Fermi Golden Rule and open quantum systems

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1 Introduction

These lecture notes are an expanded version of the lectures given by the first author in the summer school ”Open Quantum Systems” held in Grenoble, June 16—July 4, 2003. We are grateful to Stphane Attal, Alain Joye, and Claude-Alain Pillet for their hospitality and invitation to speak.

Acknowledgments. The research of both authors was partly supported by the EU Postdoctoral Training Program HPRN-CT-2002-0277 and the Polish grants SPUB127 and 2 P03A 027 25. A part of this work was done during a visit of the first author to University of Montreal and to the Schrödinger Institute in Vienna. We acknowledge useful conversations with H. Spohn, C. A. Pillet, W. A. Majewski, and especially with V. Jakšić.

1.1 Fermi Golden Rule and Level Shift Operator in an abstract setting

We will use the name “the Fermi Golden Rule” to describe the well-known second order perturbative formula for the shift of eigenvalues of a family of operators $\mathbb{L}_\lambda = \mathbb{L}_0 + \lambda\mathbb{Q}$. Historically, the Fermi Golden Rule can be traced back to the early years of Quantum Mechanics, and in particular to the famous paper by Dirac [Di]. Two “Golden Rules” describing the second order calculations for scattering amplitudes can be found in the Fermi lecture notes [Fe] on pages 142 and 148.

In its traditional form the Fermi Golden Rule is applied to Hamiltonians of quantum systems – self-adjoint operators on a Hilbert space. A nonzero imaginary shift of an eigenvalue of $\mathbb{L}_0$ indicates that the eigenvalue is unstable and that it has turned into a resonance under the influence of the perturbation $\lambda\mathbb{Q}$.

In our lectures we shall use the term Fermi Golden Rule in a slightly more general context, not restricted to Hilbert spaces. More precisely, we shall be interested in the case when $\mathbb{L}_\lambda$ is a generator of a 1-parameter group of isometries on a Banach space. For example, $\mathbb{L}_\lambda$ could be an anti-self-adjoint operator on a Hilbert space or the generator of a group of $*$-automorphisms of a W^*-algebra. These two special cases will be of particular importance for us.

Note that the spectrum of the generator of a group of isometries is purely imaginary. The shift computed by the Fermi Golden Rule may have a negative real part and this indicates that the eigenvalue has turned into a resonance. Hence, our convention differs from the traditional one by the factor of i.

In these lecture notes, we shall discuss several mathematically rigorous versions of the Fermi Golden Rule. In all of them, the central role is played by a certain operator that we call the Level Shift Operator (LSO). This operator will encode the second order shift of eigenvalues of $\mathbb{L}_\lambda$ under the influence of the perturbation. To define the LSO for $\mathbb{L}_\lambda = \mathbb{L}_0 + \lambda\mathbb{Q}$, we need to specify the projection $\mathbb{P}$ commuting with $\mathbb{L}_0$ (typically, the projection onto the point spectrum of $\mathbb{L}_0$) and a perturbation $\mathbb{Q}$. For the most part, we shall assume that $\mathbb{P}\mathbb{Q}\mathbb{P} = 0$, which guarantees the absence of the first order shift of the eigenvalues. Given the datum $(\mathbb{P}, \mathbb{L}_0, \mathbb{Q})$, we shall define the LSO as a certain operator on the range of the projection $\mathbb{P}$.

We shall describe several rigorous applications of the LSO for $(\mathbb{P}, \mathbb{L}_0, \mathbb{Q})$. One of them is the "weak coupling limit", called also the "van Hove limit". (We will not, however, use the latter name, since it often appears in a different meaning in statistical physics, denoting a special form of the thermodynamical limit). The time-dependent form of the weak coupling limit says that the reduced and rescaled dynamics $\mathrm{e}^{-t\mathbb{L}_0/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}$ converges to the semigroup generated by the LSO. The time dependent weak coupling limit in its abstract form was proven by Davies [Da1, Da2, Da3]. In our lectures we give a detailed exposition of his results.

We describe also the so-called "stationary weak coupling limit", based on the recent work [DF2]. The stationary weak coupling limit says that appropriately rescaled and reduced resolvent of $\mathbb{L}_\lambda$ converges to the resolvent of the LSO.

The LSO has a number of other important applications. It can be used to describe approximate location and multiplicities of eigenvalues and resonances of $\mathbb{L}_\lambda$ for small nonzero λ. It also gives an upper bound on the number of eigenvalues of $\mathbb{L}_\lambda$ for small nonzero λ.

1.2 Applications of the Fermi Golden Rule to open quantum systems

In these lectures, by an open quantum system we shall mean a "small" quantum system $\mathcal{S}$ interacting with a large "environment" or "reservoir" $\mathcal{R}$. The small quantum system is described by a finite dimensional Hilbert space $\mathcal{K}$ and a Hamiltonian K. The reservoir is described by a W^*-dynamical system $(\mathfrak{M}, \tau)$ and a reference state

$\omega_{\mathcal{R}}$ (for a discussion of reference states see the lecture [AJPP]). We shall assume that $\omega_{\mathcal{R}}$ is normal and $\tau_{\mathcal{R}}$-invariant.

If $\omega_{\mathcal{R}}$ is a $(\tau_{\mathcal{R}}, \beta)$-KMS state, then we say that that the reservoir at inverse temperature β and that the open quantum system is thermal. Another important special case is when $\mathcal{R}$ has additional structure, namely consists of n independent parts $\mathcal{R}_1, \cdots, \mathcal{R}_n$, which are interpreted as sub-reservoirs. If the reference state of the sub-reservoir $\mathcal{R}_j$ is β_j-KMS (for $j = 1, \cdots, n$), then we shall call the corresponding open quantum system multi-thermal.

In the literature one can find at least two distinct important applications of the Fermi Golden Rule to the study of open quantum systems.

In the first application one considers the weak coupling limit for the dynamics in the Heisenberg picture reduced to the small system. This limit turns out to be an irreversible Markovian dynamics—a completely positive semigroup preserving the identity acting on the observables of the small system $\mathcal{S}$ ($n \times n$ matrices). The generator of this semigroup is given by the LSO for the generator of the dynamics. We will denote it by M.

The weak coupling limit and the derivation of the resulting irreversible Markovian dynamics goes back to the work of Pauli, Wigner-Weisskopf and van Hove [WW, VH1, VH2, VH3] see also [KTH, Haa]. In the mathematical literature it was studied in the well known papers of Davies [Da1, Da2, Da3], see also [LeSp, AL]. Therefore, the operator M is sometimes called the Davies generator in the Heisenberg picture.

One can also look at the dynamics in the Schrödinger picture (on the space of density matrices). In the weak coupling limit one then obtains a completely positive semigroup preserving the trace. It is generated by the adjoint of M, denoted by M^*, which is sometimes called the Davies generator in the Schrödinger picture.

The second application of the Fermi Golden Rule to the study of open quantum systems is relatively recent. It has appeared in papers on the so-called return to equilibrium [JP1, DJ1, DJ2, BFS2, M]. The main goal of these papers is to show that certain W^*-dynamics describing open quantum systems has only one stationary normal state or no stationary normal states at all. This problem can be reformulated into a question about the point spectrum of the so-called Liouvillean—the generator of the natural unitary implementation of the dynamics. To study this problem, it is convenient to introduce the LSO for the Liouvillean. We shall denote it by $\mathrm{i}\Gamma$. It is an operator acting on Hilbert-Schmidt operators for the system $\mathcal{S}$—again $n \times n$ matrices.

The use of $\mathrm{i}\Gamma$ in the spectral theory hinges on analytic techniques (Mourre theory, complex deformations), which we shall not describe in our lectures. We shall take it for granted that under suitable technical conditions such applications are possible and we will focus on the algebraic properties of M, $\mathrm{i}\Gamma$ and M^*. To the best of our knowledge, some of these properties have not been discussed previously in the literature.

In Theorem 17 we give a simple characterization of the kernel of the imaginary part the operator Γ. This characterization implies that Γ has no nontrivial real eigenvalues in a generic nonthermal case. In [DJ2], this result was proven in the context of Pauli-Fierz systems and was used to show the absence of normal stationary states in a generic multithermal case. In our lectures we generalize the result of [DJ2] to a more general setting.

The characterization of the kernel of the imaginary part of Γ in the thermal case is given in Theorem 18. It implies that generically this kernel consists only of multiples of the square root of the Gibbs density matrix for the small system. In [DJ2], this result was proven in the more restrictive context of Pauli-Fierz systems and was used to show the return to equilibrium in the generic thermal case. A similar result was obtained earlier by Spohn [Sp].

The operators M, $\mathrm{i}\Gamma$ and M^* act on the same vector space (the space of $n \times n$ matrices) and have similar forms. Naively, one may expect that $\mathrm{i}\Gamma$ interpolates in some sense between M and M^*. Although this expectation is correct, its full description involves some advanced algebraic tools (the so-called noncommutative L^p-spaces associated to a von Neumann algebra), and for reasons of space we will not discuss it in these lecture notes (see [DJ4, JP6]).

In the thermal case, the relation between the operators M, $\mathrm{i}\Gamma$ and M^* is considerably simpler—they are mutually similar and in particular have the same spectrum. This result has been recently proven in [DJ3] and we will describe it in detail in our lectures.

The similarity of $\mathrm{i}\Gamma$ and M in the thermal case is closely related to the Detailed Balance Condition for M. In the literature one can find a number of different definitions of the Detailed Balance Condition applicable to irreversible quantum dynamics. In these lecture notes we shall propose another one and we will compare it with the definition due to Alicki [A] and Frigerio-Gorini-Kossakowski-Verri [FGKV].

For reason of space we have omitted many important topics in our lectures—they are treated in the review [DJ4], which is a continuation of these lecture notes. Some additional information about the weak coupling limit and the Davies generator can be also found in the lecture notes [AJPP].

2 Fermi Golden Rule in an abstract setting

2.1 Notation

Let L be an operator on a Banach space $\mathcal{X}$. $\mathrm{sp}L$, $\mathrm{sp}_{\mathrm{ess}}L$, $\mathrm{sp}_{\mathrm{p}}L$ will denote the spectrum, the essential spectrum and the point spectrum (the set of eigenvalues) of the operator L. If e is an isolated point in $\mathrm{sp}L$, then $\mathbf{1}_e(L)$ will denote the spectral projection of L onto e given by the usual contour integral. Sometimes we can also define $\mathbf{1}_e(L)$ if e is not an isolated point in the spectrum. This is well known if L is a normal operator on a Hilbert space. The definition of $\mathbf{1}_e(L)$ for some other classes of operators is discussed in Appendix, see (69), (70).

Let us now assume that L is a self-adjoint operator on a Hilbert space. Let A, B be bounded operators. Suppose that $p \in \mathbb{R}$. We define

$$A(p \pm \mathrm{i}0 - L)^{-1}B := \lim_{\epsilon \searrow 0} A(p \pm \mathrm{i}\epsilon - L)^{-1}B, \tag{1}$$

provided that the right hand side of (1) exists. We will say that $A(p \pm \mathrm{i}0 - L)^{-1}B$ exists if the limit in (1) exists.

The principal value of $p - L$

$$A\mathcal{P}(p-L)^{-1}B := \frac{1}{2}\big(A(p+\mathrm{i}0-L)^{-1}B + A(p-\mathrm{i}0-L)^{-1}B\big)$$

and the delta function of $p - L$

$$A\delta(p-L)B := \frac{\mathrm{i}}{2\pi}\big(A(p+\mathrm{i}0-L)^{-1}B - A(p-\mathrm{i}0-L)^{-1}B\big)$$

are then well defined.

$\mathcal{B}(\mathcal{X})$ denotes the algebra of bounded operators on $\mathcal{X}$. If $\mathcal{X}$ is a Hilbert space, then $\mathcal{B}^1(\mathcal{X})$ denotes the space of trace class operators and $\mathcal{B}^2(\mathcal{X})$ the space of Hilbert-Schmidt operators on $\mathcal{X}$. By a density matrix on $\mathcal{X}$ we mean $\rho \in \mathcal{B}^1(\mathcal{X})$ such that $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$. We say that ρ is nondegenerate if $\mathrm{Ker}\rho = \{0\}$.

For more background material useful in our lectures we refer the reader to Appendix.

2.2 Level Shift Operator

In this subsection we introduce the definition of the Level Shift Operator. First we describe the basic setup needed to make this definition.

Assumption 2.1 *We assume that $\mathcal{X}$ is a Banach space, $\mathbb{P}$ is projection of norm 1 on $\mathcal{X}$ and $\mathrm{e}^{t\mathbb{L}_0}$ is a 1-parameter C_0- group of isometries commuting with $\mathbb{P}$.*

We set $\mathbb{E} := \mathbb{L}_0\Big|_{\mathrm{Ran}\mathbb{P}}$ and $\widetilde{\mathbb{P}} := \mathbf{1} - \mathbb{P}$. Clearly, $\mathbb{E}$ is the generator of a 1-parameter group of isometries on $\mathrm{Ran}\mathbb{P}$. and $\mathbb{L}_0\Big|_{\mathrm{Ran}\widetilde{\mathbb{P}}}$ generates a 1-parameter group of isometries on $\mathrm{Ran}\widetilde{\mathbb{P}}$.

Later on, we will often write $\mathbb{L}_0\widetilde{\mathbb{P}}$ instead of $\mathbb{L}_0\Big|_{\mathrm{Ran}\widetilde{\mathbb{P}}}$. For instance, in (2) $((\mathrm{i}e+\xi)\widetilde{\mathbb{P}} - \mathbb{L}_0\widetilde{\mathbb{P}})^{-1}$ will denote the inverse of $(\mathrm{i}e+\xi)\mathbf{1} - \mathbb{L}_0$ restricted to $\mathrm{Ran}\widetilde{\mathbb{P}}$. This is a slight abuse of notation, which we will make often without a comment.

Most of the time we will also assume that

Assumption 2.2 *$\mathbb{P}$ is finite dimensional.*

Under Assumption 2.1 and 2.2, the operator $\mathbb{E}$ is diagonalizable and we can write its spectral decomposition:

$$\mathbb{E} = \sum_{\mathrm{i}e \in \mathrm{sp}\mathbb{E}} \mathrm{i}e\mathbf{1}_{\mathrm{i}e}(\mathbb{E}).$$

Note that $\mathbf{1}_{\mathrm{i}e}(\mathbb{E})$ are projections of norm one.

In the remaining assumptions we impose our conditions on the perturbation:

Assumption 2.3 *We suppose that* $\mathbb{Q}$ *is an operator with* $\mathrm{Dom}\mathbb{Q} \supset \mathrm{Dom}\mathbb{L}_0$ *and, for* $|\lambda| < \lambda_0$, $\mathbb{L}_\lambda := \mathbb{L}_0 + \lambda\mathbb{Q}$ *is the generator of a 1-parameter* C_0*-semigroup of contractions.*

Assumption 2.3 implies that $\widetilde{\mathbb{P}}\mathbb{Q}\mathbb{P}$ and $\mathbb{P}\mathbb{Q}\widetilde{\mathbb{P}}$ are well defined.

Assumption 2.4 $\mathbb{P}\mathbb{Q}\mathbb{P} = 0$.

The above assumption is needed to guarantee that the first nontrivial contribution for the shift of eigenvalues of $\mathbb{L}_\lambda$ is 2nd order in λ.

It is also useful to note that if Assumption 2.2 holds, then $\widetilde{\mathbb{P}}\mathbb{Q}\mathbb{P}$ and $\mathbb{P}\mathbb{Q}\widetilde{\mathbb{P}}$ are bounded. Note also that in the definition of LSO only the terms $\widetilde{\mathbb{P}}\mathbb{Q}\mathbb{P}$ and $\mathbb{P}\mathbb{Q}\widetilde{\mathbb{P}}$ will play a role and the term $\widetilde{\mathbb{P}}\mathbb{Q}\widetilde{\mathbb{P}}$ will be irrelevant.

Assumption 2.5 *We assume that for all* $\mathrm{i}e \in \mathrm{sp}\mathbb{E}$ *there exists*

$$\begin{aligned} &\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}((\mathrm{i}e+0)\widetilde{\mathbb{P}} - \mathbb{L}_0\widetilde{\mathbb{P}})^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E}) \\ := &\lim_{\xi\searrow 0} \mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}((\mathrm{i}e+\xi)\widetilde{\mathbb{P}} - \mathbb{L}_0\widetilde{\mathbb{P}})^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E}) \end{aligned} \tag{2}$$

Under Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5 we set

$$M := \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}((\mathrm{i}e+0)\widetilde{\mathbb{P}} - \mathbb{L}_0\widetilde{\mathbb{P}})^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E}) \tag{3}$$

and call it the Level Shift Operator (LSO) associated to the triple $(\mathbb{P}, \mathbb{L}_0, \mathbb{Q})$.

It is instructive to give time-dependent formulas for the LSO:

$$\begin{aligned} M &= \lim_{\xi\searrow 0} \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{i}e}(\mathbb{E}) \int_0^\infty \mathrm{e}^{-\xi s}\mathbb{Q}\mathbb{Q}(s)\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathrm{d}s \\ &= \lim_{\xi\searrow 0} \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{i}e}(\mathbb{E}) \int_0^\infty \mathrm{e}^{-\xi s}\mathbb{Q}(-s/2)\mathbb{Q}(s/2)\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathrm{d}s, \end{aligned}$$

where $\mathbb{Q}(t) := \mathrm{e}^{t\mathbb{L}_0}\mathbb{Q}\mathrm{e}^{-t\mathbb{L}_0}$.

2.3 LSO for C_0^*-dynamics

In the previous subsection we assumed that $\mathbb{L}_\lambda$ is a generator of a C_0-semigroup. In one of our applications, however, we will deal with another type of semigroups, the so-called C_0^*-semigroups (see Appendix for definitions and a discussion). In this case, we will need to replace Assumptions 2.1 and 2.3 by their "dual versions", which we state below:

Assumption 2.1* *We assume that* $\mathcal{Y}$ *is a Banach space and* $\mathcal{X}$ *is its dual, that is* $\mathcal{X} = \mathcal{Y}^*$, $\mathbb{P}$ *is a w* continuous projection of norm* 1 *on* $\mathcal{X}$ *and* $\mathrm{e}^{t\mathbb{L}_0}$ *is a 1-parameter* C_0^**- group of isometries commuting with* $\mathbb{P}$.

Assumption 2.3* *We suppose that* $\mathbb{Q}$ *is an operator with* $\mathrm{Dom}\mathbb{Q} \supset \mathrm{Dom}\mathbb{L}_0$ *and, for* $|\lambda| < \lambda_0$, $\mathbb{L}_\lambda := \mathbb{L}_0 + \lambda\mathbb{Q}$ *is the generator of a 1-parameter* C_0^**-semigroup of contractions.*

2.4 LSO for W^*-dynamics

The formalism of the Level Shift Operator will be applied to open quantum systems in two distinct situations.

In the first application, the Banach space $\mathcal{X}$ is a W^*-algebra, $\mathbb{P}$ is a normal conditional expectation and $\mathrm{e}^{t\mathbb{L}_0}$ is a W^*-dynamics.

Note that W^*-algebras are usually not reflexive and W^*-dynamics are usually not C_0-groups. However, W^*-algebras are dual Banach spaces and W^*-dynamics are C_0^*-groups.

The perturbation has the form $\mathrm{i}[V,\cdot]$ with V being a self-adjoint element of the W^*-algebra. Therefore, $\mathrm{e}^{t\mathbb{L}_\lambda}$ will be a W^*-dynamics for all real λ – again a C_0^*-group.

2.5 LSO in Hilbert spaces

In our second application, $\mathcal{X}$ is a Hilbert space. Hilbert spaces are reflexive, therefore we do not need to distinguish between C_0 and C_0^*-groups.

All strongly continuous groups of isometries on a Hilbert space are unitary groups. Therefore, the operator $\mathbb{L}_0$ has to be anti-self-adjoint (that means $\mathbb{L}_0 = \mathrm{i}L_0$, where L_0 is self-adjoint).

All projections of norm one on a Hilbert space are orthogonal. Therefore, the distinguished projection has to be orthogonal.

In our applications to open quantum systems $\mathrm{e}^{t\mathbb{L}_\lambda}$ is a unitary dynamics. This means in particular that $\mathbb{Q}$ has the form $\mathbb{Q} = \mathrm{i}Q$, where Q is hermitian.

In the case of a Hilbert space the LSO will be denoted $\mathrm{i}\Gamma$. Thus we will isolate the imaginary unit "i", which is consistent with the usual conventions for operators in Hilbert spaces, and also with the convention that we adopted in [DJ2].

Remark 1. In [DJ2] we used a formalism similar to that of Subsection 2.2 in the context of a Hilbert space. Note, however, that the terminology that we adopted there is not completely consistent with the terminology used in these lectures. In [DJ2] we considered a Hilbert space $\mathcal{X}$, an orthogonal projection P, and self-adjoint operators L_0, Q. If Γ is the LSO for the triple (P, L_0, Q) according to [DJ2], then $\mathrm{i}\Gamma$ is the LSO for $(P, \mathrm{i}L_0, \mathrm{i}Q)$ according to the present definition.

Let us quote the following easy fact valid in the case of a Hilbert space.

Theorem 1. *Suppose that $\mathcal{X}$ is a Hilbert space, Assumptions 2.1, 2.2, 2.3 and 2.5 hold and Q is self-adjoint. Then $\mathrm{e}^{\mathrm{i}t\Gamma}$ is contractive for $t > 0$.*

Proof. We use the notation $\mathbb{E} = \mathrm{i}E$, $\mathbb{L}_0 = \mathrm{i}L$, $\mathbb{Q} = \mathrm{i}Q$. We have

$$\frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*) = -\sum_{e\in\mathrm{sp}E} \mathbf{1}_e(E)Q\delta(e - L_0)Q\mathbf{1}_e(E) \leq 0$$

Therefore, $\mathrm{i}\Gamma$ is a dissipative operator and $\mathrm{e}^{\mathrm{i}t\Gamma}$ is contractive for $t > 0$. □

Note that in Theorem 5 we will show that the LSO is the generator of a contractive semigroup also in a more general situation, when $\mathcal{X}$ is a Banach space. The proof of this fact will be however more complicated and will require some additional technical assumptions.

2.6 The choice of the projection $\mathbb{P}$

In typical application of the LSO, the operators $\mathbb{L}_0$ and $\mathbb{Q}$ are given and our goal is to study the operator

$$\mathbb{L}_\lambda := \mathbb{L}_0 + \lambda\mathbb{Q}. \tag{4}$$

More precisely, we want to know what happens with its eigenvalues when we switch on the perturbation.

Therefore, it is natural to choose the projection $\mathbb{P}$ as "the projection onto the point spectrum of $\mathbb{L}_0$", that is

$$\mathbb{P} = \sum_{e\in\mathbb{R}} \mathbf{1}_{\mathrm{i}e}(\mathbb{L}_0), \tag{5}$$

provided that (5) is well defined.

More generally, if we were interested only about what happens around some eigenvalues $\{\mathrm{i}e_1, \dots, \mathrm{i}e_n\} \subset \mathrm{sp_p}\mathbb{L}_0$, then we could use the LSO defined with the projection

$$\mathbb{P} = \sum_{j=1}^{n} \mathbf{1}_{\mathrm{i}e_j}(\mathbb{L}_0). \tag{6}$$

Clearly, if $\mathcal{X}$ is a Hilbert space and $\mathbb{L}_0$ is anti-self-adjoint, then $\mathbf{1}_{\mathrm{i}e}(\mathbb{L}_0)$ are well defined for all $e \in \mathbb{R}$. Moreover, both (5) and (6) are projections of norm one commuting with $\mathbb{L}_0$, and hence they satisfy Assumption 2.1.

There is no guarantee that the spectral projections $\mathbf{1}_{\mathrm{i}e}(\mathbb{L}_0)$ are well defined in the more general case when $\mathbb{L}_0$ is the generator of a group of isometries on a Banach space. If they are well defined, then they have norm one, however, we seem to have no guarantee that their sums have norm one. In Appendix we discuss the problem of defining spectral projections onto eigenvalues in this more general case.

Note, however, that in the situation considered by us later, we will have no such problems. In fact, $\mathbb{P}$ will be always given by (5) and will always have norm one.

If $\mathbf{1}_{\mathrm{i}e}(\mathbb{L}_0)$ is well defined for all $e \in \mathbb{R}$ and we take $\mathbb{P}$ defined by (5), then $\mathbb{P}$ will be determined by the operator $\mathbb{L}_0$ itself. We will speak about "the LSO for $\mathbb{L}_\lambda$", if we have this projection in mind.

2.7 Three kinds of the Fermi Golden Rule

Suppose that Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, or 2.1*, 2.2, 2.3*, 2.4 and 2.5 are satisfied. Let $\mathbb{P}$ be given by (5) and M be the LSO for $(\mathbb{P}, \mathbb{L}_0, \mathbb{Q})$. Our main object of interest is the operator $\mathbb{L}_\lambda$.

The assumption 2.4 ($\mathbb{PQP} = 0$) guarantees that there are no first order effects of the perturbation. The operator M describes what happens with the eigenvalues of

$\mathbb{L}_0$ under the influence of the perturbation $\lambda\mathbb{Q}$ at the second order of λ. Following the tradition of quantum physics, we will use the name "the Fermi Golden Rule" to describe the second order effects of the perturbation.

The Fermi Golden Rule can be made rigorous in many ways under various technical assumptions. We can distinguish at least three varieties of the rigorous Fermi Golden Rule:

- **Analytic Fermi Golden Rule:** $\mathbb{E} + \lambda^2 M$ *predicts the approximate location (up to* $o(\lambda^2)$*) and the multiplicity of the resonances and eigenvalues of* $\mathbb{L}_\lambda$ *in a neighborhood of* $\mathrm{sp_p}\mathbb{L}_0$ *for small* λ*.*
  The Analytic Fermi Golden Rule is valid under some analyticity assumptions on $\mathbb{L}_\lambda$. It is well known and follows essentially by the standard perturbation theory for isolated eigenvalues ([Ka, RS4], see also [DF1]). The perturbation arguments are applied not to $\mathbb{L}_\lambda$ directly, but to the analytically deformed $\mathbb{L}_\lambda$. More or less explicitly, this idea was applied to Liouvilleans describing open quantum systems [JP1, JP2, BFS1, BFS2]. One can also apply it to the W^*-dynamics of open quantum systems [JP4, JP5].
  The **stationary weak coupling (or van Hove) limit** of [DF2], described in Theorem 2 and 5, can be viewed as an infinitesimal version of the Analytic Fermi Golden Rule.
- **Spectral Fermi Golden Rule:** *The intersection of the spectrum of* $\mathbb{E} + \lambda^2 M$ *with the imaginary line predicts possible location of eigenvalues of* $\mathbb{L}_\lambda$ *for small nonzero* λ*. It also gives an upper bound on their multiplicity.*
  Note that if the Analytic Fermi Golden Rule is true, then so is the Spectral Fermi Golden Rule. However, to prove the Analytic Fermi Golden Rule we need strong analytic assumption, whereas the Spectral Fermi Golden Rule can be shown under much weaker conditions. Roughly speaking, these assumptions should allow us to apply the so-called positive commutator method.
  The Spectral Fermi Golden Rule is stated in Theorem 6.7 of [DJ2], which is proven in [DJ1]. Strictly speaking, the analysis of [DJ1] and [DJ2] is restricted to Pauli-Fierz operators, but it is easy to see that their arguments extend to much larger classes of operators.
  To illustrate the usefulness of the Spectral Fermi Golden Rule, suppose that $\mathcal{X}$ is a Hilbert space, $\mathbb{L}_\lambda = \mathrm{i}L_\lambda$ with L_λ self-adjoint and $\mathrm{i}\Gamma$ is the LSO. Then the Spectral Fermi Golden Rule implies the bound
  $$\dim \mathrm{Ran}\mathbf{1}_\mathrm{p}(L_\lambda) \leq \dim \mathrm{Ker}\Gamma^\mathrm{I},$$
  where $\Gamma^\mathrm{I} := \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*)$. Bounds of this type were used in various papers related to the Return to Equilibrium [JP1, JP2, DJ2, BFS2, M].
- **Dynamical Fermi Golden Rule.** *The operator* $\mathrm{e}^{t(\mathbb{E}+\lambda^2 M)}$ *describes approximately the reduced dynamics* $\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda}\mathbb{P}$ *for small* λ*.*
  The Dynamical Fermi Golden Rule was rigorously expressed in the form of **the weak coupling** by Davies [Da1, Da2, Da3, LeSp]. Davies showed that under some weak assumptions we have

$$\lim_{\lambda\to 0} \mathrm{e}^{-t\mathbb{E}/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P} = \mathrm{e}^{tM}.$$

We describe his result in Theorems 3 and 5.

3 Weak coupling limit

3.1 Stationary and time-dependent weak coupling limit

In this section we describe in an abstract setting the weak coupling limit. We will show that, under some conditions, the dynamics restricted to an appropriate subspace, rescaled and renormalized by the free dynamics, converges to the dynamics generated by the LSO.

We will give two versions of the weak coupling limit: the time dependent and the stationary one. The time-dependent version is well known and in its rigorous form is due to Davies [Da1, Da2, Da3]. Our exposition is based on [Da3].

The stationary weak coupling limit describes the same phenomenon on the level of the resolvent. Our exposition is based on recent work [DF2]. Formally, one can pass from the time-dependent to stationary weak coupling limit by the Laplace transformation. However, one can argue that the assumptions needed to prove the stationary weak coupling limit are sometimes easier to verify. In fact, they involve the existence of certain matrix elements of the resolvent (a kind of the "Limiting Absorption Principle") only at the spectrum of $\mathbb{E}$, a discrete subset of the imaginary line. This is often possible to show by positive commutator methods.

Throughout the section we suppose that most of the assumptions of Subsection 2.2 are satisfied. We will, however, list explicitely the assumptions that we need for each particular result.

The first theorem describes the stationary weak coupling limit.

Theorem 2. *Suppose that Assumptions 2.1, 2.2, 2.3 and 2.4, or 2.1*, 2.2, 2.3* and 2.4 are true. We also assume the following conditions:*

1) *For* $\mathrm{i}e \in \mathrm{sp}\mathbb{E}$, $\xi > 0$, *we have* $\mathrm{i}e + \xi \notin \mathrm{sp}\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}$.
2) *There exists an operator* M_{st} *on* $\mathrm{Ran}\mathbb{P}$ *such that, for any* $\xi > 0$,

$$M_{\mathrm{st}} := \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \lim_{\lambda\to 0} \mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}\left((\mathrm{i}e + \lambda^2\xi)\widetilde{\mathbb{P}} - \widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E}). \tag{7}$$

(Note that a priori the right hand side of (7) may depend on ξ; we assume that it does not).

3) *For any* $\mathrm{i}e, \mathrm{i}e' \in \mathrm{sp}\mathbb{E}$, $e \neq e'$ *and* $\xi > 0$,

$$\lim_{\lambda\to 0} \lambda\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}\left((\mathrm{i}e + \lambda^2\xi)\widetilde{\mathbb{P}} - \widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e'}(\mathbb{E}) = 0,$$
$$\lim_{\lambda\to 0} \lambda\mathbf{1}_{\mathrm{i}e'}(\mathbb{E})\mathbb{Q}\left((\mathrm{i}e + \lambda^2\xi)\widetilde{\mathbb{P}} - \widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E}) = 0.$$

Then the following holds:

1. $\mathrm{e}^{tM_{\mathrm{st}}}$ *is a contractive semigroup.*
2. *For any* $\xi > 0$

$$\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \lim_{\lambda\to 0} \mathbf{1}_{\mathrm{i}e}(\mathbb{E})\left(\xi - \lambda^{-2}(\mathbb{L}_\lambda - \mathrm{i}e)\right)^{-1}\mathbb{P} = (\xi\mathbb{P} - M_{\mathrm{st}})^{-1}.$$

3. *For any* $f \in C_0([0,\infty[)$,

$$\lim_{\lambda\to 0}\int_0^\infty f(t)\mathrm{e}^{-t\mathbb{E}/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}\mathrm{d}t = \int_0^\infty f(t)\mathrm{e}^{tM_{\mathrm{st}}}\mathrm{d}t. \tag{8}$$

Next we describe the time-dependent version of the weak coupling limit for C_0-groups.

Theorem 3. *Suppose that Assumptions 2.1, 2.3 and 2.4 are true. We make also the following assumptions:*

1) $\mathbb{P}\mathbb{Q}\widetilde{\mathbb{P}}$ *and* $\widetilde{\mathbb{P}}\mathbb{Q}\mathbb{P}$ *are bounded. (Note that this assumption guarantees that* $\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}$ *is the generator of a* C_0*-semigroup on* $\mathrm{Ran}\widetilde{\mathbb{P}}$*).*
2) *Set*

$$K_\lambda(t) := \int_0^{\lambda^{-2}t} \mathrm{e}^{-s\mathbb{E}}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}\mathrm{d}s. \tag{9}$$

We suppose that for all $t_0 > 0$*, there exists* c *such that*

$$\sup_{|\lambda|<\lambda_0}\ \sup_{0\le t\le t_0} \|K_\lambda(t)\| \le c.$$

3) *There exists a bounded operator* K *on* $\mathrm{Ran}\mathbb{P}$ *such that*

$$\lim_{\lambda\to 0} K_\lambda(t) = K$$

for all $0 < t < \infty$*.*

4) *There exists an operator* M_{dyn} *such that*

$$\mathrm{s-}\lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{s\mathbb{E}}K\mathrm{e}^{-s\mathbb{E}}\mathrm{d}s = M_{\mathrm{dyn}}.$$

Then the following holds:

1. $\mathrm{e}^{tM_{\mathrm{dyn}}}$ *is a contractive semigroup.*
2. *For any* $y \in \mathrm{Ran}\mathcal{Y}$ *and* $t_0 > 0$,

$$\lim_{\lambda\to 0}\sup_{0\le t\le t_0} \|\mathrm{e}^{-\mathbb{E}t/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}y - \mathrm{e}^{tM_{\mathrm{dyn}}}y\| = 0.$$

One of possible C_0^*-versions of the above theorem is given below.

Theorem 3* *Suppose that Assumptions 2.1*, 2.3* and 2.4 are true. We make also the following assumptions:*

0) $\mathrm{e}^{t\mathbb{E}}$ *is a C_0-group. (We already know that it is a C_0^*-group).*
1) $\mathbb{P}\mathbb{Q}\widetilde{\mathbb{P}}$ *and* $\widetilde{\mathbb{P}}\mathbb{Q}\mathbb{P}$ *are w* continuous. (Note that this assumption guarantees that* $\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}$ *is a generator of a C_0^*-semigroup on* $\mathrm{Ran}\widetilde{\mathbb{P}}$*).*
2) *In the sense of a w* integral [BR1] we set*

$$K_\lambda(t) := \int_0^{\lambda^{-2}t} \mathrm{e}^{-s\mathbb{E}}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}\mathrm{d}s. \tag{10}$$

We suppose that for all $t_0 > 0$, there exists c such that

$$\sup_{|\lambda|<\lambda_0} \sup_{0\leq t\leq t_0} \|K_\lambda(t)\| \leq c.$$

3) *there exists a w* continuous operator K on* $\mathrm{Ran}\mathbb{P}$ *such that*

$$\lim_{\lambda\to 0} K_\lambda(t) = K$$

for all $0 < t < \infty$.

4) *There exists an operator M_{dyn} such that*

$$\mathrm{s}-\lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{s\mathbb{E}}K\mathrm{e}^{-s\mathbb{E}} = M_{\mathrm{dyn}}.$$

Then the same conclusions as in Theorem 3 hold.

Theorem 3 is due to Davies (we put together Theorem 5.18 and 5.11 from [Da3]). Note that, following Davies, in Theorems 3 and 3* we do not make Assumption 2.2 about the finite dimension of $\mathrm{Ran}\mathbb{P}$. Instead, we make the assumption 4) about spectral averaging. If we impose Assumption 2.2, then we can drop 4) and make some other minor simplifications, as is described below:

Theorem 4. *Suppose that Assumptions 2.1, 2.2, 2.3 and 2.4 or 2.1*, 2.2, 2.3* and 2.4 are true. Set*

$$K_\lambda(t) := \int_0^{\lambda^{-2}t} \mathrm{e}^{-s\mathbb{E}}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}\mathrm{d}s.$$

We make also the following assumptions:

1) *We suppose that for all $t_0 > 0$, there exists c such that*

$$\sup_{|\lambda|<\lambda_0} \sup_{0\leq t\leq t_0} \|K_\lambda(t)\| \leq c.$$

2) *There exists an operator K on* $\mathrm{Ran}\mathbb{P}$ *such that*

$$\lim_{\lambda\to 0} K_\lambda(t) = K$$

for all $0 < t < \infty$. We set

$$M_{\mathrm{dyn}} := \sum_{\mathrm{ie}\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{ie}}(\mathbb{E})K\mathbf{1}_{\mathrm{ie}}(\mathbb{E})$$

Then the following holds:

1. $\mathrm{e}^{tM_{\mathrm{dyn}}}$ *is a contractive semigroup.*
2. *For any* $t_0 > 0$,

$$\lim_{\lambda\to 0}\sup_{0\le t\le t_0}\|\mathrm{e}^{-\mathbb{E}t/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}-\mathrm{e}^{tM_{\mathrm{dyn}}}\|=0. \tag{11}$$

Note that if there exists an operator M_{st} satisfying (8), and an operator M_{dyn} satisfying (11), then they clearly coincide. In our last theorem of this section we will describe a connection between M_{st}, M_{dyn} and the LSO.

Theorem 5. *Suppose that Assumptions 2.1, 2.2, 2.3 and 2.4, or 2.1*, 2.2, 2.3* and 2.4 are true. Suppose also that the following conditions hold:*

1) $\int_0^\infty \sup\limits_{|\lambda|\le\lambda_0}\|\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}\|\mathrm{d}s<\infty$.

2) *For any* $s>0$, $\lim\limits_{\lambda\to 0}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}=\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_0}\mathbb{Q}\mathbb{P}$.

Then

1. *Assumption 2.5 holds, and hence the LSO for* $(\mathbb{P},\mathbb{L}_0,\mathbb{Q})$*, defined in (3) and denoted* M*, exists.*
2. e^{tM} *is a contractive semigroup.*
3. *The assumptions of Theorem 2 hold and* $M=M_{\mathrm{st}}$*, consequently, for any* $\xi>0$

$$\lim_{\lambda\to 0}\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\left(\xi-\lambda^{-2}(\mathbb{L}_\lambda-\mathrm{i}e)\right)^{-1}\mathbb{P}=(\xi\mathbb{P}-M)^{-1}.$$

4. *The assumptions of Theorem 4 hold and* $M=M_{\mathrm{dyn}}$*, consequently*

$$\lim_{\lambda\to 0}\sup_{0\le t\le t_0}\|\mathrm{e}^{-\mathbb{E}t/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}-\mathrm{e}^{tM}\|=0.$$

3.2 Proof of the stationary weak coupling limit

Proof of Theorem 2. We follow [DF2]. Let $\mathrm{i}e\in\mathrm{sp}\mathbb{E}$. Set

$$\begin{aligned}G_\lambda(\xi,\mathrm{i}e):=&\ \xi\mathbb{P}+\lambda^{-2}(\mathrm{i}e\mathbb{P}-\mathbb{E})\\ &-\mathbb{P}\mathbb{Q}\left((\lambda^2\xi+\mathrm{i}e)\widetilde{\mathbb{P}}-\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{Q}\mathbb{P}.\end{aligned}$$

By the so-called Feshbach formula (see e.g. [DJ1, BFS1]), for $\xi>0$ we have

$$G_\lambda(\xi,\mathrm{i}e)^{-1}=\mathbb{P}\left(\xi+\lambda^{-2}(\mathrm{i}e-\mathbb{L}_\lambda)\right)^{-1}\mathbb{P}$$

This and the dissipativity of $\mathbb{L}_\lambda$ implies the bound

$$\|G_\lambda(\xi,\mathrm{i}e)^{-1}\|\le\xi^{-1}. \tag{12}$$

Write for shortness G instead of $G_\lambda(\xi,\mathrm{i}e)$. For $\mathrm{i}e'\in\mathrm{sp}\mathbb{E}$, set

$$\begin{aligned}\mathbb{P}_{e'}&:=\mathbf{1}_{\mathrm{i}e'}(\mathbb{E}),\\ \mathbb{P}_{\overline{e}'}&:=\mathbb{P}-\mathbf{1}_{\mathrm{i}e'}(\mathbb{E}).\end{aligned}$$

Decompose $G=G_{\mathrm{diag}}+G_{\mathrm{off}}$ into its diagonal and off-diagonal part:

$$\begin{aligned}G_{\mathrm{diag}}&:=\sum_{\mathrm{i}e'\in\mathrm{sp}\mathbb{E}}\mathbb{P}_{e'}G\mathbb{P}_{e'},\\ G_{\mathrm{off}}&:=\sum_{\mathrm{i}e'\in\mathrm{sp}\mathbb{E}}\mathbb{P}_{e'}G\mathbb{P}_{\overline{e}'}=\sum_{\mathrm{i}e'\in\mathrm{sp}\mathbb{E}}\mathbb{P}_{\overline{e}'}G\mathbb{P}_{e'}.\end{aligned}$$

First we would like to show that for $\xi>0$ and small enough λ, G_{diag} is invertible. By an application of the Neumann series, $\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}$ is invertible on $\mathrm{Ran}\mathbb{P}_{\overline{e}}$, and we have the bound

$$\|\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}^{-1}\|\leq c\lambda^2. \tag{13}$$

It is more complicated to prove that $\mathbb{P}_eG_{\mathrm{diag}}$ is inverible on $\mathrm{Ran}\mathbb{P}_e$.

We fix $\xi>0$. We know that G is invertible and $\|G^{-1}\|\leq\xi^{-1}$. Hence we can write

$$G_{\mathrm{diag}}G^{-1}=\mathbf{1}-G_{\mathrm{off}}G^{-1}.$$

Therefore

$$\begin{aligned}\mathbb{P}_eG_{\mathrm{diag}}G^{-1}&=\mathbb{P}_e-\mathbb{P}_eG_{\mathrm{off}}\mathbb{P}_{\overline{e}}G^{-1},\\ \mathbb{P}_{\overline{e}}G_{\mathrm{diag}}G^{-1}&=\mathbb{P}_{\overline{e}}-\mathbb{P}_{\overline{e}}G_{\mathrm{off}}G^{-1}.\end{aligned} \tag{14}$$

The latter identity can be for small enough λ transformed into

$$\mathbb{P}_{\overline{e}}G^{-1}=G_{\mathrm{diag}}^{-1}\mathbb{P}_{\overline{e}}-G_{\mathrm{diag}}^{-1}\mathbb{P}_{\overline{e}}G_{\mathrm{off}}G^{-1}. \tag{15}$$

We insert (15) into the first identity of (14) to obtain

$$\mathbb{P}_eG_{\mathrm{diag}}G^{-1}=\mathbb{P}_e-\mathbb{P}_eG_{\mathrm{off}}\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}^{-1}+\mathbb{P}_eG_{\mathrm{off}}\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}^{-1}G_{\mathrm{off}}G^{-1}. \tag{16}$$

We multiply (16) from the right by $\mathbb{P}_e$ to get

$$\mathbb{P}_eG_{\mathrm{diag}}\mathbb{P}_eG^{-1}\mathbb{P}_e=\mathbb{P}_e+\mathbb{P}_eG_{\mathrm{off}}\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}^{-1}G_{\mathrm{off}}G^{-1}\mathbb{P}_e. \tag{17}$$

Now, using

$$\lim_{\lambda\to0}\lambda\|G_{\mathrm{off}}\|=0, \tag{18}$$

(12) and (13) we obtain

$$\lim_{\lambda\to0}\mathbb{P}_eG_{\mathrm{off}}\mathbb{P}_{\overline{e}}G_{\mathrm{diag}}^{-1}G_{\mathrm{off}}G^{-1}\mathbb{P}_e=0.$$

Thus, for small enough λ,

$$\mathbb{P}_eG_{\mathrm{diag}}B_1=\mathbb{P}_e,$$

where

$$B_1 := \mathbb{P}_e G^{-1}\mathbb{P}_e \left(\mathbb{P}_e + \mathbb{P}_e G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} G_{\rm off} G^{-1}\mathbb{P}_e\right)^{-1}.$$

Similarly, for small enough λ, we find B_2 such that

$$B_2\mathbb{P}_e G_{\rm diag} = \mathbb{P}_e.$$

This implies that $\mathbb{P}_e G_{\rm diag}$ is invertible on $\mathrm{Ran}\mathbb{P}_e$.

Next, we can write

$$G^{-1} = G_{\rm diag}^{-1} - G_{\rm diag}^{-1} G_{\rm off} G_{\rm diag}^{-1} + G_{\rm diag}^{-1} G_{\rm off} G_{\rm diag}^{-1} G_{\rm off} G^{-1}.$$

Hence,

$$\mathbb{P}_e G^{-1} = \mathbb{P}_e G_{\rm diag}^{-1}\left(\mathbf{1} - G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} + G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} G_{\rm off} G^{-1}\right). \tag{19}$$

Therefore, for a fixed ξ, by (12), (13) and (18) we see that as $\lambda \to 0$ we have

$$-G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} + G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} G_{\rm off} G^{-1} \to 0.$$

Therefore, for small enough λ, we can invert the expression in the bracket of (19). Consequently,

$$\begin{aligned}\mathbb{P}_e(G_{\rm diag}^{-1} - G^{-1}) = \mathbb{P}_e G^{-1}&\left(1 - G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} + G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} G_{\rm off} G^{-1}\right)^{-1}\\ &\times\left(G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} - G_{\rm off}\mathbb{P}_{\overline{e}} G_{\rm diag}^{-1} G_{\rm off} G^{-1}\right).\end{aligned} \tag{20}$$

Therefore, for a fixed ξ, by (12), (13) and (18) we see that, as $\lambda \to 0$, we have

$$\mathbb{P}_e(G_{\rm diag}^{-1} - G^{-1}) \to 0. \tag{21}$$

Hence, (12) and (21) imply that $\mathbb{P}_e G_{\rm diag}^{-1}$ is uniformly bounded as $\lambda \to 0$. We know that

$$\mathbb{P}_e G_{\rm diag} \to \mathbb{P}_e\xi - \mathbb{P}_e M_{\rm st}. \tag{22}$$

Therefore, $\xi\mathbb{P}_e - \mathbb{P}_e M_{\rm st}$ is invertible on $\mathrm{Ran}\mathbb{P}_e$ and

$$\mathbb{P}_e G_{\rm diag}^{-1} \to (\mathbb{P}_e\xi - \mathbb{P}_e M_{\rm st})^{-1}.$$

Using again (21), we see that

$$\mathbb{P}_e G^{-1} \to (\mathbb{P}_e\xi - \mathbb{P}_e M_{\rm st})^{-1}. \tag{23}$$

Summing up (23) over e, we obtain

$$\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \mathbb{P}_e G_\lambda(\xi, \mathrm{i}e)^{-1} \to (\xi\mathbb{P} - M_{\rm st})^{-1}, \tag{24}$$

which ends the proof of 2.

Let us now prove 1. We have

$$\begin{aligned}\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbb{P}_e G_\lambda(\xi,\mathrm{i}e)^{-1} &= \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\int\limits_0^\infty \mathrm{e}^{-t(\xi+\lambda^{-2}\mathrm{i}e)}\mathbb{P}_e\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}\mathrm{d}t\\ &= \int\limits_0^\infty \mathrm{e}^{-t\xi}\mathrm{e}^{-t\mathbb{E}/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}\mathrm{d}t\end{aligned} \tag{25}$$

Clearly, $\|\mathrm{e}^{-t\mathbb{E}/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}\|\leq 1$. Therefore,

$$\left\|\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbb{P}_e G_\lambda(\xi,\mathrm{i}e)^{-1}\right\|\leq \xi^{-1}.$$

Hence, by (24),

$$\left\|(\xi\mathbb{P}-M_{\mathrm{st}})^{-1}\right\|\leq\xi^{-1},$$

which proves 1.

Let $f\in C_0([0,\infty[)$ and $\delta>0$. By the Stone-Weierstrass Theorem, we can find a finite linear combination of functions of the form $\mathrm{e}^{-t\xi}$ for $\xi>0$, denoted g, such that $\|\mathrm{e}^{t\delta}f-g\|_\infty<\epsilon$. Set

$$A_\lambda(t):=\mathrm{e}^{-t\mathbb{E}/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P},\quad A_0(t):=\mathrm{e}^{tM_{\mathrm{dyn}}}.$$

Note that $\|A_\lambda(t)\|\leq 1$ and $\|A_0(t)\|\leq 1$. Now

$$\begin{aligned}\|\int f(t)(A_\lambda(t)-A_0(t))\mathrm{d}t\| \qquad &\leq \|\int \mathrm{e}^{-\delta t}g(t)(A_\lambda(t)-A_0(t))\mathrm{d}t\|\\ +\|\int(f(t)-\mathrm{e}^{-\delta t}g(t))A_\lambda(t)\mathrm{d}t\| &+\|\int(f(t)-\mathrm{e}^{-\delta t}g(t))A_0(t)\mathrm{d}t\|.\end{aligned}$$

By 2. and by the Laplace transformation, the first term on the right hand side goes to 0 as $\lambda\to 0$. The last two terms are estimated by $\epsilon\int_0^\infty \mathrm{e}^{-\delta t}\mathrm{d}t$, which can be made arbitrarily small by choosing ϵ small. This proves 3. □

3.3 Spectral averaging

Before we present the time-dependent version of the weak coupling limit, we discuss the spectral averaging of operators, following [Da3].

In this subsection, $\mathcal{Y}$ is an arbitrary Banach space and $\mathrm{e}^{t\mathbb{E}}$ is a 1-parameter C_0-group of isometries on $\mathcal{Y}$. For $K\in\mathcal{B}(\mathcal{Y})$ we define

$$K^\natural:=\mathrm{s-}\lim_{t\to\infty}t^{-1}\int_0^t \mathrm{e}^{s\mathbb{E}}K\mathrm{e}^{-s\mathbb{E}}\mathrm{d}s, \tag{26}$$

provided that the right hand side exists.

Theorem 6. *Suppose that $K^\natural$ exists. Then, for any $t_0>0$, $y\in\mathcal{Y}$,*

$$\lim_{\lambda\to 0}\sup_{0\leq t\leq t_0}\|\mathrm{e}^{-t\mathbb{E}/\lambda}\mathrm{e}^{t(\mathbb{E}+\lambda K)/\lambda}y-\mathrm{e}^{tK^\natural}y\|=0.$$

Proof. Consider the space $C([0,t_0],\mathcal{Y})$ with the supremum norm. Set $K(t) = \mathrm{e}^{t\mathbb{E}/\lambda}K\mathrm{e}^{-t\mathbb{E}/\lambda}$. For $f\in C([0,t_0],\mathcal{Y})$, define

$$B_\lambda f(t) := \int_0^t K(s/\lambda)f(s)\mathrm{d}s,$$

$$B_0 f(t) := K^\natural \int_0^t f(s)\mathrm{d}s.$$

Clearly, B_0 and B_λ are linear operators on $C([0,t_0],\mathcal{Y})$ satisfying

$$\|B_\lambda\| \leq t_0\|K\|. \tag{27}$$

Moreover

$$\lim_{\lambda\to 0} B_\lambda f = B_0 f. \tag{28}$$

To prove (28), by (27) it suffices to assume that $f\in C^1([0,t_0],\mathcal{Y})$. Now

$$\begin{aligned}B_\lambda f(t) &= \left(\int_0^t K(s/\lambda)\mathrm{d}s\right) f(t) - \int_0^t \left(\int_0^s \mathrm{d}s_1 K(s_1/\lambda)\right) f'(s)\mathrm{d}s\\ &\to tK^\natural f(t) - \int_0^t sK^\natural f'(s)\mathrm{d}s = B_0 f(t).\end{aligned}$$

We easily get

$$\|B_\lambda^n\| \leq \frac{t_0^n}{n!}\|K\|^n, \quad \|B_0^n\| \leq \frac{t_0^n}{n!}\|K\|^n. \tag{29}$$

Let $y\in\mathcal{Y}$. Set $y_\lambda(t) := \mathrm{e}^{-t\mathbb{E}/\lambda}\mathrm{e}^{t(\mathbb{E}+\lambda K)/\lambda}y$. Note that

$$y_\lambda(t) = y + B_\lambda y_\lambda(t), \quad y_0(t) = y + B_0 y_0(t).$$

Treating y as an element of $C([0,t_0],\mathcal{Y})$ – the constant function equal to y we can write

$$(1-B_\lambda)^{-1}y = \sum_{n=0}^\infty B_\lambda^n y, \quad (1-B_0)^{-1}y = \sum_{n=0}^\infty B_0^n y,$$

where both Neumann series are absolutely convergent. Therefore, in the sense of the convergence in in $C([0,t_0],\mathcal{Y})$, we get

$$y_\lambda = \sum_{n=0}^\infty B_\lambda^n y \to \sum_{n=0}^\infty B_0^n y = y_0.$$

□

Theorem 7. *Let $\mathcal{Y}$ be finite dimesional. Then $K^\natural$ exists for any $K\in\mathcal{B}(\mathcal{Y})$ and*

$$\begin{aligned}&K^\natural = \sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{i}e}(\mathbb{E})K\mathbf{1}_{\mathrm{i}e}(\mathbb{E}) = \lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{s\mathbb{E}}K\mathrm{e}^{-s\mathbb{E}}\mathrm{d}s,\\ &\lim_{\lambda\to 0}\sup_{0\leq t\leq t_0} \|\mathrm{e}^{-t\mathbb{E}/\lambda}\mathrm{e}^{t(\mathbb{E}+\lambda K)/\lambda} - \mathrm{e}^{tK^\natural}\| = 0.\end{aligned}$$

Proof. In finite dimension we can replace the strong limit by the norm limit. Moreover,

$$t^{-1}\int_0^t \mathrm{e}^{s\mathbb{E}}K\mathrm{e}^{-s\mathbb{E}}\mathrm{d}s = \sum_{\mathrm{i}e_1,\mathrm{i}e_2\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{i}e_1}(\mathbb{E})K\mathbf{1}_{\mathrm{i}e_2}(\mathbb{E})\frac{\mathrm{e}^{\mathrm{i}t(e_1-e_2)}-1}{\mathrm{i}(e_1-e_2)t}.$$

□

Remark 2. The following results generalize some aspects of Theorem 7 to the case when $\mathbb{P}$ is not necessarily finite dimensional. They are proven in [Da3]. We will not need these results.

1) If $K^\natural$ exists, then it commutes with $\mathrm{e}^{t\mathbb{E}}$.
2) If K is a compact operator and $\mathcal{Y}$ is a Hilbert space, then $K^\natural$ exists and we can replace the strong limit in (26) by the norm limit.
3) If $\mathbb{E}$ has a total set of eigenvectors, then $K^\natural$ exists as well.

3.4 Second order asymptotics of evolution with the first order term

In this subsection we consider a somewhat more general situation than in Subsection 3.1. We make the Assumptions 2.1, 2.3 and 2.4, or 2.1*, 2.3* and 2.4 but we do not assume that $\mathbb{P}$ is finite dimensional, nor that $\mathbb{PQP}=0$. Thus we allow for a term of first order in λ in the asymptotics of the reduced dynamics. We again follow [Da3].

We assume also that $\mathbb{PQ}\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}\mathbb{QP}$ are bounded or w* continuous and that $\mathbb{E}+\lambda\mathbb{PQP}$ generates a C_0- or C_0^*-group of isometries on $\mathrm{Ran}\mathbb{P}$.

Using the boundedness of off-diagonal elements $\mathbb{PQ}\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}}\mathbb{QP}$, we see that $\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}$ is the generator of a continuous semigroup.

In this subsection, the definition of $K_\lambda(t)$ slightly changes as compared with (9):

$$K_\lambda(t) := \int_0^{\lambda^{-2}t} \mathrm{e}^{-s(\mathbb{E}+\lambda\mathbb{PQP})}\mathbb{PQ}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{QP}\mathrm{d}s.$$

Theorem 8. *Suppose that the following assumptions are true:*

1) *For all* $t_0>0$*, there exists* c *such that*

$$\sup_{|\lambda|<\lambda_0}\ \sup_{0\le t\le t_0}\|K_\lambda(t)\|\le c.$$

2) *There exists a bounded (w* continuous in the* C_0^* *case) operator* K *on* $\mathrm{Ran}\mathbb{P}$ *such that*

$$\lim_{\lambda\to 0}K_\lambda(t)=K$$

for all $0<t<\infty$*.*

Then for $y\in\mathrm{Ran}\mathbb{P}$

$$\lim_{\lambda\to 0}\ \sup_{0\le t\le t_1}\left\|\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}y-\mathrm{e}^{t(\mathbb{E}+\lambda\mathbb{PQP}+\lambda^2K)/\lambda^2}y\right\|=0.$$

Proof. Set $\mathcal{Y} := \mathrm{Ran}\mathbb{P}$. Consider the space $C([0,t_0],\mathcal{Y})$. For $f \in C([0,t_0],\mathcal{Y})$ define

$$H_\lambda f(t) := \int_0^t \mathrm{e}^{(\mathbb{E}+\mathbb{PQP})(t-s)/\lambda^2} K_\lambda(t-s) f(s)\mathrm{d}s,$$

$$G_\lambda f(t) := \int_0^t \mathrm{e}^{(\mathbb{E}+\mathbb{PQP})(t-s)/\lambda^2} K f(s)\mathrm{d}s.$$

Note that H_λ and G_λ are linear operators on $C([0,t_0],\mathcal{Y})$ satisfying

$$\|H_\lambda^n\| \leq c^n t_0^n/n!, \quad \|G_\lambda^n\| \leq c^n t_0^n/n!,$$

Thus $1-H_\lambda$ and $1-G_\lambda$ are invertible. In fact, they can be defined by the Neumann series:

$$(1-H_\lambda)^{-1} = \sum_{j=0} H_\lambda^n, \qquad (1-G_\lambda)^{-1} = \sum_{j=0} G_\lambda^n.$$

Next we note that

$$\|H_\lambda^n - G_\lambda^n\| \leq \|H_\lambda - G_\lambda\| c^{n-1} t_0^{n-1}/(n-1)!, \tag{30}$$

because

$$\begin{aligned}\|H_\lambda^n - G_\lambda^n\| &\leq \sum_{j=0}^{n-1} \|H_\lambda^j\| \|G_\lambda^{n-j-1}\| \|\|H_\lambda - G_\lambda\| \\ &\leq \sum_{j=0}^{n-1} \tfrac{c^{n-1}t_0^{n-1}}{k!(n-k-1)!} \|H_\lambda - G_\lambda\| = (2ct_0)^{n-1}\|H_\lambda - G_\lambda\|/(n-1)!.\end{aligned}$$

Therefore,

$$\|(1-H_\lambda)^{-1} - (1-G_\lambda)^{-1}\| \leq c\|H_\lambda - G_\lambda\|, \tag{31}$$

Next,

$$(H_\lambda - G_\lambda)f(t) = \int_0^t \mathrm{e}^{(\mathbb{E}+\lambda\mathbb{PQP})(t-s)/\lambda^2}(K_\lambda(t-s) - K)f(s)\mathrm{d}s.$$

and hence

$$\|H_\lambda - G_\lambda\| \leq \int_0^{t_0} \|K_\lambda(s) - K\|\mathrm{d}s \to 0.$$

Thus

$$\|(1-H_\lambda)^{-1} - (1-G_\lambda)^{-1}\| \to 0. \tag{32}$$

Let $y \in \mathcal{Y}$. Define the following elements of the space $C([0,t_0],\mathcal{Y})$:

$$g_\lambda(t) := \mathrm{e}^{(\mathbb{E}+\lambda\mathbb{PQP})t/\lambda^2} y,$$

$$h_\lambda(t) := \mathbb{P}\mathrm{e}^{\mathbb{L}_\lambda t/\lambda^2}\mathbb{P}y,$$

$$g_\lambda(t) := \mathrm{e}^{(\mathbb{E}+\lambda\mathbb{PQP}+\lambda^2 K)t/\lambda^2} y.$$

Now

$$h_\lambda = g_\lambda + H_\lambda h_\lambda,$$

$$g_\lambda = g_\lambda + G_\lambda g_\lambda.$$

Thus

$$h_\lambda - g_\lambda = (1 - H_\lambda)^{-1} g_\lambda - (1 - G_\lambda)^{-1} g_\lambda \to 0.$$

□

3.5 Proof of time dependent weak coupling limit

Proof of Theorem 3 and 3*. In addition to the assumptions of Theorem 8 we suppose that $\mathbb{PQP} = 0$ and $K^\natural$ exists.

Theorem 7 implies that

$$\lim_{\lambda\to 0} \sup_{0\le t\le t_0} \|\mathrm{e}^{-\mathbb{E}t/\lambda^2} \mathrm{e}^{t(\mathbb{E}+\lambda^2 K)/\lambda^2} y - \mathrm{e}^{tK^\natural} y\| = 0.$$

Theorem 8 yields

$$\lim_{\lambda\to 0} \sup_{0\le t\le t_0} \|\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}y - \mathrm{e}^{t(\mathbb{E}+\lambda^2 K)/\lambda^2} y\| = 0.$$

Using that $\mathrm{e}^{t\mathbb{E}}$ is isometric we obtain

$$\lim_{\lambda\to 0} \sup_{0\le t\le t_0} \|\mathrm{e}^{-\mathbb{E}t/\lambda^2}\mathbb{P}\mathrm{e}^{t\mathbb{L}_\lambda/\lambda^2}\mathbb{P}y - \mathrm{e}^{tK^\natural} y\| = 0. \tag{33}$$

It is clear from (33) that $\mathrm{e}^{tK^\natural}$ is contractive. □

Proof of Theorem 4 Because of the finite dimension all operators on $\mathrm{Ran}\mathbb{P}$ are w* continuous and the strong and norm convergence coincide. Besides, we can apply Theorem 7 about the existence of $K^\natural$. □

3.6 Proof of the coincidence of M_{st} and M_{dyn} with the LSO

Proof of Theorem 5. Set

$$f(s) := \sup_{|\lambda|\le\lambda_0} \|\mathbb{PQ}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{QP}\|.$$

We know that $f(t)$ is integrable.

For any $e \in \mathbb{R}$ and $\xi \ge 0$ we can dominate the integrand in the integral

$$\begin{aligned} F_\lambda(\mathrm{i}e, \xi) &:= \int_0^\infty \mathbb{PQ}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{QP}\mathrm{e}^{-(\mathrm{i}e+\lambda^2\xi)s}\mathrm{d}s \\ &= \mathbb{PQ}\left(\widetilde{\mathbb{P}}(\mathrm{i}e + \lambda^2\xi) - \widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{QP} \end{aligned} \tag{34}$$

by $f(s)$. Hence, using the dominated convergence theorem we see that $F_\lambda(\mathrm{i}e,\xi)$ is continuous at $\lambda=0$ and $\xi\geq 0$. But

$$\begin{aligned}&\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})F_0(\mathrm{i}e,0)\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\\ =&\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\lim_{\lambda\to 0}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathbb{Q}\left(\widetilde{\mathbb{P}}(\mathrm{i}e+\lambda^2\xi)-\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}\right)^{-1}\mathbb{Q}\mathbb{P}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})=M_{\mathrm{st}}.\end{aligned}$$

Recall (9), the definition of $K_\lambda(t)$:

$$K_\lambda(t):=\int_0^{\lambda^{-2}t}\mathrm{e}^{-s\mathbb{E}}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_\lambda\widetilde{\mathbb{P}}}\mathbb{Q}\mathbb{P}\mathrm{d}s.$$

Its integrand can also be dominated by $f(s)$. Hence, using again the dominated convergence theorem, we see that, for $\lambda\to 0$, $K_\lambda(t)$ is convergent to

$$K=\int_0^\infty\mathrm{e}^{-s\mathbb{E}}\mathbb{P}\mathbb{Q}\mathrm{e}^{s\widetilde{\mathbb{P}}\mathbb{L}_0}\mathbb{Q}\mathbb{P}\mathrm{d}s.$$

Therefore,

$$\begin{aligned}K^\natural&=\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\int_0^\infty\mathbb{Q}\mathrm{e}^{s\mathbb{L}_0\widetilde{\mathbb{P}}}\mathbb{Q}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})\mathrm{e}^{-\mathrm{i}es}\mathrm{d}s\\&=\sum_{\mathrm{i}e\in\mathrm{sp}\mathbb{E}}\mathbf{1}_{\mathrm{i}e}(\mathbb{E})F_0(\mathrm{i}e,0)\mathbf{1}_{\mathrm{i}e}(\mathbb{E}).\end{aligned}$$

□

4 Completely positive semigroups

In this section we recall basic information about completely positive maps and semigroups, which are often used to describe irreversible dynamics of quantum systems. For simplicity, most of the time we restrict ourselves to the finite dimensional case.

4.1 Completely positive maps

The following facts are well known and can be e.g. found in [BR2], Notes and Remarks to Section 5.3.1.

Let $\mathcal{K}_1,\mathcal{K}_2$ be Hilbert spaces. We say that a linear map $\Xi:\mathcal{B}(\mathcal{K}_1)\to\mathcal{B}(\mathcal{K}_2)$ is positive iff $A\geq 0$ implies $\Xi(A)\geq 0$. We say that it is completely positive (c.p. for short) iff for any n, $\Xi\otimes\mathbf{1}_{\mathcal{B}(\mathbb{C}^n)}$ is positive as a map $\mathcal{B}(\mathcal{K}_1\otimes\mathbb{C}^n)\to\mathcal{B}(\mathcal{K}_2\otimes\mathbb{C}^n)$.

We will say that a positive map Ξ is Markov if $\Xi(\mathbf{1})=\mathbf{1}$.

Recall that $\mathcal{B}^1(\mathcal{K}_i)$ denotes the space of trace class operators on $\mathcal{K}_i$. We can define positive and completely positive maps from $\mathcal{B}^1(\mathcal{K}_2)$ to $\mathcal{B}^1(\mathcal{K}_1)$ repeating verbatim the definition for the algebra of bounded operators. We will say that the map is Markov if it preserves the trace.

We can also speak of positive and completely positive maps on $\mathcal{B}^2(\mathcal{K})$.

We will sometimes say that maps on the algebra $\mathcal{B}(\mathcal{K})$ are “in the Heisenberg picture”, maps on $\mathcal{B}^1(\mathcal{K})$ are “in the Schrödinger picture” and maps on $\mathcal{B}^2(\mathcal{K})$ are “in the standard picture” (see the notion of the standard representation later on and in [DJP]).

From now on, for simplicity, in this section we will assume that the spaces $\mathcal{K}_i$ are finite dimensional. Thus $\mathcal{B}(\mathcal{K}_i)$ and $\mathcal{B}^2(\mathcal{K}_i)$ and $\mathcal{B}^1(\mathcal{K}_i)$ coincide with one another as vector spaces. If Ξ is a map from matrices on $\mathcal{K}_1$ to matrices on $\mathcal{K}_2$, it is often useful to distinguish whether it is understood as a map from $\mathcal{B}(\mathcal{K}_1)$ to $\mathcal{B}(\mathcal{K}_2)$ (we then say that it is in the Heisenberg picture), as a map from $\mathcal{B}^2(\mathcal{K}_1)$ to $\mathcal{B}^2(\mathcal{K}_2)$ (we then say that it is in the standard picture) or as a map from $\mathcal{B}^1(\mathcal{K}_1)$ to $\mathcal{B}^1(\mathcal{K}_2)$ (we then say that it is in the Schrödinger picture).

Note that $\mathcal{B}^1(\mathcal{K}_i)$ and $\mathcal{B}(\mathcal{K}_i)$ are dual to one another. (This is one of the places where we use one of propertie of finite dimensional spaces. In general, $\mathcal{B}(\mathcal{K}_i)$ is only dual to $\mathcal{B}^1(\mathcal{K}_i)$ and not the other way around.) The (sesquilinear) duality between $\mathcal{B}^1(\mathcal{K}_i)$ and $\mathcal{B}(\mathcal{K}_i)$ is given by

$$\mathrm{Tr}\rho^* A, \quad \rho \in \mathcal{B}^1(\mathcal{K}_i), \quad A \in \mathcal{B}(\mathcal{K}_i).$$

If Ξ is a map “in the Heisenberg picture”, then its adjoint Ξ^*, is a map “in the Schrödinger picture” (and vice versa). Clearly, Ξ is a Markov transformation in the Heisenberg picture iff Ξ^* is Markov in the Schrödinger picture.

Note that (in a finite dimension) the definition of Ξ^* does not depend on whether we consider Ξ in the Heisenberg, standard or Schrödinger picture.

4.2 Stinespring representation of a completely positive map

By the Stinespring theorem [St], $\Xi : \mathcal{B}(\mathcal{K}_1) \to \mathcal{B}(\mathcal{K}_2)$ is completely positive iff there exists an auxilliary finite dimensional Hilbert space $\mathcal{H}$ and $W \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1 \otimes \mathcal{H})$ such that

$$\Xi(B) = W^* \, B \otimes \mathbf{1}_{\mathcal{H}} \, W, \quad B \in \mathcal{B}(\mathcal{K}_1). \tag{35}$$

In practice it can be useful to transform (35) into a slightly different form. Let us fix an orthonormal basis $(e_1, \dots, e_n)$ in $\mathcal{H}$. Then the operator W is completely determined by giving a family of operators $W_1, \dots, W_n \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ such that

$$W\Psi_2 = \sum_{j=1}^{n} (W_j \Psi_2) \otimes e_j, \quad \Psi_2 \in \mathcal{K}_2.$$

Then

$$\Xi(B) = \sum_{j=1}^{n} W_j^* B W_j. \tag{36}$$

There exists a third way of writing (35), which is sometimes useful. Let $\overline{\mathcal{H}}$ be the space conjugate to $\mathcal{H}$ and let $\mathcal{H} \ni \Phi \mapsto \overline{\Phi} \in \overline{\mathcal{H}}$ be the corresponding conjugation (see e.g. [DJ2]). We define $W^\star \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2 \otimes \overline{\mathcal{H}})$ by

$$(W^\star \Psi_1 | \Psi_2 \otimes \overline{\Phi})_{\mathcal{K}_2 \otimes \overline{\mathcal{H}}} = (\Psi_1 \otimes \Phi | W \Psi_2)_{\mathcal{K}_1 \otimes \mathcal{H}}, \tag{37}$$

(see [DJ2]). (Note that we use two different kinds of stars: $*$ for the hermitian conjugation and $\star$ for (37)). Let $\mathrm{Tr}_{\overline{\mathcal{H}}}$ denote the partial trace over $\overline{\mathcal{H}}$. Then

$$\Xi(B) = \mathrm{Tr}_{\overline{\mathcal{H}}} W^\star B W^{\star *}. \tag{38}$$

If Ξ is given by (35), then Ξ^* can be written in the following three forms:

$$\begin{aligned} \Xi^*(C) &= \mathrm{Tr}_{\mathcal{H}} W C W^* \\ &= \sum_{j=1}^n W_j C W_j^* \\ &= W^{\star *} C \otimes \mathbf{1}_{\overline{\mathcal{H}}} W^\star, \end{aligned}$$

where $C \in \mathcal{B}^1(\mathcal{K}_2)$.

4.3 Completely positive semigroups

Let $\mathcal{K}$ be a finite dimensional Hilbert space and $t \mapsto \Lambda(t)$ a continuous 1-parameter semigroup of operators on $\mathcal{B}(\mathcal{K})$. Let M be its generator, so that $\Lambda(t) = \mathrm{e}^{tM}$.

We say that $\Lambda(t)$ is a completely positive semigroup iff $\Lambda(t)$ is completely positive for any $t \geq 0$. $\Lambda(t)$ is called a Markov semigroup iff $\Lambda(t)$ is Markov for any $t \geq 0$.

$\Lambda(t)$ is a completely positive semigroup iff there exists an operator Δ on $\mathcal{K}$ and a completely positive map Ξ on $\mathcal{B}(\mathcal{K})$ such that

$$M(B) = \Delta B + B \Delta^* + \Xi(B), \quad B \in \mathcal{B}(\mathcal{K}). \tag{39}$$

Operators of the form (39) are sometimes called Lindblad or Lindblad-Kossakowski generators [GKS, L].

Let $[\cdot,\cdot]_+$ denote the anticommutator. $\Lambda(t)$ is Markov iff

$$M(B) = \mathrm{i}[\Theta, B] - \frac{1}{2}[\Xi(1), B]_+ + \Xi(B),$$

where $\Theta := \frac{1}{2}(\Delta + \Delta^*)$.

If Ξ is given by (35), then

$$\begin{aligned} M(B) &= \mathrm{i}[\Theta, B] + \tfrac{1}{2}\big(W^*(WB - B\otimes 1 W) + (BW^* - W^* B \otimes 1) W)\big) \\ &= \mathrm{i}[\Theta, B] + \tfrac{1}{2}\sum_{j=1}^n (W_j^*[W_j, B] + [B, W_j^*]W_j), \end{aligned} \tag{40}$$

and

$$\begin{aligned} M^*(B) &= \mathrm{i}[\Theta, B] - \tfrac{1}{2}[W^* W, B]_+ + \mathrm{Tr}_{\mathcal{H}} W B W^* \\ &= \mathrm{i}[\Theta, B] + \sum_{j=1}^n \left(-\tfrac{1}{2}[W_j^* W_j, B]_+ + W_j^{\star *} B \otimes 1 W_j^\star\right). \end{aligned}$$

Suppose that e^{tM} is a positive Markov semigroup in the Heisenberg picture. We say that a density matrix ρ on $\mathcal{K}$ is stationary with respect to this semiigroup iff $e^{tM^*}(\rho) = \rho$. Every positive Markov semigroup in a finite dimension has a stationary density matrix.

Markov completely positive semigroups (both in the Heisenberg and Schrödinger picture) are often used in quantum physics. In the literature, they are called by many names such as quantum dynamical or quantum Markov semigroups.

4.4 Standard Detailed Balance Condition

In the literature one can find a number of various properties that are called the Detailed Balance Condition (DBC). In the quantum context, probably the best known is the defintion due to Alicki [A] and Frigerio-Gorini-Kossakowski-Verri [FGKV], which we describe in the next subsection and call the DBC in the sense of AFGKV.

In this subsection we introduce a slightly different property that we think is the most satisfactory generalization of the DBC from the clasical to the quantum case. It is a modification of the DBC in the sense of AFGKV. To distinguish it from other kinds of the DBC, we will call it the standard Detailed Balance Condition. The name is justified by the close relationship of this condition to the standard representation. We have not seen the standard DBC in the literature, but we know that it belongs to the folklore of the subject. In particular, it was considered in the past by R. Alicki and A. Majewski (private communication).

In the literature one can also find other properties called the Detailed Balance Condition [Ma1, Ma2, MaSt]. Most of them involve the notion of the time reversal, which is not used in the case of the standard DBC or the DBC in the sense of AFGKV.

Let us assume that ρ is a nondegenerate density matrix on $\mathcal{K}$. (That means, $\rho > 0$, $\mathrm{Tr}\rho = 1$, and ρ^{-1} exists). On the space of operators on $\mathcal{K}$ we introduce the scalar product given by ρ:

$$(A|B)_\rho := \mathrm{Tr}\rho^{1/2}A^*\rho^{1/2}B. \tag{41}$$

This space equipped with the scalar product (41) will be denoted by $\mathcal{B}^2_\rho(\mathcal{K})$. Let $*\rho$ denote the hermitian conjugation with respect to this scalar product. Thus if M is a map on $\mathcal{B}(\mathcal{K})$, then $M^{*\rho}$ is defined by

$$(M^{*\rho}(A)|B)_\rho = (A|M(B))_\rho.$$

Explicitly,

$$M^{*\rho}(A) = \rho^{-1/2}M^*(\rho^{1/2}A\rho^{1/2})\rho^{-1/2}.$$

Definition 1. *Let M be the generator of a Markov c.p. semigroup on $\mathcal{B}(\mathcal{K})$. We will say that M satisfies the standard Detailed Balance Condition with respect to ρ if there exists a self-adjoint operator Θ on $\mathcal{K}$ such that*

$$\frac{1}{2\mathrm{i}}(M - M^{*\rho}) = [\Theta, \cdot]. \tag{42}$$

Theorem 9. *Let M be the generator of a Markov c.p. semigroup on $\mathcal{B}(\mathcal{K})$.*

1) *Let M satisfy the standard DBC with respect to ρ. Then*

$$\begin{aligned} M(A) &= \mathrm{i}[\Theta, A] + M_{\mathrm{d}}(A), \\ M^*(A) &= -\mathrm{i}[\Theta, A] + \rho^{1/2} M_{\mathrm{d}}(\rho^{-1/2} A \rho^{-1/2})\rho^{1/2}. \end{aligned} \tag{43}$$

where M_{d} is a generator of another Markov c.p. semigroup satisfying $M_{\mathrm{d}} = M_{\mathrm{d}}^{*\rho}$ and Θ is a self-adjoint operator on $\mathcal{K}$. Moreover, $[\Theta, \rho] = 0$, $M^*(\rho) = M_{\mathrm{d}}^*(\rho) = 0$.

2) *Let M be given by (40). If there exists a unitary operator $U : \mathcal{H} \to \overline{\mathcal{H}}$ such that*

$$[\Theta, \rho] = 0, \quad [W^*W, \rho] = 0,$$

$$W^\star = \rho^{-1/2} \otimes U\, W \rho^{1/2},$$

then M satisfies the standard DBC wrt ρ.

Proof. 1) By (42),

$$[\Theta, \cdot] = -[\Theta, \cdot]^{*\rho} = -\rho^{-1/2}[\Theta, \rho^{1/2} \cdot \rho^{1/2}]\rho^{-1/2}.$$

Using $[\Theta, 1] = 0$, we obtain $[\Theta, \rho] = 0$.

Setting $M_{\mathrm{d}} := \frac{1}{2}(M + M^{*\rho})$ we obtain the decomposition (43). Clearly, $0 = M(\mathbf{1}) = M_{\mathrm{d}}(\mathbf{1})$. Hence M_{d} is Markov. Next $0 = M_{\mathrm{d}}(\mathbf{1}) = M_{\mathrm{d}}^{*\rho}(\mathbf{1})$ gives $M_{\mathrm{d}}(\rho) = 0$.

To see 2) we note that if

$$M_{\mathrm{d}} = \frac{1}{2}[W^*W, B]_+ - W^*\; B \otimes 1\; W,$$

then

$$\begin{aligned} M_{\mathrm{d}}^{*\rho}(B) &= \rho^{-1/2}\left(\tfrac{1}{2}[W^*W, \rho^{1/2}B\rho^{1/2}]_+ - W^{\star *}\; \rho^{1/2}B\rho^{1/2} \otimes 1\; W^\star\right)\rho^{-1/2} \\ &= \tfrac{1}{2}[W^*W, B]_+ - (\rho^{1/2} \otimes 1\, W^\star \rho^{1/2})^*\; B \otimes 1\; \rho^{1/2} \otimes 1\, W^\star \rho^{-1/2}. \end{aligned}$$

□

M_{d} is called the dissipative part of the generator M.

4.5 Detailed Balance Condition in the sense of Alicki-Frigerio-Gorini-Kossakowski-Verri

In this subsection we recall the definition of Detailed Balance Condition, which can be found in [A, FGKV].

Let us introduce the scalar product

$$(A|B)_{(\rho)} := \mathrm{Tr}\rho A^* B.$$

Let $\mathcal{B}^2_{(\rho)}(\mathcal{K})$ denote the space of operators on $\mathcal{K}$ equipped with this scalar product. Let $M^{*(\rho)}$ denote the conjugate of M with respect to this scalar product. Explicitly:

$$M^{*(\rho)}(A) = \rho^{-1} M^*(\rho A).$$

Definition 2. *We will say that M satisfies the Detailed Balance Condition with respect to ρ in the sense of AFGKV if there exists a self-adjoint operator Θ such that*

$$\frac{1}{2\mathrm{i}}(M - M^{*(\rho)}) = [\Theta, \cdot].$$

Note that for DBC in the sense of AFGKV, the analog of Theorem 9 1) holds, where we replace the scalar product $(\cdot|\cdot)_\rho$ with $(\cdot|\cdot)_{(\rho)}$.

In practical applications, c.p. semigroups usually originate from the weak coupling limit of reduced dynamics, as we describe further on in our lectures. In this case the standard DBC is equivalent to DBC in the sense of AFGKV, which follows from the following theorem:

Theorem 10. *Suppose that M satisfies*

$$\rho^{1/4} M(\rho^{-1/4} A \rho^{1/4})\rho^{-1/4} = M(A).$$

Then M satisfies the DBC in the sense of (42) iff it satisfies DBC in the sense of AFGKV. Moreover, the decompositions $M = \mathrm{i}[\Theta, \cdot] + M_{\mathrm{d}}$ obtained in both cases concide.

Proof. It is enough to note that the map

$$\mathcal{B}^2_\rho(\mathcal{K}) \ni A \mapsto \rho^{-1/4} A \rho^{1/4} \in \mathcal{B}^2_{(\rho)}(\mathcal{K})$$

is unitary. □

5 Small quantum system interacting with reservoir

In this section we describe the class of W^*-dynamical systems that we consider in our notes. They are meant to describe a small quantum system $\mathcal{S}$ interacting with a large reservoir $\mathcal{R}$. Pauli-Fierz systems, considered in [DJ2], are typical examples of such systems.

In Subsect. 5.1 we recall basic elements of the theory of W^*-algebras (see [BR1, BR2, DJP] for more information). In Subsect. 5.2 we introduce the class of W^*-dynamical systems describing $\mathcal{S} + \mathcal{R}$ in purely algebraic (representation-independent) terms. In Subsect. 5.3 and 5.4 we explain the construction of two representations of our W^*-dynamical system: the semistandard and the standard representation. Both representations possess a distinguished unitary implementation of the dynamics. Its generator will be called the semi-Liouvillean in the former case and the Liouvillean in the latter case.

The standard representation and the Liouvillean can be defined for an arbitrary W^*-algebra (see next subsection, [DJP] and references therein). The semistandard representation and the semi-Liouvillean are concepts whose importance is limited to a system of the form $\mathcal{S} + \mathcal{R}$ considered in these notes. Their names were coined in [DJ2]. The advantage of the semistandard representation over the standard one is its simplicity, and this is the reason why it appears often in the literature [Da1, LeSp]. The semistandard representation is in particular well adapted to the study of the reduced dynamics.

5.1 W^*-algebras

In this subsection we recall the definitions of basic concepts related to the theory of W^*-algebras (see [BR1, BR2, DJP]).

A W^*-dynamical system $(\mathfrak{M}, \tau)$ is a pair consisting of a W^*-algebra $\mathfrak{M}$ and a 1-parameter (pointwise) σ-weakly continuous group of $*$-automorphisms of $\mathfrak{M}$, $\mathbb{R} \ni t \mapsto \tau^t$.

A standard representation of a W^*-algebra $\mathfrak{M}$ is a quadruple $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ consisting of a representation π, its Hilbert space $\mathcal{H}$, an antilinear involution J and a self-dual cone $\mathcal{H}^+$ satisfying the following conditions:

1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$;
2) $J\pi(A)J = \pi(A)^*$ for A in the center of $\mathfrak{M}$;
3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}^+$;
4) $\pi(A)J\pi(A)\mathcal{H}^+ \subset \mathcal{H}^+$ for $A \in \mathfrak{M}$.

J is called the modular conjugation and $\mathcal{H}^+$ the modular cone. Every W^*-algebra possesses a standard representation, unique up to the unitary equivalence.

Suppose that we are given a faithful state ω on $\mathfrak{M}$. In the corresponding GNS representation $\pi_\omega : \mathfrak{M} \to \mathcal{B}(\mathcal{H}_\omega)$, the state ω is given by a cyclic and separating vector Ω_ω. The Tomita-Takesaki theory yields the modular W^*-dynamics $t \mapsto \sigma_\omega^t$, the modular conjugation J_ω and the modular cone $\mathcal{H}_\omega^+ := \{AJ_\omega A\Omega_\omega \ : \ A \in \mathfrak{M}\}^{\mathrm{cl}}$, where cl denotes the closure. The state ω satisfies the -1-KMS condition for the dynamics σ_ω. The quadruple $(\pi_\omega, \mathcal{H}_\omega, J_\omega, \mathcal{H}_\omega^+)$ is a standard representation of $\mathfrak{M}$.

Until the end of this subsection, we suppose that a standard representation $(\pi, \mathcal{H}, J, \mathcal{H}^+)$ of $\mathfrak{M}$ is given.

Let ω be a state on $\mathfrak{M}$. Then there exists a unique vector in the modular cone $\Omega \in \mathcal{H}^+$ representing ω. Ω is cyclic iff Ω is separating iff ω is faithful.

Let $t \mapsto \tau^t$ be a W^*-dynamics on $\mathfrak{M}$. The Liouvillean L of τ is a self-adjoint operator on $\mathcal{H}$ uniquely defined by demanding that

$$\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}, \qquad \mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ = \mathcal{H}^+, \quad t \in \mathbb{R}.$$

(L implements the dynamics in the representation π and preserves the modular cone). It has many useful properties that make it an efficient tool in the study of the ergodic properties of the dynamics τ. In particular, L has no point spectrum iff τ has no normal invariant states, and L has a 1-dimensional kernel iff τ has a single invariant normal state.

5.2 Algebraic description

The Hilbert space of the system $\mathcal{S}$ is denoted by $\mathcal{K}$. Throughout the notes we will assume that $\dim \mathcal{K} < \infty$. Let the self-adjoint operator K be the Hamiltonian of the small system. The free dynamics of the small system is $\tau_\mathcal{S}^t(B) := \mathrm{e}^{\mathrm{i}tK}B\mathrm{e}^{-\mathrm{i}tK}$, $B \in \mathcal{B}(\mathcal{K})$. Thus the small system is described by the W^*-dynamical system $(\mathcal{B}(\mathcal{K}), \tau_\mathcal{S})$.

The reservoir $\mathcal{R}$ is described by a W^*-dynamical system $(\mathfrak{M}_\mathcal{R}, \tau_\mathcal{R})$. We assume that it has a unique normal stationary state $\omega_\mathcal{R}$ (not necessarily a KMS state). The generator of $\tau_\mathcal{R}$ is denoted by $\delta_\mathcal{R}$ (that is $\tau_\mathcal{R}^t = \mathrm{e}^{\delta_\mathcal{R} t}$).

The coupled system $\mathcal{S}+\mathcal{R}$ is described by the W^*-algebra $\mathfrak{M} := \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\mathcal{R}}$. The free dynamics is given by the tensor product of the dynamics of its constituents:

$$\tau_0^t(A) := \left(\tau_{\mathcal{S}}^t \otimes \tau_{\mathcal{R}}^t\right)(A), \quad A \in \mathfrak{M}.$$

We will denote by δ_0 the generator of τ_0.

Let V be a self-adjoint element of $\mathfrak{M}$. The full dynamics $t \mapsto \tau_\lambda^t := \mathrm{e}^{t\delta_\lambda}$ is defined by

$$\delta_\lambda := \delta_0 + \mathrm{i}\lambda[V, \cdot].$$

(One can consider also a more general situation, where V is only affilliated to $\mathfrak{M}$—see [DJP] for details).

5.3 Semistandard representation

Suppose that $\mathfrak{M}_{\mathcal{R}}$ is given in the standard form on the Hilbert space $\mathcal{H}_{\mathcal{R}}$. Let $\mathbf{1}_{\mathcal{R}}$ stand for the identity on $\mathcal{H}_{\mathcal{R}}$. We denote by $\mathcal{H}_{\mathcal{R}}^+$, $J_{\mathcal{R}}$, and $L_{\mathcal{R}}$ the corresponding modular cone, modular conjugation, and standard Liouvillean. Let $\Omega_{\mathcal{R}}$ be the (unique) vector representative in $\mathcal{H}_{\mathcal{R}}^+$ of the state $\omega_{\mathcal{R}}$. Clearly, $\Omega_{\mathcal{R}}$ is an eigenvector of $L_{\mathcal{R}}$. $|\Omega_{\mathcal{R}})(\Omega_{\mathcal{R}}|$ denotes projection on $\Omega_{\mathcal{R}}$.

Let us represent $\mathcal{B}(\mathcal{K})$ on $\mathcal{K}$ and take the representation of $\mathfrak{M}$ in the Hilbert space $\mathcal{K} \otimes \mathcal{H}_{\mathcal{R}}$. We will call it the semistandard representation and denote by $\pi^{\mathrm{semi}} : \mathfrak{M} \to \mathcal{B}(\mathcal{K} \otimes \mathcal{H}_{\mathcal{R}})$. (To justify its name, note that it is standard on its reservoir part, but not standard on the small system part). We will usually drop π^{semi} and treat $\mathfrak{M}$ as a subalgebra of $\mathcal{B}(\mathcal{K} \otimes \mathcal{H}_{\mathcal{R}})$.

Let us introduce the so-called free semi-Liouvillean

$$L_0^{\mathrm{semi}} = K \otimes 1 + 1 \otimes L_{\mathcal{R}}. \tag{44}$$

The full semi-Liouvillean is defined as

$$L_\lambda^{\mathrm{semi}} = L_0^{\mathrm{semi}} + \lambda V.$$

It is the generator of the distinguished unitary implementation of the dynamics τ_λ:

$$\tau_\lambda^t(A) = \mathrm{e}^{\mathrm{i}tL_\lambda^{\mathrm{semi}}} A \mathrm{e}^{-\mathrm{i}tL_\lambda^{\mathrm{semi}}}, \qquad A \in \mathfrak{M}, \tag{45}$$

with

$$\delta_\lambda = \mathrm{i}[L_\lambda^{\mathrm{semi}}, \cdot].$$

5.4 Standard representation

Let us recall how one constructs the standard representation for the algebra $\mathcal{B}(\mathcal{K})$. Recall that $\mathcal{B}^2(\mathcal{K})$ denotes the space of Hilbert-Schmidt operators on $\mathcal{K}$. Equipped with the inner product $(X|B) = \mathrm{Tr}(X^*B)$ it is a Hilbert space. Note that $\mathcal{B}(\mathcal{K})$ acts naturally on $\mathcal{B}^2(\mathcal{K})$ by the left multiplication. This defines a representation $\pi_{\mathcal{S}} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{B}^2(\mathcal{K}))$. Let $J_{\mathcal{S}} : \mathcal{B}^2(\mathcal{K}) \to \mathcal{B}^2(\mathcal{K})$ be defined by $J_{\mathcal{S}}(X) = X^*$, and let

$\mathcal{B}^2_+(\mathcal{K})$ be the set of all positive $X \in \mathcal{B}^2(\mathcal{K})$. The algebra $\pi_{\mathcal{S}}(\mathcal{B}(\mathcal{K}))$ is in the standard form on the Hilbert space $\mathcal{B}^2(\mathcal{K})$, and its modular cone and modular conjugation are $\mathcal{B}^2_+(\mathcal{K})$ and $J_{\mathcal{S}}$.

There exists a unique representation $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}})$ satisfying

$$\pi(B \otimes C) = \pi_{\mathcal{S}}(B) \otimes C. \tag{46}$$

The von Neumann algebra $\pi(\mathfrak{M})$ is in standard form on the Hilbert space $\mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}}$. The modular conjugation is $J = J_{\mathcal{S}} \otimes J_{\mathcal{R}}$. The modular cone can be obtained as

$$\mathcal{H}^+ := \{\pi(A)J\pi(A)\ (\rho \otimes \Omega_{\mathcal{R}})\ :\ A \in \mathfrak{M}\}^{\mathrm{cl}},$$

where ρ is an arbitrary nondegenerate element of $\mathcal{B}^2_+(\mathcal{K})$.

The Liouvillean of the free dynamics (the free Liouvillean) equals

$$L_0 = [K, \cdot\,] \otimes 1 + 1 \otimes L_{\mathcal{R}}. \tag{47}$$

and the Liouvillean of the full dynamics (the full Liouvillean) equals

$$L_\lambda = L_0 + \lambda(\pi(V) - J\pi(V)J). \tag{48}$$

Sometimes we will assume that the reservoir is thermal. By this we mean that $\omega_{\mathcal{R}}$ is a β-KMS state for the dynamics $\tau_{\mathcal{R}}$. Set

$$\Psi_0 := \mathrm{e}^{-\beta K/2} \otimes \Omega_{\mathcal{R}}.$$

Then the state $(\Psi_0|\pi(\cdot)\Psi_0)/\|\Psi_0\|^2$ is a (τ_0, β)-KMS state.

The Araki perturbation theory yields that

$$\Psi_0 \in \mathrm{Dom}(\mathrm{e}^{-\beta(L_0+\lambda\pi(V))/2}),$$

the vector

$$\Psi_\lambda := \mathrm{e}^{-\beta(L_0+\lambda\pi(V))/2}\Psi_0 \tag{49}$$

belongs to $\mathcal{H}^+ \cap \mathrm{Ker}L_\lambda$, and that $(\Psi_\lambda|\pi(\cdot)\Psi_\lambda)/\|\Psi_\lambda\|^2$ is a (τ_λ, β)-KMS state (see [BR2, DJP]). In particular, zero is always an eigenvalue of L_λ. Thus, in the thermal case, $(\mathfrak{M}, \tau_\lambda)$ has at least one stationary state.

6 Two applications of the Fermi Golden Rule to open quantum systems

In this section we keep all the notation and assumtions of the preceding section. We will describe two applications of the Fermi Golden Rule to the W^*-dynamical system $(\mathfrak{M}, \tau_\lambda)$ introduced in the previous section.

In the first application we compute the LSO for the generator of the dynamics δ_λ. We will call it the Davies generator and denote by M. In the literature, M appears in the context of the Dynamical Fermi Golden Rule. It is the generator of the semigroup obtained by the weak coupling limit to the reduced dynamics. This result can be used to partly justify the use of completely positive semigroups to describe dynamics of small quantum systems weakly interacting with environment [Da1, LeSp].

In the second application we consider the standard representation of the W^*-dynamical system in the Hilbert space $\mathcal{H}$ with the Liouvillean L. We will compute the LSO for $\mathrm{i}L_\lambda$. We denote it by $\mathrm{i}\Gamma$. In the literature, $\mathrm{i}\Gamma$ appears in the context of the Spectral Golden Rule. It is used to study the point spectrum of the Liouvillean L_λ. The main goal of this study is a proof of the uniqueness of a stationary state in the thermal case and of the nonexistence of a stationary state in the non-thermal state under generic conditions [DJ1, DJ2, DJP]. (See also [JP1, JP2, BFS2] for related results).

In Subsection 6.3, we will describe the result of [DJ3], which gives a relationship between the two kinds of LSO's in the thermal case.

In Subsections 6.4–6.6 we compute both LSO's explicitly. In the case of the Davies generator, these formulas are essentially contained in the literature, in the case of the LSO for the Liouvillean, they are generalizations of the analoguous formulas from [DJ2]. Both LSO's can be expressed in a number of distinct forms, each having a different advantage. In particular, as a result of our computations, we describe a simple characterization of the kernel of imaginary part of Γ, which can be used in the proof of the return to equilibrium. This characterization is a generalization of a result from [DJ2].

6.1 LSO for the reduced dynamics

It is easy to see that there exists a unique bounded linear map $\mathbb{P}$ on $\mathfrak{M}$ such that for $B \otimes C \in \mathfrak{M} \subset \mathcal{B}(\mathcal{K} \otimes \mathcal{H}_\mathcal{R})$

$$\mathbb{P}(B \otimes C) = \omega_\mathcal{R}(C) B \otimes \mathbf{1}_\mathcal{R}.$$

$\mathbb{P} \in \mathcal{B}(\mathfrak{M})$ is a projection of norm 1. (It is an example of a *conditional expectation*).

We identify $\mathcal{B}(\mathcal{K})$ with $\mathrm{Ran}\mathbb{P}$ by

$$\mathcal{B}(\mathcal{K}) \ni B \mapsto B \otimes \mathbf{1}_\mathcal{R} \in \mathrm{Ran}\mathbb{P}. \tag{50}$$

Note that $\delta_0\Big|_{\mathrm{Ran}\mathbb{P}}$ can be identified with $\mathrm{i}[K, \cdot]$.

We assume that $\omega_\mathcal{R}(V) = 0$. That implies $\mathbb{P}[V, \cdot]\mathbb{P} = 0$.

Note that Assumptions 2.1*, 2.2, 2.3* and 2.4 are satisfied for the Banach space $\mathfrak{M}$, the projection $\mathbb{P}$, the C_0^*-group of isometries $\mathrm{e}^{t\delta_0}$, and the perturbation $\mathrm{i}[V, \cdot]$.

Remark 3. One can ask whether the above defined projection $\mathbb{P}$ is given by the formula (5). Note that $\mathfrak{M}$ is not a reflexive Banach space, so it is even not clear if this formula makes sense.

Assume that $\delta_\mathcal{R}$ has no eigenvectors apart from scalar operators. Then the set of eigenvalues of δ_0 equals $\{\mathrm{i}(k - k') \ : \ k, k' \in \mathrm{sp}K\}$. One can also show that for any $e \in \mathbb{R}$, δ_0 is globally ergodic at $\mathrm{i}e \in \mathrm{i}\mathbb{R}$ (see Appendix) and the corresponding eigenprojection is given by

$$\mathbf{1}_{\mathrm{i}e}(\delta_0)(B\otimes C) = \sum_{k\in \mathrm{sp}K} \omega_{\mathcal{R}}(C)\left(\mathbf{1}_k(K)B\mathbf{1}_{k-e}(K)\right)\otimes \mathbf{1}_{\mathcal{R}}.$$

Therefore, in this case the answer to our question is positive and

$$\mathbb{P} = \sum_{e\in\mathbb{R}} \mathbf{1}_{\mathrm{i}e}(\delta_0),$$

as suggested in Subsection 2.6.

We make the following assumption:

Assumption 6.1 *Assumption 2.5 holds for* $(\mathbb{P}, \delta_0, \mathrm{i}[V,\cdot])$. *This means that there exists*

$$M := -\sum_{e\in\mathrm{sp}([K,\cdot])} \mathbf{1}_e([K,\cdot])[V,\cdot](\mathrm{i}e+0-\delta_0)^{-1}[V,\cdot]\mathbf{1}_e([K,\cdot]). \tag{51}$$

M is the LSO for $(\mathbb{P}, \delta_0, \mathrm{i}[V,\cdot])$. It will be called the Davies generator (in the Heisenberg picture).

To describe the physical interpretation of M, suppose that we are interested only in the evolution of the observables corresponding to system $\mathcal{S}$ (taking, however, into account the influence of $\mathcal{R}$). We also suppose that initially the reservoir is given by the state $\omega_{\mathcal{R}}$. Let X be a density matrix on the Hilbert space $\mathcal{K}$, such that the initial state of the system is described by the density matrix $X\otimes|\Omega_{\mathcal{R}})(\Omega_{\mathcal{R}}|$. Let $B\in\mathcal{B}(\mathcal{K})$ be an observable for the system $\mathcal{S}$, such that the measurement at the final time t is given by the operator $B\otimes\mathbf{1}_{\mathcal{R}}$. Then the expectation value of the measurement is given by

$$\mathrm{Tr}_{\mathcal{K}}\Big(X\otimes|\Omega)(\Omega|\ \tau_\lambda^t(B\otimes\mathbf{1}_{\mathcal{R}})\Big) \tag{52}$$

Obviously, (52) tensored with $\mathbf{1}_{\mathcal{R}}$ equals

$$\mathrm{Tr}_{\mathcal{K}}\left(X\mathbb{P}\tau_\lambda^t\mathbb{P}(B\otimes\mathbf{1}_{\mathcal{R}})\right).$$

Now under quite general conditions [Da1, Da2, Da3] we have

$$\lim_{\lambda\to 0} \mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2}\mathbb{P}\tau_\lambda^{t/\lambda^2}\mathbb{P} = \mathrm{e}^{tM}. \tag{53}$$

Thus M describes the reduced dynamics renormalized by $[K,\cdot]/\lambda^2$ in the limit of the weak coupling, where we rescale the time by λ^2.

Let us note the following fact:

Theorem 11. *Suppose Assumption 6.1 holds. Then M is the generator of a Markov c.p. semigroup and for any $z\in\mathbb{C}$,*

$$M(B) = \mathrm{e}^{zK}M(\mathrm{e}^{-zK}B\mathrm{e}^{zK})\mathrm{e}^{-zK}. \tag{54}$$

Proof. We know that LSO M commutes with $\mathbb{E} = \mathrm{i}[K, \cdot]$. This is equivalent to $\mathrm{e}^{z\mathbb{E}} M \mathrm{e}^{-z\mathbb{E}} = M$, which means (54).

The fact that M is a Lindblad-Kossakowski generator and annihilates $\mathbf{1}$ will follow immediately from explicit formulas given in Subsection 6.4.

If we can prove 53, then an alternative proof is possible: we immediately see that the left hand side of (53) is a Markov c.p. map for any t and λ, hence so is e^{tM}. □

6.2 LSO for the Liouvillean

Consider the the Hilbert space $\mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}}$ and the orthogonal projection

$$P := \mathbf{1}_{\mathcal{B}^2(\mathcal{K})} \otimes |\Omega_{\mathcal{R}})(\Omega_{\mathcal{R}}|.$$

We have $PL_0 = L_0 P = [K, \cdot]P$. We identify $\mathcal{B}^2(\mathcal{K})$ with $\mathrm{Ran}P$ by

$$\mathcal{B}^2(\mathcal{K}) \ni B \mapsto B \otimes \Omega_{\mathcal{R}} \in \mathrm{Ran}P. \tag{55}$$

We again assume that $\omega_{\mathcal{R}}(V) = 0$. This implies $P\pi(V)P = PJ\pi(V)JP = 0$.

Note that Assumptions 2.1, 2.2, 2.3 and 2.4 are satisfied for the Hilbert space $\mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}}$, the projection P, the strongly continuous unitary group $\mathrm{e}^{\mathrm{i}tL_0}$, and the perturbation $\mathrm{i}(\pi(Q) - J\pi(Q)J)$.

Remark 4. Assume that $L_{\mathcal{R}}$ has no eigenvectors apart from $\Omega_{\mathcal{R}}$. Then the set of eigenvalues of δ_0 equals $\{\mathrm{i}(k - k') \ : \ k, k' \in \mathrm{sp}K\}$ and

$$\mathbf{1}_e(L_0) B \otimes \Psi = (\Omega_{\mathcal{R}}|\Psi) \sum_{k \in \mathrm{sp}K} (\mathbf{1}_k(K) B \mathbf{1}_{k-e}(K)) \otimes \Omega_{\mathcal{R}}.$$

Therefore,

$$P = \sum_{e \in \mathbb{R}} \mathbf{1}_{\mathrm{i}e}(\mathrm{i}L_0)$$

is the spectral projection on the point spectrum of $\mathrm{i}L_0$, as suggested in Subsection 2.6.

Assumption 6.2 *Assumption 2.5 for $(P, \mathrm{i}L_0, \mathrm{i}(\pi(V) - J\pi(V)J))$ is satisfied. This means that there exists*

$$\begin{aligned} \mathrm{i}\Gamma := - \sum_{e \in \mathrm{sp}([K, \cdot])} & \mathbf{1}_e([K, \cdot])(\pi(V) - J\pi(V)J) \\ & \times (\mathrm{i}e + 0 - \mathrm{i}L_0)^{-1}(\pi(V) - J\pi(V)J)\mathbf{1}_e([K, \cdot]). \end{aligned}$$

$\mathrm{i}\Gamma$ is the LSO for $\big(P, \mathrm{i}L_0, \mathrm{i}(\pi(V) - J\pi(V)J)\big)$. We will call it the LSO for the Liouvillean. The operator Γ appeared in [DJ1], where it was used to give an upper bound on the point spectrum of L_λ for small nonzero λ.

Theorem 12. *Suppose that Assumption 6.2 holds. Then* $\mathrm{i}\Gamma$ *is the generator of a contractive c.p. semigroup and for any* $z \in \mathbb{C}$*,*

$$\Gamma(B) = \mathrm{e}^{zK}\Gamma(\mathrm{e}^{-zK}B\mathrm{e}^{zK})\mathrm{e}^{-zK}. \tag{56}$$

Proof. The proof of (56) is the same as that of (54). $\mathrm{e}^{t\mathrm{i}\Gamma}$ is contractive by Theorem 1. The proof of its complete positivity will be given later on (after (60)). □

6.3 Relationship between the Davies generator and the LSO for the Liouvillean in thermal case.

Obviously, as vector spaces, $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}^2(\mathcal{K})$ coincide. We are interested in the relation between $\mathrm{i}\Gamma$ and generator M. We will see that in the thermal case the two operators are similar to one another.

The following theorem was proven in [DJ3]:

Theorem 13. *Suppose that* $\omega_{\mathcal{R}}$ *is a* $(\tau_{\mathcal{R}}, \beta)$*-KMS state. Assumption 6.1 holds if and only if Assumption 6.2 holds. If these assumptions hold, then for* $B \in \mathcal{B}(\mathcal{K})$*, we have*

$$\begin{aligned} M(B) &= \mathrm{i}\Gamma(B\mathrm{e}^{-\beta K/2})\mathrm{e}^{\beta K/2} \\ &= \mathrm{e}^{\beta K/4}\mathrm{i}\Gamma(\mathrm{e}^{-\beta K/4}B\mathrm{e}^{-\beta K/4})\mathrm{e}^{\beta K/4}. \end{aligned} \tag{57}$$

Remark 5. Let $\rho := \mathrm{e}^{-\beta K}$ and $\gamma_\rho : \mathcal{B}(\mathcal{K}) \to \mathcal{B}^2(\mathcal{K})$ be the linear invertible map defined by

$$\gamma_\rho(B) := B\rho^{1/2}. \tag{58}$$

Then the first identity of Theorem 13 can be written as $M = \mathrm{i}\gamma_\rho^{-1} \circ \Gamma \circ \gamma_\rho$. Therefore, both $\mathrm{i}\Gamma$ and M have the same spectrum.

Theorem 13 follows from the explicit formulas for M and $\mathrm{i}\Gamma$ given in Subsections 6.4–6.6. It is, however, instructive to give an alternative, time dependent proof of Identity (57), which avoids calculating both LSO's. Strictly speaking, the identity will be proven for the "the dynamical Level Shift Operators" M_{dyn} and $\mathrm{i}\Gamma_{\mathrm{dyn}}$ which, however, according to the Dynamical Fermi Golden Rule, under broad conditions, coincide with the usual Level Shift Operators M and $\mathrm{i}\Gamma$.

Theorem 14. *Suppose that* $\omega_{\mathcal{R}}$ *is a* $(\tau_{\mathcal{R}}, \beta)$*-KMS state. Then the following statements are equivalent:*

1) *there exists an operator* M_{dyn} *satisfying*

$$\lim_{\lambda \to 0} \mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2}\mathbb{P}\tau_\lambda^{t/\lambda^2}\mathbb{P} = \mathrm{e}^{tM_{\mathrm{dyn}}}.$$

2) *there exists an operator* Γ_{dyn} *satisfying*

$$\lim_{\lambda\to 0} \mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2} P \mathrm{e}^{-\mathrm{i}tL_\lambda/\lambda^2} P = \mathrm{e}^{\mathrm{i}t\Gamma_{\mathrm{dyn}}}.$$

Moreover,

$$M_{\mathrm{dyn}} = \gamma_\rho^{-1} \circ \mathrm{i}\Gamma_{\mathrm{dyn}} \circ \gamma_\rho.$$

Proof. The Araki perturbation theory (see [DJP] and references therein) yields that the vector Ψ_λ, defined by (49), satisfies $\Psi_\lambda = \Psi_0 + O(\lambda)$ and $L_\lambda \Psi_\lambda = 0$. For $X, B \in \mathcal{B}(\mathcal{K}) = \mathcal{B}^2(\mathcal{K})$, using the identifications (50) and (55), we have

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{K}} \left(X^* \mathbb{P} \tau_0^{-t} \tau_\lambda^t (B \otimes \mathbf{1}_{\mathcal{R}}) \right) \\
&= \left(X \mathrm{e}^{\beta K/2} \otimes \Omega_{\mathcal{R}} \,\middle|\, (\mathrm{e}^{-\mathrm{i}tL_0} \mathrm{e}^{\mathrm{i}tL_\lambda}\ B \otimes \mathbf{1}_{\mathcal{R}}\ \mathrm{e}^{-\mathrm{i}tL_\lambda} \mathrm{e}^{\mathrm{i}tL_0})\ \mathrm{e}^{-\beta K/2} \otimes \Omega_{\mathcal{R}} \right) \\
&\overset{O(\lambda)}{=} (X \mathrm{e}^{\beta K/2} \otimes \Omega_{\mathcal{R}} \mid \mathrm{e}^{-\mathrm{i}tL_0} \mathrm{e}^{\mathrm{i}tL_\lambda}\ B \otimes \mathbf{1}_{\mathcal{R}}\ \mathrm{e}^{-\mathrm{i}tL_\lambda} \Psi_\lambda) \\
&\overset{O(\lambda)}{=} (X \mathrm{e}^{\beta K/2} \otimes \Omega_{\mathcal{R}} \mid \mathrm{e}^{-\mathrm{i}tL_0} \mathrm{e}^{\mathrm{i}tL_\lambda}\ B \otimes \mathbf{1}_{\mathcal{R}}\ \mathrm{e}^{-\beta K/2} \otimes \Omega_{\mathcal{R}}) \\
&= \left(X \mid (P \mathrm{e}^{-\mathrm{i}tL_0} \mathrm{e}^{\mathrm{i}tL_\lambda} \left(B \mathrm{e}^{-\beta K/2} \otimes \Omega_{\mathcal{R}} \right)) \, \mathrm{e}^{\beta K/2} \right)
\end{aligned}$$

uniformly for $t \geq 0$. Hence, since $\dim \mathcal{K} < \infty$,

$$\mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2} \mathbb{P} \tau_\lambda^t (B \otimes \mathbf{1}_{\mathcal{R}}) = \left(\mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2} P \mathrm{e}^{\mathrm{i}tL_\lambda} (B \mathrm{e}^{-\beta K/2} \otimes \Omega_{\mathcal{R}}) \right) \mathrm{e}^{\beta K/2} + O(\lambda)$$

uniformly for $t \geq 0$. We conclude that for a given t the limit

$$\lim_{\lambda\to 0} \mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2} P \mathrm{e}^{\mathrm{i}tL_\lambda/\lambda^2} P =: T^t$$

exists iff the limit

$$\lim_{\lambda\to 0} \mathrm{e}^{-\mathrm{i}t[K,\cdot]/\lambda^2} \mathbb{P} \tau_\lambda^{t/\lambda^2} \mathbb{P} =: \mathbb{T}^t$$

exists. Moreover, if the limits exist, then

$$\mathbb{T}^t = \gamma_\rho^{-1} \circ T^t \circ \gamma_\rho.$$

In particular, $\mathbb{T}^t$ is a semigroup iff T^t is a semigroup and their generators (M_{dyn} and $\mathrm{i}\Gamma_{\mathrm{dyn}}$ respectively) satisfy (57). □

It is perhaps interesting that Theorem 14 can be immediately generalized to some non-thermal cases.

Theorem 15. *Suppose that instead of assuming that* $\omega_{\mathcal{R}}$ *is KMS, we make the following stability assumption: We suppose that* ρ *is a nondegenerate density matrix on* $\mathcal{K}$*, and for* $|\lambda| \leq \lambda_0$ *there exists a normalized vector* $\Psi_\lambda \in \mathcal{H}$ *such that* $\Psi_\lambda = \rho^{1/2} \otimes \Omega_{\mathcal{R}} + o(\lambda^0)$ *and* $L_\lambda \Psi_\lambda = 0$*. Then all the statements of Theorem 14 remain true, with* ρ *replacing* $\mathrm{e}^{-\beta K}$.

Let us return to the thermal case. It is well known [A, FGKV] that in this case the Davies generator satisfies the Detailed Balance Condition. We will see that this fact is essentially equivalent to Relation (57).

Theorem 16. *Suppose that $\omega_{\mathcal{R}}$ is a $(\tau_{\mathcal{R}}, \beta)$-KMS state and Assumption 6.1 holds. Then the Davies generator M satisfies DBC for $\mathrm{e}^{-\beta K}$ both in the standard sense and in the sense of AFGKV.*

Proof. Recall that the operator γ_ρ defined in (58) is unitary from $\mathcal{B}^2_{(\rho)}(\mathcal{K})$ to $\mathcal{B}^2(\mathcal{K})$. Recall also that in the thermal case

$$M = \gamma_\rho^{-1} \circ \mathrm{i}\varGamma \circ \gamma_\rho.$$

Hence,

$$M^{*(\rho)} = -\gamma_\rho^{-1} \circ \mathrm{i}\varGamma^* \circ \gamma_\rho.$$

Thus,

$$\begin{aligned}\tfrac{1}{2\mathrm{i}}(M - M^{*(\rho)}) &= \gamma_\rho^{-1} \circ \tfrac{1}{2}(\varGamma + \varGamma^*) \circ \gamma_\rho \\ &= \gamma_\rho^{-1} \circ [\varDelta^{\mathrm{R}}, \cdot] \circ \gamma_\rho = [\varDelta^{\mathrm{R}}, \cdot],\end{aligned}$$

(where $\varDelta^{\mathrm{R}}$ will be defined in the next subsection). This proves DBC in the sense of AFGKV.

By Theorem 11 and the fact that ρ is proportional to $\mathrm{e}^{-\beta K}$, for any $z \in \mathbb{C}$ we have

$$M(B) = \rho^z M(\rho^{-z} B \rho^z)\rho^{-z}.$$

Therefore, by Theorem 10, the DBC in the sense of AFGKV is equivalent to the standard DBC. □

6.4 Explicit formula for the Davies generator

In this subsection we suppose that Assumption 6.1 is true and we describe an explicit formula for the Davies generator M.

We introduce the following notation for the set of allowed transition frequencies and the set of allowed transition frequencies from $k \in \mathrm{sp}K$:

$$\mathcal{F} := \{k_1 - k_2 \ : k_1, k_2 \in \mathrm{sp}K\} = \mathrm{sp}[K, \cdot], \quad \mathcal{F}_k := \{k - k_1 \ : \ k_1 \in \mathrm{sp}K\}.$$

Let $|\varOmega)$ denote the map

$$\mathbb{C} \ni z \mapsto |\varOmega)z := z\varOmega \in \mathcal{H}_{\mathcal{R}}.$$

Then $\mathbf{1}_{\mathcal{K}} \otimes |\varOmega) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathcal{R}})$. Set

$$v := V \ \mathbf{1}_{\mathcal{K}} \otimes |\varOmega)$$

Note that v belongs to $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathcal{R}})$. We also define

$$v^{k_1,k_2} := \mathbf{1}_{k_1}(K)\otimes\mathbf{1}_{\mathcal{R}}\ v\ \mathbf{1}_{k_2}(K);$$

$$\tilde{v}^p := \sum_{k\in\mathrm{sp}K} v^{k,k-p};$$

$$\begin{aligned}\Delta &= \sum_{k\in\mathrm{sp}K}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}\mathbf{1}\otimes(p+\mathrm{i}0-L_{\mathcal{R}})^{-1}v^{k-p,k}\\ &= \sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\mathbf{1}\otimes(p+\mathrm{i}0-L_{\mathcal{R}})^{-1}\tilde{v}^p.\end{aligned}$$

The real and the imaginary part of Δ are given by

$$\begin{aligned}\Delta^{\mathrm{R}} := \tfrac{1}{2}(\Delta+\Delta^*) &= \sum_{k\in\mathrm{sp}K}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}\mathbf{1}\otimes\mathcal{P}(p-L_{\mathcal{R}})^{-1}v^{k-p,k}\\ &= \sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\mathbf{1}\otimes\mathcal{P}(p-L_{\mathcal{R}})^{-1}\tilde{v}^p;\\ \Delta^{\mathrm{I}} := \tfrac{1}{2\mathrm{i}}(\Delta-\Delta^*) &= \pi\sum_{k\in\mathrm{sp}K}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}\mathbf{1}\otimes\delta(p-L_{\mathcal{R}})v^{k-p,k}\\ &= \pi\sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\mathbf{1}\otimes\delta(p-L_{\mathcal{R}})\tilde{v}^p;\end{aligned}$$

Note that $\Delta^{\mathrm{I}}\geq 0$. Below we give four explicit formulas for the Davies generator in the Heisenberg picture:

$$\begin{aligned}M(B) &= \mathrm{i}(\Delta B - B\Delta^*)\\ &\quad +2\pi\sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\ B\otimes\delta(p-L_{\mathcal{R}})\tilde{v}^p\\ &= \mathrm{i}\sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\mathbf{1}\otimes(p-\mathrm{i}0-L_{\mathcal{R}})^{-1}\left(\tilde{v}^pB - B\otimes\mathbf{1}_{\mathcal{R}}\tilde{v}^p\right)\\ &\quad -\mathrm{i}\sum_{p\in\mathcal{F}}\left(B(\tilde{v}^p)^* - (\tilde{v}^p)^*B\otimes\mathbf{1}_{\mathcal{R}}\right)\mathbf{1}\otimes(p+\mathrm{i}0-L_{\mathcal{R}})^{-1}\tilde{v}^p\\ &= \mathrm{i}[\Delta^{\mathrm{R}},B]\\ &\quad +\pi\sum_{p\in\mathcal{F}}(\tilde{v}^p)^*\mathbf{1}\otimes\delta(p-L_{\mathcal{R}})\left(B\otimes\mathbf{1}_{\mathcal{R}}\tilde{v}^p - \tilde{v}^pB\right)\\ &\quad +\pi\sum_{p\in\mathcal{F}}\left((\tilde{v}^p)^*B\otimes\mathbf{1}_{\mathcal{R}} - B(\tilde{v}^p)^*\right)\mathbf{1}\otimes\delta(p-L_{\mathcal{R}})\tilde{v}^p\\ &= \mathrm{i}\sum_{k\in\mathrm{sp}K}\sum_{p\in\mathcal{F}_k}\int_0^\infty \mathbf{1}_k(K)(\Omega|V\mathbf{1}_{k-p}(K)\tau_0^s(V)\Omega)\mathbf{1}_k(K)B\mathrm{d}s\\ &\quad -\mathrm{i}\sum_{k\in\mathrm{sp}K}\sum_{p\in\mathcal{F}_k}\int_{-\infty}^0 B\mathbf{1}_k(K)(\Omega|V\mathbf{1}_{k-p}(K)\tau_0^s(V)\Omega)\mathbf{1}_k(K)\mathrm{d}s\\ &\quad +2\pi\sum_{\substack{k,k'\in\mathrm{sp}K\\ p\in\mathcal{F}_k\cap\mathcal{F}_{k'}}}\int_{-\infty}^\infty \mathbf{1}_k(K)(\Omega|V\mathbf{1}_{k-p}(K)B\mathbf{1}_{k'-p}(K)\tau_0^s(V)\Omega)\mathbf{1}_{k'}(K)\mathrm{d}s.\end{aligned}$$

The first expression on the right has the standard form of a Lindblad-Kossakowski generator (39). The second expression can be used in a characterization of the kernel of M. In particular, it implies immediately that $\mathbf{1}_{\mathcal{K}} \in \mathrm{Ker} M$. The third expression shows the splitting of M into a reversible part and an irreversible part. The fourth expression uses uses time-dependent quantities and is analoguous to formulas appearing often in the physics literature.

6.5 Explicit formulas for LSO for the Liouvillean

In this subsection we suppose that Assumption 6.2 is true and we describe an explicit formula for $\mathrm{i}\Gamma$, the LSO for the Liouvillean.

Recall that π denotes the standard representation of $\mathfrak{M}$ and $L_{\mathcal{R}}$ is the Liouvillean of the free reservoir dynamics $\tau_{\mathcal{R}}$. Let $L_{\mathcal{R}}^0$ denote the Liouvillean of the modular dynamics for the state $\omega_{\mathcal{R}}$. The fact that $\omega_{\mathcal{R}}$ is stationary for $\tau_{\mathcal{R}}^t$ implies that the two Liouvilleans commute:

$$\mathrm{e}^{\mathrm{i}tL_{\mathcal{R}}}\mathrm{e}^{\mathrm{i}sL_{\mathcal{R}}^0} = \mathrm{e}^{\mathrm{i}sL_{\mathcal{R}}^0}\mathrm{e}^{\mathrm{i}tL_{\mathcal{R}}}, \;\; t, s \in \mathbb{R}.$$

The following identities follow from the modular theory and will be useful in our explicit formulas for Γ:

Proposition 1. *The following identities are true for $B \in \mathcal{B}^2(\mathcal{K})$:*

$$\pi(V)\; B \otimes \Omega_{\mathcal{R}} = vB,$$

$$J\pi(V)J\; B \otimes \Omega_{\mathcal{R}} = B \otimes \mathrm{e}^{L_{\mathcal{R}}^0/2} v.$$

Moreover, if $B_1, B_2 \in \mathcal{B}^2(\mathcal{K})$ and $\Phi \in \mathcal{H}_{\mathcal{R}}$, then

$$(B_1 \otimes \Phi | vB_2) = (\mathrm{e}^{L_{\mathcal{R}}^0/2} vB_1 | B_2 \otimes J_{\mathcal{R}}\Phi). \tag{59}$$

Proof. To prove the second identity we note that

$$J\; B \otimes \Omega_{\mathcal{R}} = B^* \otimes \Omega_{\mathcal{R}},$$

$$J\pi(V) B^* \otimes \Omega_{\mathcal{R}} = \mathrm{e}^{L_{\mathcal{R}}^0/2} B \otimes \pi(V)\Omega_{\mathcal{R}}.$$

To see (59), we note that it is enough to assume that $\Phi = A'\Omega_{\mathcal{R}}$, where $A' \in \pi(\mathfrak{M}_{\mathcal{R}})'$ and $\pi(\mathfrak{M}_{\mathcal{R}})'$ denotes the commutant of $\pi(\mathfrak{M}_{\mathcal{R}})$. Then

$$\begin{aligned}(B_1 \otimes \Phi | vB_2) &= (B_1 \otimes A'\Omega_{\mathcal{R}} | \pi(V) B_2 \otimes \Omega_{\mathcal{R}}) \\ &= (\pi(V) B_1 \otimes \Omega_{\mathcal{R}} | B_2 \otimes A'^* \Omega_{\mathcal{R}}) \\ &= (vB_1 | B_2 \otimes \mathrm{e}^{L_{\mathcal{R}}^0/2} J_{\mathcal{R}} A'\Omega_{\mathcal{R}}).\end{aligned}$$

□

Note that if we compare (59) with the definition of the $\star$-operation (37), and if we make the identification $\overline{\Phi} = J_{\mathcal{R}}\Phi$, then we see that (59) can be rewritten as

$$v^\star = \mathrm{e}^{L_{\mathcal{R}}^0/2} v.$$

The LSO for the Liouvillean equals

$$\begin{aligned} \mathrm{i}\Gamma(B) &= \mathrm{i}\Delta B - \mathrm{i}B\Delta^* \\ &+ 2\pi \sum_{p\in\mathcal{F}} (\tilde{v}^p)^*\ B\otimes\delta(p - L_{\mathcal{R}})\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p. \end{aligned} \tag{60}$$

Note that the term on the second line of (60) is completely positive. Therefore, (60) is in the Lindblad-Kossakowski form. Hence $\mathrm{e}^{\mathrm{i}t\Gamma}$ is a c.p. semigroup. This completes the proof of Theorem 12.

Let us split Γ into its real and imaginary part:

$$\Gamma^{\mathrm{R}} := \frac{1}{2}(\Gamma + \Gamma^*), \quad \Gamma^{\mathrm{I}} := \frac{1}{2\mathrm{i}}(\Gamma - \Gamma^*).$$

(Γ^* is defined using the natural scalar product in $\mathcal{B}^2(\mathcal{K})$). Then the real part is given by

$$\Gamma^{\mathrm{R}}(B) = [\Delta^{\mathrm{R}}, B]. \tag{61}$$

The imaginary part equals

$$\begin{aligned} \Gamma^{\mathrm{I}} &= \pi \sum_{p\in\mathcal{F}} (\tilde{v}^p)^* \mathbf{1}\otimes\delta(p - L_{\mathcal{R}}) \left(B\otimes\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p - \tilde{v}^p B \right) \\ &+ \pi \sum_{p\in\mathcal{F}} \left((\tilde{v}^p)^* B\otimes\mathrm{e}^{L_{\mathcal{R}}^0/2} - B(\tilde{v}^p)^* \right) \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})\tilde{v}^p. \end{aligned} \tag{62}$$

Another useful formula for Γ^{I} represents it as a quadratic form:

$$\begin{aligned} &\mathrm{Tr} B_1 \Gamma^{\mathrm{I}}(B_2) \\ &= \pi \sum_{p\in\mathcal{F}} \mathrm{Tr}(\tilde{v}^p B_1 - B_1\otimes\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p)^* \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})(\tilde{v}^p B_2 - B_2\otimes\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p). \end{aligned} \tag{63}$$

To see (63) we note the following identities:

$$\begin{aligned} (\tilde{v}^p)^* \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})\tilde{v}^p &= \mathrm{Tr}_{\mathcal{H}_{\mathcal{R}}} \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})\mathrm{e}^{L_{\mathcal{R}}^0}\tilde{v}^p(\tilde{v}^p)^*, \\ (\tilde{v}^p)^* B\otimes\delta(p - L_{\mathcal{R}})\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p &= \mathrm{Tr}_{\mathcal{H}_{\mathcal{R}}} \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})\mathrm{e}^{L_{\mathcal{R}}^0/2}\tilde{v}^p\ B(\tilde{v}^p)^*, \end{aligned}$$

which follow from (59).

The study of the kernel of Γ^{I} is important in applications based on the Spectral Fermi Golden Rule. The identity (63) is very convenient for this purpose. It was first discovered in the context of Pauli-Fierz systems in [DJ2].

In the thermal case (63) can be transformed into

$$\begin{aligned} \mathrm{Tr} B_1 \Gamma^{\mathrm{I}}(B_2) = \pi \sum_{p\in\mathcal{F}} &\mathrm{Tr} \mathrm{e}^{-\beta K}(\tilde{v}^p B_1 \mathrm{e}^{\beta K/2} - B_1 \mathrm{e}^{\beta K/2}\otimes\mathbf{1}_{\mathcal{R}}\ \tilde{v}^p)^* \\ &\times \mathbf{1}\otimes\delta(p - L_{\mathcal{R}})(\tilde{v}^p B_2 \mathrm{e}^{\beta K/2} - B_2 \mathrm{e}^{\beta K/2}\otimes\mathbf{1}_{\mathcal{R}}\ \tilde{v}^p). \end{aligned} \tag{64}$$

6.6 Identities using the fibered representation

Using the decomposition of the Hilbert space $\mathcal{H}_{\mathcal{R}}$ into the fibered integral given by the spectral decomposition of $L_{\mathcal{R}}$, we can rewrite (63) in an even more convenient form. To describe the fibered form of (63), we will not strive at the greatest generality. We will make the following assumptions (which are modelled after the version of the Jakšić-Pillet gluing condition considered in [DJ2]):

Assumption 6.3 *There exists a Hilbert space $\mathcal{G}$ and a linear isometry $U : \mathcal{G} \otimes L^2(\mathbb{R}) \to \mathcal{H}_{\mathcal{R}}$ such that $\mathrm{Ran}\ v, \mathrm{Ran}\ \mathrm{e}^{\beta L_{\mathcal{R}}^0/2} v \subset \mathcal{K} \otimes \mathrm{Ran} U$ and $U^* L_{\mathcal{R}} U$ is the operator of the multiplication by the variable in $\mathbb{R}$.*

We will identify $\mathrm{Ran} U$ with $L^2(\mathbb{R}) \otimes \mathcal{G}$. Note that $\Psi \in L^2(\mathbb{R}) \otimes \mathcal{G}$ can be identified with an almost everywhere defined function $\mathbb{R} \ni p \mapsto \Psi(p) \in \mathcal{G}$ such that

$$(L_{\mathcal{R}} \Psi)(p) = p\Psi(p),$$

(see e.g. [DJ2]). We can (at least formally) write $L_{\mathcal{R}}^0$ as the direct integral:

$$(L_{\mathcal{R}}^0 \Psi)(p) = L_{\mathcal{R}}^0(p)\Psi(p),$$

where $L_{\mathcal{R}}^0(p)$ are operators on $\mathcal{G}$.

Likewise, $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathcal{R}})$ can be interpreted as an almost everywhere defined function $\mathbb{R} \ni p \mapsto v(p) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{G})$ such that

$$(L_{\mathcal{R}} v \Phi)(p) = p v(p) \Phi, \quad \Phi \in \mathcal{K}.$$

Assumption 6.4 *$\mathbb{R} \ni p \mapsto v(p)$, $L_{\mathcal{R}}^0(p)$ are continuous at $p \in \mathcal{F}$, so that we can define unambiguously $v(p)$, $L_{\mathcal{R}}^0(p)$ for those values of p.*

Under the above two assumptions we can define

$$w^p := \tilde{v}^p(p) \quad p \in \mathcal{F}.$$

Then we can rewrite the formula (63) as

$$\begin{aligned} &\mathrm{Tr} B_1 \Gamma^{\mathrm{I}}(B_2) \\ &= \pi \sum_{p \in \mathcal{F}} \mathrm{Tr}(w^p B_1 - B_1 \otimes \mathrm{e}^{L_{\mathcal{R}}^0(p)/2} w^p)^* (w^p B_2 - B_2 \otimes \mathrm{e}^{L_{\mathcal{R}}^0(p)/2} w^p). \end{aligned} \tag{65}$$

(65) implies immediately

Theorem 17. *The kernel of Γ^{I} consists of $B \in \mathcal{B}^2(\mathcal{K})$ such that*

$$w^p\, B = B \otimes \mathrm{e}^{L_{\mathcal{R}}^0(p)/2}\, w^p, \quad p \in \mathcal{F}.$$

Note that Theorem 17 implies that generically $\mathrm{Ker}\Gamma^{\mathrm{I}} = \{0\}$. Therefore, for a generic open quantum system, if the Spectral Fermi Golden Rule can be applied, then the Liouvillean L_λ has no point spectrum for small nonzero λ. Therefore, for the same λ, the W^*-dynamical system $(\mathfrak{M}, \tau_\lambda)$ has no invariant normal states.

Identities (63), (65) and Theorem 17 are generalizations of similar statements from [DJ2]. In [DJ2] the reader will find their rigorous application to Pauli-Fierz systems.

If $\omega_{\mathcal{R}}$ is a $(\tau_{\mathcal{R}}, \beta)$-KMS state, we can transform (65) as follows:

$$\begin{aligned}\mathrm{Tr} B_1 \Gamma^{\mathrm{I}}(B_2) = \pi \sum_{p\in\mathcal{F}} \mathrm{Tr}\, \mathrm{e}^{-\beta K}(w^p\, B_1 \mathrm{e}^{\beta K/2} - B_1 \mathrm{e}^{\beta K/2} \otimes \mathbf{1}_{\mathcal{R}}\, w^p)^* \\ \times (w^p\, B_2 \mathrm{e}^{\beta K/2} - B_2 \mathrm{e}^{\beta K/2} \otimes \mathbf{1}_{\mathcal{R}}\, w^p).\end{aligned} \tag{66}$$

Following [DJ2], define

$$\mathcal{N} := \{C \ :\ w^p\, C = C \otimes \mathbf{1}_{\mathcal{R}}\, w^p,\ \ p \in \mathcal{F}\}. \tag{67}$$

Repeating the arguments of [DJ2] we get

Theorem 18. 1) *$\mathcal{N}$ is a $*$-algebra invariant wrt* $\mathrm{e}^{\mathrm{i}tK} \cdot \mathrm{e}^{-\mathrm{i}tK}$ *and containing* $\mathbb{C}1$.
2) *The kernel of Γ^{I} consists of* $\mathrm{e}^{-\beta K/2} C$ *with* $C \in \mathcal{N}$.

Theorem 18 implies that in a thermal case, generically, $\mathrm{Ker}\Gamma^{\mathrm{I}} = \{0\}$. Therefore, if the Spectral Fermi Golden Rule can be applied, for a generic open quantum system, for small nonzero λ, the Liouvillean L_λ has no point spectrum except for a nondegenerate eigenvalue at zero. Therefore, for the same λ, the W^*-dynamical system $(\mathfrak{M}, \tau_\lambda)$ has a unique stationary normal state.

Again, Identity (66) and Theorem 18 are generalizations of similar statements from [DJ2], where they were used to study the return to equilibrium for thermal Pauli-Fierz systems.

7 Fermi Golden Rule for a composite reservoir

In this section we describe a small quantum system interacting with several reservoirs. We will assume that the reservoirs $\mathcal{R}_1, \dots, \mathcal{R}_n$ do not interact directly—they interact with one another only through the small system $\mathcal{S}$. We will compute both kinds of the LSO for the composite system. We will see that it is equal to the sum of the LSO's corresponding to the interaction of $\mathcal{S}$ with a single reservoir $\mathcal{R}_i$.

Our presentation is divided into 3 subsections. The first uses the framework of Section 2, the second—that of Section 5 and the third—that of Section 6.

7.1 LSO for a sum of perturbations

Let $\mathcal{X}$ be a Banach space. Let $\mathbb{P}^1, \dots, \mathbb{P}^n$ be projections of norm 1 on $\mathcal{X}$ such that $\mathbb{P}^i\mathbb{P}^j = \mathbb{P}^i\mathbb{P}^j$. Let $\mathbb{L}_0$ be the generator of a group of isometries such that $\mathbb{L}_0\mathbb{P}^i =$

$\mathbb{P}^i\mathbb{L}_0$, $i = 1,\dots,n$. Let $\mathbb{Q}^i$ be operators such that $\mathrm{Ran}\mathbb{P}^i \subset \mathrm{Dom}\mathbb{Q}^i$ and $\mathbb{Q}^i\mathbb{P}^j = \mathbb{P}^j\mathbb{Q}^i$, $i \neq j$. Set

$$\mathbb{Q} := \sum_{j=1}^n \mathbb{Q}^j, \quad \mathbb{P} := \prod_{j=1}^n \mathbb{P}^j, \quad \mathcal{X}_j := \mathrm{Ran}\prod_{i\neq j}\mathbb{P}^i.$$

Clearly, $\mathcal{X}_j$ is left invariant by $\mathbb{L}_0, \mathbb{P}^j, \mathbb{Q}^j$. Therefore, these operators can be restricted to $\mathcal{X}_j$. We set

$$\mathbb{L}_{0,j} := \mathbb{L}_0\Big|_{\mathcal{X}_j}, \quad \mathbb{P}_j := \mathbb{P}^j\Big|_{\mathcal{X}_j} = \mathbb{P}\Big|_{\mathcal{X}_j}, \quad \mathbb{Q}_j := \mathbb{Q}^j\Big|_{\mathcal{X}_j}.$$

Clearly,

$$\mathrm{Ran}\mathbb{P} = \mathrm{Ran}\mathbb{P}_j \quad \mathbb{L}_0\Big|_{\mathrm{Ran}\mathbb{P}} = \mathbb{L}_{0,j}\Big|_{\mathrm{Ran}\mathbb{P}_j}.$$

We set $\mathbb{E} := \mathbb{L}_0\Big|_{\mathrm{Ran}\mathbb{P}}$.

Theorem 19. *Suppose that* $\mathbb{P}^j\mathbb{Q}^j\mathbb{P}^j = 0$*,* $j = 1,\dots,n$*. Then:*

1) $\mathbb{P}\mathbb{Q}\mathbb{P} = 0$, $\mathbb{P}_j\mathbb{Q}_j\mathbb{P}_j = 0$, $j = 1,\dots,n$.
2) *Suppose in addition that the LSO's for* $(\mathbb{P}_i, \mathbb{L}_{0,i}, \mathbb{Q}_i)$*, denoted* M_i*, exist. Then the LSO for* $(\mathbb{P}, \mathbb{L}_0, \mathbb{Q})$*, denoted* M*, exists as well and*

$$M = \sum_{i=1}^n M_i.$$

Proof. Set $\mathbb{J}_j := \prod_{i\neq j}\mathbb{P}^i$.

1) It is obvious that $\mathbb{P}^i\mathbb{Q}^i\mathbb{P}^i = 0$ implies $\mathbb{P}_i\mathbb{Q}_i\mathbb{P}_i = 0$.

2) We have

$$\begin{aligned} M &= \sum_{i,j=1}^n \sum_{\mathrm{ie}\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{ie}}(\mathbb{E})\mathbb{Q}^i(\mathrm{ie}+0-\mathbb{L}_0)^{-1}\mathbb{Q}^j\mathbf{1}_{\mathrm{ie}}(\mathbb{E}), \\ M_j &= \sum_{\mathrm{ie}\in\mathrm{sp}\mathbb{E}} \mathbf{1}_{\mathrm{ie}}(\mathbb{E})\mathbb{Q}_j(\mathrm{ie}+0-\mathbb{L}_{0,j})^{-1}\mathbb{Q}_j\mathbf{1}_{\mathrm{ie}}(\mathbb{E}). \end{aligned}.$$

For $i \neq j$,

$$\mathbb{P}\mathbb{Q}^i(\mathrm{ie}+0-\mathbb{L}_0)^{-1}\mathbb{Q}^j\mathbb{P} = \mathbb{P}\mathbb{Q}^i\mathbb{J}_j(\mathrm{ie}+0-\mathbb{L}_0)^{-1}\mathbb{Q}^j\mathbb{P} = 0,$$

since $\mathbb{P}\mathbb{Q}^i\mathbb{J}_j = \mathbb{P}\mathbb{P}^i\mathbb{Q}^i\mathbb{P}^i\mathbb{J}_j = 0$. Clearly,

$$\mathbb{P}\mathbb{Q}^i(\mathrm{ie}+0-\mathbb{L}_0)^{-1}\mathbb{Q}^i\mathbb{P} = \mathbb{P}\mathbb{Q}_i(\mathrm{ie}+0-\mathbb{L}_{0,i})^{-1}\mathbb{Q}_i\mathbb{P}.$$

□

7.2 Multiple reservoirs

Suppose that $(\mathfrak{M}_{\mathcal{R}_1}, \tau_{\mathcal{R}_1})$,..., $(\mathfrak{M}_{\mathcal{R}_n}, \tau_{\mathcal{R}_n})$ are W^*-dynamical systems with $\tau_{\mathcal{R}_i}^t = \mathrm{e}^{t\delta_{\mathcal{R}_i}}$. Let $\mathbf{1}_{\mathcal{R}_i}$ denote the identity on $\mathcal{H}_{\mathcal{R}_i}$. Suppose that $\mathfrak{M}_{\mathcal{R}_i}$ have a standard representation in Hilbert spaces $\mathcal{H}_{\mathcal{R}_i}$ with the modular conjugations $J_{\mathcal{R}_i}$. Let $L_{\mathcal{R}_i}$ be the Liouvillean of the dynamics $\tau_{\mathcal{R}_i}$.

Let $(\mathcal{B}(\mathcal{K}), \tau_{\mathcal{S}})$ describe the small quantum system, with $\tau_{\mathcal{S}}^t := \mathrm{e}^{\mathrm{i}t[K,\cdot]}$, as in Section 5. Define the free systems $(\mathfrak{M}_i, \tau_{0,i})$ where

$$\begin{aligned}
\mathfrak{M}_i &:= \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\mathcal{R}_i}, \\
\mathcal{H}_i &:= \mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}_i}, \\
J_i &:= J_{\mathcal{S}} \otimes J_{\mathcal{R}_i}, \\
\tau_{0,i}^t &:= \tau_{\mathcal{S}}^t \otimes \tau_{\mathcal{R}_i}^t = \mathrm{e}^{t\delta_{0,\lambda}}, \\
\delta_{0,i} &= \mathrm{i}[K, \cdot] + \delta_{\mathcal{R}_i}, \\
L_{0,i} &= [K, \cdot] + L_{\mathcal{R}_i}.
\end{aligned}$$

Let π_i be the standard representation of $\mathfrak{M}_i$ in $\mathcal{H}_i$ and J_i the corresponding conjugations.

Let $V_i \in \mathfrak{M}_i$ and define the perturbed systems $(\mathfrak{M}_i, \tau_{\lambda,i})$ where $\tau_{\lambda,i}^t := \mathrm{e}^{t\delta_{\lambda,i}}$ and

$$\begin{aligned}
\delta_{\lambda,i} &:= \delta_{0,i} + \mathrm{i}\lambda[V_i, \cdot], \\
L_{\lambda,i} &= L_{0,i} + \lambda(\pi_i(V_i) - J_i\pi_i(V_i)J_i).
\end{aligned}$$

Likewise, consider the composite reservoir $\mathcal{R}$ described by the W^*-dynamical system $(\mathfrak{M}_{\mathcal{R}}, \tau_{\mathcal{R}})$, where

$$\begin{aligned}
\mathfrak{M}_{\mathcal{R}} &:= \mathfrak{M}_{\mathcal{R}_1} \otimes \cdots \otimes \mathfrak{M}_{\mathcal{R}_n}, \\
\mathcal{H}_{\mathcal{R}} &:= \mathcal{H}_{\mathcal{R}_1} \otimes \cdots \otimes \mathcal{H}_{\mathcal{R}_n}, \\
J_{\mathcal{R}} &:= J_{\mathcal{R}_1} \otimes \cdots \otimes J_{\mathcal{R}_n}, \\
\tau_{\mathcal{R}}^t &:= \tau_{\mathcal{R}_1}^t \otimes \cdots \otimes \tau_{\mathcal{R}_n}^t = \mathrm{e}^{t\delta_{\mathcal{R}}}, \\
\delta_{\mathcal{R}} &:= \delta_{\mathcal{R}_1} + \cdots + \delta_{\mathcal{R}_n}, \\
L_{\mathcal{R}} &= L_{\mathcal{R}_1} + \cdots + L_{\mathcal{R}_n}.
\end{aligned}$$

Define the free composite system $(\mathfrak{M}, \tau_0)$ where

$$\begin{aligned}
\mathfrak{M} &:= \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\mathcal{R}}, \\
\mathcal{H} &:= \mathcal{B}^2(\mathcal{K}) \otimes \mathcal{H}_{\mathcal{R}}, \\
J &= J_{\mathcal{S}} \otimes J_{\mathcal{R}}, \\
\tau_0^t &:= \tau_{\mathcal{S}}^t \otimes \tau_{\mathcal{R}}^t = \mathrm{e}^{t\delta_0}, \\
\delta_0 &= \mathrm{i}[K, \cdot] + \delta_{\mathcal{R}}, \\
L_0 &= [K, \cdot] + L_{\mathcal{R}}.
\end{aligned}$$

Let π be the standard representation of $\mathfrak{M}$ in $\mathcal{H}$.

Set $V = V_1 + \cdots + V_n$. The perturbed composite system describing the small system $\mathcal{S}$ interacting with the composite reservoir $\mathcal{R}$ is $(\mathfrak{M}, \tau_\lambda)$, where $\tau_\lambda^t := \mathrm{e}^{t\delta_\lambda}$,

$$\begin{aligned}
\delta_\lambda &:= \delta_0 + \mathrm{i}\lambda[V, \cdot], \\
L_\lambda &:= L_0 + \lambda(\pi(V) - J\pi(V)J).
\end{aligned}$$

7.3 LSO for the reduced dynamics in the case of a composite reservoir

Suppose that the reservoir dynamics $\tau_{\mathcal{R}_i}$ have stationary states $\omega_{\mathcal{R}_i}$. We introduce a projection of norm one in $\mathfrak{M}$, denoted $\mathbb{P}^i$, such that

$$\mathbb{P}^i(B \otimes A_1 \otimes, \cdots \otimes A_i \otimes \cdots \otimes A_n) = \omega_{\mathcal{R}_i}(A_i) B \otimes A_1 \otimes \cdots \otimes \mathbf{1}_{\mathcal{R}_i} \otimes \cdots \otimes A_n.$$

Set $\mathbb{P} := \prod_{i=1}^n \mathbb{P}^i$. The projection $\mathbb{P}^i$ restricted to $\mathfrak{M}_i$ (which can be viewed as a subalgebra of $\mathfrak{M}$) is denoted by $\mathbb{P}_i$. Explicitly,

$$\mathbb{P}_i(B \otimes A_i) = \omega_{\mathcal{R}_i}(A_i) B \otimes \mathbf{1}_{\mathcal{R}_i}.$$

Assume that $\omega_{\mathcal{R}_i}(V_i) = 0$ for $i = 1, \ldots, n$.

Note that we can apply the formalism of Subsection 7.1, where the Banach space is $\mathcal{X}$ is $\mathfrak{M}$, the projections $\mathbb{P}^i$ are $\mathbb{P}^i$, the generator of an isometric dynamics $\mathbb{L}_0$ is δ_0 and the perturbations $\mathbb{Q}^i$ are $\mathrm{i}[V_i, \cdot]$. Clearly, $\mathcal{X}_i$ can be identified with $\mathfrak{M}_i$ and $\mathrm{Ran}\mathbb{P}$ with $\mathcal{B}(\mathcal{K})$.

We obtain the LSO for $(\mathbb{P}, \delta_0, \mathrm{i}[V, \cdot])$, denoted M, and the LSO's for $(\mathbb{P}_i, \delta_{0,i}, \mathrm{i}[V_i, \cdot])$, denoted M_i. By Theorem 19, we have

$$M = \sum_{i=1}^n M_i,$$

7.4 LSO for the Liovillean in the case of a composite reservoir

Let $\Omega_{\mathcal{R}_i}$ be the standard vector representative of $\omega_{\mathcal{R}_i}$. We define the orthogonal projection in $\mathcal{B}(\mathcal{H})$

$$P^i := \mathbf{1}_{\mathcal{B}^2(\mathcal{K})} \otimes \mathbf{1}_{\mathcal{R}_1} \otimes \cdots \otimes |\Omega_{\mathcal{R}_i})(\Omega_{\mathcal{R}_i}| \otimes \cdots \otimes \mathbf{1}_{\mathcal{R}_n}.$$

The projection P^i restricted to $\mathcal{H}_i$ is denoted by P_i and equals

$$P_i = 1_{\mathcal{B}^2(\mathcal{K})} \otimes |\Omega_{\mathcal{R}_i})(\Omega_{\mathcal{R}_i}|.$$

Set $P = \prod_{i=1}^n P^i$.

We can apply the formalism of Subsection 7.1, where the Banach space is $\mathcal{X}$ is $\mathcal{H}$, the projections $\mathbb{P}^i$ are P^i, the generator of an isometric dynamics $\mathbb{L}_0$ is $\mathrm{i}L_0$ and the perturbations $\mathbb{Q}^i$ are $\mathrm{i}(V_i - J_i V_i J_i)$. Clearly, $\mathcal{X}_i$ can be identified with $\mathcal{H}_i$ and $\mathrm{Ran}P$ with $\mathcal{B}^2(\mathcal{K})$ (which as a vector space coincides with $\mathcal{B}(\mathcal{K})$).

We obtain the LSO for $(P, \mathrm{i}L_0, \mathrm{i}(V - JVJ))$, denoted $\mathrm{i}\Gamma$, and the LSO for $(P_i, \mathrm{i}L_{0,i}, \mathrm{i}(V_i - J_i V_i J_i))$, denoted $\mathrm{i}\Gamma_i$. By Theorem 19, we have

$$\mathrm{i}\Gamma = \sum_{i=1}^n \mathrm{i}\Gamma_i.$$

The following theorem follows from obvious properties of negative operators:

Theorem 20. *Suppose that for some* $i \neq j$, $\dim \mathrm{Ker}\Gamma_i^{\mathrm{I}} = \dim \mathrm{Ker}\Gamma_j^{\mathrm{I}} = 1$ *and* $\mathrm{Ker}\Gamma_i^{\mathrm{I}} \neq \mathrm{Ker}\Gamma_j^{\mathrm{I}}$. *Then* $\mathrm{Ker}\Gamma = \{0\}$.

Corollary 1. *Suppose that for some* $i \neq j$, *the states* $\omega_{\mathcal{R}_i}$ *and* $\omega_{\mathcal{R}_j}$ *are* $(\tau_{\mathcal{R}_i}, \beta_i)$ *and* $(\tau_{\mathcal{R}_j}, \beta_j)$*-KMS. Let* $\mathcal{N}_i$ *and* $\mathcal{N}_j$ *be the corresponding* $*$*-algebras defined as in (67). Suppose that* $\beta_i \neq \beta_j$ *and* $\mathcal{N}_i' = \mathcal{N}_j' = \mathbb{C}1$. *Then* $\mathrm{Ker}\Gamma = \{0\}$.

If we can apply the Spectral Fermi Golden Rule, then under the assumptions of 1, for sufficiently small nonzero λ, L_λ has no point spectrum. Consequently, for the same λ, the system $(\mathfrak{M}_\lambda, \tau_\lambda)$, has no invariant normal states.

A Appendix – one-parameter semigroups

In this section we would like to discuss some concepts related to one-parameter semigroups of operators in Banach spaces, which are used in our lectures. Even though the material that we present is quite standard, we could not find a reference that presents all of it in a convenient way. Most of it can be found in [BR1]. Less pedantic readers may skip this appendix altogether.

Let $\mathcal{X}$ be a Banach space. Recall that $[0, \infty[\ni t \mapsto U(t) \in B(\mathcal{X})$ is called a 1-parameter semigroup iff $U(0) = 1$ and $U(t_1)U(t_2) = U(t_1 + t_2)$. If $[0, \infty[$ is replaced with $\mathbb{R}$, then we speak about a one-parameter group instead of a one-parameter semigroup.

We say that $U(t)$ is a strogly continuous semigroup (or a C_0-semigroup) iff for any $\Phi \in \mathcal{X}$, $t \mapsto U(t)\Phi$ is continuous. Every C_0-semigroup possesses its generator, that is the operator A defined as follows:

$$\Phi \in \mathrm{Dom}A \ \Leftrightarrow\ \lim_{t \searrow 0} t^{-1}(U(t) - \mathbf{1})\Phi =: A\Phi \ \text{ exists.}$$

The generator is always closed and densely defined and uniquely determines the semigroup. We write $U(t) = \mathrm{e}^{tA}$.

Recall also the following well known characterization of contractive semigroups:

Theorem 21. *The following conditions are equivalent:*
1) e^{tA} *is contractive for all* $t \geq 0$.
2) A *is densely defined,* $\mathrm{sp}A \subset \{z \in \mathbb{C} \ : \ \mathrm{Re}z \leq 0\}$ *and* $\|(z-A)^{-1}\| \leq (\mathrm{Re}z)^{-1}$ *for* $\mathrm{Re}z > 0$.
3) (i) A *is densely defined and for some* z_+ *with* $\mathrm{Re}z_+ > 0$, $z_+ \not\in \mathrm{sp}A$,
(ii) A *is dissipative, that is for any* $\Phi \in \mathrm{Dom}A$ *there exists* $\xi \in \mathcal{X}^*$ *with* $(\xi|\Phi) = \|\Phi\|$ *and* $(\xi|A\Phi) \leq 0$.

Moreover, if A *is bounded, then we can omit* (i) *in* 3).

There exists an obvious corollary of the above theorem for groups of isometries:

Theorem 22. *The following conditions are equivalent:*
1) e^{tA} *is isometric for all* $t \in \mathbb{R}$.
2) A *is densely defined,* $\mathrm{sp}A \subset \mathrm{i}\mathbb{R}$ *and* $\|(z-A)^{-1}\| \leq |\mathrm{Re}z|^{-1}$ *for* $\mathrm{Re}z \neq 0$.
3) (i) A *is densely defined and for some* $z_\pm$ *with* $\pm\mathrm{Re}z_\pm > 0$, $z_\pm \not\in \mathrm{sp}A$,
(ii) A *is conservative, that is for any* $\Phi \in \mathrm{Dom}A$ *there exists* $\xi \in \mathcal{X}^*$ *with* $(\xi|\Phi) = \|\Phi\|$ *and* $\mathrm{Re}(\xi|A\Phi) = 0$.

Morover, if A *is bounded, then we can omit* (i) *in* (3).

Not all semigroups considered in our lectures are C_0-semigroups. An important role in our lectures (and in applications to statistical physics) is played by somewhat less known C_0^*-semigroups. In order to discuss them, first we need to say a few words about dual Banach spaces.

Let $\mathcal{X}^*$ denote the Banach space dual to $\mathcal{X}$ (the space of continuous linear functionals on $\mathcal{X}$). We will use the sesquilinear duality between $\mathcal{X}^*$ and $\mathcal{X}$: the form $(\xi|\Phi)$ will be antilinear in $\xi \in \mathcal{X}^*$ and linear in $\Phi \in \mathcal{X}$.

The so-called weak$*$ (w$*$) topology on $\mathcal{X}^*$ is defined by the seminorms $|(\cdot|\Phi)|$, where $\Phi \in \mathcal{X}$.

The space of w$*$ continuous linear operators on $\mathcal{X}^*$ will be denoted by $\mathcal{B}_{w*}(\mathcal{X}^*)$. Note that $\mathcal{B}_{w*}(\mathcal{X}^*) \subset \mathcal{B}(\mathcal{X}^*)$. If $A \in \mathcal{B}(\mathcal{X})$, and A^* is its adjoint, then $A^* \in B_{w*}(\mathcal{X}^*)$. Conversely, if $B \in \mathcal{B}_{w*}(\mathcal{X}^*)$, then there exists a unique $A \in \mathcal{B}(\mathcal{X})$, sometimes called the preadjoint of B, such that $B = A^*$. Likewise, if A is closed and densely defined on $\mathcal{X}$, then A^* is w$*$ closed and w$*$ densely defined on $\mathcal{X}^*$.

We say that $[0,\infty[\ni t \mapsto W(t) \in \mathcal{B}_{w*}(\mathcal{X}^*)$ is a w$*$ continuous semigroup (or a C_0^*-semigroup) iff $t \ni W(t)\xi$ is w$*$ continuous for any $\xi \in \mathcal{X}^*$. Note that if $U(t)$ is a C_0 -semigroup, then $U(t)^*$ is a C_0^*-semigroup. Conversely, if $W(t)$ is a C_0^*-semigroup on $\mathcal{X}^*$, then there exists a unique C_0-semigroup $U(t)$ on $\mathcal{X}$ such that $W(t) = U(t)^*$.

Every C_0^*-semigroup $W(t)$ possesses its generator, that is the operator B defined as follows:

$$\xi \in \mathrm{Dom}B \ \Leftrightarrow \ \mathrm{w}*-\lim_{t\searrow 0} t^{-1}(W(t)-\mathbf{1})\xi =: B\xi \ \text{ exists.}$$

The generator is always w$*$-closed and w$*$-densely defined and uniquely determines the semigroup. We write $W(t) = \mathrm{e}^{tB}$. We have

$$(\mathrm{e}^{tA})^* = \mathrm{e}^{tA^*}.$$

On a reflexive Banach space, e.g. on a Hilbert space, the concepts of a C_0- and C_0^*-semigroup coincide. Unfortunately, W^*-algebras are usually not reflexive. They are, however, dual Banach spaces: they are dual to the space of normal functionals. In the context of W^*-algebras the w$*$-topology is usually called the σ-weak or ultraweak topology.

Groups of automorphisms of W^*-algebras are rarely C_0-groups. To see this note that if H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then

$$t \mapsto \mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH} \tag{68}$$

is always a C_0^*-group on $\mathcal{B}(\mathcal{H})$. It is a C_0-group (and even a norm continuous group) iff H is bounded, which is usually a very severe restriction.

In the context of W^*-algebras, C_0^* groups are usually called (pointwise) σ-weakly continuous groups. C_0^*-groups of $*$-automorphisms are often called W^*-dynamics.

So far, all the material that we recalled can be found e.g. in [BR1]. Now we would like to discuss how to define the spectral projection onto a (not necessarily isolated) eigenvalue of a generator of contractive semigroup. We will see that a fully satisfactory answer is available for purely imaginary eigenvalues in the case of a reflexive Banach spaces. For non-reflexive Banach spaces the situation is much more complicated. Our discussion is adapted from [Zs] and partly from [Da3].

Let A be the generator of a contractive C_0-semigroup on $\mathcal{X}$ and $e \in \mathbb{R}$. Following [Zs], we say that A is ergodic at $\mathrm{i}e$ iff

$$\mathbf{1}_{\mathrm{i}e}(A) := \lim_{\xi \searrow 0} \xi(\xi + \mathrm{i}e - A)^{-1} \tag{69}$$

exists.

Let B be the generator of a contractive C_0^*-semigroup on $\mathcal{X}^*$ and $e \in \mathbb{R}$. Following [Zs], we say that B is globally ergodic at $\mathrm{i}e$ iff

$$\mathbf{1}_{\mathrm{i}e}(B) := \mathrm{w}* - \lim_{\xi \searrow 0} \xi(\xi + \mathrm{i}e - \mathrm{B})^{-1} \tag{70}$$

exists and is w$*$-continuous.

As we will see from the theorem below, (69) and (70) can be called spectral projections onto the eigenvalue $\mathrm{i}e$.

Theorem 23. *Let A, B and $e \in \mathbb{R}$ be as above.*

1) *If A is ergodic at* $\mathrm{i}e$*, then $1_{\mathrm{i}e}(A)$ is a projection of norm 1 such that*

$$\mathrm{Ran}\mathbf{1}_{\mathrm{i}e}(A) = \mathrm{Ker}(A - \mathrm{i}e), \quad \mathrm{Ker}\mathbf{1}_{\mathrm{i}e}(A) = (\mathrm{Ran}\,(A - \mathrm{i}e))^{\mathrm{cl}}.$$

2) *On a reflexive Banach space, we have always the ergodic property for all generators of contractive semigroups and all* $\mathrm{i}e \in \mathrm{i}\mathbb{R}$*.*
3) *If* B *is globally ergodic at* $\mathrm{i}e$*, then* $1_{\mathrm{i}e}(B)$ *is a* $w*$*-continuous projection of norm* 1 *such that*

$$\mathrm{Ran}\mathbf{1}_{\mathrm{i}e}(B) = \mathrm{Ker}(B - \mathrm{i}e), \quad \mathrm{Ker}\mathbf{1}_{\mathrm{i}e}(B) = (\mathrm{Ran}\,(A - \mathrm{i}e))^{w*\mathrm{cl}}.$$

4) A *is ergodic at* $\mathrm{i}e$ *iff* A^* *is globally ergodic at* $-\mathrm{i}e$ *and*

$$\mathbf{1}_{\mathrm{i}e}(A)^* = \mathbf{1}_{-\mathrm{i}e}(A^*).$$

1) and 2) are proven in [Da3] Theorem 5.1 and Corollary 5.2. 3) and 4) can be proven by adapting the arguments of [Zs] Theorem 3.4 and Corollary 3.5.

As an ilustration of the above concepts consider the W^*-dynamics (68). Clearly, it is a group of isometries and the spectrum of its generator $\mathrm{i}[H, \cdot]$ is contained in $\mathrm{i}\mathbb{R}$. If H possesses only point spectrum, then $\mathrm{i}[H, \cdot]$ is globally ergodic for any $\mathrm{i}e \in \mathrm{i}\mathbb{R}$. In fact, we have the following formula for

$$\mathbf{1}_{\mathrm{i}e}(\mathrm{i}[H, \cdot])(C) = \sum_{x \in \mathbb{R}} \mathbf{1}_{x+e}(H) C \mathbf{1}_x(H).$$

Note that $\mathrm{i}[H, \cdot]$ always possesses an eigenvalue 0 and the corresponding eigenvectors are all operators commuting with H. It is never globally ergodic at 0 if H has some continuous spectrum.

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