Nonrelativistic Limit of the Abstract Chiral Quark Soliton Model and Confining Effects

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Abstract

We consider an abstract version of a Dirac operator which describes a Hamiltonian of the chiral quark soliton model (CQSM) in nuclear physics. The mass term of the Hamiltonian describing the concrete CQSM is a matrix-valued function. Hence, the abstract CQSM Hamiltonian has structures different from the standard Dirac operator. We discuss the nonrelativistic limit of the abstract CQSM Hamiltonian and show that a binding potential appears as an effective potential. As an application of this abstract result, we derive the nonrelativistic limit of the concrete CQSM.

Keywords: Dirac operator, nonrelativistic limit, chiral quark soliton model.
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1 Introduction

In the theory of the quark, physicists use a Dirac operator with a matrix-valued function mass term as an approximative theory. This model is called the chiral quark soliton model (CQSM) [5]. From this model, many computational results which are intresting from physical view points are derived ([5] and references therein). But, as far as we know, only a few mathematically rigorous analyses have been made on the CQSM [2]; we can expect that we can still find various important results on this model.

In this paper, we consider the nonrelativisitc limit of the abstract CQSM which is an abstract version of the original CQSM. Since the mass term in the CQSM is a matrix-valued function, it is very interesting to investigate whether an effective potential appears or not. In discussing the nonrelativistic limit of the standard Dirac operators, we have to renormalize the mass energy term in order to avoid divergence difficulties. But, since the mass term in the CQSM is a matrix-valued function, we can not expect that the standard renormalization is enough to treat our abstract CQSM. Hence it is also interesting to find a suitable renormalization for our model.

The present paper is organized as follows. In Sec. 2, we present some facts on self-adjoint operators in a $\mathbb{Z}_2$-graded Hilbert space. In Sec. 3, we introduce the abstract CQSM and state the main result with respect to the norelativistic limit of the abstract CQSM. We see that an effective potential which describes some kind of a binding of quark appears. In Sec. 4, we give a proof of the main result. In Sec. 5, we discuss a typical example of the abstract CQSM.

2 Preliminaries

2.1 Self-adjoint operators on a $\mathbb{Z}_2$-graded Hilbert space

In this subsection, we recall some basic properties on self-adjoint operators in a $\mathbb{Z}_2$-graded Hilbert space. For details, see [3].

Let $\mathcal{X}$ be a separable Hilbert space and Γ be a linear operator in $\mathcal{X}$. We say that the pair $(\mathcal{X}, \Gamma)$ is a $\mathbb{Z}_2$-*graded Hilbert space* if Γ is self-adjoint and unitary. The linear operator Γ is called the *grading operator in* $\mathcal{X}$. Throughout this paper, the symbol Γ always stands for a grading operator in a suitable Hilbert space. We set $P_\pm := (1 \pm \Gamma)/2$. Then we can easily check that $P_\pm$ are orthogonal projections such that $P_+ + P_- = 1$ and $P_+P_- = 0 = P_-P_+$. Hence $\mathcal{X}$ has the following $\mathbb{Z}_2$-graded structure:

$$\mathcal{X} = \operatorname{ran}(P_+) \oplus \operatorname{ran}(P_-),$$

where $\operatorname{ran}(P_\pm)$ denote the range of $P_\pm$ respectively. Let A be a linear operator with $\Gamma \operatorname{dom}(A) \subset \operatorname{dom}(A)$ ($\operatorname{dom}(A)$ denotes the domain of A). If A satisfies $\Gamma A \Gamma = A$, the linear operator A is called an *even operator*. On the other hand if $\Gamma A \Gamma = -A$, the linear operator A is called an *odd operator.* The following lemmata are fundamental:

Lemma 2.1 *Let A be an odd self-adjoint operator in a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{X}, \Gamma)$ and let f be a continuous function on $\mathbb{R}$. Then the following hold:*

(i) *If f is an even function i.e., $f(-x) = f(x)$ ($x \in \mathbb{R}$), then the linear operator $f(A)$ which is given by the functional calculus is even.*

(ii) *If f is an odd function i.e., $f(-x) = -f(x)$ ($x \in \mathbb{R}$), then the linear operator $f(A)$ is odd.*

Lemma 2.2 *Let A be an even self-adjoint operator in a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{X}, \Gamma)$ and let f be a continuous function on $\mathbb{R}$. Then $f(A)$ is even.*

2.2 Relative bounded operators

Let A and B be linear operators on a Hilbert space. If $\mathrm{dom}(A) \subset \mathrm{dom}(B)$ and there exist constants $r_1(B;A)$ and $r_2(B;A)$ such that for all $f \in \mathrm{dom}(A)$,

$$\|Bf\| \le r_1(B;A)\|Af\| + r_2(B;A)\|f\|,$$

we say that B *is A-relative bounded.* For notational simplicity, we write $B \prec A$.

Lemma 2.3 *Let B be a self-adjoint operator. Suppose that $B^n \prec A$. Then $B^j \prec A$ $(j = 1, \cdots, n-1)$. Moreover we can take $r_1(B^j;A) = r_1(B^n;A)$ $(j = 1, \cdots, n-1)$.*

Proof. By the functional calculus, we have

$$\|B^{n-1}f\| \le \|B^n f\| + \|f\|$$

for all $f \in \mathrm{dom}(A)$. Hence

$$\|B^{n-1}f\| \le r_1(B^n;A)\|Af\| + (1 + r_2(B^n;A))\|f\| \tag{1}$$

and we conclude that $B^{n-1} \prec A$. Repeating this argument, we get $B^j \prec A$ $(j = 1, \cdots, n-1)$. Note that, by (1), we can take $r_1(B^{n-1};A) = r_1(B^n;A)$. Hence we can also take $r_1(B^j;A) = r_1(B^n;A)$ $(j = 1, \cdots, n-1)$ by the repeating argument. □

Lemma 2.4 *Let A and B be self-adjoint operators. Suppose that $B^2 \prec A$. Then*

(i) *$B \prec A$ with $r_1(B;A) = \varepsilon r_1(B^2;A)$ for all $\varepsilon > 0$,*

(ii) *$B^2 \prec A^2$ with $r_1(B^2;A^2) = \gamma r_1(B^2;A)$ for all $\gamma > 0$.*

Proof. (i) For all $f \in \mathrm{dom}(A)$ and $\varepsilon > 0$,

$$\begin{aligned} \|Bf\| &\le \varepsilon\|B^2 f\| + \frac{1}{4\varepsilon}\|f\| \\ &\le \varepsilon r_1(B^2;A)\|Af\| + \Big(\varepsilon r_2(B^2;A) + \frac{1}{4\varepsilon}\Big)\|f\|. \end{aligned}$$

(ii) For all $f \in \mathrm{dom}(A^2)$ and $\gamma > 0$,

$$\begin{aligned} \|B^2 f\| &\le r_1(B^2;A)\|Af\| + r_2(B^2;A)\|f\| \\ &\le r_1(B^2;A)\Big(\gamma\|A^2 f\| + \frac{1}{4\gamma}\|f\|\Big) + r_2(B^2;A)\|f\| \\ &= \gamma r_1(B^2;A)\|A^2 f\| + \Big(\frac{r_1(B^2;A)}{4\gamma} + r_2(B^2;A)\Big)\|f\|. \quad \square \end{aligned}$$

3 Nonrelativistic limit of the abstract CQSM

Let $(\mathcal{H}, \Gamma)$ be a $\mathbb{Z}_2$-graded Hilbert space and let Q, M and G be self-adjoint opetators on $\mathcal{H}$ such that

(i) Q and G are odd, and M is even,

(ii) Q and M strongly commute, i.e., their spectral measures commute,

(iii) G and M stronlgy commute,

(iv) M is bounded and

$$m := \inf \sigma(M) > 0,$$

where $\sigma(M)$ means the spectrum of M.

A linear operator H defined by

$$H := Q + \Gamma M \, \mathrm{e}^{\mathrm{i}G}$$

is called the *abstract CQSM Hamiltonian.* This operator can be regarded as an abstract generalization of the CQSM Hamiltonian appeared in nuclear physics [5]. This section concerns the limit $c \to \infty$ of the scaled Hamiltonian

$$H(c) := cQ + \Gamma M c^2 \, \mathrm{e}^{\mathrm{i}\frac{1}{c}G}.$$

In applications to concrete CQSM Hamiltonians, this limit corresponds to the nonrelativistic limit.

First, we have to check the self-adjointness of $H(c)$.

Proposition 3.1 *For all $c > 0$, $H(c)$ is self-adjoint on* $\mathrm{dom}(Q)$.

Proof. Since G is odd, we have

$$\Gamma \mathrm{e}^{\mathrm{i}\frac{1}{c}G} \Gamma = \mathrm{e}^{-\mathrm{i}\frac{1}{c}G}$$

by Lemma 2.1. Thus we get

$$\left(\Gamma M c^2 \, \mathrm{e}^{\mathrm{i}\frac{1}{c}G} \right)^* = \mathrm{e}^{-\mathrm{i}\frac{1}{c}G} M c^2 \Gamma = \Gamma M c^2 \, \mathrm{e}^{\mathrm{i}\frac{1}{c}G},$$

i.e., $\Gamma M c^2 \, \mathrm{e}^{\mathrm{i}\frac{1}{c}G}$ is self-adjoint. Hence we obtain the desired result by the Kato-Rellich theorem [4]. □

Next we have to find a "renormalized Hamiltonian". First we consider the standard Dirac operator

$$H_D(c) = -\mathrm{i}c\alpha \cdot \nabla + mc^2 \beta + V$$

to get a key to this problem. In this case, it is well-known that, in discussing the nonrelativistic limit, we investigate $H_D(c) - mc^2$ instead of $H_D(c)$ in order to avoid divergence difficulties coming from the mass term $mc^2\beta$. In our case, unfortunately, the situation is not so simple as the standard Dirac operator. To see this reason, we expand our mass term $\Gamma Mc^2 \,\mathrm{e}^{\mathrm{i}G/c}$ with respect to $1/c$ and obtain

$$\Gamma Mc^2 \,\mathrm{e}^{\mathrm{i}\frac{1}{c}G} = \Gamma Mc^2 + \mathrm{i}\Gamma cMG - \frac{1}{2}\Gamma MG^2 + O\Big(\frac{1}{c}\Big).$$

From this equation, it is clear that $\Gamma Mc^2 \,\mathrm{e}^{\mathrm{i}G/c}$ contains two divergent terms ΓMc^2 and $\mathrm{i}\Gamma cMG$. From the consideration of the standard Dirac operator, we can expect that the term ΓMc^2 can be renormalized by a method similar to the one for standard Dirac operator. But, because of the term $\mathrm{i}\Gamma cMG$, this standard procedure is not enough, and we take the following linear operator as our renormalized Hamiltonian:

$$H_{\mathrm{ren}}(c) := H(c) - Mc^2 - \mathrm{i}c\Gamma MG.$$

To discuss the nonrelativistic limit, we need a more assumption:
(v) $G^4 \prec Q$.

Proposition 3.2 *Let* (i)-(v) *be satisfied. Then*

(a) $H_{\mathrm{ren}}(c)$ *is self-adjoint on* $\mathrm{dom}(Q)$ *for all* $c > 0$*,*

(b) $\frac{1}{2M}Q^2 - \frac{M}{2}G^2$ *is self-adjoint on* $\mathrm{dom}(Q^2)$ *(*$\frac{1}{2M} := (2M)^{-1}$*).*

Proof. By the assumption (v) and Lemma 2.3, we have $G^2 \prec Q$. Hence, by Lemma 2.4, we obtain $G \prec Q$ and $G^2 \prec Q^2$ with

$$r_1(G;Q) = r_1(G^2;Q^2) = \varepsilon r_1(G^2;Q) \tag{2}$$

for all $\varepsilon > 0$. Hence, taking ε such that $\|M\| r_1(G;Q) < 1$, we conclude (a) by the Kato-Rellich theorem. Similarly, taking ε such that $\|M\| r_1(G^2;Q^2)/m < 1$, we obtain (b) by the Kato-Rellich theorem. □

Now we state our main result of this paper.

Theorem 3.3 *Let* (i)-(v) *be satisfied. Then, for all* $z \in \mathbb{C}\backslash\mathbb{R}$*,*

$$\text{norm-}\lim_{c\to\infty} \Big(H_{\mathrm{ren}}(c) - z\Big)^{-1} = P_+\Big(\frac{1}{2M}Q^2 - \frac{M}{2}G^2 - z\Big)^{-1},$$

where norm-$\lim_{c\to\infty}$ *means limit in the operator-norm topology.*

Remark 3.4 Physically the term $(2M)^{-1}Q^2$ corresponds to an abstract version of a nonrelativistic Hamiltonian of a "quark". In this interpretation, the appearance of the effective potential $-MG^2/2$ means the binding of the quark. This may be connected with the confining effect of the quark.

4 Proof of Theorem 3.3

We say that a closed linear operator A from a Hilbert space to a Hilbert space is invertible if it is bijective. By the closed graph theorem, the inverse of an invertible operator in bounded.

Let

$$K := \frac{1}{2M} Q^2$$

with domain $\mathrm{dom}(K) = \mathrm{dom}(Q^2)$ and let

$$K_z(c) := K - z - \frac{z^2}{2c^2} M$$

for $z \in \mathbb{C} \backslash \mathbb{R}$. Suppose that

$$\frac{1}{c} < \frac{\sqrt{2m|\mathrm{Im} z|}}{|z|}. \tag{3}$$

Then it is easy to check that $K_z(c)$ is invertible with

$$K_z(c)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z^2}{2c^2}\right)^n \left((K-z)^{-1} M^{-1}\right)^n (K-z)^{-1}$$

in the operator-norm topology.

Lemma 4.1 *Let*

$$\Pi_{\pm} := cQ + \Gamma M c^2 \pm M c^2 \pm z.$$

Let (3) be satisfied. Then

$$\Pi_{\pm}^{-1} = \frac{1}{2Mc^2} \Pi_{\mp} K_z(c)^{-1}.$$

Proof. Easy or see [1, Theorem 3.3]. □

The following lemma is often useful.

Lemma 4.2 *Let $\{A(\kappa)\}_{\kappa>0}$ be a family of self-adjoint operators and B be a bounded self-adjoint operator. Suppose that $A(\kappa)$ and B are strongly commuting for all $\kappa > 0$. Then, for all κ and $z \in \mathbb{C}\backslash\mathbb{R}$ satisfying*

$$\kappa > \sqrt{2|\mathrm{Re} z| \cdot \|B\|}, \tag{4}$$

$A(\kappa) - z - \left(\frac{z}{\kappa}\right)^2 B$ *is invertible with:*

$$\left\| \left(A(\kappa) - z - \left(\frac{z}{\kappa}\right)^2 B\right)^{-1} \right\| \le \left(|\mathrm{Im} z| - \frac{2}{\kappa^2} |\mathrm{Re} z| \cdot |\mathrm{Im} z| \cdot \|B\|\right)^{-1}.$$

Proof. Given a self-adjoint operator S in a Hilbert space, we denote its spectral measure by E_S. By the assumption, $E_{A(\kappa)}$ and E_B commute for all $\kappa > 0$. Hence we can define a two-dimensional spectral measure by $E_{A(\kappa),B} := E_{A(\kappa)} \otimes E_B$. It is not hard to check the invertibility of $A(\kappa) - z - \left(\frac{z}{\kappa}\right)^2 B$. For all $\varphi \in \mathcal{H}$, we get

$$\begin{aligned}\left\| \left(A(\kappa) - z - \left(\frac{z}{\kappa}\right)^2 B \right)^{-1} \varphi \right\|^2 &= \int \left| \lambda - z - \left(\frac{z}{\kappa}\right)^2 \mu \right|^{-2} \mathrm{d}\|E_{A(\kappa),B}(\lambda,\mu)\varphi\|^2 \\ &\leq \int \left| \mathrm{Im}\left(z + \left(\frac{z}{\kappa}\right)^2 \mu \right) \right|^{-2} \mathrm{d}\|E_{A(\kappa),B}(\lambda,\mu)\varphi\|^2 \\ &= \left\| \left| \mathrm{Im}\left(z + \left(\frac{z}{\kappa}\right)^2 B \right) \right|^{-1} \varphi \right\|^2.\end{aligned}$$

For all $\kappa > 0$ satisfying (4), we obtain

$$\begin{aligned}\left\| \left| \mathrm{Im}\left(z + \left(\frac{z}{\kappa}\right)^2 B \right) \right|^{-1} \varphi \right\|^2 &= \int \left| \mathrm{Im}z + \frac{2}{\kappa^2} \mathrm{Re}z \cdot \mathrm{Im}z \cdot \mu \right|^{-2} \mathrm{d}\|E_B(\mu)\varphi\|^2 \\ &\leq \int \left| |\mathrm{Im}z| - \frac{2}{\kappa^2} |\mathrm{Re}z| \cdot |\mathrm{Im}z| \cdot \|B\| \right|^{-2} \mathrm{d}\|E_B(\mu)\varphi\|^2 \\ &= \left| |\mathrm{Im}z| - \frac{2}{\kappa^2} |\mathrm{Re}z| \cdot |\mathrm{Im}z| \cdot \|B\| \right|^{-2} \|\varphi\|^2.\end{aligned}$$

Combining these results, we have the desired assertion. □

Lemma 4.3 *Let*

$$V_{c,+} := Mc^2\Gamma\left[\cos\left(\frac{1}{c}G\right) - 1 \right].$$

Then, for sufficiently large c, $1 + V_{c,+}P_+K_z(c)^{-1}$ is invertible and

$$\left(K_z(c) + V_{c,+}P_+ \right)^{-1} = K_z(c)^{-1}\left(1 + V_{c,+}P_+K_z(c)^{-1} \right)^{-1}. \tag{5}$$

Proof. By Lemma 4.2 with $A(c) := K + V_{c,+}P_+$ and $B := M/2$, the linear operator $K_z(c) + V_{c,+}P_+$ is invertible for sufficiently large c.

Note that

$$K_z(c) + V_{c,+}P_+ = \left(1 + V_{c,+}P_+K_z(c)^{-1} \right) K_z(c).$$

This implies that $1 + V_{c,+}P_+K_z(c)^{-1}$ is bijective and that (5) holds. □

Lemma 4.4 *Let*

$$V_{c,-} := \mathrm{i}\Gamma Mc^2\left[\sin\left(\frac{1}{c}G\right) - \frac{1}{c}G \right].$$

Then the following hold:

$$\lim_{c\to\infty} \left\| V_{c,+}\frac{1}{2Mc^2}(cQ + z)(K_z(c) + V_{c,+}P_+)^{-1} \right\| = 0, \tag{6}$$

$$\lim_{c\to\infty} \left\| V_{c,-}\frac{1}{2Mc^2}(cQ + 2Mc^2P_+ + z)(K_z(c) + V_{c,+}P_+)^{-1} \right\| = 0. \tag{7}$$

Proof. Noting the following fundamental inequalities

$$|x - \sin x| \le \frac{|x|^3}{6}, \quad 1 - \cos x \le \frac{x^2}{2}$$

for all $x \in \mathbb{R}$, we obtain

$$\|V_{c,+}g\| \le \frac{1}{2}\|M\| \cdot \|G^2 g\| \text{ for all } g \in \mathrm{dom}(G^2) \tag{8}$$

and

$$\|V_{c,-}g\| \le \frac{1}{6c}\|M\| \cdot \|G^3 g\| \text{ for all } g \in \mathrm{dom}(G^3). \tag{9}$$

For simplicity, we denote $K_z(c;V) := K_z(c) + V_{c,+}P_+$. By the assumption (v) and Lemma 2.3, we have $G^2 \prec Q$. Hence, using (8), we get for all $f \in \mathcal{H}$,

$$\begin{aligned}
& \|V_{c,+}(cQ+z)M^{-1}K_z(c;V)^{-1}f\| \\
\le\ & \frac{1}{2}\|M\| \cdot \|G^2(cQ+z)M^{-1}K_z(c;V)^{-1}f\| \\
\le\ & \frac{1}{2}\|M\|\Big\{r_1(G^2;Q)\|Q(cQ+z)M^{-1}K_z(c;V)^{-1}f\| \\
& + r_2(G^2;Q)\|(cQ+z)M^{-1}K_z(c;V)^{-1}f\|\Big\} \\
\le\ & c\|M\|r_1(G^2;Q)\|KK_z(c;V)^{-1}f\| \\
& + \frac{1}{2}\|M\|\Big(r_1(G^2;Q)|z| + r_2(G^2;Q)c\Big)\|QM^{-1}K_z(c;V)^{-1}f\| \\
& + r_2(G^2;Q)\frac{|z|}{m}\|K_z(c;V)^{-1}f\|.
\end{aligned}$$

For all $\varepsilon > 0$ and all $g \in \mathrm{dom}(Q^2)$, it is easy to see that

$$\|Qg\| \le \varepsilon\|Q^2 g\| + \frac{1}{4\varepsilon}\|g\|. \tag{10}$$

Using this, we have

$$\begin{aligned}
& \|QM^{-1}K_z(c;V)^{-1}f\| \\
\le\ & \varepsilon\|Q^2M^{-1}K_z(c;V)^{-1}f\| + \frac{1}{4\varepsilon}\|M^{-1}K_z(c;V)^{-1}f\| \\
\le\ & 2\varepsilon\|KK_z(c;V)^{-1}f\| + \frac{1}{4\varepsilon m}\|K_z(c;V)^{-1}f\|.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|V_{c,+}(cQ+z)M^{-1}K_z(c;V)^{-1}f\| \\
\le\ & d_1(c)\|KK_z(c;V)^{-1}f\| + d_2(c)\|K_z(c;V)^{-1}f\|
\end{aligned}$$

where

$$d_1(c) := \|M\| r_1(G^2; Q)c + \varepsilon \|M\| \Big(r_1(G^2; Q)|z| + r_2(G^2; Q)c \Big)$$

and

$$d_2(c) := \frac{1}{m} r_2(G^2; Q)|z| + \frac{1}{8\varepsilon m} \|M\| \Big(r_1(G^2; Q)|z| + r_2(G^2; Q)c \Big).$$

By Lemma 4.5 below, we have

$$\|K K_z(c; V)^{-1} f\| \le d \Big(\|(K + V_{c,+} P_+) K_z(c; V)^{-1} f\| + \|K_z(c; V)^{-1} f\| \Big).$$

Noting that

$$\begin{aligned} \|(K + V_{c,+} P_+) K_z(c; V)^{-1} f\| &= \Big\| \Big(K_z(c; V) + z + \frac{z^2}{2c^2} M^{-1} \Big) K_z(c; V)^{-1} f \Big\| \\ &\le \|f\| + \Big(|z| + \frac{|z|^2}{2c^2 m} \Big) \|K_z(c; V)^{-1} f\|, \end{aligned}$$

we obtain

$$\|K K_z(c; V)^{-1} f\| \le d \Big\{ \|f\| + \Big(1 + |z| + \frac{|z|^2}{2c^2 m} \Big) \|K_z(c; V)^{-1} f\| \Big\}.$$

Taking $c_0 > 0$ sufficiently large,

$$q := \sup_{c > c_0} \|K_z(c; V)^{-1}\| < \infty \tag{11}$$

by Lemma 4.2. Therefore

$$\|K K_z(c; V)^{-1}\| \le d \Big\{ 1 + \Big(1 + |z| + \frac{|z|^2}{2c^2 m} \Big) q \Big\} \le \text{const.} < \infty \tag{12}$$

for $c > c_0$. Combining these results,

$$\Big\| V_{c,+} \frac{1}{2Mc^2} (cQ + z) K_z(c; V)^{-1} \Big\| \le \frac{\text{const.}}{c^2} (d_1(c) + d_2(c))$$

and we conclude (6).

Next we prove (7). Note that

$$\begin{aligned} & \Big\| V_{c,-} \frac{1}{2Mc^2} (cQ + 2Mc^2 P_+ + z) K_z(c; V)^{-1} \Big\| \\ \le\ & \Big\| V_{c,-} \frac{1}{2Mc^2} (cQ + z) K_z(c; V)^{-1} \Big\| + \Big\| V_{c,-} P_+ K_z(c; V)^{-1} \Big\| \\ =:\ & I_1(c) + I_2(c). \end{aligned}$$

$\lim_{c\to\infty} I_1(c) = 0$ can be proven in the same way as in the proof of (6). Hence we have to show $\lim_{c\to\infty} I_2(c) = 0$. By the assumption (v) and Lemma 2.3, we have $G^3 \prec Q$. Applying (9), we get for all $f \in \mathcal{H}$,

$$\begin{aligned}
\|V_{c,-}P_+K_z(c;V)^{-1}f\| &\leq \frac{\|M\|}{6c}\|G^3K_z(c;V)^{-1}f\| \\
&\leq \frac{\|M\|}{6c}\Big\{r_1(G^3;Q)\|QK_z(c;V)^{-1}f\| \\
&\qquad +r_2(G^3;Q)\|K_z(c;V)^{-1}f\|\Big\}.
\end{aligned}$$

By (10), we get

$$\begin{aligned}
\|QK_z(c;V)^{-1}f\| &\leq \varepsilon\|Q^2K_z(c;V)^{-1}f\| + \frac{1}{4\varepsilon}\|K_z(c;V)^{-1}f\| \\
&\leq 2\varepsilon\|M\|\cdot\|KK_z(c;V)^{-1}f\| + \frac{1}{4\varepsilon}\|K_z(c;V)^{-1}f\|.
\end{aligned}$$

Using (11) and (12), we get $\|KK_z(c;V)^{-1}\| \leq$ const. for $c > c_0$ and

$$I_2(c) \leq \frac{\|M\|}{6c}\text{const.}$$

Thus we get (7). □

Lemma 4.5 *There exists a constant $d > 0$ independent of c such that*

$$\|Kg\| \leq d\Big(\|(K+V_{c,+}P_+)g\| + \|g\|\Big)$$

for all $g \in \mathrm{dom}(K)$.

Proof. Applying (8), we have

$$\begin{aligned}
\|Kg\| &\leq \|(K+V_{c,+}P_+)g\| + \|V_{c,+}P_+g\| \\
&\leq \|(K+V_{c,+}P_+)g\| + \frac{\|M\|}{2}\cdot\|G^2g\|
\end{aligned}$$

for all $g \in \mathrm{dom}(K)$. For all $\varepsilon > 0$,

$$\begin{aligned}
\|G^2g\| &\leq r_1(G^2;Q)\|Qg\| + r_2(G^2;Q)\|g\| \\
&\leq \varepsilon r_1(G^2;Q)\|Q^2g\| + \Big(\frac{1}{4\varepsilon}r_1(G^2;Q) + r_2(G^2;Q)\Big)\|g\| \quad \text{(by (10))} \\
&\leq 2\|M\|r_1(G^2;Q)\varepsilon\|Kg\| + \Big(\frac{1}{4\varepsilon}r_1(G^2;Q) + r_2(G^2;Q)\Big)\|g\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|Kg\| &\leq \|(K+V_{c,+}P_+)g\| + \|M\|^2\varepsilon r_1(G^2;Q)\|Kg\| \\
&\qquad + \frac{\|M\|}{2}\Big(\frac{1}{4\varepsilon}r_1(G^2;Q) + r_2(G^2;Q)\Big)\|g\|.
\end{aligned}$$

Taking $\varepsilon > 0$ such that $1 - \|M\|^2 \varepsilon r_1(G^2; Q) > 0$, we obtain

$$\begin{aligned}\|Kg\| \;\; &\le \;\; \frac{1}{1 - \|M\|^2 \varepsilon r_1(G^2; Q)} \|(K + V_{c,+}P_+)g\| \\ &\quad + \frac{\|M\|}{2} \frac{1}{1 - \|M\|^2 \varepsilon r_1(G^2; Q)} \Big(\frac{1}{4\varepsilon} r_1(G^2; Q) + r_2(G^2; Q) \Big) \|g\|.\end{aligned}$$

Therefore we get the desired assertion by taking

$$\begin{aligned} d \;\; := \;\; &\max \Big\{ \frac{1}{1 - \|M\|^2 \varepsilon r_1(G^2; Q)}, \\ &\frac{\|M\|}{2} \frac{1}{1 - \|M\|^2 \varepsilon r_1(G^2; Q)} \Big(\frac{1}{4\varepsilon} r_1(G^2; Q) + r_2(G^2; Q) \Big) \Big\}.\end{aligned}$$

□

Lemma 4.6 *Let*

$$X_z(c) := \Big[V_{c,+} \frac{1}{2Mc^2}(cQ + z) + V_{c,-} \frac{1}{2Mc^2}(cQ + 2Mc^2 P_+ + z) \Big] (K_z(c) + V_{c,+}P_+)^{-1}.$$

Then, for sufficiently large c, the linear operator $1 + X_z(c)$ is invertible and

$$\text{norm-}\lim_{c \to \infty} (1 + X_z(c))^{-1} = 1.$$

Proof. By Lemma 4.4, $\lim_{c\to\infty} \|X_z(c)\| = 0$. Thus $1 + X_z(c)$ is invertible for sufficiently large c. Moreover,

$$(1 + X_z(c))^{-1} = 1 + \sum_{n=1}^{\infty} (-X_z(c))^n$$

in the operator-norm topology. Therefore we obtain

$$\|(1 + X_z(c))^{-1} - 1\| \to 0$$

as $c \to \infty$. □

Theorem 4.7 *For sufficiently large c,*

$$\begin{aligned} &\Big(H(c) - Mc^2 - \mathrm{i}\Gamma cMG - z \Big)^{-1} \\ = \;\; &\Big(P_+ + \frac{1}{2Mc^2}(cQ + z) \Big) (K_z(c) + V_{c,+}P_+)^{-1} (1 + X_z(c))^{-1}.\end{aligned}$$

Proof. Let $V_c := V_{c,+} + V_{c,-}$. For sufficiently large c,

$$\begin{aligned}
& \Big(H(c) - Mc^2 - \mathrm{i}\Gamma cMG - z\Big)^{-1} \\
= \ & (\Pi_- + V_c)^{-1} \\
= \ & \Pi_-^{-1}(1 + V_c\Pi_-^{-1})^{-1} \\
= \ & \frac{1}{2Mc^2}\Pi_+ K_z(c)^{-1}\Big(1 + V_c\frac{1}{2Mc^2}\Pi_+ K_z(c)^{-1}\Big) \quad \text{(by Lemma 4.1)} \\
= \ & \Big(P_+ + \frac{1}{2Mc^2}(cQ+z)\Big)K_z(c)^{-1} \\
& \times\Big\{1 + V_{c,+}P_+K_z(c)^{-1} + \frac{V_{c,+}}{2Mc^2}(cQ+z)K_z(c)^{-1} \\
& \qquad + \frac{V_{c,-}}{2Mc^2}(cQ + 2Mc^2P_+ + z)K_z(c)^{-1}\Big\}^{-1} \\
= \ & \Big(P_+ + \frac{1}{2Mc^2}(cQ+z)\Big)K_z(c)^{-1}\Big(1 + V_{c,+}P_+K_z(c)^{-1}\Big)^{-1} \\
& \times\Big\{1 + \Big[\frac{V_{c,+}}{2Mc^2}(cQ+z) + \frac{V_{c,-}}{2Mc^2}(cQ + 2Mc^2P_+ + z)\Big] \\
& \qquad \times K_z(c)^{-1}\Big(1 + V_{c,+}P_+K_z(c)^{-1}\Big)^{-1}\Big\}^{-1} \\
= \ & \Big(P_+ + \frac{1}{2Mc^2}(cQ+z)\Big)\Big(K_z(c) + V_{c,+}P_+\Big)^{-1}\Big(1 + X_z(c)\Big)^{-1} \quad \text{(by Lemma 4.3).}
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.3

First, we show that

$$\begin{aligned}
& \text{norm-}\lim_{c\to\infty}\Big(P_+ + \frac{1}{2Mc^2}(cQ+z)\Big)\Big(K_z(c) + V_{c,+}P_+\Big)^{-1} \\
= \ & P_+\Big(K - \frac{M}{2}G^2 - z\Big)^{-1}. \qquad (13)
\end{aligned}$$

Note that

$$\begin{aligned}
& \Big(K_z(c) + V_{c,+}P_+\Big)^{-1} - \Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} \\
= \ & \Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1}\Big(\frac{z^2}{2c^2}M - V_{c,+}P_+ - \frac{M}{2}G^2P_+\Big) \\
& \times\Big(K_z(c) + V_{c,+}P_+\Big)^{-1}. \qquad (14)
\end{aligned}$$

Using the fact $|x^2/2 + \cos x - 1| \le x^4/4!$ and the functional calculus, we have

$$\Big\|\Big(V_{c,+}P_+ + \frac{M}{2}G^2P_+\Big)g\Big\| \le \frac{\|M\|}{4!c^2}\|G^4g\| \qquad (15)$$

for all $g \in \mathrm{dom}(G^4)$. Hence, for all $f \in \mathcal{H}$,

$$\begin{aligned}
&\left\|\Big(V_{c,+}P_+ + \frac{M}{2}G^2P_+\Big)\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
\leq\ & \frac{\|M\|}{4!c^2}\left\|G^4\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
\leq\ & \frac{\|M\|}{4!c^2}\Big\{r_1(G^4;Q)\left\|Q\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
&\qquad + r_2(G^4;Q)\left\|\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\|\Big\} \ \text{(by (v))}.
\end{aligned}$$

By (10), we have

$$\begin{aligned}
&\left\|Q\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
\leq\ & \varepsilon\left\|Q^2\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| + \frac{1}{4\varepsilon}\left\|\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
\leq\ & 2\varepsilon\|M\| \cdot \left\|K\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| + \frac{1}{4\varepsilon}\left\|\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1} f\right\| \\
\leq\ & \text{const.}\|f\|.
\end{aligned}$$

Thus,

$$\left\|\Big(V_{c,+}P_+ + \frac{M}{2}G^2P_+\Big)\Big(K - \frac{M}{2}G^2P_+ - z\Big)^{-1}\right\| \leq \frac{\text{const.}}{c^2}.$$

By this and the fact

$$\lim_{c\to\infty}\left\|\frac{z^2M}{2c^2}\Big(K_z(c) + V_{c,+}P_+\Big)^{-1}\right\| = 0,$$

we conclude that

$$\text{norm-}\lim_{c\to\infty} P_+\Big(K_z(c) + V_{c,+}P_+\Big)^{-1} = P_+\Big(K - \frac{M}{2}G^2 - z\Big)^{-1}$$

by (14). Moreover, it is not hard to see that

$$\lim_{c\to\infty}\left\|\frac{1}{2Mc^2}(cQ+z)\Big(K_z(c) + V_{c,+}P_+\Big)^{-1}\right\| = 0.$$

Hence we get (13).

Now Theorem 3.3 immediately follows from Theorem 4.7, Lemma 4.6 and (13). □

Remark 4.8 We introduce

$$H_1(c) := cQ + \Gamma M c^2 \cos\Big(\frac{1}{c}G\Big).$$

In this case, we do not need the strange renormalization which appears in Theorem 3.3. That is, the following holds.

Theorem 4.9 *For all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\text{norm-}\lim_{c\to\infty}\Big(H_1(c) - Mc^2 - z\Big)^{-1} = P_+\Big(\frac{1}{2M}Q^2 - \frac{M}{2}G^2 - z\Big)^{-1}.$$

5 Example

Let σ_j $(j = 1, 2, 3)$ be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\alpha_j = \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix} \ \ (j = 1, 2, 3), \ \beta = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix},$$

where 0_2 and 1_2 are the 2×2 zero matrix and the 2×2 identity matrix respectively. The matrix

$$\gamma_5 := -\mathrm{i}\alpha_1\alpha_2\alpha_3$$

is Hermitian with $\gamma_5^2 = 1_4$ (the 4×4 identity matrix) satisfying the following relations:

$$[\alpha_j, \gamma_5] = 0 \ (j = 1, 2, 3), \quad \{\beta, \gamma_5\} = 0,$$

where we use the following notations $[A, B] := AB-BA$ and $\{A, B\} := AB+BA$. We set

$$\sigma := (\sigma_1, \sigma_2, \sigma_3), \quad \alpha := (\alpha_1, \alpha_2, \alpha_3).$$

Let $\nabla := (D_1, D_2, D_3)$ with the generalized partial differential operator D_j in the variable x_j, the j-th component of $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then the free Dirac operator with mass zero is defined by

$$Q := -\mathrm{i}\alpha \cdot \nabla \otimes 1_2$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^4)\otimes\mathbb{C}^2$. We take the linear operator $\beta\otimes 1_2$ as a grading operator on $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$. We note that Q is odd and self-adjoint. To introduce a mass term for Q, let $F : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function such that $F^4 \prec -\mathrm{i}\alpha \cdot \nabla$ and set

$$G := \gamma_5 \otimes \tau \cdot nF$$

where $\tau := (\tau_1, \tau_2, \tau_3)$ with $\tau_j = \sigma_j$ $(j = 1, 2, 3)$, $n := (n_1, n_2, n_3)$ with n_j a real-valued measurable function on $\mathbb{R}^3$ such that $|n(x)| = 1$ *a.e.* $x \in \mathbb{R}^3$. It is not hard to check that G is odd and self-adjoint. Let $M > 0$ be a constant, then the triplet (Q, M, G) satisfies all the conditions (i)-(v) in Sec. 2. The self-adjoint operator

$$H = Q + (\beta \otimes 1_2)M\, \mathrm{e}^{\mathrm{i}G}$$

is called the *chiral quark soliton model* [5].

Let

$$H(c) := cQ + (\beta \otimes 1_2)Mc^2\, \mathrm{e}^{\mathrm{i}\frac{1}{c}G}.$$

Then we can apply Theorem 3.3 and obtain the following.

Theorem 5.1 *For all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$
\begin{aligned}
& \text{norm-}\lim_{c\to\infty}\Big(H(c) - Mc^2 - \mathrm{i}(\beta\otimes 1_2)cMG - z\Big)^{-1} \\
= \; & P_+\Big(-\frac{\Delta}{2M} - \frac{M}{2}F^2 - z\Big)^{-1}\otimes 1_2.
\end{aligned}
$$

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