ANISOTROPIC HÖLDER AND SOBOLEV SPACES FOR HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We study spectral properties of transfer operators for diffeomorphisms $T : X \to X$ on a Riemannian manifold X: Suppose that Ω is an isolated hyperbolic subset for T, with a compact isolating neighborhood $V \subset X$. We first introduce Banach spaces of distributions supported on V, which are anisotropic versions of the usual space of C^p functions $C^p(V)$ and of the generalized Sobolev spaces $W^{p,t}(V)$, respectively. Then we show that the transfer operators associated to T and a smooth weight g extend boundedly to these spaces, and we give bounds on the essential spectral radii of such extensions in terms of hyperbolicity exponents. These bounds shed some light on those obtained by Kitaev for the radius of convergence of dynamical determinants.

1. Introduction

Let X be a d-dimensional C^∞ Riemannian manifold and let $T : X \to X$ be a diffeomorphism which is of class C^1 at least. For a given complex-valued continuous function g on X, we define the Ruelle transfer operator $\mathcal{L}_{T,g}$ by

$$\mathcal{L}_{T,g} : C^0(X) \to C^0(X), \quad \mathcal{L}_{T,g}u(x) = g(x) \cdot u \circ T(x).$$

Such operators appear naturally in the study of fine statistical properties of dynamical systems and provide efficient methods, for instance, to estimate of decay of correlations. (We refer e.g. to [1].) Typical examples are the pull-back operator

$$T^*u := \mathcal{L}_{T,1}u = u \circ T, \tag{1}$$

and the Perron-Frobenius operator

$$\mathcal{P}u := \mathcal{L}_{T^{-1},|\det DT^{-1}|}u = |\det DT^{-1}| \cdot u \circ T^{-1}. \tag{2}$$

This paper is about spectral properties of the operator $\mathcal{L}_{T,g}$.

We shall require a hyperbolicity assumption on the mapping T: Let $\Omega \subset X$ be a compact isolated invariant subset for T, with a compact isolating neighborhood V, that is, $\Omega = \cap_{m\in\mathbb{Z}} T^m(V)$. We assume that Ω is a hyperbolic subset, that is, there exists an invariant decomposition $T_\Omega M = E^u \oplus E^s$ of the tangent bundle over Ω, satisfying $\|DT^m|_{E^s}\| \le C\lambda^m$ and $\|DT^{-m}|_{E^u}\| \le C\lambda^m$, for all $m \ge 0$ and $x \in \Omega$, with constants $C > 0$ and $0 < \lambda < 1$. Up to decomposing Ω, we may suppose that the dimensions of $E^u(x)$ and $E^s(x)$ are constant.

Date: May 2005.

VB thanks Sculoa Normale Superiore Pisa for hospitality and Artur Avila for useful comments. MT thanks NCTS(Taiwan) for hospitality during his stay.

We define two local hyperbolicity exponents for each $x \in \Omega$ and each $m \geq 1$ by

$$(3)\qquad \begin{aligned} \lambda_x(T^m) &= \sup_{v \in E^s(x)\setminus\{0\}} \frac{\|DT^m(v)\|}{\|v\|} \leq C\lambda^m \quad \text{and} \\ \nu_x(T^m) &= \inf_{v \in E^u(x)\setminus\{0\}} \frac{\|DT^m(v)\|}{\|v\|} \geq C^{-1}\lambda^{-m}. \end{aligned}$$

Let ω be the Riemannian volume form on X, and let $|\det DT|$ be the Jacobian of T, that is, the function given by $T^*\omega = |\det DT| \cdot \omega$. Put, for $m \geq 1$,

$$g^{(m)}(x) = \prod_{k=0}^{m-1} g(T^k(x)).$$

For real numbers $q \leq 0 \leq p$, and for $1 \leq t \leq \infty$, we set

$$R^{p,q,t}(T, g, \Omega, m) = \sup_{\Omega} |\det DT^m_x|^{-1/t} |g^{(m)}(x)| \max\{(\lambda_x(T^m))^p, (\nu_x(T^m))^q\},$$

where we read $(\cdot)^{1/\infty} = 1$ for $t = \infty$. As $\log R^{p,q,t}(T, g, \Omega, m)$ is sub-additive with respect to m, we have

$$R^{p,q,t}(T, g, \Omega) := \lim_{m\to\infty} \sqrt[m]{R^{p,q,t}(T, g, \Omega, m)} = \inf_{m \geq 1} \sqrt[m]{R^{p,q,t}(T, g, \Omega, m)}.$$

In this paper, we introduce Banach spaces of distributions supported on V, show that the transfer operators $\mathcal{L}_{T,g}$ extend boundedly to those spaces and then give bounds for the essential spectral radii of these transfer operators, using the quantities $R^{p,q,t}(T, g, \Omega)$ introduced above. The main feature in our approach is that we work in Fourier coordinates. The definition and basic properties of the Banach spaces will be given in later sections. Here we state the main theorem as follows. Let $r_{ess}(L|_{\mathcal{B}})$ be the essential spectral radius of a bounded linear operator $L : \mathcal{B} \to \mathcal{B}$. For non-integer $s > 0$, a mapping is of class C^s if all its partial derivatives of order $[s]$ are $(s - [s])$-Hölder.

Theorem 1.1. *Suppose that T is C^r for a real number $r > 1$, and let Ω be a hyperbolic invariant set with compact isolating neighborhood V, as described above. Then, for any real numbers $q < 0 < p$ with $p - q < r - 1$, there exist Banach spaces $C^{p,q}_*(T, V)$ and $W^{p,q,t}_*(T, V)$, for $1 < t < \infty$, of distributions supported on V, such that, for any C^{r-1} function $g : X \to \mathbb{C}$ supported on V,*

(1) *(Hölder spaces) $\mathcal{L}_{T,g}$ extends boundedly to $C^{p,q}_*(T, V)$ and*

$$r_{ess}(\mathcal{L}_{T,g}|_{C^{p,q}_*(T,V)}) \leq R^{p,q,\infty}(T, g, \Omega).$$

(2) *(Sobolev spaces) $\mathcal{L}_{T,g}$ extends boundedly to $W^{p,q,t}_*(T, V)$ and*

$$r_{ess}(\mathcal{L}_{T,g}|_{W^{p,q,t}(T,V)}) \leq R^{p,q,t}(T, g, \Omega)\,, \forall\, 1 < t < \infty.$$

The Banach spaces $C^{p,q}_*(T, V)$ and $W^{p,q,t}_*(T, V)$ contain the set $C^p(V)$ of C^p functions supported on V.

Hyperbolic attractors and SRB measures. Let us see how to apply Theorem 1.1 to a hyperbolic attractor: Assume in addition to the above that Ω is an attracting hyperbolic set and take the isolating neighborhood V so that $T(V) \subset \text{interior}(V)$. Consider the pull-back operator T^* defined by (1) and the

Perron-Frobenius operator $\mathcal{P}$ defined by (2). Note that these operators are adjoint to each other:

$$\int_X T^* u \cdot v \, d\omega = \int_X u \cdot \mathcal{P} v \, d\omega. \tag{4}$$

Let $h : X \to [0,1]$ be a C^∞ function supported on V and satisfying $h \equiv 1$ on $T(V)$. Then the action of the operator $\mathcal{L}_{T^{-1},g}$ with $g(x) = |\det DT^{-1}(x)| \cdot h(x)$ coincides with that of the Perron-Frobenius operator $\mathcal{P}$ on $C_*^{p,q}(T^{-1}, V)$ and $W_*^{p,q,t}(T^{-1}, V)$, $1 < t < \infty$. Therefore Theorem 1.1 easily implies:

Theorem 1.2. *Let Ω be a hyperbolic attractor for a C^r diffeomorphism $T : X \to X$, and let V be a compact neighborhood of Ω such that $T(V) \subset \mathrm{interior}(V)$ and $\cap_{m \geq 0} T^m(V) = \Omega$. Then, for real numbers $q < 0 < p$ with $p - q < r - 1$, the Perron-Frobenius operator $\mathcal{P}$ extends boundedly to $\mathcal{P} : C_*^{p,q}(T^{-1}, V) \to C_*^{p,q}(T^{-1}, V)$ and also to $\mathcal{P} : W_*^{p,q,t}(T^{-1}, V) \to W_*^{p,q,t}(T^{-1}, V)$, and it holds*

$$r_{ess}(\mathcal{P}|_{C_*^{p,q}(T^{-1},V)}) \leq R^{-q,-p,1}(T, 1, \Omega) < 1$$

and

$$r_{ess}(\mathcal{P}|_{W_*^{p,q,t}(T^{-1},V)}) \leq R^{-q,-p,t/(t-1)}(T, 1, \Omega) < 1 \quad \text{for } 1 < t < \infty.$$

If we assume further that $T : X \to X$ is an Anosov diffeomorphism on a closed manifold X, then, for real numbers $q < 0 < p$ with $p - q < r - 1$, the pull-back operator T^* extends boundedly to $T^* : C_*^{p,q}(T, X) \to C_*^{p,q}(T, X)$ and also to $T^* : W_*^{p,q,t}(T, X) \to W_*^{p,q,t}(T, X)$, and it holds

$$r_{ess}(T^*|_{C_*^{p,q}(T,X)}) \leq R^{p,q,\infty}(T, 1, X) < 1,$$

and

$$r_{ess}(T^*|_{W_*^{p,q,t}(T,X)}) \leq R^{p,q,t}(T, 1, X) < 1, \quad \text{for } 1 < t < \infty.$$

Remark 1.3. From (4) and Theorem 1.2, if T is Anosov, the Perron-Frobenius operator $\mathcal{P}$ acts naturally on the (strong) dual spaces of $C_*^{p,q}(T, X)$ and $W_*^{p,q,t}(T, X)$, and we have for instance

$$r_{ess}(\mathcal{P}|_{(C_*^{p,q}(T,X))^*}) \leq R^{p,q,\infty}(T, 1, X) < 1,$$

for real numbers $q < 0 < p$ with $p - q < r - 1$.

Once we have the estimates in Theorem 1.2, it is not difficult to see that the spectral radii of the Perron-Frobenius operator $\mathcal{P}$ on $C_*^{p,q}(T^{-1}, V)$ and $W_*^{p,q,t}(T^{-1}, V)$ are equal to one. If (T, Ω) is topologically mixing in addition, then 1 is the unique eigenvalue on the unit circle, it is a simple eigenvalue, and the fixed vector gives rise to the SRB measure μ on Ω: This corresponds to exponential decay of correlations for C^p observables and μ. (See Blank–Keller–Liverani [4, §3.2] for example.)

Spectral stability and kneading theory. We point out that, in the setting of Theorem 1.1, there is $\epsilon > 0$ so that if $\widetilde{T}$ and $\widetilde{g}$, respectively, are ϵ-close to T and g, respectively, in the C^r, resp. C^{r-1}, topology, then the associated operator $\mathcal{L}_{\widetilde{T},\widetilde{g}}$ has same spectral properties than $\mathcal{L}_{T,g}$ on *the same Banach spaces.* Spectral stability can then be proved, as it has been done [4] or [5] for the norms defined there (see also the historical comments below).

Furthermore, we note that controlling the essential spectral radius on a scale of Sobolev spaces may be useful in view of the kneading approach relating transfer operators and dynamical determinants (see [3]).

Organization of the paper. After defining a version of our norms in $\mathbb{R}^d$ in Section 2, we proceed in the usual way: prove compact embeddings in Section 5 and a Lasota-Yorke type estimate in Section 6. In Section 8, we prove Theorem 1.1 by reducing to the model from Sections 2–6 starting from a C^r diffeomorphism on a manifold, and applying Hennion's [6] theorem. For the Hölder spaces, our proof is elementary: it only uses integration by parts. For the Sobolev spaces, we require in addition a standard L^t estimate (Theorem 3.1) for (operator-valued) pseudodifferential operators with C^∞ symbols $P(\xi)$ depending only on ξ.

Historical Remarks. The first estimates on the essential spectral radius of transfer operators were obtained for one-sided subshifts of finite type (Sinai, Ruelle, Bowen, ...) and expanding (or piecewise expanding) endomorphisms (Lasota-Yorke, Ruelle, Hofbauer-Keller, ...). In order to study Anosov diffeomorphisms, a reduction to the expanding case was used in the eighties (Pollicott, Haydn, Ruelle, ...). Since this essentially involves quotienting along a dynamical foliation, and since these foliations are in general only Hölder (even if $r = \infty$), this severely limited the sharpness of the bounds. In the early nineties, Rugh, and then Fried, introduced some ideas which allow to bypass this reduction to the expanding case, but only for analytic Anosov (or Axiom A) diffeomorphisms, and for a transfer operator associated to a"model" for the dynamics. Since there are no partitions of unity in the analytic category, translating the results for the model back into the manifold is not trivial. Precise statements and references for the, by now "classical," results mentioned in this paragraph may be found in the book [1].

Under the assumptions of Theorem 1.1 Kitaev [9] proved that the following "dynamical Fredholm determinant"

$$d(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x)=x} \frac{1}{|\det(DT^n(x) - \mathrm{Id})|} \tag{5}$$

extends to a holomorphic function in the disc $\{z : |z| \cdot R^{p,q,\infty}(T, 1, \Omega) < 1\}$ for all $q < 0 < p$ with $p - q < r - 1$. Kitaev's bounds are in fact slightly more general (see Remark 8.1), but no spectral interpretation of his result is given in [9].

The first article which analyses the essential spectrum of the transfer operator $\mathcal{P}$ associated to an Anosov diffeomorphism and acting on a space of distributions on the manifold is due to Blank, Keller, and Liverani [4]. Another breakthrough of that paper is that perturbation theory was made possible, because the same Banach space could be used for nearby Anosov diffeomorphisms. The methods in [4], however, only allowed to exploit very limited smoothness of the diffeomorphism in the Lasota-Yorke bounds, so that the estimates there are far from being sharp.

In 2004, Gouëzel and Liverani [5, v1] obtained bounds for the essential spectral radius of $\mathcal{P}$ for an Anosov diffeomorphism T, acting on Banach spaces $\mathcal{B}^{p,q}$, for *integers* $q < 0 < p$ with $p - q < r - 1$, using methods different from ours. The space $\mathcal{B}^{p,q}$ is similar in spirit to (but not quite as simple as) the dual of our Hölder space $C^{p,q}_*(T, V)$. Although the upper bounds for $r_{ess}(\mathcal{P}|_{\mathcal{B}^{p,q}})$ in [5, v1] are not sharp, they tend to zero as $(p - q) \to \infty$ if $r = \infty$, and this is one of the important contributions of [5, v1].

Also in 2004, if $r = \infty$ and *under the very strong additional assumption* that either the stable or the unstable foliation of the Anosov diffeomorphism T is C^∞, the first named author of the present paper introduced tangential pseudodifferential methods [2] to define "foliated" Sobolev spaces $\widetilde{W}^{p,q,t}$ for $1 < t < \infty$, and to prove e.g. the upper bounds $R^{-q,-p,t/(1-t)}(T,1,X)$ on the essential spectral radius of $\mathcal{P}$ acting on $\widetilde{W}^{p,q,t}(T^{-1},X)$. (The proof in [2] requires pseudodifferential operators with symbols $\Psi(x,\xi)$ which are not C^∞ in x.) As mentioned in [2], this foliated approach should extend to the case when either the stable or the unstable foliation is $C^{1+\epsilon}$. However this generalization requires paradifferential techniques.

As we were finishing to write the present paper, Gouëzel and Liverani substantially improved their bounds in a second version [5, v2] of their article, still in the Anosov case. Their bounds now look similar to ours: In our notation, they prove that if $p-q<r-1$, where $p\geq 1$ is an integer, the essential spectral radius of $\mathcal{P}$ on their Banach space $\mathcal{B}^{p,q}$ is bounded by $R^{-q,-p,\infty}(T^{-1},1,X)=R^{p,q,\infty}(T,1,X)$. They do not have an analogue of our Sobolev spaces.

2. Definition of the anisotropic norms.

We recall a few facts on Sobolev and Hölder norms, which motivate our definition of anisotropic norms.

Fix an integer $d\geq 1$ and a C^∞ function $\chi:\mathbb{R}\to[0,1]$ with

$$\chi(s)=1,\quad \text{for } s\leq 1,\qquad \chi(s)=0,\quad \text{for } s\geq 2.$$

For $n\in\mathbb{Z}_+$, define $\chi_n:\mathbb{R}^d\to[0,1]$ as $\chi_n(x)=\chi(2^{-n}|\xi|)$ and, setting $\chi_{-1}\equiv 0$,

$$\psi_n:\mathbb{R}^d\to[0,1],\qquad \psi_n(\xi)=\chi_n(\xi)-\chi_{n-1}(\xi).$$

We have $1=\sum_{n=0}^\infty \psi_n(\xi)$, and $\mathrm{supp}(\psi_n)\subset\{\xi\mid 2^{n-1}\leq|\xi|\leq 2^{n+1}\}$. Also $\psi_n(\xi)=\psi_1(2^{-n+1}\xi)$ for $n\geq 1$. Thus, for every multi-index α, there exists a constant C_α such that $\|\partial^\alpha\psi_n\|_{L^\infty}\leq C_\alpha 2^{-n|\alpha|}$ for all $n\geq 0$, and the inverse Fourier transform of ψ_n,

$$\widehat{\psi}_n(x)=(2\pi)^{-d}\int e^{ix\xi}\psi_n(\xi)d\xi,$$

decays rapidly, satisfies $\widehat{\psi}_n(x)=2^{d(n-1)}\widehat{\psi}_1(2^{n-1}x)$ for $n\geq 1$ and all x, and is bounded uniformly in n with respect to the L^1-norm.

We decompose each C^∞ function $u:\mathbb{R}^d\to\mathbb{C}$ with compact support as $u=\sum_{n\geq 0}u_n$ by defining for integer $n\in\mathbb{Z}_+$,

$$u_n(x)=\psi_n(D)u(x):=(2\pi)^{-d}\int e^{i(x-y)\xi}\psi_n(\xi)u(y)dyd\xi=\widehat{\psi}_n*u. \tag{6}$$

Remark 2.1. The operator $\psi_n(D)$ in (6) is the "pseudodifferential operator with symbol ψ_n." We refer to the books [7] and [10] for more about pseudodifferential operators, although our text is self-contained, except for Theorem 3.1.

¿From now on, we fix a compact subset $K\subset\mathbb{R}^d$ with non-empty interior. Let $C^\infty(K)$ be the space of complex-valued C^∞ functions on $\mathbb{R}^d$ supported on K. For a real number p and $1<t<\infty$, we define on $C^\infty(K)$ the norms

$$\|u\|_{C^p_*}=\sup_{n\geq 0}2^{pn}\|u_n\|_{L^\infty}\quad\text{and}\quad \|u\|_{W^{p,t}_*}=\left\|\left(\sum_{n\geq 0}4^{pn}|u_n|^2\right)^{1/2}\right\|_{L^t}.$$

It is known that the norm $\|u\|_{C^p_*}$ is equivalent to the Hölder norm

$$\|u\|_{C^p}=\max\left\{\max_{|\alpha|\leq[p]}\sup_{x\in\mathbb{R}^d}|\partial^\alpha u(x)|,\ \max_{|\alpha|=[p]}\sup_{x\in\mathbb{R}^d}\sup_{y\in\mathbb{R}^d/\{0\}}\frac{|\partial^\alpha u(x+y)-\partial^\alpha u(x)|}{|y|^{p-[p]}}\right\}$$

provided that $p > 0$ is not an integer, and $\|u\|_{W_*^{p,t}}$ is equivalent to the generalized Sobolev norm

$$\|u\|_{W^{p,t}} = \left\|(1+\Delta)^{p/2}u\right\|_{L^t}$$

for any $p \in \mathbb{R}$ and $1 < t < \infty$. (See [11, Appendix A] for a brief account.)

The Hölder space $C_*^p(K)$ is the completion of $C^\infty(K)$ with respect to the norm $\|\cdot\|_{C_*^p}$ (by the above, it coincides with the space of C^p functions supported on K, if $p > 0$ is not an integer [1]). The generalized Sobolev space $W_*^{p,t}(K)$ for $1 < t < \infty$ is the completion of $C^\infty(K)$ with respect to the norm $\|\cdot\|_{W_*^{p,t}}$.

We are going to introduce anisotropic versions of the norms and spaces above. Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be closed cones in $\mathbb{R}^d$ with nonempty interiors such that $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$. Let then $\varphi_+, \varphi_- : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_+; \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_-, \end{cases} \qquad \varphi_-(\xi) = 1 - \varphi_+(\xi). \tag{7}$$

We shall work with combinations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ as above. For another such combination $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$, we write $\Theta' < \Theta$ if

$$\mathrm{closure}(\mathbb{R}^d \setminus \mathbf{C}_+) \subset \mathrm{interior}(\mathbf{C}'_-) \cup \{0\}$$

(This implies $\mathbf{C}'_+ \subset \mathbf{C}_+$ and $\mathbf{C}'_- \supset \mathbf{C}_-$ in particular.) For $n \in \mathbb{Z}_+$ and $\sigma \in \{+,-\}$, we define

$$\psi_{\Theta,n,\sigma}(\xi) = \begin{cases} \psi_n(\xi)\varphi_\sigma(\xi/|\xi|), & \text{if } n > 0; \\ \chi_n(\xi)/2, & \text{if } n = 0. \end{cases}$$

Note that the $\psi_{\Theta,n,\sigma}(\xi)$ enjoy similar properties as those of the ψ_n, in particular the L^1-norm of the rapidly decaying function $\widehat{\psi}_{\Theta,n,\sigma}$ is bounded uniformly in n. For a C^∞ function $u : \mathbb{R}^d \to \mathbb{C}$ with compact support, an integer $n \in \mathbb{Z}_+$, $\sigma \in \{+,-\}$, and a combination $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$, we define

$$u_{\Theta,n,\sigma} = \psi_{\Theta,n,\sigma}(D)u = \widehat{\psi}_{\Theta,n,\sigma} * u.$$

Since $1 = \sum_{n=0}^\infty \sum_{\sigma=\pm} \psi_{\Theta,n,\sigma}(\xi)$ by definition, we have $u = \sum_{n\ge 0}\sum_{\sigma=\pm} u_{\Theta,n,\sigma}$.

Let p and q be real numbers. For $u \in C^\infty(K)$, we define the anisotropic Hölder norm $\|u\|_{C_*^{\Theta,p,q}}$ by

$$\|u\|_{C_*^{\Theta,p,q}} = \max\left\{ \sup_{n\ge 0} 2^{pn}\|u_{\Theta,n,+}\|_{L^\infty}, \ \sup_{n\ge 0} 2^{qn}\|u_{\Theta,n,-}\|_{L^\infty} \right\}, \tag{8}$$

and the anisotropic Sobolev norm $\|u\|_{W_*^{\Theta,p,q,t}}$ for $1 < t < \infty$ by

$$\|u\|_{W_*^{\Theta,p,q,t}} = \left\| \left(\sum_{n\ge 0} 4^{pn}|u_{\Theta,n,+}|^2 + 4^{qn}|u_{\Theta,n,-}|^2 \right)^{1/2} \right\|_{L^t}. \tag{9}$$

Let $C_*^{\Theta,p,q}(K)$ be the completion of $C^\infty(K)$ with respect to the norm $\|\cdot\|_{C_*^{\Theta,p,q}}$. Likewise, for $1 < t < \infty$, let $W_*^{\Theta,p,q,t}(K)$ be the completion of $C^\infty(K)$ with respect to the norm $\|\cdot\|_{W_*^{\Theta,p,q,t}}$. We will call these spaces $C_*^{\Theta,p,q}(K)$ and $W_*^{\Theta,p,q,t}(K)$ of distributions the anisotropic Hölder and Sobolev space respectively. In Section 8,

[1] If p is an integer, we get a Zygmund class: our "Hölder" terminology is slightly unprecise, but convenient.

we will construct the Banach spaces in Theorem 1.1 by patching these Hölder and Sobolev spaces using local coordinates.

3. Preliminaries

In studying the anisotropic Hölder and Sobolev norms, it is convenient to work in different "coordinates" that we introduce next. Let $\Gamma = \{(n,\sigma) \mid n \in \mathbb{Z}_+, \sigma \in \{+,-\}\}$ and put

$$\mathbb{C}^\Gamma = \{(f_{n,\sigma})_{(n,\sigma)\in\Gamma} \mid f_{n,\sigma} \in \mathbb{C}\}.$$

For real numbers p and q, and for $\mathbf{f} = (f_{n,\sigma})_{(n,\sigma)\in\Gamma}$ and $\mathbf{g} = (g_{n,\sigma})_{(n,\sigma)\in\Gamma}$ in $\mathbb{C}^\Gamma$, we define a norm associated to a scalar product

$$|\mathbf{f}|_{\mathcal{W}^{p,q}} = \sqrt{(\mathbf{f},\mathbf{f})_{\mathcal{W}^{p,q}}}, \qquad (\mathbf{f},\mathbf{g})_{\mathcal{W}^{p,q}} = \sum_{n=0}^{\infty}\left(4^{pn} f_{n,+}\cdot\overline{g_{n,+}} + 4^{qn} f_{n,-}\cdot\overline{g_{n,-}}\right),$$

and a norm $|\mathbf{f}|_{\mathcal{C}^{p,q}} = \max\left\{\sup_{n\geq 0} 2^{pn}|f_{n,+}|, \sup_{n\geq 0} 2^{qn}|f_{n,-}|\right\}$. We then set

$$\mathcal{W}^{p,q} = \{\mathbf{f}\in\mathbb{C}^\Gamma \mid |\mathbf{f}|_{\mathcal{W}^{p,q}} < \infty\} \quad\text{and}\quad \mathcal{C}^{p,q} = \{\mathbf{f}\in\mathbb{C}^\Gamma \mid |\mathbf{f}|_{\mathcal{C}^{p,q}} < \infty\}.$$

Recall that $K\subset\mathbb{R}^d$ is a fixed compact set. Consider the operator $\mathcal{Q}_\Theta : C^\infty(K) \to C^\infty(\mathbb{R}^d, \mathbb{C}^\Gamma)$ defined by

$$\mathcal{Q}_\Theta u = (\psi_{\Theta,n,\sigma}(D)u)_{(n,\sigma)\in\Gamma}.$$

By definition, this operator extends to

$$\mathcal{Q}_\Theta : C_*^{\Theta,p,q}(K) \to L^\infty(\mathbb{R}^d, \mathcal{C}^{p,q}), \quad \mathcal{Q}_\Theta : W_*^{\Theta,p,q,t}(K) \to L^t(\mathbb{R}^d, \mathcal{W}^{p,q}).$$

If we define norms

$$\|\mathbf{u}\|_{p,q,\infty} = \||\mathbf{u}|_{\mathcal{C}^{p,q}}(x)\|_{L^\infty} \quad \text{for } \mathbf{u}\in L^\infty(\mathbb{R}^d,\mathcal{C}^{p,q})$$

and

$$\|\mathbf{u}\|_{p,q,t} = \||\mathbf{u}|_{\mathcal{W}^{p,q}}(x)\|_{L^t} \quad \text{for } \mathbf{u}\in L^t(\mathbb{R}^d,\mathcal{W}^{p,q}),$$

respectively, the anisotropic Hölder norm and the Sobolev norms coincide with their respective pull-backs by $\mathcal{Q}_\Theta$:

$$\|u\|_{C_*^{\Theta,p,q}} = \|\mathcal{Q}_\Theta u\|_{p,q,\infty} \quad\text{and}\quad \|u\|_{W_*^{\Theta,p,q,t}} = \|\mathcal{Q}_\Theta u\|_{p,q,t}. \tag{10}$$

The pseudodifferential operator $\psi(D)$ with symbol $\psi\in C_0^\infty(\mathbb{R}^d)$ extends to a continuous operator $\psi(D): L^t(\mathbb{R}^d)\to L^t(\mathbb{R}^d)$ for $1\leq t\leq\infty$ whose operator norm is bounded by $\|\widehat{\psi}\|_{L^1}$, because

$$\|\psi(D)u\|_{L^t} = \|\widehat{\psi} * u\|_{L^t} \leq \|\widehat{\psi}\|_{L^1}\|u\|_{L^t} \tag{11}$$

by Young's inequality. We will use the following more general result on operator-valued pseudodifferential operators:

Theorem 3.1 ([11, Theorem 0.11.F]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ be the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ equipped with the operator norm. If $P(\cdot)\in C^\infty(\mathbb{R}^d,\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))$ satisfies*

$$\|D_\xi^\alpha P(\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq C_\alpha(1+|\xi|^2)^{-|\alpha|/2} \tag{12}$$

for each multi-index α, then for each $1<t<\infty$ the operator

$$P(D): L^t(\mathbb{R}^d,\mathcal{H}_1)\to L^t(\mathbb{R}^d,\mathcal{H}_2)$$

is bounded.

Remark 3.2. The operator-valued pseudodifferential operator $P(D)$ is defined by

$$P(D)u = (2\pi)^{-d} \int e^{i(x-y)\xi} P(\xi)u(y) d\xi dy.$$

Remark 3.3. The proof of Theorem 3.1 does not need much knowledge on the theory of pseudodifferential operators and, in fact, is rather simple. Since the case $t = 2$ is proved by using Parseval's identity, one only has to check that the arguments in Sections 0.2 and 0.11 of [11] extend straightforwardly to the operator-valued case.

By "integration by parts on w", we will mean application, for $f \in C^2(\mathbb{R}^d)$ and $g \in C_0^1(\mathbb{R}^d)$, of the formula

$$\begin{aligned}\int e^{if(w)} g(w) dw &= -\sum_{k=1}^{d} \int i(\partial_k f(w)) e^{if(w)} \cdot \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^{d} (\partial_j f(w))^2} dw \\ &= i \cdot \int e^{if(w)} \cdot \sum_{k=1}^{d} \partial_k \left(\frac{\partial_k f(w) \cdot g(w)}{\sum_{j=1}^{d} (\partial_j f(w))^2} \right) dw,\end{aligned}$$

where $w = (w_k)_{k=1}^d \in \mathbb{R}^d$, and ∂_k denotes partial differentiation with respect to w_k.

4. A pseudolocal property

Although the pseudodifferential operators $\psi_{\Theta,n,\sigma}(D)$ are not local operators, i.e., $\psi_{\Theta,n,\sigma}(D)u$ does not necessarily vanish outside of the support of $u \in C^\infty(K)$, we have the following rapid decay property, which will be used in Sections 5 and 7:

Lemma 4.1. *For all positive real numbers b, c, ϵ and each $1 < t \le \infty$, there exists a constant $C = C(b, c, \epsilon, t) > 0$ such that*

$$|u_{\Theta,n,\sigma}(x)| \le \frac{C \sum_{\tau=\pm} \sum_{\ell \ge 0} 2^{-c\max\{n,\ell\}} \|u_{\Theta,\ell,\tau}\|_{L^t}}{d(x, \mathrm{supp}(u))^b},$$

for all $n \ge 1$, all $u \in C^\infty(K)$, and all $x \in \mathbb{R}^d$ satisfying $d(x, \mathrm{supp}(u)) > \epsilon$.

Note that the numerator of the right hand side above is bounded by $C\|u\|_{C_*^{\Theta,p,q}}$ in the case $t = \infty$, and by $C\|u\|_{W_*^{\Theta,p,q,t}}$ in the case $1 < t < \infty$ provided $c > p$.

Proof. Choose a C^∞ function $\rho : \mathbb{R}^d \to [0, 1]$ supported in the disk of radius $\epsilon/4$ centered at the origin and so that $\int \rho(x)dx = 1$. Fix $u \in C^\infty(K)$. Let $U(\epsilon)$ be the ϵ-neighborhood of $\mathrm{supp}(u)$. Put $\chi_0(x) = \int \mathbf{1}_{U(\epsilon/4)}(y) \cdot \rho(x-y) dy$, where $\mathbf{1}_Z$ denotes the indicator function of a subset $Z \subset \mathbb{R}^d$. Then χ_0 is supported in $U(\epsilon/2)$, with $0 \le \chi_0(x) \le 1$ for any $x \in \mathbb{R}^d$, and $\chi_0(x) = 1$ for $x \in \mathrm{supp}(u)$. Since $\|\chi_0\|_{C_*^c}$ is bounded by a constant depending only on c and ϵ, we have

$$\|\psi_j(D)\chi_0\|_{L^\infty} \le C(c, \epsilon) 2^{-cj}. \tag{13}$$

Furthermore, integrating several times by parts on ξ in

$$\psi_j(D)\chi_0(y) = (2\pi)^{-d} \int e^{i(y-w)\xi} \psi_j(\xi)\chi_0(w) d\xi dw,$$

we can see that for any $y \in \mathbb{R}^d$ satisfying $d(y, \mathrm{supp}(\chi_0)) \ge \epsilon/4$

$$|\psi_j(D)\chi_0(y)| \le C(b, c, \epsilon) \cdot 2^{-cj} d(y, \mathrm{supp}(\chi_0))^{-b}. \tag{14}$$

We assume $d(x, \mathrm{supp}(u)) > \epsilon$ henceforth and estimate

$$\psi_{\Theta,n,\sigma}(D)u(x) = \psi_{\Theta,n,\sigma}(D)(\chi_0 u)(x) = \sum_{(\ell,\tau)\in\Gamma} \widehat{\psi}_{\Theta,n,\sigma} * (\chi_0 u_{\Theta,\ell,\tau})(x).$$

By the Hölder inequality, we have

$$\begin{aligned}(15)\qquad |\widehat{\psi}_{\Theta,n,\sigma} * (\chi_0 u_{\Theta,\ell,\tau})(x)| &\le \|\mathbf{1}_{U(\epsilon/2)} \cdot \widehat{\psi}_{\Theta,n,\sigma}(x-\cdot)\|_{L^{t'}} \|\chi_0 u_{\Theta,\ell,\tau}\|_{L^t} \\ &\le C(b,c,\epsilon,t') \cdot 2^{-cn} \cdot d(x, \mathrm{supp}(u))^{-b} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t}\end{aligned}$$

for any n and ℓ, where t' is the conjugate exponent of t, i.e. $t^{-1} + (t')^{-1} = 1$.

Suppose that $\ell \ge n+3$. Then we have

$$\psi_{\Theta,n,\sigma}(D)((\psi_j(D)\chi_0) \cdot u_{\Theta,\ell,\tau}) = 0 \quad \text{for } j < \ell - 2,$$

because $\mathrm{supp}(\psi_{\Theta,n,\sigma})$ does not meet $\mathrm{supp}(\psi_j) + \mathrm{supp}(\psi_{\Theta,\ell,\tau})$ which supports the Fourier transform of $(\psi_j(D)\chi_0) \cdot u_{\Theta,\ell,\tau}$. Thus

$$\psi_{\Theta,n,\sigma}(D)(\chi_0 u_{\Theta,\ell,\tau}) = \sum_{j \ge \ell-2} \widehat{\psi}_{\Theta,n,\sigma} * ((\psi_j(D)\chi_0) \cdot u_{\Theta,\ell,\tau}).$$

For each $j \ge \ell - 2$ with $\ell \ge n+3$, we can see from (13-14) that

$$\begin{aligned}&|\widehat{\psi}_{\Theta,n,\sigma} * ((\psi_j(D)\chi_0) \cdot u_{\Theta,\ell,\tau})(x)| \\ &\quad \le \|\widehat{\psi}_{\Theta,n,\sigma}\|_{L^\infty} \cdot \|\mathbf{1}_{\mathbb{R}^d \setminus U(\delta)} \cdot \psi_j(D)\chi_0\|_{L^{t'}} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t} \\ &\qquad + \|\mathbf{1}_{U(\delta)} \cdot \widehat{\psi}_{\Theta,n,\sigma}(x-\cdot)\|_{L^{t'}} \cdot \|\psi_j(D)\chi_0\|_{L^\infty} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t} \\ &\quad \le C(b,c,\epsilon,t) \cdot 2^{-cj} \cdot d(x, \mathrm{supp}(u))^{-b} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t},\end{aligned}$$

where $\delta = \epsilon/2 + d(x, \mathrm{supp}(u))/4$. (We decomposed the domain of integration in the convolution into $U(\delta)$ and its complement.) Hence, if $\ell \ge n+3$, we have

$$|\psi_{\Theta,n,\sigma}(D)(\chi_0 u_{\Theta,\ell,\tau})(x)| \le C(b,c,\epsilon,t) \cdot 2^{-c\ell} d(x, \mathrm{supp}(u))^{-b} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t}.$$

With this and (15) we conclude the proof of the lemma. □

5. Compact embeddings

Recall that $K \subset \mathbb{R}^d$ is a compact subset with non-empty interior. If $p' \le p$ and $q' \le q$, we have the obvious continuous inclusions

$$(16)\qquad C_*^{\Theta,p,q}(K) \subset C_*^{\Theta,p',q'}(K), \quad W_*^{\Theta,p,q,t}(K) \subset W_*^{\Theta,p',q',t}(K) \quad \text{for } 1 < t < \infty.$$

Here we prove:

Proposition 5.1. *If $p' < p$ and $q' < q$, the inclusions (16) are compact.*

Proof. Take any sequence $u^{(k)}$, $k \ge 1$, in $C^\infty(K)$ such that $\|u^{(k)}\|_{C_*^{\Theta,p,q}} < E$ (respectively $\|u^{(k)}\|_{W_*^{\Theta,p,q,t}} < E$) for some positive constant $E > 0$. We show that there exists a subsequence $\{k(j)\}$ such that $\{u^{(k(j))}\}$ is a Cauchy sequence in the norm $\|\cdot\|_{C_*^{\Theta,p',q'}}$ (respectively $\|\cdot\|_{W_*^{\Theta,p',q',t}}$). For each $(n,\sigma) \in \Gamma$, the Fourier transform $\hat{u}^{(k)}_{\Theta,n,\sigma}$ of $u^{(k)}_{\Theta,n,\sigma}$ is a C^∞ function supported on $\{\xi \mid 2^{n-1} \le |\xi| \le 2^{n+1}\}$, and its first order derivatives are bounded uniformly for $k \ge 1$ and $\xi \in \mathbb{R}^d$ since $(1+|x|)u^{(k)}_{\Theta,n,\sigma}(x)$ are uniformly bounded in L^1norm from Lemma 4.1. Hence, by Ascoli-Arzelá's theorem and by the diagonal argument, we can choose a subsequence $\{k(j)\}$ such that the sequences $\{\hat{u}^{(k(j))}_{\Theta,n,\sigma}\}_{j=0}^\infty$ are all Cauchy sequences with respect to the L^∞-norm and so is the sequence $\{u^{(k(j))}_{\Theta,n,\sigma}\}_{j=0}^\infty$. This is the subsequence with the required property. Indeed, for given $\epsilon > 0$, we can choose an integer $N > 0$ so that $\sum_{n>N}(2^{(q'-q)n} + 2^{(p'-p)n})E < \epsilon/2$, and then we have

$$\|u^{(k(j))} - u^{(k(j'))}\|_{C_*^{\Theta,p',q'}} \le \epsilon/2 + \sum_{n\le N}\left(2^{p'n}\left\|u^{(k(j))}_{\Theta,n,+} - u^{(k(j'))}_{\Theta,n,+}\right\|_{L^\infty} + 2^{q'n}\left\|u^{(k(j))}_{\Theta,n,-} - u^{(k(j'))}_{\Theta,n,-}\right\|_{L^\infty}\right),$$

(respectively the same inequality with the norms $\|\cdot\|_{C_*^{\Theta,p',q'}}$ and $\|\cdot\|_{L^\infty}$ replaced by $\|\cdot\|_{W_*^{\Theta,p',q',t}}$ and $\|\cdot\|_{L^t}$). The right hand side is $< \epsilon$ for large enough j, j'. $\square$

6. A Lasota-Yorke type inequality

Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighborhood W of K. Let $T : W \to K'$ be a C^r diffeomorphism onto its image. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a C^{r-1} function such that $\mathrm{supp}(g) \subset K$. In this section we study the transfer operator on $\mathbb{R}^d$:

$$L : C^{r-1}(K') \to C^{r-1}(K), \qquad Lu(x) = g(x)\cdot u\circ T(x).$$

For two fixed combinations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$ as in Section 2, we make the following *cone-hyperbolicity* assumption on T:

$$DT_x^{tr}(\mathbb{R}^d \setminus \mathrm{interior}\,(\mathbf{C}_+)) \subset \mathrm{interior}\,(\mathbf{C}'_-) \cup \{0\} \quad \text{for all } x \in W, \tag{17}$$

where DT_x^{tr} denotes the transpose of the derivative of T at x. We put

$$\|T\|_+ = \sup_{x\in\mathrm{supp}(g)} \sup_{0\ne DT_x^{tr}(\xi)\notin\mathbf{C}'_-} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|}, \quad \|T\|_- = \inf_{x\in W}\inf_{0\ne\xi\notin\mathbf{C}_+} \frac{\|DT_x^{tr}(\xi)\|}{\|\xi\|}.$$

Theorem 6.1. *Fix Θ and Θ' and assume (17). For any $q < 0 < p$ such that $p - q < r - 1$, the operator L extends to continuous operators*

$$L : C_*^{\Theta,p,q}(K') \to C_*^{\Theta',p,q}(K), \qquad L : W_*^{\Theta,p,q,t}(K') \to W_*^{\Theta',p,q,t}(K)$$

for $1 < t < \infty$. Furthermore, for any $0 \le p' < p$ and $q' < q$ such that $p - q' < r - 1$, we have the following Lasota-Yorke type inequalities:

Hölder case: *There exist a constant C, that does not depend on T or g, and a constant $C(T, g)$, that may depend on T and g, such that for any $u \in C_*^{\Theta,p,q}(K)$*

$$\|Lu\|_{C_*^{\Theta',p,q}} \le C\|g\|_\infty \cdot \max\{\|T\|_+^p, \|T\|_-^q\}\|u\|_{C_*^{\Theta,p,q}} + C(T,g)\|u\|_{C_*^{\Theta,p',q'}}.$$

Sobolev case: *For each $1 < t < \infty$, there exist a constant $C(t)$, that does not depend on T or g, and a constant $C(T, g, t)$, that may depend on T and g, such that for any $u \in W_*^{\Theta,p,q,t}(K)$*

$$\|Lu\|_{W_*^{\Theta',p,q,t}} \le C(t)\|g\|_\infty \cdot \frac{\max\{\|T\|_+^p, \|T\|_-^q\}}{\inf|\det DT|^{1/t}}\|u\|_{W_*^{\Theta,p,q,t}} + C(T,g,t)\|u\|_{W_*^{\Theta,p',q',t}}.$$

For the proof of Theorem 6.1, we need more notation. By (17) there exist closed cones $\widetilde{\mathbf{C}}_+$, $\widetilde{\mathbf{C}}_-$ such that

$$\widetilde{\mathbf{C}}_+ \subset \mathrm{interior}\,(\mathbf{C}_+) \text{ and } \widetilde{\mathbf{C}}_- \subset \mathrm{interior}\,(\mathbf{C}_-),$$

and that

$$DT_x^{tr}(\mathbb{R}^d \setminus \mathrm{interior}(\widetilde{\mathbf{C}}_+)) \subset \mathrm{interior}(\mathbf{C}'_-) \cup \{0\} \quad \text{for all } x \in \mathrm{supp}(g). \tag{18}$$

Let $\tilde{\varphi}_+$, $\tilde{\varphi}_- : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions satisfying

$$\tilde{\varphi}_+(\xi) = \begin{cases} 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_-; \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_-. \end{cases}, \qquad \tilde{\varphi}_-(\xi) = \begin{cases} 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_+; \\ 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_+. \end{cases}$$

Recall the function χ we fixed in the beginning. Put, for $\ell \geq 0$,

$$\tilde{\psi}_\ell(\xi) = \begin{cases} \chi(2^{-\ell-1}|\xi|) - \chi(2^{-\ell+2}|\xi|), & \text{if } \ell \geq 1; \\ \chi(2^{-1}|\xi|), & \text{if } \ell = 0; \end{cases}$$

and then define, for $(\ell,\tau) \in \Gamma$,

$$\tilde{\psi}_{\Theta,\ell,\tau}(\xi) = \tilde{\psi}_\ell(\xi)\tilde{\varphi}_\tau(\xi/|\xi|).$$

Note that $\tilde{\psi}_{\Theta,\ell,\tau}(\xi) = 1$ if $\xi \in \operatorname{supp}(\psi_{\Theta,\ell,\tau})$.

Next, take integers $h_{\max}$, $h_{\min}$, $h^-_{\min}$, and $h^+_{\max}$ such that for all $x \in W$

$$\begin{aligned} 2^{h_{\min}+4}\|\xi\| \leq \|DT_x^{tr}(\xi)\| &\leq 2^{h_{\max}-4}\|\xi\| && \text{for any } \xi \in \mathbb{R}^d, \\ \|DT_x^{tr}(\xi)\| &\leq 2^{h^+_{\max}-4}\|\xi\| && \text{if } DT_x^{tr}(\xi) \notin \mathbf{C}'_-, \\ 2^{h^-_{\min}+4}\|\xi\| \leq \|DT_x^{tr}(\xi)\| & && \text{if } \xi \notin \widetilde{\mathbf{C}}_+. \end{aligned}$$

By modifying the cones $\widetilde{\mathbf{C}}_\pm$ if necessary, we may and do assume

$$2^{h^-_{\min}} > \|T\|_- - 5, \qquad 2^{h^+_{\max}} < \|T\|_+ + 5.$$

We write $(\ell,\tau) \hookrightarrow (n,\sigma)$ if either

- $(\tau,\sigma) = (+,+)$ and $n \leq \ell + h^+_{\max}$, or
- $(\tau,\sigma) = (-,-)$ and $\ell + h^-_{\min} \leq n$, or
- $(\tau,\sigma) = (+,-)$ and $\ell + h_{\min} \leq n \leq \ell + h_{\max}$.

We write $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ otherwise. By the definition of $\not\hookrightarrow$, and by (18), there exists an integer $N(T) > 0$ such that for all $x \in \operatorname{supp}(g)$

$$d(\operatorname{supp}(\psi_{\Theta',n,\sigma}), DT_x^{tr}(\operatorname{supp}(\tilde{\psi}_{\Theta,\ell,\tau}))) \geq 2^{\max\{n,\ell\}-N(T)} \quad \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma). \tag{19}$$

Proof of Theorem 6.1. For $v := Lu$, we have

$$v_{\Theta',n,\sigma} = \sum_{(\ell,\tau)\in\Gamma} \psi_{\Theta',n,\sigma}(D)L(u_{\Theta,\ell,\tau}).$$

We define $\mathbf{S}$ as the formal matrix of operators

$$S^{\ell,\tau}_{n,\sigma}u = \begin{cases} \psi_{\Theta',n,\sigma}(D)L\,u, & \text{if } (\ell,\tau) \hookrightarrow (n,\sigma), \\ \psi_{\Theta',n,\sigma}(D)L\,\tilde{\psi}_{\Theta,\ell,\tau}(D)u & \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma), \end{cases}$$

for $((\ell,\tau),(n,\sigma)) \in \Gamma \times \Gamma$. That is, we set

$$\mathbf{S}\left((u_{\Theta,\ell,\tau})_{(\ell,\tau)\in\Gamma}\right) = \left(\sum_{(\ell,\tau)\in\Gamma} S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}\right)_{(n,\sigma)\in\Gamma}.$$

Since $\tilde{\psi}_{\Theta,\ell,\tau}(D)u_{\Theta,\ell,\tau} = u_{\Theta,\ell,\tau}$, we have the commutative relation $\mathbf{S}\circ\mathcal{Q}_\Theta = \mathcal{Q}_{\Theta'}\circ L$. For the proof of Theorem 6.1, it is enough to show

$$\|\mathbf{S}\mathbf{u}\|_{p,q,\infty} < C2^{\max\{ph^+_{\max},qh^-_{\min}\}}\|g\|_{L^\infty}\|\mathbf{u}\|_{p,q,\infty} + C(T,g)\|\mathbf{u}\|_{p',q',\infty}$$

and that, for $1 < t < \infty$,

$$\|\mathbf{S}\mathbf{u}\|_{p,q,t} < \frac{C(t)2^{\max\{ph^+_{\max}, qh^-_{\min}\}}\|g\|_{L^\infty}}{\inf|\det DT|^{1/t}}\|\mathbf{u}\|_{p,q,t} + C(T,g,t)\|\mathbf{u}\|_{p',q',t}.$$

To prove the above inequalities, we split the matrix of operator $\mathbf{S}$ into two parts:

$$\mathbf{S}_0 = (\widetilde{S}^{\ell,\tau}_{n,\sigma}), \quad \widetilde{S}^{\ell,\tau}_{n,\sigma} = \begin{cases} S^{\ell,\tau}_{n,\sigma}, & \text{if } (\ell,\tau) \hookrightarrow (n,\sigma); \\ 0 & \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma), \end{cases}$$

and $\mathbf{S}_1 = \mathbf{S} - \mathbf{S}_0$. We first consider $\mathbf{S}_0$. This is the composition $\Phi(D) \circ \Psi \circ \mathbf{L}$ of

- the operator $\mathbf{L}$ defined by $\mathbf{L}(\mathbf{u})(x) = g(x) \cdot \mathbf{u} \circ T(x)$,
- the operator Ψ defined by
$$\Psi\left((f_{\ell,\tau}(x))_{(\ell,\tau)\in\Gamma}\right)_{(n,\sigma)} = \sum_{(\ell,\tau)\hookrightarrow(n,\sigma)} f_{\ell,\tau}(x),$$
where $\sum_{(\ell,\tau)\hookrightarrow(n,\sigma)}$ is the sum over $(\ell,\tau) \in \Gamma$ such that $(\ell,\tau) \hookrightarrow (n,\sigma)$,
- the pseudodifferential operator $\Phi(D)$ with symbol $\Phi : \mathbb{R}^d \to \mathcal{L}(\mathbb{C}^\Gamma, \mathbb{C}^\Gamma)$,
$$\Phi(\xi)\left((f_{\ell,\tau})_{(\ell,\tau)\in\Gamma}\right)_{(n,\sigma)} = \psi_{\Theta',n,\sigma}(\xi) f_{n,\sigma}.$$

Clearly $\|\mathbf{L}\|_{p,q,\infty} \le \|g\|_\infty$ and $\|\mathbf{L}\|_{p,q,t} \le \|g\|_\infty \sup|\det DT|^{-1/t}$ for $1 < t < \infty$. Also we can prove

$$\|\Psi\|_{p,q,t} \le C2^{\max\{ph^+_{\max}, qh^-_{\min}\}} \quad \text{for } 1 < t \le \infty \tag{20}$$

as follows. Set $c(+) = p$, $c(-) = q$ and observe that there is C so that

$$\sum_{(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{2c(\sigma)n - 2c(\tau)\ell} \le C2^{2\max\{ph^+_{\max}, qh^-_{\min}\}}, \ \forall (n,\sigma), \tag{21}$$

For $\mathbf{f}(x) = (f_{n,\sigma}(x))_{(n,\sigma)\in\Gamma}$, we have, at each point $x \in \mathbb{R}^d$,

$$|\Psi(\mathbf{f})|_{C^{p,q}}(x) \le C2^{\max\{ph^+_{\max}, qh^-_{\min}\}}|\mathbf{f}|_{C^{p,q}}(x) \tag{22}$$

and also, by Cauchy-Schwartz and (21),

$$|\Psi(\mathbf{f})|_{\mathcal{W}^{p,q}}(x) \le C2^{\max\{ph^+_{\max}, qh^-_{\min}\}}|\mathbf{f}|_{\mathcal{W}^{p,q}}(x). \tag{23}$$

Taking the supremum and L^t norm of the both sides respectively, we obtain (20). The operator $\Phi(D)$ is bounded with respect to the norm $\|\cdot\|_{p,q,t}$ for $1 < t \le \infty$: If $t = \infty$, this follows from (11) since $\widehat{\psi}_{\Theta',n,\sigma}$ is bounded uniformly for $(n,\sigma) \in \Gamma$ in L^1-norm, and the case $1 < t < \infty$ follows from Theorem 3.1. Thus we conclude

$$\|\mathbf{S}_0\|_{p,q,\infty} \le C\|g\|_\infty \cdot 2^{\max\{ph^+_{\max}, qh^-_{\min}\}},$$

and

$$\|\mathbf{S}_0\|_{p,q,t} \le \frac{C(t)\|g\|_\infty \cdot 2^{\max\{ph^+_{\max}, qh^-_{\min}\}}}{\inf|\det DT|^{1/t}} \quad \text{for } 1 < t < \infty.$$

Next we consider $\mathbf{S}_1$. It only remains to show the following two estimates:

$$\|\mathbf{S}_1\mathbf{u}\|_{p,q,\infty} < C(T,g)\|\mathbf{u}\|_{p',q',\infty} \quad \text{and} \quad \|\mathbf{S}_1\mathbf{u}\|_{p,q,t} < C(T,g,t)\|\mathbf{u}\|_{p',q',t}, \forall 1 < t < \infty.$$

For this, it is enough to prove that for $1 < t \le \infty$,

$$\|S^{\ell,\tau}_{n,\sigma} u\|_{L^t} \le C(T,g)2^{-(r-1)\max\{n,\ell\}}\|u\|_{L^t} \quad \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma). \tag{24}$$

Indeed, (24) with $t=\infty$ implies the first estimate:

$$\begin{aligned}\|\mathbf{S}_1\mathbf{u}\|_{p,q,\infty} &\le \sup_{(n,\sigma)\in\Gamma} \sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} 2^{pn}\|S^{\ell,\tau}_{n,\sigma} u_{\Theta,\ell,\tau}\|_{L^\infty}\\ &\le C(T,g)\cdot \sup_{(n,\sigma)\in\Gamma}\left(\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} 2^{pn-q'\ell-(r-1)\max\{n,\ell\}}\right)\|\mathbf{u}\|_{p',q',\infty},\end{aligned}$$

and the sum in $(\cdot)$ above is finite from the assumption $p-q'<r-1$. Next, the Cauchy-Schwartz inequality implies that

$$(|\mathbf{S}_1\mathbf{u}|_{\mathcal{W}^{p,q}}(x))^2 = \sum_{(n,\sigma)\in\Gamma} 2^{2c(\sigma)n}\left|\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} S^{\ell,\tau}_{n,\sigma} u_{\Theta,\ell,\tau}(x)\right|^2$$

is bounded by

$$\begin{aligned}&\sup_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\in\Gamma}\left(2^{c(\sigma)n-c(\tau)\ell-(r-1)\max\{n,\ell\}}\right)^2\\ &\qquad\cdot\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)}\left(2^{c(\tau)\ell+(r-1)\max\{n,\ell\}}|S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}(x)|\right)^2.\end{aligned}$$

The assumption $p-q<p-q'<r-1$ implies that the sum in the second line above is finite. Setting $c'(+)=p'$, and $c'(-)=q'$, we have

$$\begin{aligned}&\left\|\left(\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)}\left(2^{c(\tau)\ell+(r-1)\max\{n,\ell\}}|S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}(x)|\right)^2\right)^{1/2}\right\|_{L^t(\mathbb{R}^d)}\\ &\quad\le \sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} 2^{c(\tau)\ell+(r-1)\max\{n,\ell\}}\|S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}\|_{L^t}\\ &\quad\le C(T,g)\cdot\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} 2^{c(\tau)\ell}\|u_{\Theta,\ell,\tau}\|_{L^t}\quad\text{by (24) for } 1<t<\infty\\ &\quad\le C(T,g)\cdot\left(\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\leftrightarrow(n,\sigma)} 2^{(c(\tau)-c'(\tau))\ell}\right)\|\mathbf{u}\|_{p'q',t}.\end{aligned}$$

Since the sum in $(\cdot)$ in the last line is finite, this implies the second estimate.

To show (24), we rewrite the operator $S^{\ell,\tau}_{n,\sigma}$ as

$$(S^{\ell,\tau}_{n,\sigma}u)(x) = (2\pi)^{-2d}\int V^{\ell,\tau}_{n,\sigma}(x,y)\cdot u\circ T(y)|\det DT(y)|dy,$$

where

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int e^{i(x-w)\xi+i(T(w)-T(y))\eta} g(w)\psi_{\Theta',n,\sigma}(\xi)\tilde{\psi}_{\Theta,\ell,\tau}(\eta)dwd\xi d\eta. \tag{25}$$

Since $\|u\circ T\cdot|\det DT|\|_{L^t}\le C(T,g)\|u\|_{L^t}$, the inequality (24) follows if we show that there exists $C(T)$ such that for all $(\ell,\tau)\not\leftrightarrow(n,\sigma)$ and all $1<t\le\infty$ the operator norm of the integral operator

$$H^{\ell,\tau}_{n,\sigma}: v\mapsto\int V^{\ell,\tau}_{n,\sigma}(x,y)v(y)dy$$

acting on $L^t(\mathbb{R}^d)$ is bounded by $C(T,g)\cdot 2^{-(r-1)\max\{n,\ell\}}$.

Define the positive-valued integrable function $b : \mathbb{R}^d \to \mathbb{R}$ by

$$b(x) = 1 \quad \text{if } |x| \le 1, \qquad b(x) = |x|^{-d-1} \quad \text{if } |x| > 1. \tag{26}$$

The required estimate on $H^{\ell,\tau}_{n,\sigma}$ follows if we show

$$|V^{\ell,\tau}_{n,\sigma}(x,y)| \le C(T,g) 2^{-(r-1)\max\{n,\ell\}} \cdot 2^{d\min\{n,\ell\}} b(2^{\min\{n,\ell\}}(x-y)) \tag{27}$$

for some $C(T,g) > 0$ and all $(\ell,\tau) \not\hookrightarrow (n,\sigma)$. Indeed, as the right hand side of (27) is written as a function of $x-y$, say $B(x-y)$, we have, by Young's inequality,

$$\|H^{\ell,\tau}_{n,\sigma} v\|_{L^t} \le \|B * v\|_{L^t} \le \|B\|_{L^1} \|v\|_{L^t} \le C(T) 2^{-(r-1)\max\{n,\ell\}} \cdot \|b\|_{L^1} \cdot \|v\|_{L^t}.$$

Below we prove the estimate (27). We may assume that r is not an integer, up to replacing r by $r' \in (p-q+1, r]$.

Integrating (25) by parts $[r]-1$ times on w, we obtain

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta} F(\xi,\eta,w) \psi_{\Theta',n,\sigma}(\xi) \tilde{\psi}_{\Theta,\ell,\tau}(\eta) dw d\xi d\eta,$$

where $F(\xi,\eta,w)$ is a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. Using (19), we can see that for all α, β

$$\|\partial_\xi^\alpha \partial_\eta^\beta F\|_{C^{r-[r]}} \le C_{\alpha,\beta}(T,g) 2^{-n|\alpha| - \ell|\beta| - ([r]-1)\max\{n,\ell\}}.$$

Put $F_k(\xi,\eta,w) = \psi_k(D) F(\xi,\eta,w)$, where the pseudodifferential operator $\psi_k(D)$ acts on $F(\xi,\eta,w)$ as a function of w. Then each $F_k(\xi,\eta,w)$ is C^∞ and satisfies

$$\|\partial_\xi^\alpha \partial_\eta^\beta \partial_w^\gamma F_k\|_{L^\infty} \le C_{\alpha,\beta,\gamma}(T,g) \cdot 2^{-n|\alpha| - \ell|\beta| - ([r]-1)\max\{n,\ell\} - (r-[r]-|\gamma|)k} \tag{28}$$

for any α, β, and γ. (Note that $\partial_\xi^\alpha \partial_\eta^\beta \partial_w^\gamma F$, as a function of w, lies to $C_*^{r-[r]-|\gamma|}(\mathbb{R}^d)$ with $r-[r]-|\gamma| \in \mathbb{R}$.) Correspondingly, we decompose $V^{\ell,\tau}_{n,\sigma}(x,y)$ into

$$W^{(k)}(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta} G_k(\xi,\eta,w) dw d\xi d\eta \quad \text{for } k \ge 0, \tag{29}$$

where $G_k(\xi,\eta,w) = F_k(\xi,\eta,w) \psi_{\Theta',n,\sigma}(\xi) \tilde{\psi}_{\Theta,\ell,\tau}(\eta)$.

Recall the choice of $N(T)$ from (19). We first estimate $W^{(k)}(x,y)$ for $k > \max\{n,\ell\} - N$. Decompose $W^{(k)}(x,y)$ into

$$\int e^{i(x-w)\xi + i(T(w)-T(y))\eta} G_k(\xi,\eta,w) \psi_i(2^n(x-w)) \psi_j(2^\ell(T(w)-T(y))) dw d\xi d\eta$$

for $i \ge 0$ and $j \ge 0$, each of which is denoted by $W^{(k)}_{i,j}(x,y)$. Integrating by parts on ξ for $d+1$ times if $i > 0$, and integrating by parts on η for $d+1$ times if $j > 0$, we can see

$$|W^{(k)}_{i,j}(x,y)| \le C(T,g) 2^{d\min\{n-i,\ell-j\} - ([r]-1)\max\{n,\ell\} - (r-[r])k - i - j}. \tag{30}$$

In fact the case $i > 0$ and $j > 0$ can be shown as follows and the other cases are similar: The result of the integration by parts is a sum of terms of the form

$$\pm \int e^{i(x-w)\xi + i(T(w)-T(y))\eta} \partial_\xi^\alpha \partial_\eta^\beta G_k(\xi,\eta,w) G_{\alpha,\beta}(x,w,y) dw d\xi d\eta,$$

where $|\alpha| = |\beta| = d+1$, and

$$G_{\alpha,\beta}(x,w,y) = \frac{(x-w)^\alpha (T(w)-T(y))^\beta \cdot \psi_i(2^n(x-w)) \psi_j(2^\ell(T(w)-T(y)))}{|x-w|^{2|\alpha|} |T(w)-T(y)|^{2|\beta|}}.$$

Note that $|G_{\alpha,\beta}(x,w,y)| \leq C(T) \cdot 2^{(d+1)(n-i)+(d+1)(\ell-j)}$ and that $G_{\alpha,\beta}(x,w,y) \neq 0$ only if $|x-w| < 2^{-n+i+1}$ and $|T(w)-T(y)| < 2^{-\ell+j+1}$. Together with (28), we see

$$\int |\partial_\xi^\alpha \partial_\eta^\beta G_k(\xi,\eta,w) G_{\alpha,\beta}(x,w,y)| dw \leq$$
$$C(T,g) 2^{-n(d+1)-\ell(d+1)-([r]-1)\max\{n,\ell\}-d\max\{n-i,\ell-j\}+(d+1)(n-i)+(d+1)(\ell-j)-(r-[r])k}$$
$$= C(T,g) 2^{-nd-\ell d-([r]-1)\max\{n,\ell\}+d\min\{n-i,\ell-j\}-(r-[r])k-i-j}.$$

Since $\partial_\xi^\alpha \partial_\eta^\beta G_k(\xi,\eta,w) \neq 0$ only if $\xi \in \operatorname{supp}(\psi_{\Theta',n,\sigma})$ and $\eta \in \operatorname{supp}(\tilde{\psi}_{\Theta,\ell,\tau})$, we get (30).

Since $r > [r]$, for all x, y and i, j, (30) implies

$$\sum_{k > \max\{n,\ell\}-N} |W_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{d\min\{n-i,\ell-j\}-(r-1)\max\{n,\ell\}-i-j}. \tag{31}$$

Thus, for all x and y,

$$\sum_{k > \max\{n,\ell\}-N} \sum_{i \geq 0} \sum_{j \geq 0} |W_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{-(r-1)\max\{n,\ell\}+d\min\{n,\ell\}}. \tag{32}$$

If $|x-y| \geq 2^{-\min\{n,\ell\}}$ we have better estimates: Let $M = M(T)$ be so that $|T(x)-T(y)| \geq 2^{-M}|x-y|$ for all $x, y \in W$. Let $q_0 = q_0(x,y) \leq \min\{n,\ell\}$ be the smallest integer satisfying $|x-y| > 2^{-q_0}$. If $\min\{n-i,\ell-j\} \geq q_0 + M + 4$, we have $W_{i,j}^{(k)}(x,y) = 0$ since $\psi_i(2^n(x-z))\psi_j(2^\ell(T(z)-T(y))) \equiv 0$. Therefore, we get from (31) that

$$\sum_{k > \max\{n,\ell\}-N} \sum_{i \geq 0} \sum_{j \geq 0} |W_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{-(r-1)\max\{n,\ell\}-\min\{n,\ell\}+(d+1)q_0}. \tag{33}$$

To finish, we consider the case $k \leq \max\{n,\ell\} - N$. Integrate (29) by parts on w once, obtaining

$$W^{(k)}(x,y) = \int e^{i(x-w)\xi + i(T(w)-T(y))\eta} \widetilde{G}_k(\xi,\eta,w) dw d\xi d\eta \quad \text{for } k \geq 0.$$

Decompose $W^{(k)}(x,y)$ into the sum of $\widetilde{W}_{i,j}^{(k)}(x,y)$ for $i \geq 0$ and $j \geq 0$, each of which is defined in the same manner as $W_{i,j}^{(k)}(x,y)$, but with $G_k(\xi,\eta,w)$ replaced by $\widetilde{G}_k(\xi,\eta,w)$. Integrate $\widetilde{W}_{i,j}^{(k)}(x,y)$ by parts on ξ for $d+1$ times if $i > 0$, on η for $d+1$ times if $j > 0$. Then we obtain, by (28) for $|\gamma| = 1$,

$$|\widetilde{W}_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{d\min\{n-i,\ell-j\}-[r]\max\{n,\ell\}+([r]+1-r)k-i-j},$$

and hence, since $[r]+1 > r$, we find for any x, y, i and j,

$$\sum_{k \leq \max\{n,\ell\}-N} |\widetilde{W}_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{d\min\{n-i,\ell-j\}-(r-1)\max\{n,\ell\}-i-j}.$$

Therefore

$$\sum_{k \leq \max\{n,\ell\}-N} \sum_{i \geq 0} \sum_{j \geq 0} |\widetilde{W}_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{-(r-1)\max\{n,\ell\}+d\min\{n,\ell\}}. \tag{34}$$

By the same argument as above, we see that, for $|x-y| > 2^{-q_0} \geq 2^{-\min\{n,\ell\}}$,

$$\sum_{k \leq \max\{n,\ell\}-N} \sum_{i \geq 0} \sum_{j \geq 0} |\widetilde{W}_{i,j}^{(k)}(x,y)| \leq C(T,g) 2^{-(r-1)\max\{n,\ell\}-\min\{n,\ell\}+(d+1)q_0}. \tag{35}$$

The inequalities (32,33,34,35) imply (27) for noninteger r. □

7. Partitions of unity

Let $r > 0$ and recall $K \subset \mathbb{R}^d$ is compact with nonempty interior. A C^r partition of unity on K is by definition a finite family of C^r functions $g_i : \mathbb{R}^d \to [0,1]$, $1 \le i \le I$, such that $\sum_i g_i(x) = 1$ for $x \in K$ and $\sum_i g_i(x) \le 1$ for $x \in \mathbb{R}^d$. The intersection multiplicity of a partition of unity is $\nu := \sup_x \#\{i \mid g_i(x) > 0\}$. For $u \in C^\infty(K)$, we set $u_i := g_i u$ so that $u = \sum_i u_i$. In this section, we compare the norms of u and those of the u_i's. (This will be useful to refine partitions in the proof of Theorem 1.1 in the next section.)

Lemma 7.1. *Let $q \le 0 \le p$ satisfy $p - q < r$, and let p' and q' be real numbers with $p' < p$ and $q' < q$. For every C^r partition of unity $\{g_i\}$ whose intersection multiplicity is ν, there are constants C and $C(t)$ (that may depend on the g_i's) so that for any $u \in C^\infty(K)$*

$$\|u\|_{C_*^{\Theta,p,q}} \le \nu \cdot \max_{1\le i\le I} \|u_i\|_{C_*^{\Theta,p,q}} + C \sum_{1\le i\le I} \|u_i\|_{C_*^{\Theta,p',q'}}$$

and

$$\|u\|_{W_*^{\Theta,p,q,t}} \le \nu \cdot \left[\sum_{1\le i\le I} \|u_i\|^t_{W_*^{\Theta,p,q,t}}\right]^{1/t} + C(t) \sum_{1\le i\le I} \|u_i\|_{W_*^{\Theta,p',q',t}}, \forall 1 < t < \infty.$$

Proof. Let $U(i,\epsilon)$ be the ϵ-neighborhood of the support of g_i. Take $\epsilon > 0$ so small that the intersection multiplicity of the sets $U(i,\epsilon)$ is ν. Decompose $\mathcal{Q}_\Theta u_i$ (recall Section 3) into

$$\mathbf{u}_i^{\text{body}} = \mathbf{1}_{U(i,\epsilon)} \cdot \mathcal{Q}_\Theta u_i \quad \text{and} \quad \mathbf{u}_i^{\text{tail}} = \mathcal{Q}_\Theta u_i - \mathbf{u}_i^{\text{body}}.$$

On the one hand, Lemma 4.1 implies

$$\|\mathbf{u}_i^{\text{tail}}\|_{p,q,\infty} \le C\|u_i\|_{C_*^{\Theta,p',q'}} \quad \text{and} \quad \|\mathbf{u}_i^{\text{tail}}\|_{p,q,t} \le C\|u_i\|_{W_*^{\Theta,p',q',t}}.$$

On the other hand, since the bound on the intersection multiplicity is ν, we have

$$\left\|\sum_i \mathbf{u}_i^{\text{body}}\right\|_{p,q,\infty} \le \nu \cdot \max_i \left\|\mathbf{u}_i^{\text{body}}\right\|_{p,q,\infty},$$

and, using the Hölder inequality,

$$\left\|\sum_i \mathbf{u}_i^{\text{body}}\right\|_{p,q,t} \le \nu^{1/t'} \cdot \left[\sum_i \left\|\mathbf{u}_i^{\text{body}}\right\|^t_{p,q,t}\right]^{1/t}.$$

Therefore we obtain the estimates in the lemma by using (10). □

The next proposition gives bounds in the opposite direction.

Proposition 7.2. *Let $q \le 0 \le p$, and let p' and q' be real numbers with $p' < p$, $q' < q$ and $p - q' < r$. If $\Theta' < \Theta$, there are constants C_0 and $C_0(t)$ so that for every C^r partition of unity $\{g_i\}$ there are constants $C(\{g_i\})$ and $C(\{g_i\},t)$ (which may depend on the g_i's) so that for all $u \in C^\infty(K)$*

$$\max_{1\le i\le I} \|u_i\|_{C_*^{\Theta',p,q}} \le C_0\|u\|_{C_*^{\Theta,p,q}} + C(\{g_i\})\|u\|_{C_*^{\Theta,p',q'}},$$

and, for $1 < t < \infty$,

$$\left[\sum_{1\le i\le I} \|u_i\|^t_{W_*^{\Theta',p,q,t}}\right]^{1/t} \le C_0(t)\|u\|_{W_*^{\Theta,p,q,t}} + C(\{g_i\},t)\|u\|_{W_*^{\Theta,p',q',t}}.$$

Proof. We revisit the proof of Theorem 6.1, setting $T = id$. (Note that assumption (17) holds since we are assuming $\Theta' < \Theta$.) Recall $\Phi(D)$ and Ψ there, and let $\mathbf{S}^{(i)}$, $\mathbf{S}_0^{(i)}$, $\mathbf{S}_1^{(i)}$ and $\mathbf{L}^{(i)}$ be the operators defined in the same way as $\mathbf{S}$, $\mathbf{S}_0$, $\mathbf{S}_1$ and $\mathbf{L}$ respectively with g replaced by g_i. Obviously $|\mathbf{L}_i(\mathbf{f})|_{C^{p,q}}(x) \le |g_i(x)||\mathbf{f}|_{C^{p,q}}(x)$ and $|\mathbf{L}_i(\mathbf{f})|_{\mathcal{W}^{p,q}}(x) \le |g_i(x)||\mathbf{f}|_{\mathcal{W}^{p,q}}(x)$ at each point x. These and (22-23) imply

$$\max_i \|\Psi \circ \mathbf{L}^{(i)}(\mathbf{u})\|_{p,q,\infty} \le C_1\|\mathbf{u}\|_{p,q}$$

and

$$\left[\sum_i \|\Psi \circ \mathbf{L}^{(i)}(\mathbf{u})\|^t_{p,q,t}\right]^{1/t} \le C_1\|\mathbf{u}\|_{p,q,t}, \forall 1 < t < \infty$$

for $\mathbf{u} = (u_{\Theta,n,\sigma})_{(n,\sigma)\in\Gamma}$. By boundedness of $\Phi(D)$, the same estimates holds with $\Psi \circ \mathbf{L}^{(i)}$ replaced by $\mathbf{S}_0^{(i)} = \Phi(D) \circ \Psi \circ \mathbf{L}^{(i)}$. The conclusion of the proposition then follows from those estimates and the estimates on the operators $\mathbf{S}_1^{(i)}$ parallel to that on $\mathbf{S}_0$ in the proof of Theorem 6.1. □

8. Transfer operators for hyperbolic diffeomorphisms

In this section we prove Theorem 1.1 by reducing to the model of Sections 2–7.

Proof of Theorem 1.1. We first define the spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$, by using local charts to patch the anisotropic Hölder and Sobolev spaces from Section 2. Fix a finite system of C^∞ local charts $\{(V_j,\kappa_j)\}_{j=1}^J$ that cover the compact isolating neighborhood V of Ω, and a finite system of pairs of closed cones [2] $\{(\mathbf{C}_{j,+},\mathbf{C}_{j,-})\}_{j=1}^J$ in $\mathbb{R}^d$ with the properties that for all $1 \le j,k \le J$:

(a) Closure($\kappa_j(V_j)$) is a compact subset of $\mathbb{R}^d$.
(b) The cones $\mathbf{C}_{j,\pm}$ are transversal to each other: $\mathbf{C}_{j,+} \cap \mathbf{C}_{j,-} = \{0\}$.
(c) If $x \in V_j \cap \Omega$, the cones $(D\kappa_j)^*(\mathbf{C}_{j,+})$ and $(D\kappa_j)^*(\mathbf{C}_{j,-})$ in the cotangent space contain the normal subspaces of $E^s(x)$ and $E^u(x)$, respectively.
(d) If $T^{-1}(V_k) \cap V_j \ne \emptyset$, setting $U_{jk} = \kappa_j(T^{-1}(V_k) \cap V_j)$, the map in charts

$$T_{jk} := \kappa_k \circ T \circ \kappa_j^{-1} : U_{jk} \to \mathbb{R}^d$$

enjoys the cone-hyperbolicity condition:

$$DT^{tr}_{jk,x}(\mathbb{R}^d \setminus \text{interior }(\mathbf{C}_{j,+})) \subset \text{interior}(\mathbf{C}_{k,-}) \cup \{0\}, \quad \forall x \in U_{jk}. \tag{36}$$

Choose C^∞ functions $\varphi_j^+, \varphi_j^- : \mathbf{S}^{d-1} \to [0,1]$ for $1 \le j \le J$ which satisfy condition (7) with $\mathbf{C}_\pm = \mathbf{C}_{j,\pm}$, giving combinations $\Theta_j = (\mathbf{C}_{j,+},\mathbf{C}_{j,-},\varphi_j^+,\varphi_j^-)$ as in Section 2. Choose finally a C^∞ partition of the unity $\{\phi_j\}$ on V subordinate to the covering $\{V_j\}_{j=1}^J$, that is, the support of each $\phi_j : X \to [0,1]$ is contained in V_j and we have $\sum_{j=1}^J \phi_j \equiv 1$ on V.

[2] We regard $\mathbf{C}_{j,\pm}$ as constant cone fields in the *cotangent* bundle $T^*\mathbb{R}^d$.

We define the Banach spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$ for $1 < t < \infty$, respectively, to be the completion of $C^\infty(V)$ for the norm

$$\|u\|_{C_*^{p,q}(T,V)} := \max_{1\le j\le J} \|(\phi_j \cdot u)\circ \kappa_j^{-1}\|_{C_*^{\Theta_j,p,q}}$$

and

$$\|u\|_{W_*^{p,q,t}(T,V)} := \max_{1\le j\le J} \|(\phi_j \cdot u)\circ \kappa_j^{-1}\|_{W_*^{\Theta_j,p,q,t}}. \tag{37}$$

By this definition, we have that $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$, respectively, contain $C^p(V) \subset C_*^p(V)$ and $W^{p,t}(V)$, respectively, as dense subsets. Take and fix real numbers $0 \le p' < p$ and $q' < q$ such that $p - q' < r - 1$. By Lemma 5.1 and a finite diagonal argument over $\{1,\dots,J\}$, we can see that the inclusions $C_*^{p,q}(T,V) \subset C_*^{p',q'}(T,V)$ and $W_*^{p,q,t}(T,V) \subset W_*^{p',q',t}(T,V)$ are compact.

For $m \ge 1$ and j, k so that

$$V_{m,jk} := T^{-m}(V_k) \cap V_j \cap \left(\cap_{i=0}^m T^{-i}(V)\right) \ne \emptyset,$$

we may consider the map in charts

$$T_{jk}^m = \kappa_k \circ T^m \circ \kappa_j^{-1} : \kappa_j(V_{m,jk}) \to \mathbb{R}^d.$$

Note that (36) implies that

$$(DT_{jk,x}^m)^{tr}(\mathbb{R}^d \setminus \text{interior } (\mathbf{C}_{j,+})) \subset \text{interior}(\mathbf{C}_{k,-}) \cup \{0\}, \quad \forall x \in \kappa_j(V_{m,jk}). \tag{38}$$

For $1 < t \le \infty$, we set

$$\Lambda_{m,t} = \max_j \max_k \sup_{x\in\kappa_j(V_{m,jk})} \frac{|g^{(m)} \circ \kappa_j^{-1}(x)| \cdot \max\{(\|T_{jk}^m\|_+(x))^p, (\|T_{jk}^m\|_-(x))^q\}}{|\det DT_x^m|^{1/t}}$$

where

$$\|T_{jk}^m\|_+(x) = \sup\left\{ \frac{\|(DT_{jk}^m)_x^{tr}(\xi)\|}{\|\xi\|} \;;\; 0 \ne (DT_{jk}^m)_x^{tr}(\xi) \notin \mathbf{C}_{k,-} \right\},$$

and

$$\|T_{jk}^m\|_-(x) = \inf\left\{ \frac{\|(DT_{jk}^m)_x^{tr}(\xi)\|}{\|\xi\|} \;;\; 0 \ne \xi \notin \mathbf{C}_{j,+} \right\}.$$

Then a standard argument on hyperbolic sets gives a constant $C(t) > 1$ that does not depend on $m > 0$ such that

$$C(t)^{-1} R^{p,q,t}(T,g,\Omega,m) \le \Lambda_{m,t} \le C(t) R^{p,q,t}(T,g,\Omega,m). \tag{39}$$

The definition of $\Lambda_{m,t}$ involves first taking a maximum and a product, and then taking the supremum over x. We shall apply Theorem 6.1 in a moment: the upper bound there corresponds to taking a supremum first. Since different points in $\kappa_j(V_{m,jk})$ may have very different itineraries, it is necessary to refine our partition of unity, depending on m. This will not cause problems since we can take arbitrarily fine finite C^∞ partitions of unity on $\mathbb{R}^d$, with intersection multiplicities bounded uniformly by a constant depending only on d. Using such a partition of unity, we decompose the function $u_{jk} = (\phi_k(\phi_j \circ T^{-m}) \cdot u) \circ \kappa_k^{-1}$ into $u_{jk,i}$ for $1 \le i \le I_{jk}$. Take combinations $\Theta'_k > \Theta_k$ (close to Θ_k) so that the iterated cone-hyperbolicity condition (38) holds with Θ_k replaced by Θ'_k. For each m, by taking a sufficiently fine partition of unity, we can apply Theorem 6.1 to obtain, for $1 \le i \le I_{jk}$,

$$\|g^{(m)} \circ \kappa_j^{-1} \cdot u_{jk,i} \circ T_{jk}^m\|_{C_*^{\Theta_j,p,q}} \le 2\Lambda_{m,\infty} \cdot \|u_{jk,i}\|_{C_*^{\Theta'_k,p,q}} + C\|u_{jk,i}\|_{C_*^{\Theta'_k,p',q'}}.$$

Then, using Lemma 7.1 and Proposition 7.2, we get

$$\|g^{(m)} \circ \kappa_j^{-1} \cdot u_{jk} \circ T_{jk}^m\|_{C_*^{\Theta_j,p,q}} \le C_1 \cdot \Lambda_{m,\infty} \cdot \|u_{jk}\|_{C_*^{\Theta_k,p,q}} + C_1(m) \cdot \|u_{jk}\|_{C_*^{\Theta_k,p',q'}},$$

where C_1 is a constant that does not depend on m. Thus, using again Lemma 7.1 and Proposition 7.2, we obtain the following Lasota-Yorke type inequalities:

$$\|\mathcal{L}_{T,g}^m u\|_{C_*^{p,q}(T,V)} \le C_2 \cdot J \cdot \Lambda_{m,\infty} \cdot \|u\|_{C_*^{p,q}(T,V)} + C_2(m)\|u\|_{C_*^{p',q'}(T,V)},\ m \ge 1.$$

Likewise, we obtain for $1 < t < \infty$

$$\|\mathcal{L}_{T,g}^m u\|_{W_*^{p,q,t}(T,V)} \le C_2(t) \cdot J \cdot \Lambda_{m,t} \cdot \|u\|_{W_*^{p,q,t}(T,V)} + C_2(m,t)\|u\|_{W_*^{p',q',t}(T,V)}.$$

Finally Hennion's theorem [6] gives the claimed upper bounds

$$\liminf_{m\to\infty} (C(t)\Lambda_{m,t})^{1/m} = R^{p,q,t}(T, g, \Omega)$$

for the essential spectral radius of $\mathcal{L}_{T,g}$. □

Remark 8.1. The proof above applies to (hyperbolic) mixed transfer operators [9].

Remark 8.2. Though it is not explicit in our notation, the definition of the spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$ depends on the system of charts $\{(V_j, \kappa_j)\}_{j=1}^J$, the set of combinations $\{(\mathbf{C}_{j,+}, \mathbf{C}_{j,-}, \varphi_{j,+}, \varphi_{j,-})\}_{j=1}^J$, and the partition of unity $\{\phi_j\}_{j=1}^J$. Choosing a different system of local charts, a different set of combinations, or a different partition of unity, does not a priori give rise to equivalent norms, though Theorem 6.1 gives relations. This is a little unpleasant, but does not cause problems.

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