(\psi_N^\kappa, S_1^2 \dots S_k^2\psi_N^\kappa) + \sum_{\ell=1}^{k-1} \sum_{\boldsymbol{\alpha}_\ell : |\boldsymbol{\alpha}_\ell| = k} N^\ell(\psi_N^\kappa, S_1^{2\alpha_1} \dots S_\ell^{2\alpha_\ell}\psi_N^\kappa) \\ &\leq N^k(\varphi^\kappa, (1+p^2)^{1/2}\varphi^\kappa)^k + C(k,\kappa)N^{k-1} \\ &\leq C^k N^k \left(1 + \frac{C(k,\kappa)}{C^k}N^{-1}\right)\end{aligned}$$

where the constant C only depends on $\|\varphi\|_{H^1}$. For fixed k,κ we can now choose N_0 large enough such that $C(k,\kappa)N^{-1}\leq 1$. Then for $N>N_0$ we have $(\psi_N^\kappa,(S_1^2+\dots+S_N^2)^k\psi_N^\kappa)\leq (2C)^k N^k$, which proves the claim. $\square$

Using the bound (4.21), the conservation of the energy, and the lower bound for $\widetilde{H}_N^k$ given in Proposition 4.1, we finally arrive at the a-priori bound (4.1).

Theorem 4.4 (A-Priori Estimate). *Fix a one-particle wave function $\varphi\in H^1(\mathbb{R}^3)$, with $\|\varphi\|=1$. For $\kappa>0$ let $\varphi^\kappa=\exp(-\kappa|p|/N)\varphi$. Define $\psi_N^\kappa(\mathbf{x})=\prod_{j=1}^N\varphi^\kappa(x_j)$. Let $\psi_{N,t}^{\kappa,\varepsilon}$ be the solution of the Schrödinger equation (1.11) with modified interaction and initial data ψ_N^κ. Then, for all fixed $\varepsilon,\kappa>0$ and $k\geq 1$ there exists $N_0(\varepsilon,\kappa,k)$ such that*

$$(\psi_{N,t}^{\varepsilon,\kappa},\,S_1^2\dots S_k^2\,\psi_{N,t}^{\varepsilon,\kappa})\leq C^k\qquad\text{and}\qquad(\psi_{N,t}^{\varepsilon,\kappa},\,S_1^4S_2^2\dots S_{k-1}^2\,\psi_{N,t}^{\varepsilon,\kappa})\leq C^k\,N \tag{4.22}$$

for all $N\geq N_0$ and for all $t\in\mathbb{R}$. The constant C only depends on λ and $\|\varphi\|_{H^1}$. Denoting by $\widetilde{\gamma}_{N,t}^{(k)}$ the k-particle marginal distribution associated to $\psi_{N,t}^{\varepsilon,\kappa}$, (4.22) is equivalent to

$$\mathrm{Tr}\,\left|S_1\dots S_k\,\widetilde{\gamma}_{N,t}^{(k)}\,S_k\dots S_1\right|\leq C^k\qquad\text{and}\qquad\mathrm{Tr}\,\left|S_1^2S_2\dots S_{k-1}\,\widetilde{\gamma}_{N,t}^{(k)}S_{k-1}\dots S_2S_1^2\right|\leq C^kN$$

for all $N\geq N_0$.

Proof. From Proposition 4.2 we have

$$(\psi_N^\kappa,\widetilde{H}_N^k\psi_N^\kappa)\leq C^k(\psi_N^\kappa,(S_1^2+\dots+S_N^2)^k\psi_N^\kappa)+C(k,\varepsilon,a)N^a\sum_{\ell=1}^{k-1}\sum_{\boldsymbol{\alpha}_\ell:|\boldsymbol{\alpha}_\ell|=k}N^\ell\,(\psi_N^\kappa,S_1^{2\alpha_1}\dots S_\ell^{2\alpha_\ell}\psi_N^\kappa)\,.$$

By Lemma 4.3, part ii) and iii) we find

$$(\psi_N^\kappa,\widetilde{H}_N^k\psi_N^\kappa)\leq C^kN^k+C(k,\varepsilon,\kappa,a)N^aN^{k-1}\leq C^kN^k\left(1+\frac{C(k,\varepsilon,\kappa,a)}{C^k}N^{a-1}\right)$$

for N large enough, depending on k and κ. Hence, fixing $a<1$, we find

$$(\psi_N^\kappa,\widetilde{H}_N^k\psi_N^\kappa)\leq C_1^kN^k$$

for all N large enough, depending on k,ε,κ, and with a constant C_1 only depending on $\|\varphi\|_{H^1}$ and λ. By the conservation of the energy and by Proposition 4.1, we get

$$\begin{aligned}C_1^kN^k&\geq(\psi_N^\kappa,\,\widetilde{H}_N^k\,\psi_N^\kappa)=(\psi_{N,t}^{\varepsilon,\kappa},\,\widetilde{H}_N^k\,\psi_{N,t}^{\varepsilon,\kappa})\\&\geq C_2^kN^k(\psi_{N,t}^{\varepsilon,\kappa},\,S_1^2\dots S_k^2\,\psi_{N,t}^{\varepsilon,\kappa})+C_2^kN^{k-1}(\psi_{N,t}^{\varepsilon,\kappa},\,S_1^4S_2^2\dots S_{k-1}^2\,\psi_{N,t}^{\varepsilon,\kappa})\end{aligned}$$

for all N large enough. $\square$

5 Compactness of the sequence $\widetilde{\Gamma}_{N,t}$.

Recall that we defined $\widetilde{\Gamma}_{N,t}=\{\widetilde{\gamma}_{N,t}^{(k)}\}_{k=1}^N$ as the marginal densities associated to the solution $\psi_{N,t}^{\varepsilon,\kappa}$ of the regularized Schödinger equation (1.11) with initial data ψ_N^κ. They satisfy the regularized BBGKY hierarchy

$$\begin{aligned}i\partial_t\widetilde{\gamma}_{N,t}^{(k)}=&\sum_{j=1}^k\left[(1-\Delta_{x_j})^{1/2},\widetilde{\gamma}_{N,t}^{(k)}\right]+\lambda N^{-1}\sum_{1\leq i<j\leq k}\left[V^\varepsilon(x_i-x_j),\widetilde{\gamma}_{N,t}^{(k)}\right]\\&+\lambda(1-\frac{k}{N})\sum_{j=1}^k\mathrm{Tr}_{k+1}\left[V^\varepsilon(x_j-x_{k+1}),\widetilde{\gamma}_{N,t}^{(k+1)}\right]\,,\end{aligned}\tag{5.1}$$

where $V^{\varepsilon}(x) = (|x| + \varepsilon N^{-1})^{-1}$.

Theorem 5.1. *Fix $k \geq 1$ and $T \geq 0$. Then $\widetilde{\gamma}^{(k)}_{N,t} \in C([0,T], \mathcal{H}_k)$ for all N large enough (depending on k, ε and κ). Moreover the sequence $\widetilde{\gamma}^{(k)}_{N,t}$ is compact in $C([0,T], \mathcal{H}_k)$ with respect to the metric $\widehat{\rho}_k$ (defined in (2.3)), and, if $\gamma^{(k)}_{\infty,t} \in C([0,T], \mathcal{H}_k)$ denotes an arbitrary limit point of the sequence $\widetilde{\gamma}^{(k)}_{N,t}$ with respect to the metric $\widehat{\rho}_k$, we have*

$$\|\gamma^{(k)}_{\infty,t}\|_{\mathcal{H}_k} = \mathrm{Tr}\, |S_1 \dots S_k \gamma^{(k)}_{\infty,t} S_k \dots S_1| \leq C^k \tag{5.2}$$

for all $t \in [0,T]$ (the constant C is independent of k, ε, and κ).

Proof. We prove that the sequence $\widetilde{\gamma}^{(k)}_{N,t}$ is equicontinuous in t with respect to the metric $\widehat{\rho}_k$. To this end we rewrite the BBGKY hierarchy (5.1) in the integral form

$$\begin{aligned}
\widetilde{\gamma}^{(k)}_{N,t} = \widetilde{\gamma}^{(k)}_{N,s} - i \sum_{j=1}^{k} \int_s^t \mathrm{d}\tau \, [S_j^2, \widetilde{\gamma}^{(k)}_{N,\tau}] - i \frac{\lambda}{N} \sum_{i<j}^{k} \int_s^t \mathrm{d}\tau \, [V^{\varepsilon}(x_i - x_j), \widetilde{\gamma}^{(k)}_{N,\tau}] \\
- i(1 - \frac{1}{N}) \lambda \sum_{j=1}^{k} \int_s^t \mathrm{d}\tau \, \mathrm{Tr}_{k+1} [V^{\varepsilon}(x_j - x_{k+1}), \widetilde{\gamma}^{(k+1)}_{N,\tau}]
\end{aligned}$$

where Tr_{k+1} denotes the partial trace over the $k+1$'th variable. Next we choose $J^{(k)} \in \mathcal{K}_k$ such that $\|S_j J^{(k)} S_j^{-1}\|$ is bounded for all $j = 1, \dots k$, and we compute

$$\begin{aligned}
\mathrm{Tr}^{(k)} \, J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,t} - \widetilde{\gamma}^{(k)}_{N,s} \right) = & -i \sum_{j=1}^{k} \int_s^t \mathrm{d}\tau \, \mathrm{Tr}^{(k)} \, \left(S_j^{-1} J^{(k)} S_j - S_j J^{(k)} S_j^{-1} \right) \, S_j \widetilde{\gamma}^{(k)}_{N,\tau} S_j \\
& - i \frac{\lambda}{N} \sum_{i<j}^{k} \int_s^t \mathrm{d}\tau \, \mathrm{Tr}^{(k)} \, \Big(S_j^{-1} J^{(k)} S_j S_j^{-1} V^{\varepsilon}(x_i - x_j) S_j^{-1} S_j \widetilde{\gamma}^{(k)}_{N,\tau} S_j \\
& \qquad - S_j J^{(k)} S_j^{-1} S_j \widetilde{\gamma}^{(k)}_{N,\tau} S_j S_j^{-1} V^{\varepsilon}(x_i - x_j) S_j^{-1} \Big) \\
& - i(1 - \frac{1}{N}) \lambda \sum_{j=1}^{k} \int_s^t \mathrm{d}\tau \, \mathrm{Tr}^{(k+1)} J^{(k)} [S_{k+1}^{-1} V(x_j - x_{k+1}) S_{k+1}^{-1}, S_{k+1} \widetilde{\gamma}^{(k)}_{N,\tau} S_{k+1}]
\end{aligned}$$

where we used the fact that $S_{k+1} = (1 + p_{k+1}^2)^{1/4}$ commutes with $J^{(k)}$. Since we assumed $\|S_j J^{(k)} S_j^{-1}\|$ to be bounded for all $j = 1, \dots k$ and since $\|S_j^{-1} V^{\varepsilon}(x_i - x_j) S_j^{-1}\| \leq C$ (see Lemma 9.1), we find, using Theorem 4.4,

$$\left| \mathrm{Tr}^{(k)} \, J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,t} - \widetilde{\gamma}^{(k)}_{N,s} \right) \right| \leq C_k \, |t - s|$$

for a constant C_k which depends on k and $J^{(k)}$, but is independent of $t, s \in [0,T]$ and of N, for N large enough (depending on k, ε and κ). Since the set of all $J^{(k)} \in \mathcal{K}_k$, with $\|S_j J^{(k)} S_j^{-1}\|$ bounded for $j = 1, \dots k$, is a dense subset of $\mathcal{A}_k$, it follows by Lemma 9.4 that the sequence $\widetilde{\gamma}^{(k)}_{N,t}$ is equicontinuous with respect to the metric ρ_k (defined in (2.2)). This in particular implies that $\widetilde{\gamma}^{(k)}_{N,t} \in C([0,T], \mathcal{H}_k)$ for all N large enough. Moreover, since the sequence $\gamma^{(k)}_{N,t} \in C([0,T], \mathcal{H}_k)$ is uniformly bounded, it follows by the Arzela-Ascoli Theorem (see [11], Theorem I.28) that it is compact with respect to the metric $\widehat{\rho}_k$, and that any limit point $\gamma^{(k)}_{\infty,t}$ satisfies (5.2). $\square$

6 Convergence to the infinite hierarchy

In this section we consider limit points of the sequence $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$, where $\widetilde{\gamma}^{(k)}_{N,t}$ are the marginals associated with the solution $\psi^{\varepsilon,\kappa}_{N,t}$ of the modified Schrödinger equation (1.11) with regularized initial data ψ^{κ}_N. We prove that any limit point of this sequence, as $N \to \infty$, is a solution of the infinite hierarchy of equations (1.6).

Theorem 6.1. *Suppose $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1} \in C([0,T],\mathcal{H})$ is a limit point of the sequence $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ with respect to the product of the topologies induced by the metrics $\widehat{\rho}_k$ (defined in (2.3)). Then $\Gamma_{\infty,t}$ satisfies the infinite hierarchy (1.6) with initial data*

$$\gamma^{(k)}_{\infty,0}(\mathbf{x}_k;\mathbf{x}'_k) = \gamma^{(k)}_0(\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^k \varphi(x_j)\overline{\varphi(x_j)}\,. \tag{6.1}$$

Proof. Without loss of generality we can assume that $\widetilde{\Gamma}_{N,t} \to \Gamma_{\infty,t} \in C([0,T],\mathcal{H})$ with respect to the product of the topologies induced by the metrics $\widehat{\rho}_k$. Then, for every fixed $k \geq 1$ and $t \in [0,T]$, we have $\widetilde{\gamma}^{(k)}_{N,t} \to \gamma^{(k)}_{\infty,t} \in \mathcal{H}_k$ with respect to the weak* topology of $\mathcal{H}_k$ (because the sequence $\gamma^{(k)}_{N,t}$ is uniformly bounded in $\mathcal{H}_k$ and the metric ρ_k is equivalent to the weak* topology for uniformly bounded sequences). Hence, for every $t \in [0,T]$, $k \geq 1$, and every $J^{(k)} \in \mathcal{A}_k$, we have

$$\mathrm{Tr}\; J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,t} - \gamma^{(k)}_{\infty,t}\right) \to 0 \quad \text{for } N \to \infty\,. \tag{6.2}$$

We choose $J^{(k)} \in \mathcal{K}_k$ such that $S_j J^{(k)} S_j^{-1}$ is bounded for all $j = 1,\dots k$, and we obtain, from the regularized BBGKY hierarchy (5.1), rewritten in integral form,

$$\begin{aligned}\mathrm{Tr}^{(k)}\; J^{(k)}\widetilde{\gamma}^{(k)}_{N,t} = &\;\mathrm{Tr}^{(k)} J^{(k)}\mathcal{U}^{(k)}(t)\widetilde{\gamma}^{(k)}_{N,0} \\ &- i\lambda(1-k/N)\sum_{j=1}^k \int_0^t \mathrm{d}s\; \mathrm{Tr}^{(k+1)}\; J^{(k)}\mathcal{U}^{(k)}(t-s)\; [V^{\varepsilon}_{j,k+1}, \widetilde{\gamma}^{(k+1)}_{N,s}]\end{aligned} \tag{6.3}$$

where

$$\mathcal{U}^{(k)}(t)\gamma^{(k)} = e^{-itH^{(k)}}\gamma^{(k)}e^{itH^{(k)}} \qquad \text{with} \quad H^{(k)} = \sum_{j=1}^k S_j^2 + \frac{\lambda}{N}\sum_{i<j}^k V^{\varepsilon}_{i,j}$$

and where we use the notation $V^{\varepsilon}_{i,j} = V^{\varepsilon}(x_i - x_j)$. Next we consider the limit $N \to \infty$ of (6.3). From (6.2), the l.h.s. of (6.3) converges to $\mathrm{Tr}\; J^{(k)}\gamma^{(k)}_{\infty,t}$. As for the first term on the r.h.s. of (6.3) we find

$$\begin{aligned}\mathrm{Tr}^{(k)}\; J^{(k)}\; \left(\mathcal{U}^{(k)}(t)\widetilde{\gamma}^{(k)}_{N,0} - \mathcal{U}^{(k)}_0(t)\gamma^{(k)}_{\infty,0}\right) = &\;\mathrm{Tr}^{(k)}\; J^{(k)}\; \left(\mathcal{U}^{(k)}(t) - \mathcal{U}^{(k)}_0(t)\right)\widetilde{\gamma}^{(k)}_{N,0} \\ &+ \mathrm{Tr}^{(k)}\; J^{(k)}\,\mathcal{U}^{(k)}_0(t)\left(\widetilde{\gamma}^{(k)}_{N,0} - \gamma^{(k)}_{\infty,0}\right)\end{aligned} \tag{6.4}$$

where we used the notation

$$\mathcal{U}^{(k)}_0(t)\gamma^{(k)} = \exp\left(-it\sum_{j=1}^k S_j^2\right)\gamma^{(k)}\exp\left(it\sum_{j=1}^k S_j^2\right)$$

for the free evolution of the first k particles. The first contribution on the r.h.s. of (6.4) can be handled as follows:

$$\mathrm{Tr}^{(k)}\, J^{(k)} \left(\mathcal{U}^{(k)}(t) - \mathcal{U}_0^{(k)}(t)\right) \widetilde{\gamma}^{(k)}_{N,0} = \frac{-i\lambda}{N} \sum_{i,j}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k)}\, J^{(k)} \mathcal{U}^{(k)}(t-s) V^{\varepsilon}_{i,j}\, \mathcal{U}_0^{(k)}(s) \widetilde{\gamma}^{(k)}_{N,0}$$

and thus, using the permutation symmetry of $\widetilde{\gamma}^{(k)}_{N,0}$, and the a-priori estimate from Theorem 4.4,

$$\begin{aligned} \left|\mathrm{Tr}^{(k)}\, J^{(k)} \left(\mathcal{U}^{(k)}(t) - \mathcal{U}_0^{(k)}(t)\right) \widetilde{\gamma}^{(k)}_{N,0}\right| &\leq C N^{-1}\, t\, k^2 \|J^{(k)}\|\, \|V^{\varepsilon}_{1,2} S_1^{-1} S_2^{-1}\|\, \mathrm{Tr}^{(k)}\, S_1 S_2 \widetilde{\gamma}^{(k)}_{N,0} S_2 S_1 \\ &\leq C N^{-1} \end{aligned}$$

with a constant C depending on k, on the observable $J^{(k)}$ and on t. As for the second term on the r.h.s. of (6.4) we have

$$\mathrm{Tr}^{(k)}\, J^{(k)} \mathcal{U}_0^{(k)}(t) \left(\widetilde{\gamma}^{(k)}_{N,0} - \gamma^{(k)}_{\infty,0}\right) = \mathrm{Tr}^{(k)} \left(\mathcal{U}_0^{(k)}(-t) J^{(k)}\right) \left(\widetilde{\gamma}^{(k)}_{N,0} - \gamma^{(k)}_{\infty,0}\right) \to 0$$

for $N \to \infty$, because, if $J^{(k)} \in \mathcal{K}_k$, then also $\mathcal{U}_0^{(k)}(-t) J^{(k)} \in \mathcal{K}_k \subset \mathcal{A}_k$, and thus (6.2) can be used.

Next we prove that the difference

$$\begin{aligned} (1-k/N) \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)} \mathcal{U}^{(k)}(t-s)\, [V^{\varepsilon}_{j,k+1}, \widetilde{\gamma}^{(k+1)}_{N,s}] \\ - \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)}\, \mathcal{U}_0^{(k)}(t)\, [V_{j,k+1}, \gamma^{(k+1)}_{\infty,s}] \end{aligned}$$

converges to zero, for $N \to \infty$. Here and henceforth, $V_{i,j} = V(x_i - x_j) = |x_i - x_j|^{-1}$. To this end, we rewrite it as the sum of four terms

$$\begin{aligned} &- k/N \sum_{j=1}^{k} \int_0^t \mathrm{d}s \mathrm{Tr}^{(k+1)}\, J^{(k)}\, \mathcal{U}^{(k)}(t)\, [V^{\varepsilon}_{j,k+1}, \widetilde{\gamma}^{(k)}_{N,s}] \\ &+ \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)} \left(\mathcal{U}^{(k)}(t-s) - \mathcal{U}_0^{(k)}(t-s)\right) [V^{\varepsilon}_{j,k+1}, \widetilde{\gamma}^{(k)}_{N,s}] \\ &+ \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)}\, \mathcal{U}_0^{(k)}(t-s) [V^{\varepsilon}_{j,k+1} - V_{j,k+1}, \widetilde{\gamma}^{(k)}_{N,s}] \\ &+ \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}_{k+1}\, J^{(k)}\, \mathcal{U}_0^{(k)}(t-s) \left[V_{j,k+1}, \left(\widetilde{\gamma}^{(k)}_{N,s} - \gamma^{(k)}_{\infty,s}\right)\right] . \end{aligned}$$

The first term converges to zero, for $N \to \infty$, because

$$\begin{aligned} &\left| k/N \sum_{j=1}^{k} \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)}\, \mathcal{U}^{(k)}(t)\, [V^{\varepsilon}_{j,k+1}, \widetilde{\gamma}^{(k+1)}_{N,s}] \right| \\ &\qquad \leq C\, N^{-1} k^2 t\, \|J^{(k)}\|\, \|V^{\varepsilon}_{1,k+1} S_1^{-1} S_{k+1}^{-1}\| \sup_{s\in[0,t]} \mathrm{Tr}^{(k+1)}\, |S_1 S_{k+1} \gamma^{(k+1)}_{N,s}| \\ &\qquad \leq C_{k,t} N^{-1} \sup_{s\in[0,t]} \mathrm{Tr}^{(k+1)}\, S_1 S_{k+1} \gamma^{(k+1)}_{N,s} S_{k+1} S_1 \leq C_{k,t} N^{-1} \end{aligned}$$

for a constant $C_{k,t}$ depending on k, on t, and on the observable $J^{(k)}$. To control the second term we note that

$$\sum_{j=1}^{k}\int_0^t \mathrm{d}s\,\mathrm{Tr}^{(k+1)}\; J^{(k)}\;\left(\mathcal{U}^{(k)}(t-s)-\mathcal{U}_0^{(k)}(t-s)\right)\left[V^{\varepsilon}_{j,k+1},\widetilde{\gamma}^{(k+1)}_{N,s}\right]$$
$$= -i\lambda N^{-1}\sum_{j=1}^{k}\sum_{\ell<m}^{k}\int_0^t \mathrm{d}s\int_0^{t-s}\mathrm{d}\tau\,\mathrm{Tr}^{(k+1)}\; J^{(k)}\,\mathcal{U}_0^{(k)}(t-s-\tau)\left[V^{\varepsilon}_{\ell,m},\mathcal{U}^{(k)}(r)\left[V^{\varepsilon}_{j,k+1},\widetilde{\gamma}^{(k+1)}_{N,s}\right]\right]\,.$$

Writing down the four terms arising from the two commutators and using the permutation symmetry, we get the bound

$$\begin{aligned}\left|\sum_{j=1}^{k}\int_0^t \mathrm{d}s\,\mathrm{Tr}^{(k+1)}\; J^{(k)}\;\left(\mathcal{U}^{(k)}(t-s)-\mathcal{U}_0^{(k)}(t-s)\right)\left[V^{\varepsilon}_{j,k+1},\widetilde{\gamma}^{(k+1)}_{N,s}\right]\right| \\ \leq Ct^2k^3N^{-1}\sup_{\ell\leq k}\left(\|S_\ell J^{(k)}S_\ell^{-1}\|+\|J^{(k)}\|\right)\,\|S_1^{-1}V^{\varepsilon}_{1,2}\|\|V^{\varepsilon}_{1,k+1}S_1^{-1}S_{k+1}^{-1}\| \\ \times\sup_{s\in[0,T]}\left(\mathrm{Tr}\,|S_1S_2\widetilde{\gamma}^{(k+1)}_{N,s}S_3|+\mathrm{Tr}\,|S_1S_2S_3\widetilde{\gamma}^{(k+1)}_{N,s}|\right) \\ \leq C_{\varepsilon,t,k}N^{-1/2}\sup_{s\in[0,T]}\mathrm{Tr}\,S_1S_2S_3\widetilde{\gamma}^{(k+1)}_{N,s}S_1S_2S_3\leq CN^{-1/2}\end{aligned}$$

where the constant C depends on ε,k,t and on $J^{(k)}$. Here we used that $\|S_\ell^{-1}V^{\varepsilon}_{\ell,m}\|\leq C\varepsilon^{-1/2}N^{1/2}$, the symmetry of $\widetilde{\gamma}^{(k+1)}_{N,s}$, and Theorem 4.4. To bound the third term in (6), we note that

$$V^{\varepsilon}_{j,k+1}-V_{j,k+1}=\frac{\varepsilon N^{-1}}{|x_j-x_{k+1}|(|x_j-x_{k+1}|+\varepsilon N^{-1})}.$$

Hence

$$\begin{aligned}\left|\sum_{j=1}^{k}\int_0^t \mathrm{d}s\,\mathrm{Tr}^{(k+1)}\; J^{(k)}\,\mathcal{U}_0^{(k)}(t-s)\left[V^{\varepsilon}_{j,k+1}-V_{j,k+1},\widetilde{\gamma}^{(k+1)}_{N,s}\right]\right| \\ \leq kt\varepsilon N^{-1}\,\|S_1^{-1}J^{(k)}S_1\|\,\|S_1^{-1}\frac{1}{|x_1-x_2|(|x_1-x_2|+\varepsilon N^{-1})}S_1^{-1}S_2^{-2}\|\sup_{s\in[0,T]}\mathrm{Tr}\,|S_1S_2\widetilde{\gamma}^{(k+1)}_{N,s}S_1| \\ \leq C_{k,t}N^{-1/2}\sup_{s\in[0,T]}\mathrm{Tr}\,S_1S_2\widetilde{\gamma}^{(k+1)}_{N,s}S_1S_2\leq CN^{-1/2}\end{aligned}$$

for a constant $C_{k,t}$ depending on k, t, and $J^{(k)}$. Finally we consider the last term in (6)

$$\sum_{j=1}^{k}\int_0^t \mathrm{d}s\,\mathrm{Tr}_{k+1}\; J^{(k)}\,\mathcal{U}_0^{(k)}(t-s)\left[V_{j,k+1},\left(\widetilde{\gamma}^{(k+1)}_{N,s}-\gamma^{(k+1)}_{\infty,s}\right)\right] \tag{6.5}$$

This term converges to zero, because, since $J^{(k)}\in\mathcal{K}_k\subset\mathcal{A}_k$, we have $(\mathcal{U}_0^{(k)}(s-t)J^{(k)})V_{j,k+1}\in\mathcal{A}_{k+1}$ for every $j\leq k$, every s and t. In fact

$$\begin{aligned}\|S_1^{-1}\dots S_{k+1}^{-1}(\mathcal{U}_0^{(k)}(s-t)J^{(k)})V_{j,k+1}S_1^{-1}\dots S_{k+1}^{-1}\| \\ \leq\|S_1^{-1}\dots S_k^{-1}J^{(k)}S_k^{-1}\dots S_1^{-1}\|\,\|S_{k+1}^{-1}V_{j,k+1}S_{k+1}^{-1}\|\leq C\|J^{(k)}\|_{\mathcal{A}_k}.\end{aligned}$$

This proves that the integrand in (6.5) converges to zero as $N \to \infty$, for every $s \in [0,t]$ and for every $j = 1, \dots, k$. Since the integrand is uniformly bounded in s, it follows that (6.5) converges to zero for $N \to \infty$.

We have proven that, for every fixed $t \in [0,T]$, $k \geq 1$ and $J^{(k)} \in \mathcal{K}_k$ such that $S_j J^{(k)} S_j^{-1}$ is finite for all j, we have

$$\mathrm{Tr}^{(k)}\, J^{(k)} \gamma^{(k)}_{\infty,t} = \mathrm{Tr}\, J^{(k)} \mathcal{U}_0^{(k)}(t) \gamma^{(k)}_{\infty,0} - i\lambda \sum_{j=1}^k \int_0^t \mathrm{d}s\, \mathrm{Tr}^{(k+1)}\, J^{(k)} \mathcal{U}_0^{(k)}(t-s) \left[V_{j,k+1}, \gamma^{(k+1)}_{\infty,s} \right] . \tag{6.6}$$

Since the set of $J^{(k)} \in \mathcal{K}_k$ such that $\sup_{j \leq k} \|S_j^{-1} J^{(k)} S_j\| < \infty$ is a dense subset of $\mathcal{A}_k$, it follows by a simple approximation argument, that (6.6) holds true for all $J^{(k)} \in \mathcal{A}_k$. Thus

$$\gamma^{(k)}_{\infty,t} = \mathcal{U}_0^{(k)}(t) \gamma^{(k)}_{\infty,0} - i\lambda \sum_{j=1}^k \int_0^t \mathrm{d}s\, \mathcal{U}_0^{(k)}(t-s)\, \mathrm{Tr}_{k+1} \left[V_{j,k+1}, \gamma^{(k+1)}_{\infty,s} \right]$$

for all $t \in [0,T]$. Finally we have to prove that $\gamma^{(k)}_{\infty,0}$ is given by (6.1). Recall that

$$\widetilde{\gamma}^{(k)}_{N,0}(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi^\kappa(x_j) \overline{\varphi^\kappa}(x'_j)$$

with $\varphi^\kappa = \exp(-\kappa |p|/N) \varphi$. Hence, for any $J^{(k)} \in \mathcal{K}_k$, we have

$$\mathrm{Tr}\, J^{(k)} \left(\gamma_0^{(k)} - \gamma^{(k)}_{\infty,0} \right) = \mathrm{Tr}\, J^{(k)} \left(\gamma_0^{(k)} - \widetilde{\gamma}^{(k)}_{N,0} \right) + \mathrm{Tr}\, J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,0} - \gamma^{(k)}_{\infty,0} \right) . \tag{6.7}$$

The second term converges to zero, for $N \to \infty$. As for the first one, we have

$$\left| \mathrm{Tr}\, J^{(k)} \left(\gamma_0^{(k)} - \widetilde{\gamma}^{(k)}_{N,0} \right) \right| \leq \|J^{(k)}\|\, \mathrm{Tr} \left| \gamma_0^{(k)} - \widetilde{\gamma}^{(k)}_{N,0} \right| \leq Ck\, \|\varphi - \varphi^\kappa\| \leq CkN^{-1}$$

where we used that $\|\varphi^\kappa\| \leq \|\varphi\| = 1$, and that

$$\|\varphi - \varphi^\kappa\|^2 = \int \mathrm{d}p\, (1 - e^{-\kappa|p|/N})^2 |\varphi(p)|^2 \leq \kappa^2 N^{-2} \|\varphi\|_{H^1} .$$

Since the choice of N on the right side of (6.7) is arbitrary, we find

$$\mathrm{Tr}\, J^{(k)} \left(\gamma_0^{(k)} - \gamma^{(k)}_{\infty,0} \right) = 0$$

for all $J^{(k)} \in \mathcal{K}_k$. Moreover, using the fact that $\gamma_0^{(k)}$ and $\gamma^{(k)}_{\infty,0}$ have finite $\mathcal{H}_k$ norm, a simple approximation argument shows that the last equation is true for all $J^{(k)} \in \mathcal{A}_k$. This proves that $\gamma^{(k)}_{\infty,0} = \gamma_0^{(k)}$. □

7 Uniqueness of the solution of the infinite hierarchy

In this section we prove the uniqueness of the solution of the infinite hierarchy (1.6). We already know that $\Gamma_t = \{\gamma_t^{(k)}\}_{k \geq 1}$, with

$$\gamma_t^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^k \varphi_t(x_j) \overline{\varphi_t}(x_j)$$

and φ_t the solution of the nonlinear Hartree equation (1.3), is a solution of (1.6). Since, by Theorem 6.1, we know that every limit point of $\widetilde{\Gamma}_{N,t}$ is a solution of (1.6), it follows that Γ_t is the only limit point of $\widetilde{\Gamma}_{N,t}$.

Theorem 7.1. *Fix $T > 0$. Let $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1} \in \mathcal{H} = \bigoplus_{k\geq 1} \mathcal{H}_k$. Moreover, suppose that there exists $C > 0$ such that*

$$\|\gamma_0^{(k)}\|_{\mathcal{H}_k} = \mathit{Tr}\, |S_1 \dots S_k \gamma_0^{(k)} S_k \dots S_1| \leq C^k$$

for all $k \geq 1$. Then there exists at most one solution $\Gamma_t \in C([0,T], \mathcal{H})$ of the infinite hierarchy (1.6) such that $\Gamma_{t=0} = \Gamma_0$ and

$$\|\gamma_t^{(k)}\|_{\mathcal{H}_k} \leq C^k$$

for all $t \in [0,T]$.

Proof. Suppose $\Gamma_t = \{\gamma_t^{(k)}\}_{k\geq 1}$ is a solution of the infinite BBGKY hierarchy (1.6), so that $\Gamma_{t=0} = \Gamma_0$ and

$$\|\gamma_t^{(k)}\|_{\mathcal{H}_k} \leq C^k \tag{7.1}$$

for all $t \in [0,T]$ and $k \geq 1$. Rewriting (1.6) in integral form we find

$$\gamma_t^{(k)} = \mathcal{U}_0^{(k)}(t)\gamma_0^{(k)} - i\lambda \sum_{j=1}^k \int_0^t \mathrm{d}s\, \mathcal{U}_0^{(k)}(t-s)\, \mathrm{Tr}_{k+1}\, [V_{j,k+1}, \gamma_s^{(k+1)}]\,, \tag{7.2}$$

where we use the notation $V_{i,j} = V(x_i - x_j) = |x_i - x_j|^{-1}$ and where the free evolution $\mathcal{U}_0^{(k)}(t)$ is defined by

$$\mathcal{U}_0^{(k)}(t)\gamma^{(k)} = \exp\left(-it\sum_{j=1}^k S_j^2\right)\gamma^{(k)} \exp\left(it\sum_{j=1}^k S_j^2\right)\,.$$

We iterate (7.2), and find

$$\gamma_t^{(k)} = \mathcal{U}_0^{(k)}(t)\gamma_0^{(k)} + \sum_{m=1}^{n-1} \zeta(k,t,m) + \widetilde{\zeta}(k,t,n) \tag{7.3}$$

where

$$\begin{aligned}\zeta(k,t,m) = (-i\lambda)^m \sum_{j_1=1}^k \cdots \sum_{j_m=1}^{k+m-1} \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{m-1}} \mathrm{d}s_m\, \mathcal{U}_0^{(k)}(t-s_1)\, \mathrm{Tr}_{k+1}\, \Big[V_{j_1,k+1}, \mathcal{U}_0^{(k+1)}(s_1 - s_2) \\ \cdots \times \mathcal{U}_0^{(k+m-1)}(s_{m-1} - s_m)\, \mathrm{Tr}_{k+m}\, \Big[V_{j_m,k+m}, \mathcal{U}_0^{(k+m)}(s_m)\gamma_0^{(k+m)}\Big] \dots \Big]\end{aligned}$$

and the error term $\widetilde{\zeta}(k,n,t)$ is given by

$$\begin{aligned}\widetilde{\zeta}(k,t,n) = (-i\lambda)^n \sum_{j_1=1}^k \cdots \sum_{j_n=1}^{k+n-1} \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{n-1}} \mathrm{d}s_n\, \mathcal{U}_0^{(k)}(t-s_1)\, \mathrm{Tr}_{k+1}\, \Big[V_{j_1,k+1}, \mathcal{U}_0^{(k+1)}(s_1 - s_2) \\ \cdots \times \mathcal{U}_0^{(k+n-1)}(s_{n-1} - s_n)\, \mathrm{Tr}_{k+n}\, \Big[V_{j_n,k+n}, \gamma_{s_n}^{(k+n)}\Big] \dots \Big]\,.\end{aligned} \tag{7.4}$$

In order to bound the error term, we note that, for any $\ell \geq 1$, $j = 1, \dots \ell$, $s \in \mathbb{R}$, and $\gamma^{(\ell+1)} \in \mathcal{H}_{\ell+1}$, we have

$$\begin{aligned}
\|\mathcal{U}_0^{(\ell)}(s)\mathrm{Tr}_{\ell+1}[V_{j,\ell+1},\gamma^{(\ell+1)}]\|_{\mathcal{H}_\ell} \\
&= \mathrm{Tr}^{(\ell)} \left| S_1 \dots S_\ell \mathcal{U}_0^{(\ell)}(s) \left(\mathrm{Tr}_{\ell+1} \left[V_{j,\ell+1}, \gamma^{(\ell+1)} \right] \right) S_\ell \dots S_1 \right| \\
&\leq \mathrm{Tr}^{(\ell)} \left| S_1 \dots S_\ell \left(\mathrm{Tr}_{\ell+1} V_{j,\ell+1} \gamma^{(\ell+1)} \right) S_\ell \dots S_1 \right| \\
&\quad + \mathrm{Tr}^{(\ell)} \left| S_1 \dots S_\ell \left(\mathrm{Tr}_{\ell+1} \gamma^{(\ell+1)} V_{j,\ell+1} \right) S_\ell \dots S_1 \right|
\end{aligned} \tag{7.5}$$

because the free evolution $\mathcal{U}_0^{(\ell)}$ commutes with the operators S_j. We consider the first term. By the cyclicity of the partial trace and because

$$\mathrm{Tr}^{(\ell)}|\mathrm{Tr}_{\ell+1}A| \leq \mathrm{Tr}^{(\ell+1)}|A|,$$

(see Proposition 9.4 in [4]), we have

$$\begin{aligned}
\mathrm{Tr}^{(\ell)} \left| S_1 \dots S_\ell \left(\mathrm{Tr}_{\ell+1} V_{j,\ell+1} \gamma^{(\ell+1)} \right) S_\ell \dots S_1 \right| \\
&= \mathrm{Tr}^{(\ell)} \left| S_1 \dots S_\ell \left(\mathrm{Tr}_{\ell+1} S_{\ell+1}^{-1} V_{j,\ell+1} \gamma^{(\ell+1)} S_{\ell+1} \right) S_\ell \dots S_1 \right| \\
&\leq \mathrm{Tr}^{(\ell+1)} |S_1 \dots S_\ell S_{\ell+1}^{-1} V_{j,\ell+1} \gamma^{(\ell+1)} S_{\ell+1} S_\ell \dots S_1| \\
&= \mathrm{Tr}^{(\ell+1)} \left| \left(S_j S_{\ell+1}^{-1} V_{j,\ell+1} S_j^{-1} S_{\ell+1}^{-1} \right) S_1 \dots S_{\ell+1} \gamma^{(\ell+1)} S_{\ell+1} \dots S_1 \right| \\
&\leq \|S_j S_{\ell+1}^{-1} V_{j,\ell+1} S_j^{-1} S_{\ell+1}^{-1}\| \, \|\gamma^{(\ell+1)}\|_{\mathcal{H}_{\ell+1}} \leq \widetilde{C} \|\gamma^{(\ell+1)}\|_{\mathcal{H}_{\ell+1}}
\end{aligned}$$

for a constant $\widetilde{C}$, independent of ℓ. In the last inequality we used Lemma 9.1, part (ii). The second term on the r.h.s. of (7.5) can be bounded similarly. We get

$$\|\mathcal{U}^{(\ell)}(s)\mathrm{Tr}_{\ell+1}[V_{j,\ell+1}, \gamma^{(\ell+1)}]\|_{\mathcal{H}_\ell} \leq 2\widetilde{C}\|\gamma^{(\ell+1)}\|_{\mathcal{H}_{\ell+1}} .$$

Applying this bound iteratively to (7.4) we find

$$\|\widetilde{\zeta}(k,n,t)\|_{\mathcal{H}_k} \leq 2^n \lambda^n \widetilde{C}^n t^n \binom{k+n}{n} \sup_{s \in [0,t]} \|\gamma_s^{(k+n)}\|_{\mathcal{H}_{k+n}} .$$

By (7.1) we have,

$$\|\widetilde{\zeta}(k,n,t)\| \leq D^k D^n t^n \,, \tag{7.6}$$

where D depends only on the constant C in the bound (7.1) (and on λ). Now suppose that $\Gamma_{1,t} = \{\gamma_{1,t}^{(k)}\}_{k\geq 1}$ and $\Gamma_{2,t} = \{\gamma_{2,t}^{(k)}\}_{k\geq 1}$ are two solutions of the infinite BBGKY hierarchy with $\Gamma_{1,t=0} = \Gamma_{2,t=0} = \Gamma_0$ and satisfying (7.1). Then, for $t \leq 1/(2D)$, we have

$$\|\gamma_{1,t}^{(k)} - \gamma_{2,t}^{(k)}\|_{\mathcal{H}_k} = \|\widetilde{\zeta}_1(k,t,n) - \widetilde{\zeta}_2(k,t,n)\|_{\mathcal{H}_k} \leq 2D^k 2^{-n}$$

for any $n > 1$ (note that the terms $\zeta(k,t,m)$ in the sum over m in (7.3) depend only on the initial data $\gamma_0^{(k)}$, and thus do not contribute to the difference $\gamma_{1,t}^{(1)} - \gamma_{2,t}^{(2)}$). Since $n \geq 1$ is arbitrary, we find

$$\gamma_{1,t}^{(k)} = \gamma_{2,t}^{(k)}$$

for all $k \geq 1$ and for all $t \leq 1/(2D)$. Since the bound (7.1) holds uniformly in t, for $t \in [0,T]$, the argument can be iterated to prove that $\Gamma_{1,t} = \Gamma_{2,t}$, for all $t \in [0,T]$. $\square$

8 Removal of the cutoffs

From Theorem 6.1 and Theorem 7.1, we know that $\widetilde{\Gamma}_{N,t}$ converges to $\Gamma_t = \{\gamma_t^{(k)}\}_{k\geq 1} \in C([0,T],\mathcal{H})$ for $N \to \infty$, with respect to the product of the topologies $\widehat{\rho}_k$ (defined in 2.3). In this section, we show how to remove the cutoffs ε and κ, which are used to regularize the interaction and the initial data. To remove the cutoff ε we need to compare two dynamics, the one generated by the modified Hamiltonian (1.8), and the one generated by the original Hamiltonian (1.1): this task is accomplished in the next proposition.

Proposition 8.1. *Suppose $\psi_{N,t}^{\varepsilon,\kappa}$ is a solution of the modified Schrödinger equation (1.11) with initial data ψ_N^κ, and let $\psi_{N,t}^\kappa$ be the solution of the original Schrödinger equation (1.4), with the same initial data. Then there exists a constant C, independent of N, κ, ε and t such that*

$$\|\psi_{N,t}^{\kappa,\varepsilon} - \psi_{N,t}^{\kappa}\| \leq C\, t\, \varepsilon^{1/4}$$

for all $t \geq 0$, and for all N large enough (depending on ε and κ).

Proof. We use the notation $\phi_{N,t} = \psi_{N,t}^\kappa$ and $\widetilde{\phi}_{N,t} = \psi_{N,t}^{\varepsilon,\kappa}$. We compute

$$\begin{aligned}\partial_t \|\phi_{N,t} - \widetilde{\phi}_{N,t}\|^2 &= i(H_N\phi_{N,t} - \widetilde{H}_N\widetilde{\phi}_{N,t}, \phi_{N,t} - \widetilde{\phi}_{N,t}) - i(\phi_{N,t} - \widetilde{\phi}_{N,t}, H_N\phi_{N,t} - \widetilde{H}_N\widetilde{\phi}_{N,t}) \\ &= -\mathrm{Im}\,(\phi_{N,t} - \widetilde{\phi}_{N,t}, (H_N - \widetilde{H}_N)\widetilde{\phi}_{N,t})\end{aligned}$$

by the self-adjointess of H_N. Hence

$$\pm\partial_t\|\phi_{n,t} - \widetilde{\phi}_{N,t}\|^2 \leq \|\phi_{N,t} - \widetilde{\phi}_{N,t}\|\, \|(H_N - \widetilde{H}_N)\widetilde{\phi}_{N,t}\|\,. \tag{8.1}$$

We have

$$H_N - \widetilde{H}_N = \frac{\lambda}{N}\sum_{i<j}^N \left(\frac{1}{|x_i - x_j|} - \frac{1}{|x_i - x_j| + \varepsilon N^{-1}}\right) = \frac{\lambda}{N}\sum_{i<j}^N \frac{\varepsilon N^{-1}}{|x_i - x_j|(|x_i - x_j| + \varepsilon N^{-1})}.$$

Therefore, using the symmetry with respect to permutations

$$\begin{aligned}\|(H_N - \widetilde{H}_N)\widetilde{\phi}_{N,t}\|^2 &= (\widetilde{\phi}_{N,t}, (H_N - \widetilde{H}_N)^2\widetilde{\phi}_{N,t}) \\ &\leq \lambda^2\varepsilon^2\, (\widetilde{\phi}_{N,t}, \frac{1}{|x_1 - x_2|(|x_1 - x_2| + \varepsilon N^{-1})}\frac{1}{|x_3 - x_4|(|x_3 - x_4| + \varepsilon N^{-1})}\,\widetilde{\phi}_{N,t}) \\ &+ \lambda^2\varepsilon^2 N^{-1}\, (\widetilde{\phi}_{N,t}, \frac{1}{|x_1 - x_2|(|x_1 - x_2| + \varepsilon N^{-1})}\frac{1}{|x_2 - x_3|(|x_2 - x_3| + \varepsilon N^{-1})}\,\widetilde{\phi}_{N,t}) \\ &+ \lambda^2\varepsilon^2 N^{-2}\, (\widetilde{\phi}_{N,t}, \frac{1}{|x_1 - x_2|^2(|x_1 - x_2| + \varepsilon N^{-1})^2}\,\widetilde{\phi}_{N,t})\,.\end{aligned} \tag{8.2}$$

The first term on the r.h.s. of the last equation can be estimated by

$$\begin{aligned}(\widetilde{\phi}_{N,t}, &\frac{1}{|x_1 - x_2|(|x_1 - x_2| + \varepsilon N^{-1})}\frac{1}{|x_3 - x_4|(|x_3 - x_4| + \varepsilon N^{-1})}\,\widetilde{\phi}_{N,t}) \\ &\leq (\widetilde{\phi}_{N,t}, \frac{1}{|x_1 - x_2|^2\,|x_3 - x_4|^2}\,\widetilde{\phi}_{N,t}) \\ &\leq C\,(\widetilde{\phi}_{N,t},\, S_1^2 S_2^2 S_3^2 S_4^2\,\widetilde{\phi}_{N,t}) \leq C\end{aligned}$$

for all $t \in \mathbb{R}$, and for every N large enough (depending on ε, κ, see Theorem 4.4). The second term can be bounded, using again Theorem 4.4, by

$$\begin{aligned}
(\widetilde{\phi}_{N,t}, \frac{1}{|x_1-x_2|(|x_1-x_2|+\varepsilon N^{-1})} \frac{1}{|x_2-x_3|(|x_2-x_3|+\varepsilon N^{-1})} \widetilde{\phi}_{N,t}) \\
&\leq \varepsilon^{-1} N (\widetilde{\phi}_{N,t}, \frac{1}{|x_1-x_2|^2 \, |x_2-x_3|} \widetilde{\phi}_{N,t}) \\
&\leq C \, \varepsilon^{-1} N (\widetilde{\phi}_{N,t}, S_3 \frac{1}{|x_1-x_2|^2} S_3 \, \widetilde{\phi}_{N,t}) \\
&\leq C \, \varepsilon^{-1} N (\widetilde{\phi}_{N,t}, \, S_1^2 S_2^2 S_3^2 \, \widetilde{\phi}_{N,t}) \\
&\leq C \, \varepsilon^{-1} N
\end{aligned}$$

for all N large enough. Finally, we estimate the last term on the r.h.s. of (8.2) as follows

$$\begin{aligned}
(\widetilde{\phi}_{N,t}, \frac{1}{|x_1-x_2|^2(|x_1-x_2|+\varepsilon N^{-1})^2} \widetilde{\phi}_{N,t}) &\leq \varepsilon^{-3/2} N^{3/2} (\widetilde{\phi}_{N,t}, \frac{1}{|x_1-x_2|^{5/2}} \widetilde{\phi}_{N,t}) \\
&\leq C \varepsilon^{-3/2} N^{3/2} (\widetilde{\phi}_{N,t}, \, S_1^{5/2} S_2^{5/2} \, \widetilde{\phi}_{N,t})
\end{aligned} \tag{8.3}$$

by Lemma 9.1, part (i). Next we note that, by Theorem 4.4,

$$(\widetilde{\phi}_{N,t}, S_1^3 S_2^3 \widetilde{\phi}_{N,t}) \leq (\widetilde{\phi}_{N,t}, S_1^4 S_2^2 \widetilde{\phi}_{N,t}) \leq CN$$

for all N large enough, and for all t (the constant C is independent of ε). Since $(S_1 S_2)^{1/2} \leq N^{1/2} + N^{-1/2} S_1 S_2$, we find

$$(\widetilde{\phi}_{N,t}, S_1^{5/2} S_2^{5/2} \widetilde{\phi}_{N,t}) \leq N^{1/2} (\widetilde{\phi}_{N,t}, S_1^2 S_2^2 \phi_{N,t}) + N^{-1/2} (\widetilde{\phi}_{N,t}, S_1^3 S_2^3 \widetilde{\phi}_{N,t}) \leq C N^{1/2} \,.$$

From (8.3) we get

$$(\widetilde{\phi}_{N,t}, \frac{1}{|x_1-x_2|^2(|x_1-x_2|+\varepsilon N^{-1})^2} \widetilde{\phi}_{N,t}) \leq C \varepsilon^{-3/2} N^2$$

for all N large enough. From (8.2) it follows that

$$\|(H_N - \widetilde{H}_N) \widetilde{\phi}_{N,t}\|^2 \leq C \varepsilon^{1/2}$$

for all N large enough (depending on ε), and for all t (the constant C only depends on λ and $\|\varphi\|_{H^1}$, see Theorem 4.4). Hence, by (8.1) and by Gronwall's Lemma,

$$\pm \partial_t \|\phi_{N,t} - \widetilde{\phi}_{N,t}\|^2 \leq C \varepsilon^{1/4} \|\phi_{N,t} - \widetilde{\phi}_{N,t}\| \quad \Rightarrow \quad \|\phi_{N,t} - \widetilde{\phi}_{N,t}\| \leq C \varepsilon^{1/4} t$$

for all $t \geq 0$ and for all N large enough. $\square$

Finally we have to remove the cutoff κ from the initial wave function ψ_N^κ.

Proposition 8.2. *Suppose $\varphi \in H^1(\mathbb{R}^3)$, with $\|\varphi\| = 1$. For $\kappa > 0$ put $\varphi^\kappa = \exp(-\kappa |p|/N) \varphi$. Suppose $\psi_{N,t}^\kappa$ and $\psi_{N,t}$ are the solutions of the Schrödinger equation (1.4) with initial data $\psi_N^\kappa(\mathbf{x}) = \prod_{j=1}^N \varphi^\kappa(x_j)$, and, respectively, $\psi_N(\mathbf{x}) = \prod_{j=1}^k \varphi(x_j)$. Then*

$$\|\psi_{N,t}^\kappa - \psi_{N,t}\| \leq \kappa \|\varphi\|_{H^1}$$

for all $t \in \mathbb{R}$ and all $N \geq 1$.

Proof. By the unitarity of the time evolution, we have

$$\begin{aligned} \|\psi^\kappa_{N,t} - \psi_{N,t}\| = \|\psi^\kappa_N - \psi_N\| &= \left\| \prod_{j=1}^N \varphi^\kappa(x_j) - \prod_{j=1}^N \varphi(x_j) \right\| \\ &\leq \sum_{j=1}^N \|\varphi^\kappa(x_1) \dots \varphi^\kappa(x_{j-1})(\varphi^\kappa(x_j) - \varphi(x_j))\varphi(x_{j+1}) \dots \varphi(x_N)\| \\ &\leq N \|\varphi^\kappa - \varphi\| \,. \end{aligned} \tag{8.4}$$

Here we used that the L^2 norm of φ^κ is bounded by $\|\varphi^\kappa\| \leq \|\varphi\| = 1$. Since $\varphi^\kappa = e^{-\kappa|p|/N}\varphi$ and $1 - e^{-\kappa|p|/N} \leq \kappa N^{-1}|p|$, we have

$$\|\varphi^\kappa - \varphi\|^2 = \int \mathrm{d}p \, \left(e^{-\kappa|p|/N} - 1 \right)^2 |\widehat{\varphi}(p)|^2 \leq \kappa^2 N^{-2} \int \mathrm{d}p \, |p|^2 |\widehat{\varphi}(p)|^2 = \kappa^2 N^{-2} \|\varphi\|^2_{H^1} \,.$$

By (8.4), we find

$$\|\psi^\kappa_{N,t} - \psi_{N,t}\| \leq \kappa \|\varphi\|_{H^1} \,.$$

□

9 Some Technical Results

In this section we collect technical results used throughout the paper.

In the first lemma we show how to control singularities like $|x_1 - x_2|^{-a}$, for $a < 3$ in terms of the operators $S_j = (1 + p_j^2)^{1/4} = (1 - \Delta_j)^{1/4}$, for $j = 1, 2$.

Lemma 9.1. *Let $V(x) = |x|^{-1}$ and $V^\varepsilon(x) = (|x| + \varepsilon N^{-1})^{-1}$. Moreover, we set $V_{i,j} = V(x_i - x_j)$ and $V^\varepsilon_{i,j} = V^\varepsilon(x_i - x_j)$.*

i) For all $a < 3$ there exists $C(a)$ such that

$$\frac{1}{|x_1 - x_2|^a} \leq C(a) \, S_1^\alpha S_2^\beta \qquad \text{for all } \alpha, \beta > 0 \quad \text{with } \alpha + \beta = 2a \,.$$

If $a = 1$, we have the tight bound

$$V_{1,2} \leq \frac{\pi}{2} \, S_1^\alpha S_2^\beta \qquad \text{for all } \alpha, \beta > 0 \quad \text{with } \alpha + \beta = 2 \,.$$

ii) The operator

$$S_1 S_2^{-1} V_{1,2} S_1^{-1} S_2^{-1}$$

is bounded. Moreover

$$\|S_1 S_2^{-1} V^\varepsilon_{1,2} S_1^{-1} S_2^{-1}\| \leq C \tag{9.1}$$

uniformly in ε.

iii) For all $\delta > 0$, there exists $C_\delta < \infty$ such that

$$\|S_1^{-2} S_2^{-1} V^\varepsilon_{1,2} S_2\| \leq C_\delta \varepsilon^{-\delta} N^\delta$$

Proof. To prove i) we use that

$$\frac{1}{|x_1 - x_2|} \leq \frac{\pi}{2} S_i^2, \quad \text{and} \quad \frac{1}{|x_1 - x_2|^a} \leq C(a) S_i^{2a} \tag{9.2}$$

for some constant $C(a) < \infty$, and for $i = 1, 2$. This is proven in [8]. The statement i) can now be shown by the following general result. Suppose $A \geq 0$, and $B, C \geq 1$ are three operators, with

$$A \leq B \qquad \text{and} \qquad A \leq C$$

and such that B commutes with C. Then

$$A \leq B^\alpha C^\beta \quad \text{for all } \alpha, \beta \quad \text{with } \alpha + \beta = 1\,. \tag{9.3}$$

In fact, $A \leq B$ implies that $B^{-1/2} A B^{-1/2} \leq 1$, and, by the operator monotonicity of powers smaller than one, this implies that $(B^{-1/2} A B^{-1/2})^\alpha \leq 1$ for all $\alpha \leq 1$. On the other hand $A \leq C$ implies $B^{-1/2} A B^{-1/2} \leq B^{-1} C$ and also $(B^{-1/2} A B^{-1/2})^\beta \leq B^{-\beta} C^\beta$ for all $\beta \leq 1$. Hence

$$\begin{aligned} B^{-1/2} A B^{-1/2} &= (B^{-1/2} A B^{-1/2})^{\beta/2} (B^{-1/2} A B^{-1/2})^\alpha (B^{-1/2} A B^{-1/2})^{\beta/2} \\ &\leq (B^{-1/2} A B^{-1/2})^\beta \leq B^{-\beta} C^\beta\,. \end{aligned}$$

Multiplying both sides by $B^{1/2}$ gives (9.3). Part i) follows by (9.2) and by (9.3), taking $A = |x_1 - x_2|^{-a}$, $B = S_1$ and $C = S_2$. Next we prove ii). We will make use of following fact (known as the Holmgren-Shur inequality). Suppose that the set of mutually orthognal projections $\{P_n\}_{n=1}^\infty$ resolves the identity in a strong sense, that is

$$s - \lim_{N \to \infty} \sum_{n=1}^{N} P_n = 1\,.$$

Then for any operator A we have the norm inequality

$$\|A\| \ \leq \max \left(\sup_k \sum_{n=1}^\infty \|P_k \, A \, P_n\|\,,\, \sup_n \sum_{k=1}^\infty \|P_k \, A \, P_n\| \right)\,. \tag{9.4}$$

We choose

$$P_k := \chi(16^{k-1} \leq p_1^2 + 1 < 16^k)\,,$$

then the boundedness of $S_1 S_2^{-1} V_{1,2} S_1^{-1} S_2^{-1}$ follows by (9.4) if we prove that

$$\|P_k \, S_1^{-1} \, S_2^{-1} \, \frac{1}{|x_1 - x_2|} \, S_1 \, S_2^{-1} \, P_n\| \ \leq C \, 2^{-|k-n|/2} \ . \tag{9.5}$$

for a constant C independent of n, k. To show the estimate (9.5) we consider two possibilities: $k \geq n$ and $n > k$.

When $k \geq n$ we can bound the l.h.s. of (9.5) by

$$\|P_k \, S_1^{-1}\| \, \|S_2^{-1} \, \frac{1}{|x_1 - x_2|} \, S_2^{-1}\| \, \|S_1 \, P_n\|\,. \tag{9.6}$$

Since, by part i),

$$\| \, S_2^{-1} \, \frac{1}{|x_1 - x_2|} \, S_2^{-1}\| \ \leq \ \frac{\pi}{2}\,,$$

and

$$\|P_k\, S_1^{-1}\| \le\ 2^{-k+1}\,, \quad \|S_1\, P_n\| \ \le\ 2^n\,,$$

we obtain (9.5) (in fact a stronger estimate). If $n > k$ we write

$$\begin{aligned}
P_k\, S_1^{-1}\, S_2^{-1}\, &\frac{1}{|x_1-x_2|}\, S_1\, S_2^{-1}\, P_n \\
&= P_k\, S_1^{-1}\, S_2^{-1}\,(p_1^2+1)(p_1^2+1)^{-1}\,\frac{1}{|x_1-x_2|}\, S_1\, S_2^{-1}\, P_n \\
&= P_k\, S_1^{-1}\, S_2^{-1}\,(p_1^2+1)\,\frac{1}{|x_1-x_2|}\,(p_1^2+1)^{-1}\, S_1\, S_2^{-1}\, P_n \\
&\qquad + P_k\, S_1^{-1}\, S_2^{-1}\,(p_1^2+1)\,\left[(p_1^2+1)^{-1}\,,\frac{1}{|x_1-x_2|}\right]\, S_1\, S_2^{-1}\, P_n\,.
\end{aligned} \tag{9.7}$$

The norm of the first contribution can be now bounded by

$$\|P_k\, S_1^3\|\,\|S_2^{-1}\,\frac{1}{|x_1-x_2|}\,S_2^{-1}\|\,\|S_1^{-3}\,P_n\| \ \le\ C\,2^{-3\,|k-n|}\,,$$

hence we are left with the task of checking that also the norm of second contribution in (9.7) satisfies a similar bound. Note now that

$$\left[(p_1^2+1)^{-1}\,,\frac{1}{|x_1-x_2|}\right] \ = \ (p_1^2+1)^{-1}\,\left\{p_1\cdot\frac{x_1-x_2}{|x_1-x_2|^3}\ +\ \frac{x_1-x_2}{|x_1-x_2|^3}\cdot p_1\right\}\,(p_1^2+1)^{-1}\,,$$

therefore it suffices to estimate

$$\|P_k\, S_1^{-1}\, S_2^{-1}\, p_1\cdot\frac{x_1-x_2}{|x_1-x_2|^3}\,(p_1^2+1)^{-1}\ S_1\, S_2^{-1}\, P_n\| \tag{9.8}$$

and

$$\|P_k\, S_1^{-1}\, S_2^{-1}\,\frac{x_1-x_2}{|x_1-x_2|^3}\cdot p_1\,(p_1^2+1)^{-1}\ S_1\, S_2^{-1}\, P_n\|\,. \tag{9.9}$$

We bound (9.8), using the result of part i), by

$$\|P_k\, S_1^2\|\,\|S_1^{-3}\, S_2^{-1}\, p_1\cdot\frac{x_1-x_2}{|x_1-x_2|^3}\, S_2^{-1} S_1^{-1}\|\,\|S_1^{-2}\, P_n\| \ \le\ C\,2^{-2\,|k-n|}\,.$$

As for (9.9) we estimate it by

$$\begin{aligned}
\|P_k\, S_1^{1/2}\|\,\|S_1^{-3/2}\, S_2^{-1}\,&\frac{x_1-x_2}{|x_1-x_2|^3}\cdot p_1\, S_2^{-1}\, S_1^{-5/2}\|\,\|S_1^{-1/2}\, P_n\| \\
&\le C\,2^{-|k-n|/2}\,\|S_1^{-3/2} S_2^{-1}\frac{1}{|x_1-x_2|^{5/4}}\|\,\|\frac{1}{|x_1-x_2|^{3/4}} S_1^{-1/2} S_2^{-1}\| \\
&\le C\,2^{-|k-n|/2}\,,
\end{aligned}$$

hence the result. The same line of reasoning can clearly be applied also to prove (9.1), uniformly in $\varepsilon > 0$. Finally, we show part iii). To this end, we write

$$S_1^{-2} S_2^{-1} V_{1,2}^{\varepsilon} S_2 = S_1^{-2} V_{1,2}^{\varepsilon} + \int_0^\infty \mathrm{d}s\, s^{-1/4}\,\frac{S_1^{-2}}{s+1+p_2^2}\,\left(p_2\cdot\nabla V_{1,2}^{\varepsilon} + \nabla V_{1,2}^{\varepsilon}\cdot p_2\right)\,\frac{S_2}{s+1+p_2^2}\,.$$

The first term is bounded, by part i), uniformly in ε. As for the integral, for any given $\delta > 0$ its norm can be bounded by

$$
\begin{aligned}
&(\varepsilon^{-1}N)^{\delta} \int_0^{\infty} \frac{\mathrm{d}s}{s^{1/4}} \left\{ \left\| \frac{|p_2|}{s+1+p_2^2} \right\| \left\| S_1^{-2} \frac{1}{|x_1-x_2|} \right\| \left\| \frac{1}{|x_1-x_2|^{1-\delta}} \, \frac{1}{(|p_2|+1)^{1-\delta}} \right\| \left\| \frac{S_2(|p_2|+1)^{1-\delta}}{s+1+p_2^2} \right\| \right. \\
&\qquad + \left\| \frac{(|p_2|+1)^{1/2-\delta}}{s+1+p_2^2} \right\| \left\| (|p_2|+1)^{-1/2+\delta} S_1^{-2} \frac{1}{|x_1-x_2|^{3/2-\delta}} \right\| \\
&\qquad\qquad \left. \times \left\| \frac{1}{|x_1-x_2|^{1/2}} \, \frac{1}{(|p_2|+1)^{1/2}} \right\| \left\| \frac{S_2 |p_2| (|p_2|+1)^{1/2}}{s+1+p_2^2} \right\| \right\} \\
&\leq D_{\delta} \varepsilon^{-\delta} N^{\delta} \int_0^{\infty} \mathrm{d}s \, s^{-1/4} \frac{1}{(s+1)^{3/4+\delta/2}} \\
&\leq C_{\delta} \varepsilon^{-\delta} N^{\delta} \, .
\end{aligned}
$$

□

The following lemma is used in the proof of Proposition 4.2, in order to control terms of the form $V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon}$ by powers of S_1 and S_2.

Lemma 9.2. *For every $n \geq 1$, $\varepsilon > 0$, and $0 < a < 1$, there exists $D(n, \varepsilon, a)$, independent of N, such that*

$$
V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon} \leq D(n, \varepsilon, a) \, N^a \sum_{m=1}^{n} S_1^{2(n-m+2)} S_2^2 N^{m-1}
$$

Proof. Suppose first that $n = 2\ell$ is even. Then

$$
S_1^{2n} = S_1^{4\ell} = (1+p_1^2)^{\ell} \leq 2^{\ell} \, (1 + (p_1^2)^{\ell}) \leq 2^{\ell} + 6^{\ell} \sum_{\alpha=1}^{3} p_{1,\alpha}^{2\ell} \, ,
$$

where $p_{1,\alpha}$ denotes the α's component of the vector p_1. Hence

$$
V_{1,2}^{\varepsilon} \, S_1^{4\ell} \, V_{1,2}^{\varepsilon} \leq 2^{\ell} \, (V_{1,2}^{\varepsilon})^2 + 6^{\ell} \sum_{\alpha=1}^{3} V_{1,2}^{\varepsilon} \, p_{1,\alpha}^{2\ell} \, V_{1,2}^{\varepsilon} \, . \tag{9.10}
$$

Using the commutator expansion

$$
V_{1,2}^{\varepsilon} \, p_{1,\alpha}^{\ell} = p_{1,\alpha}^{\ell} \, V_{1,2}^{\varepsilon} + \sum_{j=1}^{\ell} \binom{\ell}{j} p_{1,\alpha}^{\ell-j} \, \underbrace{[\dots [V_{1,2}^{\varepsilon}, p_{1,\alpha}], p_{1,\alpha}], \dots p_{1,\alpha}]}_{j \text{ commutators}}
$$

and a Schwarz inequality, we obtain the bound

$$
V_{1,2}^{\varepsilon} \, p_{1,\alpha}^{2\ell} \, V_{1,2}^{\varepsilon} \leq 2 \, p_{1,\alpha}^{\ell} \, (V_{1,2}^{\varepsilon})^2 \, p_{1,\alpha}^{\ell} + 2^{\ell} \sum_{j=1}^{\ell} \binom{\ell}{j} p_{1,\alpha}^{\ell-j} \, |\underbrace{[\dots [V_{1,2}^{\varepsilon}, p_{1,\alpha}], \dots p_{1,\alpha}] \dots]}_{j \text{ commutators}}|^2 \, p_{1,\alpha}^{\ell-j} \, . \tag{9.11}
$$

Next we use $(V_{1,2}^{\varepsilon})^2 \leq C \, S_1^2 S_2^2$ and

$$
\begin{aligned}
|\underbrace{[\dots [V_{1,2}^{\varepsilon}, p_{1,\alpha}], \dots p_{1,\alpha}] \dots]}_{j \text{ commutators}}|^2 &\leq C (j!)^2 \, \frac{1}{(|x_1-x_2| + \varepsilon N^{-1})^{2j+2}} \\
&\leq C (j!)^2 (\varepsilon^{-1} N)^{2j-1+a} \frac{1}{|x_1-x_2|^{3-a}} \\
&\leq D_1(j, \varepsilon, a) \, N^{2j-1+a} \, S_1^4 S_2^2
\end{aligned}
$$

for some $0 < a < 1$ (the constant $D_1(j,\varepsilon,a)$ is proportional to $(j!)^2$, to $\varepsilon^{-2j+1-a}$, and diverges logarithmically in a, for $a \to 0$). From (9.11) we find

$$V_{1,2}^{\varepsilon}\, p_{1,\alpha}^{2\ell}\, V_{1,2}^{\varepsilon} \leq C\, p_{1,\alpha}^{\ell}\, S_1^2 S_2^2\, p_{1,\alpha}^{\ell} + D_2(\ell,\varepsilon,a) \sum_{j=1}^{\ell} p_{1,\alpha}^{2(\ell-j)} S_1^4 S_2^2\, N^{2j-1+a}$$

for a new constant $D_2(\ell,\varepsilon,a)$ independent of N. Inserting the last equation into (9.10), summing over α, and using again $(V_{1,2}^{\varepsilon})^2 \leq C S_1^2 S_2^2$, it follows that

$$V_{1,2}^{\varepsilon} S_1^{4\ell} V_{1,2}^{\varepsilon} \leq D(\ell,\varepsilon,a) \left(S_1^{4\ell+2} S_2^2 + \sum_{j=1}^{\ell} S_1^{4(\ell-j)} S_1^4 S_2^2\, N^{2j-1+a} \right).$$

Replacing the indices ℓ and j by $n = 2\ell$ and $m = 2j$, we have

$$V_{1,2}^{\varepsilon} S_1^{2n} V_{1,2}^{\varepsilon} \leq C(n,\varepsilon,a) \sum_{m=1}^{n} S_1^{2(n-m+2)} S_2^2\, N^{m-1+a}$$

which proves the claim if n is even (since all terms in the sum over m are positive, we can also allow m to be odd).

If n is odd, the proof is a little bit more difficult. Let $n = 2\ell + 1$. Then

$$\begin{aligned}
V_{1,2}^{\varepsilon} S_1^{4\ell+2} V_{1,2}^{\varepsilon} &= V_{1,2}^{\varepsilon} (1+p_1^2)^{\ell} S_1^2 V_{1,2}^{\varepsilon} \\
&\leq 2^{\ell} V_{1,2}^{\varepsilon} S_1^2 V_{1,2}^{\varepsilon} + 6^{\ell} \sum_{\alpha=1}^{3} V_{1,2}^{\varepsilon} p_{1,\alpha}^{2\ell} S_1^2 V_{1,2}^{\varepsilon} \\
&\leq C(\ell) \Bigg(S_1 (V_{1,2}^{\varepsilon})^2 S_1 + |[V_{1,2}^{\varepsilon}, S_1]|^2 + \sum_{\alpha=1}^{3} p_{1,\alpha}^{\ell} S_1 (V_{1,2}^{\varepsilon})^2 S_1 p_{1,\alpha}^{\ell} + p_{1,\alpha}^{\ell} |[V_{1,2}^{\varepsilon}, S_1]|^2 p_{1,\alpha}^{\ell} \\
&\qquad + \sum_{\alpha=1}^{3} \sum_{j=1}^{\ell} p_{1,\alpha}^{\ell-j} S_1 \,|\underbrace{[\dots[V_{1,2}^{\varepsilon}, p_{1,\alpha}], \dots], p_{1,\alpha}]}_{j \text{ commutators}}|^2 S_1 p_{1,\alpha}^{\ell-j} \\
&\qquad + \sum_{\alpha=1}^{3} \sum_{j=1}^{\ell} p_{1,\alpha}^{\ell-j} \,|[\underbrace{[\dots[V_{1,2}^{\varepsilon}, p_{1,\alpha}] \dots], p_{1,\alpha}]}_{j \text{ commutators}}, S_1]|^2 p_{1,\alpha}^{\ell-j} \Bigg).
\end{aligned} \tag{9.12}$$

Next we note that, for any integer $r \geq 0$,

$$\left[\frac{\partial^r}{\partial x_{1,\alpha}^r} V_{1,2}^{\varepsilon}, S_1 \right] = \sum_{\beta=1}^{3} \int_0^{\infty} \mathrm{d}s\, s^{1/4} \frac{1}{s+1+p_1^2} \left(p_{1,\beta} \frac{\partial^{r+1}}{\partial x_{1,\beta} \partial x_{1,\alpha}^r} V_{1,2}^{\varepsilon} + \text{h.c.} \right) \frac{1}{s+1+p_1^2}.$$

Using

$$\left| \frac{\partial^{r+1} V_{1,2}^{\varepsilon}}{\partial x_{1,\beta} \partial x_{1,\alpha}^r} \right| \leq C\, \frac{(r+1)!}{(|x_1 - x_2| + \varepsilon N^{-1})^{r+2}},$$

we find, for every $r \geq 0$, $\varepsilon > 0$ and $a > 0$, a constant $C(r,\varepsilon,a)$, such that

$$\left\| \left[\frac{\partial^r}{\partial x_{1,\alpha}^r} V_{1,2}^{\varepsilon}, S_1 \right] S_1^{-2} S_2^{-1} \right\| \leq C(r,\varepsilon,a) N^{r+(a/2)}$$

and thus, for every $a > 0$, we have the operator inequality

$$\left\| \left[\frac{\partial^r}{\partial x_{1,\alpha}^r} V_{1,2}^{\varepsilon}, S_1 \right] \right\|^2 \leq C(r, \varepsilon, a) S_1^4 S_2^2 \, N^{2r+a} \, .$$

This can be used in the second, fourth and last term on the r.h.s. of (9.12). The other terms can be handled as in the case of even n. □

In the next lemma we give a proof of the fact that a solution φ_t of the Hartree equation (3.3) has $H^{1/2}$-norm uniformly bounded in time. We assume here that the initial data $\varphi \in H^{1/2}$ (we apply this result in the proof of Theorem 3.1, Step 5, where we have the stronger condition $\varphi \in H^1(\mathbb{R}^3)$).

Lemma 9.3. *Suppose $\varphi \in H^{1/2}(\mathbb{R}^3)$, with $\|\varphi\| = 1$, and let φ_t be the solution of the nonlinear Hartree equation*

$$i\partial_t \varphi_t = (1 - \Delta)^{1/2} \varphi_t + \lambda (\frac{1}{|\cdot|} * |\varphi_t|^2) \varphi_t$$

with initial data $\varphi_{t=0} = \varphi$. Then, if $\lambda > -4/\pi$, there exists a constant C, depending only on λ such that

$$(\varphi_t, (1 - \Delta)^{1/2} \varphi_t) \leq C \|\varphi\|_{H^{1/2}}$$

for all $t \in \mathbb{R}$.

Proof. The L^2 norm of φ is conserved, so $\|\varphi_t\| = 1$. Also the Hartree energy

$$E(\varphi) = \int \mathrm{d}x \, |(1 - \Delta)^{1/4} \varphi(x)|^2 + \frac{\lambda}{2} \int \mathrm{d}x \mathrm{d}y \, \frac{1}{|x - y|} \, |\varphi(x)|^2 |\varphi(y)|^2$$

is conserved by the time evolution. Note that

$$\begin{aligned} \int \mathrm{d}x \mathrm{d}y \, \frac{1}{|x - y|} \, |\varphi(x)|^2 |\varphi(y)|^2 &\leq \sup_x \int \mathrm{d}y \, \frac{1}{|x - y|} |\varphi(y)|^2 \, \|\varphi\|^2 \\ &\leq \frac{\pi \|\varphi\|^2}{2} \int \mathrm{d}y \, |(1 - \Delta_y)^{1/4} \varphi(y)|^2 \end{aligned}$$

where we used the operator inequality $|x - y|^{-1} \leq (\pi/2)(1 - \Delta_y)^{1/2}$ for every $x \in \mathbb{R}^3$. For $\lambda < 0$, we find

$$(1 + \frac{\pi}{4} \lambda)(\varphi, (1 - \Delta)^{1/2} \varphi) \leq E(\varphi) \leq (\varphi, (1 - \Delta)^{1/2} \varphi) \, .$$

For $\lambda > 0$, on the other hand, we have

$$(\varphi, (1 - \Delta)^{1/2} \varphi) \leq E(\varphi) \leq (1 + \frac{\pi}{4} \lambda)(\varphi, (1 - \Delta)^{1/2} \varphi) \, .$$

Hence, for all $\lambda > -4/\pi$, we have

$$(\varphi_t, (1 - \Delta)^{1/2} \varphi_t) \leq C E(\varphi_t) = C E(\varphi) \leq C \|\varphi\|_{H^{1/2}}^2$$

for a constant C only depending on λ. □

Finally, in the next lemma, we give a criterium for the equicontinuity of a sequence of time dependent density matrices $\gamma_{N,t}^{(k)} \in \mathcal{H}_k$, with respect to the metric ρ_k. This result is used in the proof of Theorem 5.1, in Step 3, to show the compactness of the sequence $\widetilde{\gamma}_{N,t}^{(k)} \in C([0, T], \mathcal{H}_k)$ with respect to the metric $\widehat{\rho}_k$.

Lemma 9.4. *A sequence of time-dependent density matrices $\gamma^{(k)}_{N,t}$, $N = 1, 2, \dots$, defined for $t \in [0, T]$ and satisfying*

$$\sup_{t\in[0,T]} \|\gamma^{(k)}_{N,t}\|_{\mathcal{H}_k} \leq C \tag{9.13}$$

for all N, is equicontinuous in $C([0, T], \mathcal{H}_k)$ with respect to the metric ρ_k (defined in (2.2)), if and only if, for all $J^{(k)}$ in a dense subset of $\mathcal{A}_k$, and for every $\eta > 0$ there exists a $\delta > 0$ such that

$$\left| Tr\, J^{(k)} \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \eta \tag{9.14}$$

for all N, whenever $|t - s| \leq \delta$.

Proof. Equicontinuity w.r.t. the metric ρ_k means that, for any $\eta > 0$ there exists $\delta > 0$ (independent of N), such that

$$\rho_k(\gamma^{(k)}_{N,t}, \gamma^{(k)}_{N,s}) = \sum_{j=1}^{\infty} 2^{-j} \left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \varepsilon \tag{9.15}$$

if $|t - s| \leq \delta$. Recall that $\{J^{(k)}_j\}_{j\geq 1}$ was chosen as a dense countable subset of the unit ball of $\mathcal{A}_k$. Using (9.13), one can approximate any $J^{(k)} \in \mathcal{A}_k$ by an appropriate finite linear combinations of the $J^{(k)}_j$ and thus one can easily prove that (9.15) implies (9.14).

Next we prove the opposite implication. By a standard approximation argument, one can prove that, if (9.14) holds for all $J^{(k)}$ in a dense subset of $\mathcal{A}_k$, then it holds for every $J^{(k)}$ in $\mathcal{A}_k$. In particular, for every $j \geq 1$, and $\eta > 0$, we can find $\delta(j, \eta) > 0$ such that

$$\left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \eta$$

for all N, if $|t - s| \leq \delta(j, \eta)$. Moreover, using (9.13), we note that, given $\eta > 0$, we have

$$\sum_{j>m} 2^{-j} \left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \sum_{j>m} 2^{-j} \|J^{(k)}_j\|_{\mathcal{A}_k} \left(\|\gamma^{(k)}_{N,t}\|_{\mathcal{H}_k} + \|\gamma^{(k)}_{N,s}\|_{\mathcal{H}_k} \right) \leq C \sum_{j>m} 2^{-j} \leq \eta/2$$

if m is sufficiently large (independently of N and of $t, s \in [0, T]$). Hence

$$\sum_{j\geq 1} 2^{-j} \left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \eta/2 + \sum_{j\leq m} 2^{-j} \left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right|.$$

With $\delta = \min_{j\leq m} \delta(j, \eta/2)$, we find

$$\sum_{j\geq 1} 2^{-j} \left| \mathrm{Tr}\, J^{(k)}_j \left(\gamma^{(k)}_{N,t} - \gamma^{(k)}_{N,s} \right) \right| \leq \eta$$

for all $t, s \in [0, T]$ with $|t - s| \leq \delta$ and for all N. This proves that (9.14) implies (9.15). □

References

[1] C. Bardos, F. Golse and N. Mauser: *Weak coupling limit of the N-particle Schrödinger equation.* Methods Appl. Anal. **7** (2000) 275–293.

[2] A. Elgart, L. Erdős, B. Schlein, and H.-T. Yau: *Gross-Pitaevskii Equation as the Mean Field Limit of Weakly Coupled Bosons.* arXiv:math-ph/0410038. To appear in Arch. Rat. Mech. Anal.

[3] L. Erdős, B. Schlein and H.-T. Yau: *Derivation of the Gross-Pitaevskii Equation for the Dynamics of Bose-Einstein Condensate.* arXiv:math-ph/0410005.

[4] L. Erdős and H.-T. Yau: *Derivation of the nonlinear Schrödinger equation from a many-body Coulom system* Adv. Theor. Math. Phys. **5** (6) (2001), 1169–1205.

[5] J. Fröhlich and E. Lenzmann: *Mean field limit of quantum Bose gases and nonlinear Hartree equation.* Preprint arXiv:math-ph/0409019.

[6] J. Ginibre and G. Velo: *The classical field limit of scattering theory for non-relativistic many-boson systems. I and II.* Commun. Math. Phys. **66**, 37–76 (1979) and **68**, 45-68 (1979).

[7] K. Hepp: *The classical limit for quantum mechanical correlation functions.* Commun. Math. Phys. **35**, 265–277 (1974).

[8] I. Herbst: *Spectral theory of the operator* $(p^2+m^2)^{1/2} - Ze^2/r$. Comm. Math. Phys. **53** (3) (1977), 285–294.

[9] E. Lenzmann. In preparation.

[10] H. Lieb and H.-T. Yau: *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics.* Comm. Math. Phys. **112** (1) (1987), 147–174.

[11] M. Reed and B. Simon: Methods of mathematical physics. Vol. I. Academic Press, 1975.

[12] W. Rudin: Functional analysis. McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1973.

[13] H. Spohn: *Kinetic Equations from Hamiltonian Dynamics.* Rev. Mod. Phys. **52** no. 3 (1980), 569–615.