Spectral theory and distributional Borel summability for the quantum Hénon-Heiles model

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Abstract

The Borel summability in the distributional sense is established of the divergent perturbation theory for the ground state resonance of the quantum Hénon-Heiles model.

1 Introduction and statement of the results

A standard model for transition to chaos is given by the Hénon-Heiles Hamiltonian (see e.g. [16], [12], [18]) defined by

$$H(\beta) = p_1^2 + p_2^2 + q_1^2 + q_2^2 + \beta \left(q_1^2 q_2 - \frac{1}{3} q_2^3 \right), \quad \beta \in \mathbf{R}. \tag{1.1}$$

The quantum counterpart of (1.1) (see e.g. [15], [2], [1], [11], [3], [14]) is represented by the Schrödinger operator in $L^2(\mathbf{R}^2)$ formally given by

$$H(\beta) = -\Delta + x^2 + \beta \left(x_1^2 x_2 - \frac{1}{3} x_2^3 \right) := H(0) + \beta V, \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, $x^2 = |x|^2 = x_1^2 + x_2^2$, Δ is the $2-$dimensional Laplace operator, V is the multiplication operator by the function $V(x) = x_1^2 x_2 - \frac{1}{3} x_2^3$ and $H(0)$ is the operator corresponding to the harmonic oscillator. Its spectral properties have been extensively investigated numerically (see e.g. [13]). However the mathematical analysis of the problem is somewhat tricky. The purpose of this paper is to solve one of the mathematical problems involved, namely the meaning of perturbation theory, as announced in [5], in order to obtain results analogous to those obtained in [7] and [4] for the one dimensional odd anharmonic oscillator. Indeed, as in the one dimensional case, the minimal operator generated by (1.2), with $C_0^\infty(\mathbf{R}^2)$ as domain, is not essentially selfadjoint; in fact, it has infinitely many selfadjoint extensions, none of which with discrete spectrum. Therefore the numerically observed eigenvalues have to be interpreted as (real part of) resonances [19], which are complex objects: the real part is the location, and the imaginary part the width. Another singularity in this problem is due to the fact that the Rayleigh-Schrödinger perturbation expansion (from now on denoted RSPE) near the lowest unperturbed eigenvalue $E_0 = 2$ not only diverges but has real coefficients with constant signs. As is well known, this prevents the series from being Borel summable in the ordinary sense. The aim of this paper is then to prove that, as in the Stark effect (see [9]), the perturbation series near E_0 is Borel summable in the distributional sense to the real part of the resonance. On the other hand, if β is taken to be purely imaginary, then $H(\beta)$ shows a less singular bahavior (see [19] and [6]). In fact $H(\beta)$ is closable on $C_0^\infty(\mathbf{R}^2)$ and, if $D(H(\beta))$ denotes the domain of its closure, then $H(\beta)$ is $\mathcal{PT}$-symmetric, i.e.

$$\overline{(H(\beta)u)(-x)} = H(\beta)\overline{u(-x)}, \quad \forall u \in D(H(\beta)).$$

Furthermore the RSPE near any eigenvalue has real coefficients with alternating signs and is Borel summable to the corresponding (real) eigenvalue of $H(\beta)$ (see [19]). Thus, the natural way of dealing with $H(\beta)$ is to start with β complex and then look for a continuation to β real. More precisely we will prove the following.

Theorem 1.1 *(a) $H(\beta)$ represents an analytic family of type A of closed operators with compact resolvents, with domain*

$$D(H(\beta)) = D(H(0)) \cap D(V), \quad \textit{for } 0 < \arg\beta < \pi.$$

(b) Let $E_0 = 2$ denote the ground state energy level of $H(0)$. Then for any $\delta > 0$ there is $B(\delta) > 0$ such that for $|\beta| < B(\delta)$, $0 < \arg\beta < \pi$, $H(\beta)$ has exactly one eigenvalue $E(\beta)$ near E_0, which admits an analytic continuation across the real axis to the sector

$$S_\delta := \left\{ \beta \in \mathbf{C} : 0 < |\beta| < B(\delta), \ -\frac{\pi}{4} + \delta < \arg\beta < \frac{5}{4}\pi - \delta \right\}. \tag{1.3}$$

Moreover, $\lim_{\substack{\beta\to 0\\ \beta\in S_\delta}} E(\beta) = E_0$

In [19] weaker results concerning the spectral properties of $H(\beta)$ are obtained, which are not sufficient to establish our main result, stated in Theorem 1.2 below, i.e. the distributional Borel summability (from now on denoted DBS) of the RSPE around E_0. More precisely in order to specify the domain of $H(\beta)$ and guarantee the compacteness of its resolvents (as stated in Theorem 1.1-(a)), we need a quadratic estimate on $H(\beta)$, proved in Appendix B. We can now state the main result of this paper.

Theorem 1.2 *Let $\beta \in \mathbf{R}$. Then*

(a) the RSPE near E_0 is Borel-Leroy summable of order $\frac{1}{2}$ in the ordinary sense to $E(\beta)$ for $0 < \arg \beta < \pi$ and in the distributional sense to $\Re E(\beta)$ for $\beta \in \mathbf{R}$, $|\beta|$ suitably small;

(b) $\Re E(\beta) = \Re E(-\beta)$, $\Im E(\beta) = -\Im E(-\beta)$, for $\beta \in \mathbf{R}$.

The notion of distributional Borel-Leroy summability and the corresponding criterion are recalled in Appendix A for the convenience of the reader.

Remark 1.3 In order to extend the result stated in Theorem 1.2 to any unperturbed eigenvalue $E_l = 2(l + 1)$, $l = 1, 2, \ldots$ of $H(0)$, one needs to extend the definition (and corresponding criterion) of distributional Borel summability to the degenerate case, as Hunziker and Pillet did in [17] for the notion of ordinary Borel summability. We plan to do this in a forthcoming paper.

The proof of Theorem 1.1 is obtained in Section 2, following [4] where analogous results were obtained for the one dimensional add anharmonic oscillator. We will include most of the details in order to make the paper self-contained. The proof of Theorem 1.2 requires the verification of the analogue of the Nevanlinna criterion for the DBS, as stated in Appendix A and proved in [8]. In particular we need to extend the analyticity of $E(\beta)$ to a suitable Nevanlinna disc. This result is achieved in Section 3 by using a refinement of the Hunziker-Vock stability technique introduced in [9], [10] and [4] where analogous results are obtained for the resonances of the Stark effect, of double-well oscillators and of the $1-$dimensional odd anharmonic oscillator, respectively. However such technique cannot be directly applied to the present $2-$dimensional problem. Indeed, the method is based on a control of the numerical range of $H(\beta)$, whose distance from any complex number $z \notin \sigma(H(0))$ must be bounded from below by a positive constant as $|x| \to \infty$ and $\beta \to 0$. It is possible, however, to overcome this difficulty by passing to polar coordinates, where the angular coordinate can be easily dealt with and the problem essentially reduces to a $1-$dimensional one.

2 The operator $H(\beta)$ for $\Im\beta \neq 0$ and the analytic continuation of the eigenvalues

In order to prove Theorem 1.1 we need some preliminary results.

Lemma 2.1 *Let $\beta = |\beta|e^{i\alpha}$ with $\alpha \in]0,\pi[$ and Ω be a compact subset of*

$$\{\gamma \in \mathbf{C} \setminus \{0\} : |\gamma|^2 \geq 4|\beta|\sin\alpha, \ -\pi+\alpha < \arg\beta < \alpha\}.$$

Then there exist $a, b > 0$ such that

$$\|\Delta u\|^2 + |\gamma|^2\|x^2 u\|^2 + |\beta|^2\|Vu\|^2 \leq a\|(-\Delta + \gamma x^2 + \beta V)u\|^2 + b\|u\|^2 \tag{2.1}$$

$\forall u \in C_0^\infty(\mathbf{R}^2)$, $\gamma \in \Omega$, $0 < |\beta| \leq 1$, a and b independent of γ in Ω and α in a closed interval contained in $]0,\pi[$.

Proof. See Appendix B.

Corollary 2.2 *Let $\gamma, \beta \in \mathbf{C}$ satisfy the conditions of Lemma 2.1. Then the operator $T(\gamma,\beta)$ defined in $L^2(\mathbf{R}^2)$ by*

$$T(\gamma,\beta)u = -\Delta u + \gamma x^2 u + \beta V u, \quad \forall u \in D(T(\gamma,\beta))$$

on the domain $D(T(\gamma,\beta)) = D(H(0)) \cap D(V)$ is closed and has $C_0^\infty(\mathbf{R}^2)$ as a core.

Lemma 2.3 *Let $\gamma, \beta \in \mathbf{C}$ satisfy the conditions of Lemma 2.1, i.e. $\alpha = \arg\beta \in]0,\pi[$, $-\pi+\alpha < \arg\gamma < \alpha$, $|\gamma|^2 > 4|\beta|\sin\alpha$.*
Then there exists $\xi > 0$ such that

$$\xi \Re\,[e^{-i(\alpha-\pi/2)} < u, T(\gamma,\beta)u >] \geq < u, -\Delta u > \quad \forall u \in D(T(\gamma,\beta)). \tag{2.2}$$

Proof. It is enough to prove (2.2) for $u \in C_0^\infty(\mathbf{R}^2)$. We have

$$\begin{aligned}
&\Re\,[e^{-i(\alpha-\pi/2)} < u, (-\Delta + \gamma x^2 + \beta V)u >] \\
&\quad = \cos\left(\alpha - \frac{\pi}{2}\right) < u, -\Delta u > + |\gamma|\cos\left(\frac{\pi}{2} - \alpha + \arg\gamma\right) < u, x^2 u > \\
&\quad = \sin\alpha < u, -\Delta u > + |\gamma|\sin(\alpha - \arg\gamma) < u, x^2 u > \geq \sin\alpha < u, -\Delta u >
\end{aligned}$$

since $\sin(\alpha - \arg\gamma) > 0$ because $0 < \alpha - \arg\gamma < \pi$ by assumption. Now, since $0 < \alpha < \pi$, the lemma is proved with $\xi = (\sin\alpha)^{-1}$.

Corollary 2.4 *The numerical range of $T(\gamma,\beta)$ is contained in the half-plane $\{z \in \mathbf{C} : -\pi+\alpha \leq \arg z \leq \alpha\}$.*

Set $H(\beta) := T(1, \beta)$, for $\alpha = \arg \beta \in]0, \pi[$. By the above results we have the following theorem, which corresponds to Theorem 1.1-(a).

Theorem 2.5 *$H(\beta)$ represents an analytic family of type A of operators with compact resolvents, with $D(H(\beta)) = D(H(0)) \cap D(V)$, for $0 < \arg \beta < \pi$.*

We will prove that the (discrete) spectrum of $H(\beta)$ is non-empty (see also [19]). In order to prove Theorem 1.2, i.e. the DBS of the RSPE near E_0 we need to prove that it is stable with respect to the family $H(\beta)$, as $\beta \to 0, \Im\beta > 0$ and that the corresponding eigenvalue $E(\beta)$ of $H(\beta)$ can be analytically continued to a wider sector than $0 < \arg \beta < \pi$. To this end we start by making use of standard dilation analyticity techniques, i.e. we introduce the operator

$$H(\beta, \theta) := -e^{-2\theta}\Delta + e^{2\theta}x^2 + \beta e^{3\theta}V(x) := e^{-2\theta}K(\beta, \theta) \tag{2.3}$$

which, for $\theta \in \mathbf{R}$, is unitarily equivalent to $H(\beta), \Im\beta > 0$, via the dilation operator

$$(U(\theta)u)(x) = e^{\theta}u(e^{\theta}x), \qquad \forall u \in L^2(\mathbf{R}^2)$$

First of all notice that $K(\beta, \theta) = -\Delta + e^{4\theta}x^2 + \beta e^{5\theta}V(x)$ corresponds to $T(\gamma, \beta')$ with $\gamma = e^{4\theta}, \beta' = \beta e^{5\theta}$. From now on we will assume $|\Re\theta| < 1$ and $|\beta| < \frac{e^{-3}}{4}$ so that the conditions $|\beta'| < 1$ and $|\gamma|^2 > 4|\beta'| \sin(\arg \beta')$ required in Lemma 2.1 are automatically satisfied.

Next observe that the further conditions on β' and γ so far required:

$$\begin{cases} -\pi + \arg \beta' < \arg \gamma < \arg \beta' \\ 0 < \arg \beta' < \pi \end{cases} \tag{2.4}$$

are equivalent to the following conditions on β and θ:

$$\begin{cases} 0 < \arg \beta + \Im\theta < \pi \\ 0 < \arg \beta + 5\Im\theta < \pi \end{cases} \tag{2.5}$$

In complete analogy with Theorem 2.5 of [4] we can now prove the following

Theorem 2.6 *Let $s = \arg \beta, t = \Im\theta$. Then $H(\beta, \theta)$ is a holomorphic family of type A of closed operators on $D(H(\beta, \theta)) = D(H_0) \cap D(V)$ with compact resolvents for β and θ such that s and t vary in the parallelogram P of the $(s, t)-$plane defined by*

$$P = \{(s, t) \in \mathbf{R}^2 : 0 < t + s < \pi, 0 < 5t + s < \pi\}.$$

Remark 2.7 1. By Corollary 2.4, the numerical range of $K(\beta,\theta)$ is contained in the half-plane $-\pi+\alpha \le \arg z \le \alpha$ with $\alpha = \arg\beta + 5\Im\theta$; thus, $H(\beta,\theta)$ has numerical range contained in the half-plane

$$\Pi = \{z \in \mathbf{C} :\ -\pi + \arg\beta + 3\Im\theta \le \arg z \le \arg\beta + 3\Im\theta\}\,. \tag{2.6}$$

Moreover the (discrete) spectrum of $H(\beta,\theta)$ is contained in Π and $\forall z \notin \Pi$, $\|(z - H(\beta,\theta))^{-1}\| \le \operatorname{dist}(z,\Pi)^{-1}$. Finally, the analyticity of $H(\beta,\theta)$ in the region defined by P allows β to be extended to the sector $S = \{\beta :\ 0 < |\beta| < \beta_0, -\frac{\pi}{4} < \arg\beta < \frac{5}{4}\pi\}$

2. If we start from the operator $H(\beta)$ with $\Im\beta < 0$, analogous results can be obtained for the operator family $H(\beta,\theta)$ for β and θ satisfying the conditions:

$$\begin{cases} -\pi < \Im\theta + \arg\beta < 0 \\ -\pi < 5\Im\theta + \arg\beta < 0 \end{cases} \tag{2.7}$$

Furthermore $H(\beta,\theta)^* = H(\bar\beta,\bar\theta)$.

By standard dilation analyticity arguments the eigenvalues $E_l = 2(l+1)$, $l = 0,1,\ldots$, of $H(0,\theta) = -e^{-2\theta}\Delta + e^{2\theta}x^2, D(H(0,\theta)) = D(H(0))$, are independet of θ for $-\frac{\pi}{4} < \Im\theta < \frac{\pi}{4}$. By an argument similar to that used to prove Theorem 2.7 in [4] (see also [19]) we can prove the stability in the sense of Kato of each eigenvalue E_l with respect to the family $\{H(\beta,\theta) :\ |\beta| > 0\}$, β and θ in the region defined by P. More precisely we can state the following

Theorem 2.8 *Let β and θ satisfy conditions (2.5). We have*

(a) if $\lambda \notin \sigma(H(0,\theta))$, then $\lambda \in \mathcal{D}$, where

$$\mathcal{D} = \{z \in \mathbf{C} :\ z \notin \sigma(H(\beta,\theta)) \textit{ and } (z - H(\beta,\theta))^{-1} \textit{ is uniformly bounded as } |\beta| \to 0\}$$

(b) if $\lambda \in \sigma(H(0,\theta)) = \{2(l+1) :\ l = 0,1,\ldots\}$, then λ is stable with respect to the family $H(\beta,\theta)$, i.e. if $r > 0$ is sufficiently small, so that the only eigenvalue of $H(0,\theta)$ enclosed in $\Gamma_r = \{z \in \mathbf{C} :\ |z - \lambda| = r\}$ is λ, then there is $B > 0$ such that for $|\beta| < B, \dim P(\beta,\theta) = \dim P(0,\theta)$, where

$$P(\beta,\theta) = (2\pi i)^{-1} \oint_{\Gamma_r} (z - H(\beta,\theta))^{-1} dz$$

is the spectral projection of $H(\beta,\theta)$ corresponding to the part of the spectrum enclosed in $\Gamma_r \subset \mathbf{C} \setminus \sigma(H(0,\theta))$.

Remark 2.9 It can be immediately checked that all the results so far obtained, in particular the analyticity of the family $H(\beta,\theta)$ and the stability of the eigenvalues of the harmonic oscillator $H(0,\theta)$ with respect to $H(\beta,\theta)$ as $|\beta|\to 0$, hold unifomly in β and θ such that $(\arg\beta,\theta)$ varies in any compact subset of P.

Now we specialize the result obtained in Theorem 2.8 to the ground state energy level $E_0 = 2$ of $H(0)$. More precisely, for any $\delta > 0$ there exists $B(\delta) > 0$ such that for $|\beta| < B(\delta), -\frac{\pi}{4}+\delta < \arg\beta < \frac{5}{4}\pi-\delta, H(\beta,\theta)$ has one and only one eigenvalue $E(\beta)$, independent of θ if $(\arg\beta,\Im\theta)\in P$, which converges to E_0 as $|\beta|\to 0$.
By Theorem 2.6, $E(\beta)$ is analytic in the sector

$$S_\delta = \left\{\beta\in\mathbf{C} :\ 0<|\beta|<B(\delta),\ -\frac{\pi}{4}+\delta<\arg\beta<\frac{5}{4}\pi-\delta\right\} \tag{2.8}$$

and is an eigenvalue of $H(\beta)$ for $0<\Im\beta<\pi$. For future reference we state this result in the following

Theorem 2.10 *For any $\delta > 0$, there is $B(\delta) > 0$ such that for $|\beta| < B(\delta), 0 < \arg\beta < \pi$, $H(\beta)$ has exactly one eigenvalue $E(\beta)$ near E_0, which admits an analytic continuation across the real axis to the sector S_δ. Moreover $\lim_{\substack{\beta\to 0\\ \beta\in S_\delta}} E(\beta) = E_0$.*

3 Analyticity of $E(\beta)$ in a Nevanlinna disc and DBS

The basic analyticity result needed to establish the DBS of the RSPE near E_0 for $\beta > 0$ is obtained in the following theorem.

Theorem 3.1 *There exists $R > 0$ such that the eigenvalue $E(\beta)$ of $H(\beta) = -\Delta + x^2+\beta(x_1^2x_2-\frac{1}{3}x_2^3)$ near E_0 for $|\beta|$ small is analytic in the Nevanlinna disc $C_R=\{\beta : \Re\beta^{-2}>R^{-1}\}$ in the β^2-plane.*

Remark 3.2 (I) The sector $S(\delta)$ given by (2.8) can be rewritten in terms of the variable β^2 as:

$$S(\delta) = \left\{\beta :\ 0<|\beta|<B(\delta),\ -\frac{\pi}{2}+2\delta<\arg\beta^2<\frac{\pi}{2}+2\pi-2\delta\right\}. \tag{3.1}$$

(II) The function $E(\beta)$, analytic in any sector $S(\delta)$ and for which we want to prove analyticity in C_R, represents an eigenvalue of $H(\beta,\theta)$ if $(\arg\beta,\Im\theta)\in P$. In particular, for $-\frac{\pi}{4}<\arg\beta<0$ we can choose the path inside P given by the straight line of equation

$$\Im\theta = -\frac{1}{3}\arg\beta+\frac{\pi}{6}\,.$$

Then, if we set

$$\arg\beta = -\frac{\pi}{4} + \frac{\varepsilon}{2}, \quad \text{i.e. } \arg\beta^2 = -\frac{\pi}{2} + \varepsilon, \quad \varepsilon \to 0^+ \tag{3.2}$$

we obtain $\Im\theta = \frac{\pi}{4} - \frac{\varepsilon}{6}$, and the operator $H(\beta, \theta)$ takes the form

$$A(\rho) = -\theta_0^{-2} e^{-i(\frac{\pi}{2} - \frac{\varepsilon}{3})} \Delta + \theta_0^2 e^{i(\frac{\pi}{2} - \frac{\varepsilon}{3})} x^2 + i\rho\theta_0^3 V(x) \tag{3.3}$$

with $\rho = |\beta|$ and $\theta_0 = e^{\Re\theta}$.

(III) For $\beta = \rho e^{i\arg\beta}$ and $\arg\beta = -\frac{\pi}{4} + \frac{\varepsilon}{2}$, the boundary of C_R has the equation

$$\sin\varepsilon = \frac{\rho^2}{R}. \tag{3.4}$$

(IV) Since the disc C_R is the union of the boundaries of discs of smaller radius, the proof of Theorem 3.1 reduces to a stability argument for the eigenvalue E_0 with respect to the family $\{A_\rho\}_{\rho>0}$ as $\rho \to 0^+$. In view of Remark 3.2 (IV) it is convenient to move to polar coordinates (r, φ) as follows:

$$\begin{cases} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi \end{cases} \tag{3.5}$$

Then $A(\rho)$ is equivalent to the operator $A_1(\rho)$ formally given by

$$\begin{aligned} A_1(\rho) &= \theta_0^{-2} e^{-i(\frac{\pi}{2} - \frac{\varepsilon}{3})} \left\{ -\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4r^2} \right\} + \theta_0^2 e^{i(\frac{\pi}{2} - \frac{\varepsilon}{3})} r^2 \\ &\quad + i\rho\theta_0^3 r^3 \left(\cos^2\varphi \sin\varphi - \frac{1}{3}\sin^3\varphi \right) \end{aligned} \tag{3.6}$$

in $L^2(\mathbf{R}^+ \times \mathbf{T}^1)$ with suitable conditions at the origin.
More precisely $A_1(\rho)$ is the operator generated by the quadratic form

$$\begin{aligned} a_\rho[u] &:= \int_0^\infty \int_0^{2\pi} \left\{ \theta_0^{-2} e^{-i(\frac{\pi}{2} - \frac{\varepsilon}{3})} \left[|p_r u|^2 + \frac{1}{r^2}|p_\varphi u|^2 - \frac{1}{4r^2}|u|^2 \right] \right. \\ &\quad \left. + \left[\theta_0^2 e^{i(\frac{\pi}{2} - \frac{\varepsilon}{3})} r^2 + i\rho\theta_0^3 r^3 \left(\cos^2\varphi\sin\varphi - \frac{1}{3}\sin^3\varphi \right) \right] |u|^2 \right\} dr d\theta \end{aligned} \tag{3.7}$$

defined on the maximal domain with the condition at the origin $u(r, \varphi) \simeq r^{1/2}$ uniformly in φ. Here we adopt the notation $p_r = -i\frac{\partial}{\partial r}, p_\varphi = -i\frac{\partial}{\partial\varphi}$. Now, as in [9], let $\mathcal{U}$ be a transformation in the set of L^2 functions which are translation analytic in some suitable strip $|\Im r| < \delta_0$, defined by:

$$(\mathcal{U}\psi)(r, \theta) = \xi'(r)^{1/2} \psi(\xi(r), \theta) \tag{3.8}$$

where, setting $r_0 = \frac{a_0}{\rho}$ for a suitable $a_0 > 0$, we define $\xi(r) \in C_0^\infty(\mathbf{R}^+)$ so that, for $\eta_0 \in]0,1[$ fixed,

$$\xi(r) = r - 2i\eta_0[1 - (1+r^3)^{-1/6}], \qquad 0 < r \le r_0 \tag{3.9}$$

$$\xi(r) = r, \qquad r \ge r_0 + \eta_0 \tag{3.10}$$

Setting $f(r) = \frac{1}{\xi'(r)}$ and $\omega = e^{-i(\frac{\pi}{2} - \frac{\varepsilon}{3})}$, the transformed operator $\mathcal{U}A(\rho)\mathcal{U}^{-1}$ is given by

$$\begin{aligned} H_\rho &= \omega\theta_0^{-2}\left[p_r f^2 p_r + \frac{1}{4}(f^2)'' - \frac{1}{\xi^2}p_\varphi^2 - \frac{1}{4\xi^2}\right] \\ &\quad + \omega^{-1}\theta_0^2\xi^2 + i\rho\theta_0^3\xi^3\left(\cos^2\varphi\sin\varphi - \frac{1}{3}\sin^3\varphi\right). \end{aligned} \tag{3.11}$$

The quadratic form which ganerates H_ρ is defined by:

$$\begin{aligned} h_\rho[u] &= \int_0^\infty\int_0^{2\pi}\left\{\omega\theta_0^{-2}\left[f^2|p_r u|^2 + \frac{1}{\xi^2}|p_\varphi u|^2\right]\right. \\ &\quad + \left[\frac{\omega\theta_0^{-2}}{4}(f_2)'' - \frac{\omega\theta_0^{-2}}{4\xi^2} + \omega^{-1}\theta_0^2\xi^2\right. \\ &\quad \left.\left. + i\rho\theta_0^3\xi^3\left(\cos^2\varphi\sin\varphi - \frac{1}{3}\sin^3\varphi\right)\right]|u|^2\right\}drd\varphi \end{aligned} \tag{3.12}$$

on the maximal domain with the condition at the origin $u(r,\varphi) \simeq r^{1/2}$, uniformly in φ. The limit in the strong resolvent sense as $\rho \to 0^+$ of H_ρ is defined by

$$H_0 = -i\theta_0^{-2}\left\{p_r f_0^2 p_r + \frac{1}{4}(f_0^2)'' - \frac{1}{4\xi_0^2} - \frac{1}{\xi_0^2}p_\varphi^2\right\} + i\theta_0^2\xi_0^2 \tag{3.13}$$

where $f_0 = (\xi_0')^{-1}$ and $\xi_0(r)$ is defined by (3.9) $\forall r > 0$. From Remark 3.2-(IV) it follows that in order to prove Theorem 3.1 it suffices to prove the stability of the eigenvalue E_0 of H_0 with respect to the family H_ρ as $\rho \to 0^+$. As in [9] this result is achieved by means of some preliminary results.

Lemma 3.3 *Let* $\sin\varepsilon = \frac{\rho^2}{R}$. *There exists* $\rho_0 > 0, n_0 \in \mathbf{N}$ *and positive real constants* a_1, a_2, c, c_1, c_2 *such that*

$$\begin{aligned} \Re h_\rho[u] &\ge a_1\int_0^{2\pi}\int_1^{r_0}\left\{\sin\left(\frac{\varepsilon}{3}\right)[1 - r^4(1+r^3)^{-7/3}] + \eta_0 r^2(1+r^3)^{-7/6}\right\}|p_r u|^2 drd\varphi \\ &\quad + a_2\int_0^{2\pi}\int_{r_0}^\infty \sin\left(\frac{\varepsilon}{3}\right)|p_r u|^2 drd\varphi - c\|u\|^2 \end{aligned} \tag{3.14}$$

$\forall\rho \in]0,\rho_0], \forall u \in D(h_p)$. A similar estimate holds with h_ρ replaced by h_0 $(r_0 = +\infty, \varepsilon = 0)$.
Moreover, $\forall u \in D(h_\rho)$ such that supp $u \subset (n,+\infty)$

$$\Re h_\rho[u] \geq (c_1 R^{-1} - c_2)\|u\|^2 \tag{3.15}$$

$\forall n \geq n_0$ and c_1, c_2 independent of R, $\forall \rho \in [0,\rho_0]$.

Proof. We have

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\infty}\left\{f^2|p_r u|^2 - \frac{1}{4\xi^2}|u|^2\right\}drd\varphi\\ &\quad = \int_0^{2\pi}d\varphi\int_0^{\infty}f^2\left[|p_r u|^2 - \frac{1}{4r^2}|u|^2\right]dr + \frac{1}{4}\int_0^{2\pi}d\varphi\int_0^{\infty}\left(\frac{f^2}{r^2} - \frac{1}{\xi^2}\right)|u|^2 dr\,.\end{aligned} \tag{3.16}$$

Since the function $\frac{f^2}{r^2} - \frac{1}{\xi^2}$ is bounded we obtain

$$\begin{aligned}&\Re\int_0^{2\pi}\int_0^{\infty}\omega\left\{f^2|p_r u|^2 - \frac{1}{4\xi^2}|u|^2 drd\varphi\right\}\\ &\quad\geq \int_0^{2\pi}\int_0^{\infty}\Re(\omega f^2)\left[|p_r u|^2 - \frac{1}{4r^2}|u|^2\right]drd\varphi - (\text{const.})\|u\|^2\\ &\quad= \int_0^{2\pi}\int_0^{1}(\cdots)drd\varphi + \int_0^{2\pi}\int_1^{r_0}(\cdots)drd\varphi + \int_0^{2\pi}\int_{r_0}^{\infty}(\cdots)drd\varphi - (\text{const.})\|u\|^2\,.\end{aligned} \tag{3.17}$$

Let us now denote I_1, I_2 and I_3 the first, second and third integral respectively, in the right hand side of (3.17). For $r \leq r_0$ we have

$$\begin{aligned}\Re f^2 &\geq 4^{-1}\{1 - r^4(1+r^3)^{-7/3}\}\\ \Im f^2 &\geq 2^{-1}\eta_0 r^2(1+r^3)^{-7/6}\,.\end{aligned} \tag{3.18}$$

Hence,

$$\Re(\omega f^2) \geq 4^{-1}\sin\left(\frac{\varepsilon}{3}\right)[1 - r^4(1+r^3)^{-7/3}] + \frac{\eta_0}{2}\cos\left(\frac{\varepsilon}{3}\right)r^2(1+r^3)^{-7/6}\,. \tag{3.19}$$

Now, by Sobolev's inequality $I_1 \geq 0$; moreover

$$I_3 \geq \int_0^{2\pi}\int_0^{\infty}\sin\left(\frac{\varepsilon}{3}\right)|p_r u|^2 drd\varphi - (\text{const.})\,\|u\|^2\,, \tag{3.20}$$

$$\begin{aligned}I_2 \geq\ & a_1\int_0^{2\pi}\int_1^{r_0}\left\{\sin\left(\frac{\varepsilon}{3}\right)[1 - r^4(1+r^3)^{-7/3}]\right.\\ &\left.+\eta_0 r^2(1+r^3)^{-7/6}\right\}|p_r u|^2 drd\varphi\,.\end{aligned} \tag{3.21}$$

Let us now estimate the remaining terms appearing in $\Re h_\rho[u]$. We have

$$\Re \int_0^{2\pi}\int_0^\infty \xi^{-2}|p_\varphi u|^2 dr d\varphi = \int_0^{2\pi}\int_0^\infty \frac{r^2-4\eta^2(r)}{(r^2+\eta(r)^2)^2}|p_\varphi u|^2 dr d\varphi \tag{3.22}$$

where

$$\eta(r) = \begin{cases} \eta_0[1-(1+r^3)^{-1/6}], & \text{for } r < r_0 \\ 0 & \text{for } r > r_0+\eta_0 \ . \end{cases} \tag{3.23}$$

By choosing η_0 suitably small (e.g. $\eta_0 < 2^{-6/5}$) we have $(r^2-4\eta^2)(r^2+\eta^2)^2 \geq 0$ and therefore (3.22) is non negative. Next notice that $(f^2)''$ is bounded; thus,

$$\Re \int_0^{2\pi}\int_0^\infty \frac{\omega\theta_0^{-2}}{4}(f^2)''|u|^2 dr d\varphi \geq -\,(\text{const.})\,\|u\|^2 \,. \tag{3.24}$$

We can now estimate the potential term $V_\rho(r,\varphi) := \alpha^{-1}\theta_0^2\xi^2 + i\rho\theta_0^3\xi^3(\cos^2\varphi\sin\varphi - \frac{1}{3}\sin^3\varphi)$. We have

$$\begin{aligned} \Re(V_\rho(r,\varphi)) &= \sin\left(\frac{\varepsilon}{3}\right)(r^2-\eta(r)^2)\theta_0^2 + 4\cos\left(\frac{\varepsilon}{3}\right) r\eta(r)\theta_0^2 \\ &\quad + 6\rho r^2\theta_0^3\eta(r)\left(\cos^2\varphi\sin\varphi - \frac{1}{3}\sin^3\varphi\right) . \end{aligned} \tag{3.25}$$

Fixing $R>0$, for any $b>0$ there exists $k>0$ such that for $r \in (0,b)$:

$$\Re V_\rho(r,\varphi) \geq -k \,. \tag{3.26}$$

Moreover:

(I) For $r \geq r_0+\eta_0$, recalling that $r_0 = a_0/\rho$, we have

$$\Re V_\rho(r,\varphi) \geq k_1\frac{\rho^2}{R}r_0^2 = \frac{k_1 a_0^2}{R} \tag{3.27}$$

for a suitable constant $k_1>0$.

(II) Finally, for $b < r < r_0+\eta_0$ ($b>0$, independent of $\rho>0$) we have $0 \leq \eta(b) \leq \eta(r) \leq \eta_0$. Thus,

$$\begin{aligned} \Re V_\rho(r,\varphi) &\geq -k_2 + 4\cos\left(\frac{\varepsilon}{3}\right) r\theta_0^2\eta(r) - 8r^2\rho\eta_0\theta_0^3 \\ &\geq -k_2 + 2r\eta(b)\theta_0^2 - 8\eta_0 r^2\rho\theta_0^3 \end{aligned} \tag{3.28}$$

for some $k_2>0$, if $\rho>0$ is sufficiently small so that $\cos(\frac{\varepsilon}{3}) > \frac{1}{2}$. By suitably choosing $a_0>0$ (e.g. $a_0 = \eta(b)/8\eta_0\theta_0, b=1$) the term $2r\eta(b) - 8\rho\eta_0 r^2$ attains its maximum at $r = r_0 = \frac{a_0}{\rho}$ and its minimum at $r=b$ in (b,r_0). Thus, for $r\in(b,r_0)$ we have

$$\Re V_\rho(r,\varphi) \geq -k_2 + 2b^2\eta(b)\theta_0^2 - 8\rho\eta_0 b^2\theta_0^3 \geq -k_2, \tag{3.29}$$

and this concludes the proof of (3.14).

As for (3.15) notice that the kinetic part is $\geq -(\text{const.})\,\|u\|^2$, and for the potential part we have:

(I') For $r \geq r_0 + \eta_0$ we can repeat the argument used in (I).

(II') For $n_0 \leq n \leq r \leq r_0$ we proceed as in (II) with $n_0 \geq b\,(b=1), \eta(r) \geq \eta(b)$; thus,

$$\Re V_\rho(r,\varphi) \geq -k_2 + 2r\eta(b)\theta_0^2 - 8\eta_0 r^2 \rho\theta_0^3\,. \tag{3.30}$$

Again $2r\eta(b)\theta_0^2 - 8\eta_0\rho r^2\theta_0^3$ attains its maximum in (n, r_0) at r_0 and its minimum at $r = n$. Hence

$$\begin{aligned} \Re V_\rho(r,\varphi) &\geq -k_2 + 2n\eta(b)\theta_0^2 - 8n^2\rho\eta_0\theta_0^3 \\ &\geq k_3 n - k_4 \geq \frac{k_5}{R} - k_6. \end{aligned} \tag{3.31}$$

This concludes the proof of the Lemma.

Corollary 3.4 *Let $\chi_n(r) = \chi(r/n)$, $\chi \in C^\infty(\mathbf{R}^+)$, $\chi(r) = 1$ for $r \leq 1, \chi(r) = 0$ if $r \geq \frac{3}{2}$. Then there exists $c_3 > 0$ such that*

$$\|[H_\rho, \chi_n]u\| \leq c_3 n^{-1/4}(\|H_\rho u\| + \|u\|) \quad \forall u \in D(H_\rho),\, 0 \leq \rho < \rho_0. \tag{3.32}$$

Proof. It is enough to prove (3.32) for $\rho > 0$, since for $\rho = 0$ the argument is simpler. For simplicity we set $\|u\| = 1$. Let $\gamma_{2n}(r)$ be the characteristic function of the interval $[1, 2n]$ in the r variable. Then we have:

$$[H_\rho, \chi_n] = -\theta_0^{-2}\gamma_{2n}\omega\left\{2in^{-1}f^2\chi'\left(\frac{r}{n}\right) + n^{-2}\chi''\left(\frac{r}{n}\right) + 2ff'\chi'\left(\frac{r}{n}\right)\right\}. \tag{3.33}$$

Notice that the term $-\frac{\omega\theta_0^{-2}}{\xi^2}p_\varphi^2$ in H_ρ gives no contribution to the commutator $[H_\rho, \chi_n]$ since $\chi(r)$ does not depend on φ. Now:

$$\begin{aligned} \|[H_\rho, \chi_n]u\| &\leq c_4 n^{-1}\left\{\left(\int_0^{2\pi}\int_1^{2n} d\varphi dr |p_r u|^2\right)^{1/2} + 1\right\} \\ &\leq c_5 n^{-1}\left\{\frac{(1+8n^3)^{7/12}}{2n}\left[\int_0^{2\pi} d\varphi \int_1^{2n} dr |p_r u|^2 \eta_0 r^2 (1+r^3)^{-7/6}\right]^{1/2} + 1\right\} \\ &\leq c_6 n^{-1/4}\left\{\left[\int_0^{2\pi} d\varphi \int_1^{r_0} \eta_0 r^2 (1+r^3)^{-7/6}|p_r u|^2 dr\right]^{1/2} + 1\right\} \\ &\leq c_7 n^{-1/4}\{[\Re < H_\rho u, u > + c_8]^{1/2} + 1\} \end{aligned} \tag{3.34}$$

where the last inequality follows from Lemma 3.3. Now, taking $c_8 = c_9 + 1$ with $\Re < H_\rho u, u > + c_9 \geq 0$ we obtain:

$$\|[H_\rho, \chi_n]u\| \leq c_{10} n^{-1/4}\{\Re < H_\rho u, u > + c_{11}\} \tag{3.35}$$

where $c_{11} = c_8 + 1$. This concludes the proof.

Remark 3.5 A similar argument can be used to obtain the analogous estimate for the adjoint operator H_ρ^*:

$$\|[H_\rho^*, \chi_n]u\| \le c_3 n^{-1/4}(\|H_\rho^* u\| + \|u\|) \quad \forall u \in D(H_\rho^*), \quad 0 \le \rho < \rho_0 \,. \tag{3.36}$$

Proposition 3.6 *Let $M_n = 1 - \chi_n$, where χ_n is defined as in Corollary 3.4.*
If

$$d_n(\lambda, \rho) = \inf\{\|(\lambda - H_\rho)M_n u\| : \|M_n u\| = 1, \, u \in D(H_\rho)\}$$

then $\forall \lambda \in \mathbf{C}$, $\exists R, n_0, \rho_0, \delta > 0$ such that

$$d_n(\lambda, \rho) \ge \delta > 0, \quad \forall n \ge n_0, \quad \forall \rho \le \rho_0 \,. \tag{3.37}$$

Proof. First of all notice that

$$d_n(\lambda, \rho) \ge \mathrm{dist}(\lambda, E_n(\rho)) \tag{3.38}$$

where

$$E_n(\rho) = \{< M_n u, H_\rho M_n u >: \, u \in D(H_\rho), \, \|M_n u\| = 1\}$$

Moreover, by (3.15) we have:

$$\Re < M_n \rho, H_\rho M_n u > \ge c_1 R^{-1} - c_2 \,. \tag{3.39}$$

Now, (3.37) follows from (3.39): since c_1 and c_2 are independent of R we can take $R > 0$ suitably small so that (3.37) is satisfied with $\delta = \frac{c_1}{R} - c_2 - |\lambda|$.

Lemma 3.7 *Let $\rho_m > 0$ and $u_m \in D(H_{\rho_m})$ be two sequences such that $\rho_m \to 0^+$, $\|H_{\rho_m} u_m\|$ is bounded and $\|u_m\| = 1$, $u_m \xrightarrow{w} 0$. Then the sequences $\rho_{m(n)}, M_m u_{m(n)}$ satisfy the same properties for suitable $m = m(n)$, if $R > 0$ is chosen sufficiently small.*

Proof. By Corollary 3.4 the boundedness of $\|H_{\rho_m} u_m\|$ implies that of $\|H_{\rho_{m(n)}} M_n u_{m(n)}\|$. Thus it is enough to prove that

$$\lim_{n \to \infty} \|\chi_n u_{m(n)}\| = 0, \quad \forall n \,. \tag{3.40}$$

To prove (3.40), let $H'_\rho = \omega^{-1} H_\rho$ and $\lambda \in \mathbf{C} \setminus \sigma(H'_0)$ be fixed. Then

$$\|\chi_n u_m\|^2 \le c(\|\chi_n R'_o(H'_0 - H'_{\rho_m})u_m\|^2 + \|\chi_n R'_0(H'_{\rho_m} - \lambda)u_m\|^2) \,. \tag{3.41}$$

The second term in the right hand side of (3.41) tends to zero as $n \to \infty$, because $R'_0 := (H'_0 - \lambda)^{-1}$ is compact and $(H'_{\rho_m} - \lambda)u_m \xrightarrow{w} 0$. The first term in the right hand side of (3.41) can be bounded, up to a constant factor, by

$$\|R'_0 \chi_n (H'_0 - H'_{\rho_m})u_m\|^2 + \|[R'_0, \chi_n](H'_0 - H'_{\rho_m})u_m\|^2 \,. \tag{3.42}$$

Now, the first term in (3.42) can be bounded as follows

$$\begin{aligned}\|R_0'\chi_n(H_0'-H_{\rho_m}')u_m\|^2 &\leq c\|R_0'\|^2\int_0^{2\pi}d\varphi\int_0^{2n}|(H_0'-H_{\rho_m}')u_m|^2dr\\ &\leq c\|R_0'\|^2\int_0^{2\pi}d\varphi\int_0^{2n}(|\xi(r)|^2|1-e^{-\frac{2}{3}i\varepsilon_m}|+\rho_m|\xi(r)|^3)^2|u_m|^2dr\\ &\leq c'(\varepsilon_m n^2+\rho_m^2n^3)^2\|u_m\|^2 \qquad (3.43)\end{aligned}$$

where $\sin(\varepsilon_m)=\rho_m^2/R$ by hypothesis. In the second inequality we have used the fact that, for $r\in(0,2n)$, $\xi(r)=\xi_0(r)=r-2i\eta(r)$, $\eta(r)=\eta_0[1-(1+r^3)^{-1/6}]\leq\eta_0$. Since (3.43) tends to zero as $n\to\infty$, let us estimate the second term in (3.42):

$$\|[R_0',\chi_n](H_0'-H_{\rho_m}')u_m\|\leq\|R_0'[H_0',\chi_n]R_0'(H_0'-H_{\rho_m}')u_m\|\leq c''n^{-1/4}\,. \qquad (3.44)$$

Indeed, the operator $R_0'[H_0',\chi_n]=([\chi_n,(H_0')^*](R_0')^*)^*$, when applied to the bounded sequence $R_0'(H_0'-H_{\rho_m}')u_m$ satisfies the inequality (3.44) by (3.36).

Proof of Theorem 3.1. Since $\lim_{\rho\to0^+}H_\rho u=H_0u$, $\lim_{\rho\to0^+}H_\rho^*u=H_0^*u$, $\forall u\in C_0^\infty(\mathbf{R}^+\times\mathbf{T}^1)$, we can use Corollary 3.4, Proposition 3.6 and Lemma 3.7 in order to apply Theorem A.1 of [10] which provides the following stability result:

(i) if $\lambda\notin\sigma(H_0)$ then $(\lambda-H_\rho)^{-1}$ is uniformly bounded as $\rho\to0^+$;

(ii) if $\lambda\in\sigma(H_0)$ then λ is stable with respect to the family $\{H_\rho\}_{\rho>0}$.

Now the proof of the theorem is a consequence of Remark 3.2-(IV).

Proof of Theorem 1.2. Taking into account the result obtained in Theorem 3.1, the proofs of Theorems 3.13 and 1.3 of [4] can now be taken over directly without change in order to prove (a) and (b) respectively.

Remark 3.8 (1) For $\beta\in C_R, E(\beta)$ and $\bar{E}(\bar{\beta})$ are the so called "upper sum" and "lower sum" respectively of the RSPE (see Remark A.3 below), while the distributional Borel sum is given by $f(\beta)=\frac{1}{2}(E(\beta)+\bar{E}(\bar{\beta}))$ and $d(\beta)=E(\beta)-\bar{E}(\bar{\beta})$ is the discontinuity with zero asymptotic expansion. The result obtained in Theorem 1.2 can be interpreted in terms of resonances of the problem as explained in Remark 3.14 of [4].

(2) Similar results can be obtained if we now start from $\Im\beta<0$, instead of $\Im\beta>0$. We can establish a relationship between the resonance $E_1(\beta)$ obtained in this case and $E(\beta)$ following again [4]. Indead we have $E_1(\beta)=\bar{E}(\beta)$ for $\beta\in\mathbf{R}$.

A Appendix

To make the paper self-contained, in this appendix we first recall the notion of distributional Borel-Leroy summability of order q as introduced in [8].

Definition A.1 *Let q be a rational number, $(a_s)_{s\in\mathbf{N}}$ a sequence of real numbers and $R > 0$. We say that the formal series $\sum_{s=0}^{\infty} a_s\beta^s$ is Borel- Leroy summable of order q in the distributional sense to $f(\beta)$ for $0 < \beta < R$ if the following conditions are satisfied.*

(a) Set

$$B(t) = \sum_{s=0}^{\infty} \frac{a_s}{\Gamma(qs+1)} t^s \,. \tag{A.1}$$

Then $B(t)$ is holomorphic in some circle $|t| < \Lambda$; moreover $B(t)$ admits a holomorphic continuation to the intersection of some neighborhood of $\mathbf{R}^+ := \{t \in \mathbf{R} : t > 0\}$ with $\mathbf{C}^+ := \{t \in \mathbf{C} : \Im t > 0\}$.

(b) The boundary value distribution $B(t + i0)$ exists $\forall t \in \mathbf{R}^+$, and the following representation holds:

$$f(\beta) = \frac{1}{q\beta}\int_0^{\infty} PP(B(t))e^{-(t/\beta)^{1/q}} \left(\frac{t}{\beta}\right)^{-1+1/q} dt \tag{A.2}$$

for β belonging to the Nevanlinna disc of the $\beta^{1/q}-$plane $C_R := \{\beta : \Re\beta^{-1/q} > R^{-1}\}$, where $PP(B(t)) = \frac{1}{2}(B(t+i0) + \overline{B(t+i0)})$.
If $q = 1$ the series is called Borel summable in the distributional sense to $f(\beta)$.

Let us now recall the criterion for the distributional Borel-Leroy summability (see [8]). As for the ordinary Borel sum, it shows that the representation (A.2) is unique among all real functions admitting the prescribed formal power series expansion and fulfilling suitable analyticity requirements and remainder estimates. For the sake of simplicity we limit ourselves to the case $q = 1$.

Theorem A.2 *Let $f(\beta)$ be bounded and analytic in the Nevanlinna disc $C_R = \{\beta : \Re\beta^{-1} > R^{-1}\}$ and let $f(\beta) = (\Phi(\beta) - \bar{\Phi}(\bar{\beta}))/2$, with $\Phi(\beta)$ analytic in C_R and such that*

$$\left|\Phi(\beta) - \sum_{s=0}^{N-1} a_s\beta^s\right| \le A\sigma(\varepsilon)^N N!|\beta|^N, \quad \forall N = 1, 2, \ldots \tag{A.3}$$

uniformly in $C_{R,\varepsilon} = \{\beta \in C_R : \arg\beta \ge -\pi/2 + \varepsilon\}, \forall\varepsilon > 0$. Then the series $\sum_{s=0}^{\infty}(a_s/s!)u^s$ is convergent for small $|u|$ and it admits an analytic continuation $B(u) = B_1(u) + B_2(u)$, where $B_1(u)$ is analytic in $C_d^1 = \{u : dist(u, \mathbf{R}^+) < d^{-1}\}$ and $B_2(u)$

is analytic in $C_d^2 = \{u : (\Im u > 0, \Re u > -d^{-1})\ or |u| < d^{-1}\}$ *for some* $d > 0$. $B(u)$ *satisfies*

$$|B(t+i0)| \le A'(\eta_0)^{-1}e^{t/R} \tag{A.4}$$

uniformly for $t > 0$*, for any* η_0 *such that* $0 < \eta_0 < d^{-1}$*. Moreover, setting* $PP(B(t)) = (B(t+i0) + \overline{B(t+i0)})/2$*,* $f(\beta)$ *admits the integral representation*

$$f(\beta) = \beta^{-1}\int_0^\infty PP(B(t))e^{-t/\beta}dt, \qquad \beta \in C_R \tag{A.5}$$

i.e. $f(\beta)$ *is the distributional Borel sum of* $\sum_{s=0}^\infty a_s\beta^s$ *for* $0 < \beta < R$ *in the sense of Definition A.1. Conversely, if* $B(u) = \sum_{s=0}^\infty (a_s/s!)u^s$ *is convergent for* $|u| < d^{-1}$ *and admits the decomposition* $B(u) = B_1(u) + B_2(u)$ *with the above quoted properties, then the function defined by (A.5) is real-analytic in* C_R *and* $\Phi(\beta) = \beta^{-1}\int_0^\infty B(t+io)e^{-t/\beta}dt$ *is analytic and satisfies (A.3) in* C_R*.*

Remark A.3 The function $\phi(\beta) = \beta^{-1}\displaystyle\int_0^\infty B(t+i0)e^{-t/\beta}dt$ is called "the upper sum" and $\overline{\phi(\overline{\beta})} = \beta^{-1}\displaystyle\int_0^\infty \overline{B(t+i0)}e^{-t/\beta}dt$ "the lower sum" of the series. It follows that, for $\beta > 0$, $f(\beta) = \mathrm{Re}\phi(\beta)$. On the other hand with this method we can single out a unique function with zero asymptotic power series expansion, that is the "discontinuity"

$$d(\beta) = \beta^{-1}\int_0^\infty (B(t+i0) - \overline{B(t+i0)})e^{-t/\beta}dt = \phi(\beta) - \overline{\phi(\overline{\beta})}\,.$$

Thus, $d(\beta) = 2i\mathrm{Im}\phi(\beta)$, for $\beta > 0$.

B Appendix

Proof of Lemma 2.1. We shall proof the following estimate, equivalent to (2.1):

$$\|\Delta u\|^2 + |\sigma|^2\|x^2u\| + |\beta|^2\|Vu\|^2 \le a\|(-e^{-i\alpha}\Delta + \sigma x^2 + |\beta|V)u\|^2 + b\|u\|^2 \tag{B.1}$$

$\forall u \in D(H_0)\cap D(V)$, with $\sigma = \gamma e^{-i\alpha}$ varying in a compact subset of $\{\sigma \in \mathbf{C}\setminus\{0\} : |\sigma|^2 > 4|\beta|\sin\alpha,\ -\pi < \arg\pi < 0\}$. From now on we shall use the notation $-\Delta = p_1^2 + p_2^2$, where $p_j = -\frac{\partial^2}{\partial x_j^2}$, $j = 1, 2$. As quadratic forms on $C_0^\infty(\mathbf{R}^2)\otimes C_0^\infty(\mathbf{R}^2)$ we have

$$\begin{aligned}
&(-e^{i\alpha}\Delta + \bar\sigma x^2 + |\beta|V(x))(-e^{-i\alpha}\Delta + \sigma x^2 + |\beta|V(x)) \\
&\quad = (-e^{i\alpha}\Delta + |\beta|V(x))(-e^{-i\alpha}\Delta + |\beta|V(x)) + |\sigma|^2|x|^4 \\
&\qquad + \Re\sigma[(-e^{i\alpha}\Delta + |\beta|V(x))x^2 + x^2(-e^{-i\alpha}\Delta + |\beta|V(x))] \\
&\qquad + i\Im\sigma[(-e^{i\alpha}\Delta + |\beta|V(x))x^2 - x^2(-e^{-i\alpha}\Delta + |\beta|V(x))] \\
&\quad = \left|\frac{\Re\sigma}{\sigma}\right|(-e^{-i\alpha}\Delta + |\beta|V(x) \pm |\sigma|x^2)(-e^{-i\alpha}\Delta + |\beta|V(x) \pm |\sigma|x^2)
\end{aligned}$$

$$
\begin{aligned}
&+\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[(-e^{i\alpha}\Delta+|\beta|V(x))(-e^{-i\alpha}\Delta+|\beta|V(x))+|\sigma|^2|x|^4] \\
&+i\Im\sigma(-e^{i\alpha}\Delta x^2+e^{-i\alpha}x^2\Delta) \\
\geq&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[(-e^{i\alpha}\Delta+|\beta|V(x))(-e^{-i\alpha}\Delta+|\beta|V(x))+|\sigma|^2|x|^4] \\
&+i\Im\sigma\cos\alpha[-\Delta,x^2]-\Im\sigma\sin\alpha(-\Delta x^2-x^2\Delta) \\
=&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[\cdots]+2\Im\sigma\cos\alpha(p_1x_1+x_1p_1+p_2x_2+x_2p_2) \\
&-\Im\sigma\sin\alpha(p_1^2x_1^2+x_1^2p_1^2+p_2^2x_2^2+x_2^2p_2^2+2x_1^2p_2^2+2x_2^2p_1^2) \\
&(\text{since } \Im\sigma<0, \sin\alpha>0 \text{ and } x_1^2p_2^2+x_2^2p_1^2\geq 0) \\
\geq&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[\cdots]+2\Im\sigma\cos\alpha(p_1x_1+x_1p_1+p_2x_2+x_2p_2) \\
&-\Im\sigma\sin\alpha(p_1^2x_1^2+x_1^2p_1^2+p_2^2x_2^2+x_2^2p_2^2) \\
=&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[\cdots]\pm 2\Im\sigma|\cos\alpha|(p_1x_1+x_1p_1+p_2x_2+x_2p_2) \\
&-\Im\sigma\sin\alpha(-4+2p_1x_1^2p_1+2p_2x_2^2p_2) \\
&(\text{since } p_jx_j^2p_j\geq 0, j=1,2) \\
\geq&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[\cdots]-2\Im\sigma|\cos\alpha|[(p_1\mp x_1)^2-p_1^2-x_1^2+(p_2\mp x_2)^2-p_2^2-x_2^2] \\
&+4\Im\sigma\sin\alpha \\
\geq&\left(1-\left|\frac{\Re\sigma}{\sigma}\right|\right)[\cdots]+2\Im\sigma|\cos\alpha|(p_1^2+x_1^2+p_2^2+x_2^2)+4\Im\sigma\sin\alpha\,. \qquad \text{(B.2)}
\end{aligned}
$$

We shall prove below that the term inside square brackets in (B.2) satisfies the following estimate:

$$
\begin{aligned}
&(-e^{i\alpha}\Delta+|\beta|V(x))(-e^{-i\alpha}\Delta+|\beta|V(x))+|\sigma|^2|x|^4 \\
&\quad\geq a_1[(p_1^2+p_2^2)^2+|\beta|^2V(x)^2]+\frac{|\sigma|^2}{2}|x|^4-b_1. \qquad \text{(B.3)}
\end{aligned}
$$

for suitable constants $a_1, b_1>0$, independent of $\gamma\in\Omega$ and α in a compact subset of $]0,\pi[$.

Now, using (B.3) and setting $A=(1-\left|\frac{\Re\sigma}{\sigma}\right|)a_1$, $B=\frac{1}{2}(1-\left|\frac{\Re\sigma}{\sigma}\right|)$ and $b_2=(1-\left|\frac{\Re\sigma}{\sigma}\right|)b_1-4\Im\sigma\sin\alpha$, (B.2) can be bounded from below by:

$$
\begin{aligned}
&A[(p_1^2+p_2^2)^2+|\beta|^2V(x)^2]+B|\sigma|^2|x|^4-b_2 \\
&\quad+2\Im\sigma|\cos\alpha|(p_1^2+x_1^2+p_2^2+x_2^2) \\
&\quad=\left[Aa'(p_1^2+p_2^2)^2+2\Im\sigma|\cos\alpha|(p_1^2+p_2^2)-b_2+\frac{b'}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+A|\beta|^2V(x)^2+\left[a'|B||\sigma|^2|x|^4+2\Im\sigma|\cos\alpha|(x_1^2+x_2^2)+\frac{b'}{2}\right]\\
&+A(1-a')(p_1^2+p_2^2)^2+B(1-a')|\sigma|^2|x|^4-b'\,. \qquad \text{(B.4)}
\end{aligned}
$$

Since the terms inside square brackets in (B.4) are positive for a suitable choice of the constants $a', b' > 0, a' < 1$, we finally obtain

$$
\begin{aligned}
&(-e^{i\alpha}\Delta+\bar{\sigma}x^2+|\beta|V(x))(-e^{-i\alpha}\Delta+\sigma x^2+|\beta|V(x))\\
&\quad\geq A(1-a')(p_1^2+p_2^2)^2+A|\beta|^2V(x)^2+B(1-a')|\sigma|^2|x|^4-b'\,. \qquad \text{(B.5)}
\end{aligned}
$$

Now (B.1) follows from (B.5) with $a = \min(A(1-a'), B(1-a'))$ and $b = \frac{b'}{a}$. In order to complete the proof of the lemma we need to prove (B.3). We have

$$
\begin{aligned}
&(-e^{i\alpha}\Delta+|\beta|V(x))(-e^{-i\alpha}\Delta+|\beta|V(x))+\frac{|\sigma|^2}{2}|x|^4\\
&=(p_1^2+p_2^2)^2+|\beta|^2V(x)^2+|\beta|\cos\alpha[(p_1^2+p_2^2)V(x)+V(x)(p_1^2+p_2^2)]\\
&\quad+i|\beta|\sin\alpha[(p_1^2+p_2^2),V(x)]+\frac{|\sigma|^2}{2}|x|^4\\
&=(p_1^2+p_2^2)^2+|\beta|^2V(x)^2\pm|\beta||\cos\alpha|[(p_1^2+p_2^2)V(x)+V(x)(p_1^2+p_2^2)]\\
&\quad-2\sin\alpha|\beta|\left[x_2(p_1x_1+x_1p_1)+x_1(p_2x_2+x_2p_2)+\frac{1}{2}(x_2^2p_2+p_2x_2^2)\right]+\frac{|\sigma|^2}{2}|x|^4\\
&=|\cos\alpha|(p_1^2+p_2^2\pm|\beta|V(x))^2+(1-|\cos\alpha|)[(p_1^2+p_2^2)^2+|\beta|^2V(x)^2]\\
&\quad-2\sin\alpha|\beta|\left[x_2(p_1x_1+x_1p_1)+x_1(p_2x_2+x_2p_2)+\frac{1}{2}(x_2^2p_2+p_2x_2^2)\right]+\frac{|\sigma|^2}{2}|x|^4\\
&\geq(1-|\cos\alpha|)[(p_1^2+p_2^2)^2+|\beta|^2V(x)^2]+\frac{|\sigma|^2}{2}|x|^4+2|\beta|\sin\alpha\Big[(p_1-x_1x_2)^2\\
&\quad+(p_2-x_1x_2)^2+\frac{1}{2}(p_2-x_2^2)^2-\left(p_1^2+\frac{3}{2}p_2^2-2x_1^2x_2^2-\frac{1}{2}x_2^4\right)\Big]\\
&\geq(1-|\cos\alpha|)[(p_1^2+p_2^2)^2+|\beta|^2V(x)^2]-2|\beta|\sin\alpha\left(p_1^2+\frac{3}{2}p_2^2+2x_1^2x_2^2+\frac{1}{2}x_2^4\right)\\
&\quad+\frac{|\sigma|^2}{2}|x|^4\\
&=\left[(1-|\cos\alpha|)a_2(p_1^2+p_2^2)^2-2|\beta|\sin\alpha\left(p_1^2+\frac{3}{2}p_2^2\right)+b_1\right]\\
&\quad+(1-|\cos\alpha|)(1-a_2)(p_1^2+p_2^2)^2+(1-|\cos\alpha|)|\beta|^2V(x)^2-b_1\\
&\quad+\frac{|\sigma|^2}{2}|x|^4-|\beta|\sin\alpha(4x_1^2x_2^2+x_2^4). \qquad \text{(B.6)}
\end{aligned}
$$

Now, for a suitable choice of the constants $0 < a_2 < 1, b_3 > 0$, the term in square brackets in (B.6) is positive and therefore (B.6) can be bounded from below by:

$$
\begin{aligned}
&(1-|\cos\alpha|)(1-a_2)(p_1^2+p_2^2)\\
&\quad+(1-|\cos\alpha|)|\beta|^2V(x)^2-b_1+\frac{|\sigma|^2}{2}|x|^4-|\beta|\sin\alpha(4x_1^2x_2^2+x_2^4). \qquad \text{(B.7)}
\end{aligned}
$$

Next notice that, for $|\beta|\sin\alpha < \frac{|\sigma|^2}{4}$, we have

$$\begin{aligned}&\frac{|\sigma|^2}{2}|x|^4 - 4|\beta|\sin\alpha\, x_1^2 x_2^2 - |\beta|\sin\alpha\, x_2^4 \\ &\quad = \frac{|\sigma|^2}{2}x_1^4 + \left(\frac{|\sigma|^2}{2} - |\beta|\sin\alpha\right)x_2^4 + (|\sigma|^2 - 4|\beta|\sin\alpha)x_1^2x_2^2 \geq 0\,.\end{aligned}$$

Thus, we finally obtain

$$\begin{aligned}&(-e^{i\alpha}\Delta + |\beta|V(x))(-e^{-i\alpha}\Delta + |\beta|V(x)) + |\sigma|^2|x|^4 \\ &\quad \geq (1-|\cos\alpha|)(1-a_2)(p_1^2+p_2^2)^2 + (1-|\cos\alpha|)V(x)^2 \\ &\qquad + \frac{|\sigma|^2}{2}|x|^4 - b_1\end{aligned} \tag{B.8}$$

which corresponds to (B.3) with $a_1 = (1-|\cos\alpha|)(1-a_2)$.

References

[1] B. C. Bag, D. S. Ray: J. Stat. Phys. **96** (1999) 271

[2] C. M. Bender, G. V. Dunne, P. N. Meisinger, M. Sinsek: Phys. Lett. A **281** (2001) 311

[3] M. Brak, R. K. Bhaduri, J. Law, M. V. Murthy: Phys. Rev. Lett. **70** (5) (1993) 568

[4] E. Caliceti: J. Phys. A **33** (2000) 3753

[5] E. Caliceti: Czech. J. Phys. **54** (2004), n. 1, 29

[6] E. Caliceti, S. Graffi: J. Phys. A **37** (2004) 2239

[7] E. Caliceti, S. Graffi, M. Maioli: Commun. Math. Phys. **75** (1980) 51

[8] E. Caliceti, V. Grecchi, M. Maioli: Commun. Math. Phys. **104** (1986) 163

[9] E. Caliceti, V. Grecchi, M. Maioli: Commun. Math. Phys. **157** (1993) 347

[10] E. Caliceti, V. Grecchi, M. Maioli: Commun. Math. Phys. **176** (1996) 1

[11] L. Carlson, W. C. Schieve: Phys. Rev. A **40** (1989) 5896

[12] G. Casati, B. V. Chirikov, F M. Izrailev, J. Ford: *Stochastic Behaviour in Classical and Quantum Hamiltonian System*, Lecture Notes in Physics, Vol. 93, edited by G. Casati and J. Ford (Springer-Verlag, NewYork, 1979) p 334

[13] J. B. Delos: *Chaos in Atomic and Molecular Theory, (From Abstract Mathematics to Patented Devices)*, http://www.pa.uky.edu/mike/tamoc/frontiers/html/delosf.html (Refereed web publication)

[14] L. Faddeev, P. Von Moerbeke, F. Lambert (Eds.): *Bilinear Integrable Systems, from Classical to Quantum, Continuos to Discrete.* Proceedings of the NATO Advanced Research Workshop, St. Petersburg 15-19/9/2002. Series: Nato Sciences Series II: Mathematics, Physics and Chemistry, vol. 201 (Springer 2005)

[15] M. Feingold, N. Moiseyev, A. Peres: Phys. Rev. A **30**, n. 1 (1984) 509

[16] H. Henon, C. Heiles: Astron. J. **69** (1964) 73

[17] W. Hunziker, C. A. Pillet: Commun. Math. Phys. **90** (1983) 219

[18] A. J. Lichtenberg, M. Lieberman: *Regular and Stochastic Motion*, Springer, New York, 1983

[19] F. Nardini: Boll. U.M.I. B **4** (1985) 473