DISTRIBUTION OF RESONANCES FOR OPEN QUANTUM MAPS

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1. Introduction

1.1. **Statement of the results.** In this note we analyze simple models of classical chaotic open systems and of their quantizations. They provide a numerical confirmation of the *fractal Weyl law* for the density of quantum resonances of such systems. The exponent in that law is related to the dimension of the classical repeller of the system. In a simplified model, a rigorous argument gives the full resonance spectrum, which satisfies the fractal Weyl law. Our model is similar to models recently studied in atomic and mesoscopic physics (see §2.4 below). Before stating the main result we remark that in this paper we use mathematician's notation h for what the physicists call $\hbar$. That is partly to stress that our h is a small parameter in asymptotic analysis, not necessarily interpreted as the Planck constant.

Theorem 1. *There exist families of symplectic relations, $\widetilde{B} \subset \mathbb{T}^2 \times \mathbb{T}^2$, and of their (subunitary) quantization, $\widetilde{B}_h \in \mathcal{L}(\mathbb{C}^N)$, $N = (2\pi h)^{-1}$, such that*

$$\#\big\{ \operatorname{Spec}(\widetilde{B}_h) \setminus D(0,r) \big\} = c(r)\, h^{-\nu} + o(h^{-\nu})\,,$$
$$r > 0\,, \quad h = h_k \to 0, \quad k \to \infty\,,$$
$$\nu = \dim \big(\Gamma_-(\widetilde{B}) \cap W_+(\widetilde{B}) \big)\,, \quad c(r) = (2\pi)^{-\nu} \chi_{[0, r_0(\widetilde{B})]}(r)\,, \quad 0 < r_0(\widetilde{B}) < 1\,,$$

where the decreasing sequence $\{h_k\}_{k\in\mathbb{N}}$ is chosen depending on $\widetilde{B}$. Here $\Gamma_-(\widetilde{B}) \subset \mathbb{T}^2$ is the forward trapped set of $\widetilde{B}$ and $W_+(\widetilde{B})$ is the unstable manifold of $\widetilde{B}$ at any point of $\Gamma_-(\widetilde{B})$.

The asymptotics are actually much more precise and include uniform angular distribution. The resonances lie on a lattice (see Fig. 1) and some of this structure is also seen in numerically computed more generic situations (see Fig. 5). In the models used in Theorem 1 we can compute the conductance and the shot noise power (or the closely related Fano factor) — see §2.4.3 and references given there for physics background and §7 for precise definitions.

Theorem 2. *For models considered in Theorem 1, with openings consisting of two "leads" of equal widths (see §7.1), the quantum conductance between the two leads* (2.17) *satisfies*

$$G(h) = \frac{1}{2}\, M(h) \big(1 + o(1) \big)\,, \quad h = h_k \to 0\,, \tag{1.1}$$

where $M(h) \sim h^{-1}$ is the number of scattering channels in each lead. The Fano factor (2.18) *is given by*

$$F(h) = \frac{11}{80} \frac{M(h)^\nu}{G(h)} \big(1 + o(1)\big) , \quad h = h_k \to 0 . \tag{1.2}$$

The theorem should be interpreted as follows. In (1.1) we see that for a model of scattering through a chaotic cavity, approximately one half of the scattering channels get transmitted from one lead to the other, the other half being reflected back (this is natural and well known). Asymptotics in (1.2) are more interesting. We see that the fractal Weyl law, $h^{-\nu}$, appears in the expression for the Fano factor. In the interpretation of the Fano factor in terms of "shot noise" (see §2.4.3), 11/80 gives the average "shot noise" per "nonclassical transmission channel". This number is close to the random matrix theory prediction for this quantity, namely 1/8 [23, 49]. In fact, had (1.2) come from a physical experiment rather than an asymptotic computation, it would be regarded as being in a very good agreement with random matrix theory[†].

Much of the paper is devoted to rigorous definitions of the objects appearing in the statements of the two theorems. In this section we give some general indications, with detailed references to previous works appearing below.

We consider $\mathbb{T}^2 = [0,1) \times [0,1)$, the two-torus, as our classical phase space. Classical observables are functions on $\mathbb{T}^2$ and classical dynamics is given in terms of relations, $B \subset \mathbb{T}^2 \times \mathbb{T}^2$, which are unions of truncated graphs of symplectic (area preserving) maps $\mathbb{T}^2 \to \mathbb{T}^2$. An example is given by the baker's relation

$$(\rho'; \rho) = (q', p'; q, p) \in B \iff \begin{cases} q' = 3q \,, & p' = p/3 \,, \quad\quad\quad 0 \leq q \leq 1/3 \\ q' = 3q - 2 \,, & p' = (p+2)/3 \,, \quad 2/3 \leq q < 1 \,. \end{cases} \tag{1.3}$$

This is a "rectangular horseshoe" modeling a Poincaré map of a chaotic open system: some points (here $\{\, 1/3 < q < 2/3 \,\}$) are thrown out at each iteration.

For relations such as B we can define the *incoming and outgoing tails* (see (2.4) for the definition in the case of flows):

$$\begin{aligned} \rho \in \Gamma_- \ &\Leftrightarrow \ \exists \{\rho_j\}_{j=0}^{\infty} \,, \quad\ \ \rho_0 = \rho \,, \ \ (\rho_j ; \rho_{j-1}) \in B \,, \ j > 0 \,, \\ \rho \in \Gamma_+ \ &\Leftrightarrow \ \exists \{\rho_j\}_{j=-\infty}^{0} \,, \ \ \rho_0 = \rho \,, \ \ (\rho_j ; \rho_{j-1}) \in B \,, \ j \leq 0 \,. \end{aligned}$$

In the example (1.3), $\Gamma_- = C \times [0,1)$, $\Gamma_+ = [0,1) \times C$, where C is the usual $\frac{1}{3}$–Cantor set.

We also define the *trapped set* $K = \Gamma_+ \cap \Gamma_-$ and, at points of K, the stable and unstable manifolds, $W_\pm$. In the case of the above baker's relation,

$$\frac{1}{2} \dim K = \dim \Gamma_+ \cap W_- = \dim \Gamma_- \cap W_+ = \frac{\log 2}{\log 3} \,,$$

but for general (possibly multivalued) relations these equalities do not hold.

[†]We are grateful to Yan Fyodorov for this amusing comment.

A quantization (in the sense made rigorous in §4.4) of B is given by

$$B_h = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/3} \end{pmatrix}, \quad h = (2\pi N)^{-1}, \ 3|N, \tag{1.4}$$

where $\mathcal{F}_M$ is the discrete Fourier transform on $\mathbb{C}^M$.

$N = 3^k$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$
$k = 1$	5	5	5	5	5	4	3	3
$k = 2$	14	14	10	9	8	8	7	6
$k = 3$	32	26	23	19	16	16	14	5
$k = 4$	63	53	45	40	33	33	30	6
$k = 5$	124	103	85	78	71	65	63	11
$k = 6$	237	196	161	150	142	131	128	12

TABLE 1. Number of eigenvalues of B_h in the regions $\{\,|\lambda| > r\,\}$, for $2\pi h = 1/N$, N given by powers of 3.

Table 2 shows the analogies between the eigenvalues of this subunitary quantum map and the resonances of a Schrödinger operator for a scattering situation (see §2.1).

For B_h given by (1.4) we are unable to prove the fractal Weyl law presented in the last line of Table 2, but numerical results in §5 strongly support its validity. A striking illustration is given by tripling N in which case the number of eigenvalues approximately doubles, in agreement with the fractal Weyl law — see Table 1.

The family of subunitary quantum maps in the main theorem is obtained by simplifying B_h, and is described explicitly in (6.2). It is a quantization of a more complicated multivalued relation for which $\Gamma_+ = \mathbb{T}^2$, $\Gamma_- = C \times [0, 1)$, and $\Gamma_- \cap W_+ \simeq C$ — see Proposition 6.1. Theorem 1 follows from the more precise Proposition 6.4.

1.2. **Organization of the paper.** In §2 we present related results from recent mathematical, numerical, and physics literature. In particular, in §2.4.3 we give the physical motivation for the objects appearing in Theorem 2 above. §3 is devoted to the review of classical dynamics used in our models, stressing the dynamics of open baker's relations.

In §4 we first review the quantization of tori. We assume the knowledge of semiclassical quantization in $T^*\mathbb{R}^n$ (pseudodifferential operators) but otherwise the presentation is self contained. The definitions of Lagrangian states associated to smooth and singular Lagrangian submanifolds is based on the ideas of Guillemin, Hörmander, Melrose, and Uhlmann in microlocal analysis but, partly due to technical differences, we give direct

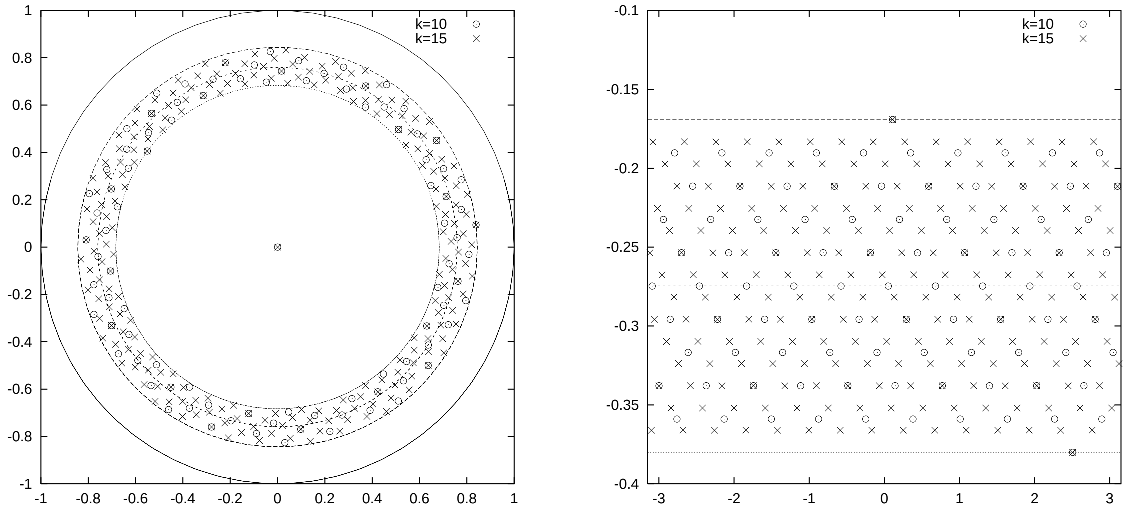

FIGURE 1. The illustration of Theorem 1: on the left, eigenvalues $\lambda \in \mathrm{Spec}(\widetilde{B}_h)$ for $(2\pi h)^{-1} = N = 3^{10}$ (circles) and for $N = 3^{15}$ (crosses). The four circles correspond to radii $1, |\lambda_+|, |\lambda_+\lambda_-|^{\frac{1}{2}}|, |\lambda_-|$, where $\lambda_\pm$ are given in Proposition 6.5. The critical radius in the statement of the theorem is here $r_0(\widetilde{B}) = |\lambda_+\lambda_-|^{\frac{1}{2}}$. On the right, we plot the phase (horizontal) and modulus (vertical, log scale) of these eigenvalues. The eigenvalues appear with high, varying, multiplicities which peak on the circle $|\lambda| = r_0(\widetilde{B})$, but the angular distribution is asymptotically uniform.

proofs of the properties we need in this paper. These properties are used to analyze the quantizations of the baker's relation coming from the work of Balazs, Voros, Saraceno and Vallejos.

In §5 we present numerical results for the quantization of the open baker's relation and in §6 we discuss a simplified toy model in the language of Walsh-Fourier transform. It contains the proof of Theorem 1. Finally in §7 we give precise definitions of objects appearing in Theorem 2 and in a lengthy computation we give its proof.

Acknowledgments. We are grateful to Christof Thiele and Terry Tao for pointing out the "Walsh" interpretation of our toy model. The first author thanks Marcos Saraceno for his insights on that model, and André Voros for interesting questions. He is also grateful to UC Berkeley for the hospitality in April 2004. Generous support of both authors by the National Science Foundation under the grant DMS-0200732 is also gratefully acknowledged.

2. Motivation and background

In this section we discuss motivating topics in mathematics and theoretical physics, and survey related results.

2.1. **Schrödinger operators.** The original motivation comes from the study of resonances in potential scattering. The simplest case is given by considering the following quantum Hamiltonian:

$$H = -h^2\Delta + V(q)\,, \quad V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R})\,. \tag{2.1}$$

By assuming that the potential vanishes near infinity and that it is infinitely differentiable, we eliminate the need for technical assumptions — see [20], and [46] for more general settings, in the analytic and $\mathcal{C}^\infty$ categories respectively. For instance, as noted in [43, (c.32)-(c.33)] the theory of [20] applies to arbitrary homogeneous polynomial potentials at non-degenerate energy levels.

$h \to 0$	$N = (2\pi h)^{-1} \to \infty$
$\chi \exp\big(-it(-h^2\Delta + V)/h\big)\chi\,, \quad t \geq 0\,,$ χ a cut-off to an interaction region	B_h^t, $t = 0, 1, \cdots$ B_h a subunitary matrix
$e^{-itz/h}$, z a resonance of $H = -h^2\Delta + V$	λ^t, λ an eigenvalue of $B_h \in \mathcal{L}(\mathbb{C}^N)$
$z \in [E-h, E+h] - i[0, \gamma h]$	$1 \geq \lvert\lambda\rvert > r > 0$
$\#\{z \in [E-h, E+h] - i[0,\gamma h]\} \simeq \; C(\gamma)\, h^{-\mu_E}$	$\#\{\lambda, \lvert\lambda\rvert > r\} \simeq \; C(r)\, N^{\frac{\log 2}{\log 3}}$

Table 2. Analogies between Schrödinger propagators and quantum maps.

Before discussing open systems we recall the well known results for closed systems, obtained for instance by considering H above on a bounded domain $\Omega \subset \mathbb{R}^n$ and imposing a self-adjoint boundary condition at $\partial\Omega$ (Dirichlet or Neumann). Then the spectrum, $\mathrm{Spec}(H)$, of H is discrete and, at a non-degenerate energy level E its density is described by the celebrated *Weyl law*:

$$\# \{ \operatorname{Spec}(H) \cap [E-\delta, E+\delta] \} = \frac{1}{(2\pi h)^n} \iint_{|p^2+V(q)-E|<\delta} dq\, dp + \mathcal{O}(h^{1-n}) \,, \tag{2.2}$$

see [13, 22] and references given there. We note that this implies a precise upper bound

$$\# \{ \operatorname{Spec}(H) \cap [E-Ch, E+Ch] \} = \mathcal{O}(h^{1-n}) \,, \tag{2.3}$$

which can be improved further by making assumptions on the classical flow of the Hamiltonian $\xi^2 + V(x)$ on Ω, see [13, 22].

For open systems, with the simplest example given by the Hamiltonian in (2.1), real eigenvalues are replaced by complex *resonances*. The simplest definition (easily made rigorous in the case (2.1)) comes from considering the meromorphic continuation of the resolvent. Defining the Green's function $G(z, q', q)$ for $\operatorname{Im} z > 0$ through

$$(z-H)^{-1} u(q') = \int_{\mathbb{R}^n} G(z, q', q)\, u(q)\, dq \,, \quad u \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n) \,,$$

then $G(z, q', q)$ admits a meromorphic continuation in z across the real axis. Its poles for $\operatorname{Im} z < 0$ (which do not depend on (q', q)) are the *quantum resonances* of H.

Counting of resonances is affected by the dynamical structure of the scatterer much more dramatically than counting of eigenvalues of closed systems. Since we are now counting points in the complex plane we need to make geometric choices dictated by dynamical and physical considerations. Here we consider scatterers and energies exhibiting a hyperbolic classical flow, and regions in the lower half-plane which lie at a distance proportional to h from the real axis. This choice is motivated as follows. Quantum mechanics interprets a resonance at $z = E - i\gamma$ in terms of a *metastable state*, which decays proportionally to $\exp(-t\gamma/h)$. Hence for $\gamma \gg h$ the decay is so rapid that the state is invisible. On the other hand, for many chaotic scatterers there are no resonances with $\gamma \ll h$. One class for which this is known rigorously consists in the Laplacian on co-compact quotients $\mathbb{H}^n/\Gamma$, $H = -h^2 \Delta_{\mathbb{H}^n/\Gamma}$, when the dimension of the limit set satisfies $\delta(\Gamma) < (n-1)/2$. This follows from the work of Patterson and Sullivan — see the discussion below and [34].

After a complex deformation (see [46] and references given there) the long living quantum states should semiclassically concentrate on the set of phase space points which do not escape to infinity, that is on the *trapped set* K_E defined as follows: let Ξ_H be the Hamilton vector field of the Hamiltonian $H(q,p) = p^2/2 + V(q)$:

$$\Xi_H = \sum_{j=1}^n 2p_j \partial_{q_j} - \partial_{q_j} V(q) \partial_{p_j} \,.$$

Then

$$\begin{gathered} K_E \stackrel{\rm def}{=} \Gamma_+(E) \cap \Gamma_-(E) \,, \\ \Gamma_\pm(E) \stackrel{\rm def}{=} \{ \rho \in \Sigma_E \; : \; \exp t\, \Xi_H(\rho) \not\to \infty \,, \; \mp t \to \infty \} \,. \end{gathered} \tag{2.4}$$

Suppose that the flow generated by Ξ_H is hyperbolic near $K_{E'}$ for E' close to a non-degenerate energy E. That means that the field Ξ_H does not vanish on the energy surfaces $\Sigma_{E'} = \{p^2 + V(q) = E'\} \subset \mathbb{R}^n \times \mathbb{R}^n$ for $E' \approx E$, and that for $\rho \in \Sigma_{E'}$ near $K_{E'}$,

$$\begin{aligned} &T_\rho \Sigma_{E'} = \mathbb{R}\, \Xi_H(\rho) \oplus E_+(\rho) \oplus E_-(\rho)\,, \\ &\Sigma_{E'} \ni \rho \longmapsto E_\pm(\rho) \subset T_\rho \Sigma_{E'} \ \text{ is continuous,} \\ &d(\exp t\, \Xi_H)(E_\pm(\rho)) = E_\pm(\exp t\, \Xi_H(\rho))\,, \\ &\| d(\exp t\, \Xi_H)(X) \| \leq C e^{\pm \lambda t} \|X\|\,, \quad \text{for all } X \in E_\pm(\rho),\ \mp t \geq 0. \end{aligned} \tag{2.5}$$

Weaker assumptions are possible — see [43, §5] and [46, §7].

Typically, the set K_E has a fractal structure and in the semiclassical estimates the Minkowski dimension naturally appears:

$$\dim K_E = 2n - 1 - \sup \left\{ c \ : \ \limsup_{\epsilon \to 0} \epsilon^{-c} \operatorname{vol}\{\rho \in \Sigma_E \ : \ \operatorname{dist}(K_E, \rho) < \epsilon\} < \infty \right\}.$$

We say that K_E is of *pure dimension* if the supremum is attained. For simplicity of the presentation we assume that this is the case.

Under these assumptions the estimate (2.3) has an analogue for chaotic open systems [46]. For $C_0 > 0$ there exists C_1 such that

$$\# \left\{ \operatorname{Res}(H) \cap \{ z \ : \ |z - E| < C_0 h \} \right\} \leq C_1 h^{-\mu_E}\,, \quad \dim K_E = 2\mu_E + 1\,. \tag{2.6}$$

We notice that for a closed system the trapped set is the entire energy surface, so that in that case $\mu_E = n - 1$, hence (2.6) is consistent with (2.3). In this note we use open quantum maps to provide the first evidence that this precise estimate is optimal.

2.2. **Survey of related results.** The first indication that fractal dimensions enter into counting laws for quantum resonances of chaotic open systems appears in a result of Sjöstrand [43]:

$$\begin{gathered} \# \left\{ \operatorname{Res}(H) \cap \{ z \ : \ |z - E| < \delta\,, \ \operatorname{Im} z > -\gamma \} \right\} \leq C_1 \delta \left(\frac{h}{\gamma} \right)^{-n} \gamma^{-\frac{1}{2}\widetilde{m}}\,, \\ Ch \leq \gamma \leq 1/C\,, \quad \max(h^{\frac{1}{2}}, h/\gamma) \leq \delta \leq 2/C\,, \end{gathered} \tag{2.7}$$

where $\widetilde{m}$ is any number greater than the dimension of the trapped set in the shell $H^{-1}(E - 1/C_2, E + 1/C_2)$. In a homogeneous situation, such as for instance obstacle scattering, the dimension of K_E, $2\mu_E + 1$, is independent of E, so that $\widetilde{m} > 2(\mu_E + 1)$.

The improvement in [46] quoted in (2.6) lies in providing a bound for the number of resonances in a smaller region $D(E, Ch)$. Heuristic arguments suggesting that the estimate (2.7) should be optimal were given in [28] and [29].

Another class of Hamiltonians with chaotic classical flows and fractal trapped sets is given by Laplacians on convex co-compact quotients, $\mathbb{H}/\Gamma$, where Γ is a discrete subgroup of isometries of $\mathbb{H}$, the hyperbolic plane, such that

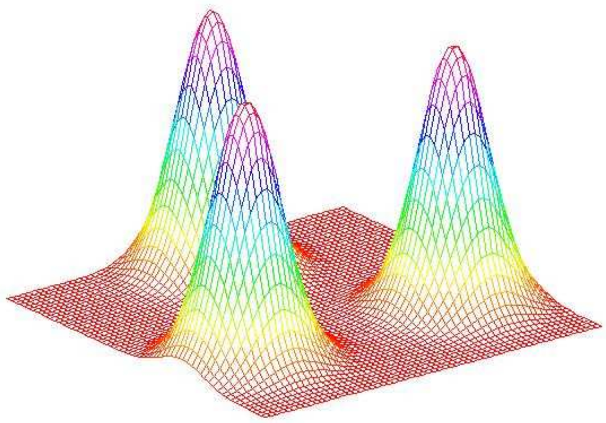

FIGURE 2. A three bump potential exhibiting hyperbolic dynamics in a range of energies.

- All elements $\gamma \in \Gamma$ are hyperbolic, which means that their action on $\mathbb{H}$ can be represented as

$$\alpha \circ \gamma \circ \alpha^{-1}(x, y) = e^{\ell(\gamma)}(x, y)\,, \quad (x, y) \in \mathbb{H} \simeq \mathbb{R}_+ \times \mathbb{R}\,, \quad \alpha \in \mathrm{Aut}(\mathbb{H})\,. \tag{2.8}$$

- If $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, and $\Lambda(\Gamma) \subset \partial\mathbb{H}$ is the limit set of Γ, that is the set of limit points of $\{\gamma(z) : \gamma \in \Gamma\}$, $z \in \mathbb{H}$, then $\pi(\text{convex hull } \Lambda(\Gamma))$ is compact.

An example is shown in Fig. 3. The trapped set is determined by $\Lambda(\Gamma)$: trapped trajectories are given by geodesics connecting two points of $\Lambda(\Gamma)$ at infinity, and

$$\dim K_E = 2\delta_\Gamma + 1\,, \quad \delta_\Gamma = \dim \Lambda(\Gamma)\,.$$

The limit set is always of pure dimension, which coincides with its Hausdorff dimension.

A nice feature of this model is the exact correspondence between the resonances of

$$H = h^2(-\Delta_{\mathbb{H}/\Gamma} - 1/4)\,,$$

and the zeros of the Selberg zeta function, $Z_\Gamma(s)$[‡]:

$$z \in \mathrm{Res}(H) \iff Z_\Gamma(s) = 0\,, \quad z = h^2(s(1-s) - 1/4)\,, \quad \mathrm{Re}\, s \leq \delta_\Gamma\,, \tag{2.9}$$

[‡]We refer to [35] for this and a general treatment. The term $\frac{1}{4}$ in the definition of the Hamiltonian H comes from requiring that the bottom of the spectrum of H is 0, so that Green's function $(H - \lambda^2)^{-1}$ is meromorphic in $\lambda \in \mathbb{C}$

where the multiplicities of zeros and resonances agree. The Selberg zeta function is defined by the analytic continuation of

$$Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k \geq 0} \left(1 - e^{-(s+k)\,\ell(\gamma)}\right) , \quad \operatorname{Re} s > \delta_\Gamma ,$$

where $\{\gamma\}$ denotes a conjugacy class of a primitive element $\gamma \in \Gamma$ (an element which is not a power of another element), and we take a product over distinct primitive conjugacy classes (each of which corresponds to a primitive closed orbit). The length $\ell(\gamma)$ of the corresponding closed orbit appears in (2.8). The exact analogue of (2.6) is given by

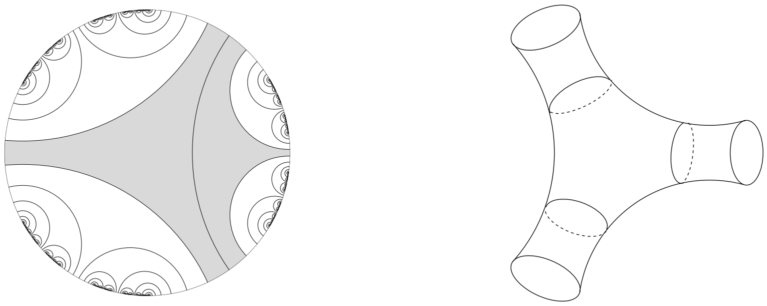

FIGURE 3. An example of $\mathbb{H}^2/\Gamma$ where Γ is generated by compositions of reflections in three discs.

$$\# \{ s \;:\; Z_\Gamma(s) = 0 , \quad \operatorname{Re} s > -C_0 , \quad r < \operatorname{Im} s < r + C_1 \} \leq C_2 \, r^{\delta_\Gamma} , \tag{2.10}$$

which is a consequence of an estimate established by Guillopé-Lin-Zworski [18] in a more general setting of convex co-compact Schottky groups in any dimension,

$$|Z_\Gamma(s)| \leq C_K \, e^{C_K \, |s|^{\delta_\Gamma}} , \quad \operatorname{Re} s \geq -K , \quad \text{for any } K. \tag{2.11}$$

This improved earlier estimates of [51], the proof of which was largely based on [43].

In the (non-quantum) context of rational maps on the complex plane, similar results were obtained concerning the zeros of associated zeta functions [10, 47]. Take f a uniformly expanding rational map on $\mathbb{C}$ (for instance $z \mapsto z^2 + c$, $c < -2$), and call f^n its n-fold composition. The zeta function associated with this map is given by

$$Z(s) = \exp\left(- \sum_{n=1}^{\infty} n^{-1} \sum_{f^n(z) = z} \frac{|(f^n)'(z)|^{-s}|}{1 - |(f^n)'(z)|^{-1}} \right) . \tag{2.12}$$

Then the number or resonances in a strip is also given by a law of the type (2.10), where δ_Γ is replaced by the dimension of the Julia set:

$$J = \overline{\bigcup_{n \geq 1} \{ z \;:\; f^n(z) = z \}} .$$

Note that this set is also made of "trapped orbits".

2.3. **Survey of numerical results.** The first model investigated numerically was perhaps the hardest to give definitive results. Lin [27, 28] studied the semiclassical Schrödinger Hamiltonian (2.1) with the potential given in Fig. 2. The semiclassical resonances were computed using the method of complex scaling and were counted in boxes of type $[E-\delta, E+\delta]-i[0,h]$ with δ fixed. The purpose was to verify optimality of Sjöstrand's estimate (2.7) with these parameters. The results were encouraging but not conclusive. Since for small values of h the method of [27] required the use of large matrices to discretize the Hamiltonians, the range of h's was rather limited.

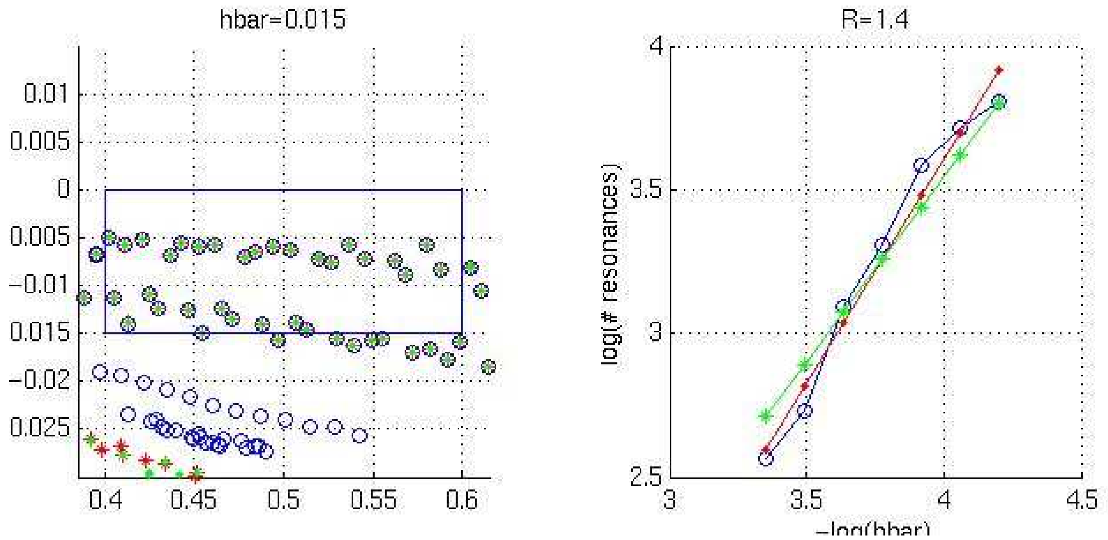

FIGURE 4. A sample of results of [27]: the plot on the left shows resonances for $h = 0.015$, and the plot on the right is the log-log plot of the number of resonances vs. h with $\circ$ denoting numerical data, $*$ the density predicted by the fractal Weyl law, and $\bullet$, the least square interpolants.

A different point of view was taken by Lu-Sridhar-Zworski [29] where resonances for the three discs scatterer in the plane were computed using the semiclassical zeta function of Eckhardt-Cvitanović, Gaspard, and others (see for instance [11, 16, 50] and references therein). The zeta function is computed using the cycle expansion method loosely based on the Ruelle theory of dynamical zeta functions. Although it is not rigorously known if the resonances computed by this method approximate resonances of the Dirichlet Laplacian in the exterior of the discs, or even if the semiclassical zeta function has an analytic continuation, proceeding this way is widely accepted in the physics literature. Resonances $z = h^2 k^2$ were counted in regions

$$\{ k \in \mathbb{C} \ : \ 1 \leq \operatorname{Re} k \leq r \,, \quad \operatorname{Im} k \geq -\gamma \} \,, \quad r \to \infty \,, \tag{2.13}$$

which under semiclassical rescaling correspond to counting in $[1/2, 2] - i[0, \gamma h/2]$, $h \to 0$. Let us denote the number of resonances (zeros of the semiclassical zeta function) in (2.13) by $N(r,\gamma)$. The fractal Weyl law corresponds to the claim that for γ large enough,

$$N(r,\gamma) \sim C(\gamma) r^{\mu}\,, \quad r \to \infty\,, \tag{2.14}$$

where $2\mu+1$ is the dimension of the trapped set in the three dimensional energy shell (for such homogeneous problems, all energy shells are equivalent). In [29] the prediction (2.14) was tested by linear fitting of $\log N(r,\gamma)$ as a function of $\log r$:

$$\log N(r,\gamma) = \alpha(\gamma)\log r + \mathcal{O}(1)\,.$$

We found that the coefficient $\alpha(\gamma)$ was independent of γ for γ large enough, and that it agreed with μ. The counting was done for three different equilateral disc configurations, parametrized by $\rho = R/a$ where a is the radius of each disc, and R the distance between them. We also noticed that if γ_ρ is the classical rate of decay for the ρ configuration, then

$$\frac{\alpha_\rho(x\gamma_\rho/2)}{\mu_\rho}$$

is essentially independent of ρ for $1 < x < 1.5$. This corresponds to a numerical observation that for each ρ the distribution of resonance widths (imaginary parts) peaks near $\gamma = \gamma_\rho/2$.

Encouraged by the results of [29], the cycle method was used in [18] to count the zeros of the Selberg zeta function for the quotient depicted in Fig. 3, but the results were not definitive. For the dynamical zeta function (2.12) with $f(z) = z^2 + c$, $c < -2$ the resonances were computed by Strain-Zworski [47], using a different method based on the theory of the transfer operator on Hilbert spaces of holomorphic functions introduced in [18]. The zeros were counted in a region of the same type as in (2.13),

$$\{s \;:\; \operatorname{Re} s > -K\,,\ 0 \le \operatorname{Im} s \le r\}\,,$$

where real parts and imaginary parts exchange their meaning due to different conventions[§]. By reaching very high values of r we saw a very good agreement of the log-log fit with the fractal Weyl, with μ given by the dimension of the Julia set.

In the model considered in this paper we can verify the optimality of the fractal Weyl law on much smaller scales. That could not be seen in the other approaches.

2.4. **Related models in physics.** The behaviour of quantum open systems has been recently investigated in situations where the classical dynamics has chaotic features. The physical motivation can originate from nuclear or atomic physics (study the lifetime statistics of metastable states, possibly leading to ionization), mesoscopic physics (study the conductance, conductance fluctuations, shot noise in quantum dots or quantum wires), and from waveguides (optical wave propagation in an optical fiber with some dissipation, microwave propagation in an open microwave cavity).

[§]Although frustrating, the different conventions of semiclassical, obstacle, and hyperbolic scattering show how the same phenomenon appears in historically different fields.

2.4.1. *Kicked rotator with absorbing boundaries.* In [3] Casati *et al.* use a kicked rotator with absorbtion to model the process of ionization. The classical kicked rotator is Chirikov's *standard map* on the cylinder, which is a paradigmatic model for transitions from regular to chaotic motion [8]. Quantizing the map on $L^2(\mathbb{T}^1)$ results in a unitary operator U, a first instance of *quantum map*. To model the ionization process which takes place at some threshold momentum $p_{\rm ion}$, the authors truncate the map U to the subspace $\mathcal{H}_{\rm ion} = \operatorname{span}\{\, |p_j\rangle \,:\, |p_j| \leq p_{\rm ion}\,\}$: a particle reaching that threshold is ionized, or equivalently "escapes to infinity". Here the discrete values $p_j = 2\pi h j$ are the eigenvalues of the momentum operator on $L^2(\mathbb{T}^1)$; the space $\mathcal{H}_{\rm ion}$ is thus of dimension $N \approx p_{\rm ion}/\pi h$. This projection leads to a *open quantum map*, namely the sub-unitary propagator $U_{\rm ion} = \Pi_{\rm ion} U$, where $\Pi_{\rm ion}$ is the orthogonal projector on $\mathcal{H}_{\rm ion}$. For the parameters used by [3], the classical dynamics is diffusive, meaning that a particle starting from $p = 0$ will need many kicks to reach the ionization threshold.

The matrix $U_{\rm ion}$ was numerically diagonalized for various values of h with $p_{\rm ion}$ fixed, and the distribution of the N level widths $\gamma_i = -2\ln|\lambda_i|$, $\lambda_i \in \operatorname{Spec}(U_{\rm ion})$ was found approximately independent of h, such that the number of resonances

$$n(N,\gamma) = \#\{\gamma_i \leq \gamma\}$$

scales like $C(\gamma)N$ in this case. In subsequent works [40, 52, 15], this distribution was shown to correspond to an ensemble of subunitary random matrices.

2.4.2. *Quasi-bound states in an open quantum map.* Recently, Schomerus and Tworzydło [42] have performed a similar study for the quantized kicked rotator on the torus (obtained from the map of the former section by periodizing the momentum variable). They also "opened" the map by assuming that particles reaching a certain position window $q \in L$ "escape to infinity". The quantum projector associated with these "escape window" is denoted by Π_L, so that the remaining subunitary quantum map reads $U_{\rm op} = (I - \Pi_L)U$. The main difference with the case studied in the previous section lies in the strongly chaotic motion (as opposed to diffusive), due to a different choice of parameters. The map has a positive Lyapunov exponent λ, and a typical trajectory will escape after a few kicks: the average "dwell time", called τ_D, is of order unity.

The eigenmodes associated with eigenvalues bounded away from zero are called "quasi-bound states" , as opposed to the "instantaneous decay modes" associated with very small eigenvalues. The authors provide numerical and heuristic evidence that, in the semiclassical limit, the number of quasi-bound states grows like $N_{eff} = N^{1-1/(\lambda\tau_D)}$. This shows that most eigenvalues of $U_{\rm op}$ are very close to zero, while only a small fraction N_{eff}/N remains bounded away from zero. The authors also plot the distribution of the $\sim N_{eff}$ quasi-bound eigenvalues: it resembles the spectrum of a random subunitary matrix of dimension N_{eff}.

The quantized baker's relation we will study in §5–6 will be of similar nature. For the map (4.31), the fractal dimension d given in (3.6) can be shown to be close to the formula

$1-1/(\lambda\tau_D)$, in the limit when the dwell time τ_D is large compared to unity (limit of small opening).

2.4.3. *Conductance through an open chaotic cavity.* Beenakker *et al.* [49] use the same quantum kicked rotator in order to understand the fluctuations of conductance through a *quantum dot*. The evolution inside the closed dot is represented by the same unitary matrix U as in last subsection, and its opening L is split into two intervals, L_2 and L_1, which represent the two "leads" bringing in and taking out the charge carriers from the dot. The orthogonal projector corresponding to these openings reads $\Pi_L = \Pi_{L_1} \oplus \Pi_{L_2}$. The conductance can then be analyzed from the *scattering matrix* of the dot:

$$\tilde{S}(\vartheta) = \Pi_L \sum_{n\geq 0} \big(e^{i\vartheta} U(1-\Pi_L)\big)^n e^{i\vartheta} U\,\Pi_L \tag{2.15}$$

$$= \Pi_L \{e^{-i\vartheta} - U(1-\Pi_L)\}^{-1} U \Pi_L\,. \tag{2.16}$$

Here $\vartheta \in [0,2\pi)$ is called the *quasi-energy*. In terms of this parameter, the "physical half-plane" corresponds to $\operatorname{Im}\vartheta > 0$: the matrix $\tilde{S}(\vartheta)$ has no singularity in this region. On the opposite, the resonances analyzed in the previous section, which are the *poles* of $\tilde{S}(\vartheta)$, are situated in the region $\operatorname{Im}\vartheta < 0$.

While $\tilde{S}(\vartheta)$ is unitary, its subblock,

$$t \stackrel{\text{def}}{=} \Pi_{L_2} \tilde{S}(\vartheta) \Pi_{L_1}\,,$$

describes the *transmission* from the lead L_1 to the lead L_2. The dimensionless conductance (which depends on ϑ) is given by the Landauer-Büttiker formula

$$G = \operatorname{tr}(tt^*)\,. \tag{2.17}$$

The eigenvalues of tt^* (called "transmission eigenvalues") can be either close to 1 (corresponding to a total transmission), or close to 0 (corresponding to a total reflection), or inbetween. The latter correspond to genuinely quantum transmission eigenmodes, which are partly transmitted, partly reflected, due to interference phenomena inside the dot. The "quantum shot noise" is associated with these intermediate transmission eigenvalues. A simple measure of that noise is given by the Fano factor [5]

$$F = \operatorname{tr}(tt^*(1-tt^*))/\operatorname{tr} tt^*\,. \tag{2.18}$$

Using similar arguments as in the former section, the authors show that the number of such intermediate eigenvalues also scales like N_{eff}, and thereby estimate the Fano factor, by assuming that these eigenvalues are distributed according to the prediction of random matrix theory.

In §7 we will analytically compute both the conductance and the Fano factor in the case of a Walsh-quantized baker's relation.

3. Classical dynamics

3.1. **Symplectic geometry on tori.** We consider the simplest class of compact symplectic manifolds, the tori,

$$\mathbb{T}^{2n} \stackrel{\text{def}}{=} \mathbb{R}^{2n}/\mathbb{Z}^{2n} \simeq (\mathbb{I} \times \mathbb{I})^n \,, \quad \omega = \sum_{\ell=1}^{n} dq_\ell \wedge dp_\ell \,, \quad (q,p) \in \mathbb{T}^{2n} \,.$$

Here and in what follows, we identify the interval $\mathbb{I} = [0,1)$ with the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. A Lagrangian (submanifold) $\Lambda \subset \mathbb{T}^{2n}$ is a n dimensional embedded submanifold of $\mathbb{T}^{2n}$ such that $\omega\restriction_\Lambda = 0$. We recall the following well known fact (see for instance [21, Theorem 21.3.2]):

Proposition 3.1. *Suppose that $\Lambda \subset \mathbb{T}^{2n}$ is a Lagrangian submanifold, and that $(q_0, p_0) \in \Lambda$. Then, after a possible permutation of indices, there exists k, $0 \leq k \leq n$, and a splitting of coordinates:*

$$q = (q', q'') \,, \quad p = (p', p'') \,, \quad q' = (q_1, \cdots q_k) \,, \quad p'' = (p_{k+1}, \cdots , p_n) \,,$$

such that

$$\Lambda \ni (q,p) \longmapsto (q'', p') \in \mathbb{I}^{n-k} \times \mathbb{I}^k$$

is bijective from a neighbourhood V of (q_0, p_0) to a neighbourhood W of (q_0'', p_0'). Consequently there exists a function, $S = S(q'', p')$ defined on W, such that $\Lambda \cap V$ is generated by the function S, that is,

$$\Lambda \cap V = \Big\{ \Big(d_{p'} S(q'', p'), q''; p', -d_{q''} S(q'', p') \Big) \,, \quad (q'', p') \in W \Big\} \,.$$

In this paper we will also consider *singular Lagrangian manifolds* obtained by taking finite unions of Lagrangians with piecewise smooth boundaries.

3.2. Symplectic relations.

3.2.1. *Symplectic maps.* A symplectic (or "canonical") diffeomorphism on the torus $\mathbb{T}^{2n}$ is an invertible map $\kappa : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ which leaves invariant the symplectic form on $\mathbb{T}^{2n}$: $\kappa^*\omega = \omega$. An equivalent characterization of such a map is through its *graph* Γ, which is the $2n$-dimensional embedded submanifold of $\mathbb{T}^{2n} \times \mathbb{T}^{2n}$, defined as

$$\Gamma_\kappa = \big\{ \, (\rho'; \rho) \ : \ \rho = (q,p) \in \mathbb{T}^{2n}, \ \rho' = \kappa(\rho) \, \big\} \,.$$

Using the identification $\mathbb{I}^n = \mathbb{R}^n/\mathbb{Z}^n$, we setup the reflection map $\mathbb{I}^n \ni p \mapsto -p \in \mathbb{I}^n$, and define the *twisted graph* [21, §25.2]

$$\Gamma'_\kappa = \{(q', q; p', -p) \ : \ (q', p'; q, p) \in \Gamma_\kappa\} \subset \mathbb{T}^{4n} \,. \tag{3.1}$$

Then the diffeomorphism κ is symplectic iff Γ'_κ is a Lagrangian submanifold of $\mathbb{T}^{4n}$ (equipped with the symplectic form $\sum_{j=1}^n dq'_j \wedge dp'_j + dq_j \wedge dp_j$). For this reason, we will sometimes denote Γ'_κ by Λ_κ.

The definition of the twisted graph is clearly dependent on the choice of the splitting of variables (q,p), which will be related to a choice of *polarization* in the quantization process.

More generally, one can consider invertible maps on $\mathbb{T}^{2n}$ which are smooth and symplectic except on a negligible set of singularities (say, discontinuities on a hypersurface). The twisted graph of such a map is then a singular Lagrangian submanifold of $\mathbb{T}^{4n}$.

Example. Let us consider the usual "baker's map", which is the following piecewise-linear transformation κ on $\mathbb{T}^2$:

$$\kappa(q,p) \stackrel{\text{def}}{=} \begin{cases} (2q, p/2) & \text{if } 0 \leq q < 1/2 \\ (2q-1, p/2+1/2) & \text{if } 1/2 \leq q < 1\,. \end{cases}$$

The twisted graph of κ:

$$\Lambda_\kappa \stackrel{\text{def}}{=} \{\, (q', q; p', -p) \ : \ (q,p) \in \mathbb{T}^2,\ (q',p') = \kappa(q,p) \,\}$$

is a Lagrangian submanifold of $\mathbb{T}^4$. It can be decomposed into $\Lambda_\kappa = \Lambda_0 \cup \Lambda_1$, with the components

$$\begin{aligned} \Lambda_j &= \left\{ (2q-j, q; \frac{p+j}{2}, -p) \ : \ j/2 \leq q < j/2 + 1/2,\ p \in \mathbb{I} \right\} \\ &= \{\, (2q-j, q; p', -2p'+j) \ : \ j/2 \leq q, p' < j/2 + 1/2 \,\}\,. \end{aligned}$$

Each Λ_j is locally Lagrangian in $\mathbb{T}^4$ and, as a manifold with corners, it is diffeomorphic to a 2-dimensional square.

3.2.2. *Multivalued symplectic map.* A canonical (or symplectic) *relation* is an arbitrary subset $\mathcal{C} \subset \mathbb{T}^{2n} \times \mathbb{T}^{2n}$, such that

$$\mathcal{C}' = \{\, (q', q; p', -p) \ : \ (q', p'; q, p) \in \mathcal{C} \,\}$$

is a Lagrangian submanifold of $\mathbb{T}^{4n}$. Clearly, $\mathcal{C}$ is a canonical map if it is the graph of a map and $\mathcal{C}'$ is Lagrangian.

We are interested in symplectic relations coming from multivalued symplectic maps. Such a relation is obtained as the union of finitely many components κ_j, where κ_j is a canonical diffeomorphism between an open subset $\mathcal{S}_j$ with a piecewise smooth boundary of $\mathbb{T}^{2n}$ and its image $\mathcal{S}'_j = \kappa_j(\mathcal{S}_j) \in \mathbb{T}^{2n}$. A priori, the sets $\mathcal{S}_j$ (respectively $\kappa_j(\mathcal{S}_j)$) can overlap, and their union can be a proper subset of $\mathbb{T}^{2n}$.

Each component κ_j is associated to its graph

$$\Gamma_j = \{\, (\kappa_j(\rho); \rho) \ : \ \rho \in \mathcal{S}_j \,\}\,,$$

and the symplectic relation can now be defined through its graph

$$\Gamma = \bigcup_j \Gamma_j\,,$$

or equivalently its twisted graph Γ' (defined from Γ as in (3.1)). The latter is a singular Lagrangian in $\mathbb{T}^{4n}$, that is, a finite union of Lagrangians with possible boundaries or corners.

The inverse relation can be defined through its graph

$$\Gamma^{-1} \stackrel{\text{def}}{=} \{ (\rho;\rho') \ : \ (\rho';\rho) \in \Gamma \} = \bigcup_j \{ (\kappa_j^{-1}(\rho);\rho) \ : \ \rho \in \mathcal{S}_j' \} ,$$

and the composition of two relations by

$$\tilde{\Gamma} \circ \Gamma \stackrel{\text{def}}{=} \Big\{ (\rho'';\rho) \in \mathbb{T}^{4n} \ : \ \exists\, \rho' \in \mathbb{T}^{2n} ,\ (\rho';\rho) \in \Gamma \ \text{ and } \ (\rho'';\rho') \in \tilde{\Gamma} \Big\} .$$

We can obviously iterate a relation Γ, defining a multivalued dynamical system $\{\Gamma^n : n \in \mathbb{Z}\}$ on $\mathbb{T}^{2n}$.

3.3. **Open baker's relation.** The dynamics we will consider takes place on the 2-torus phase space,

$$\mathbb{T}^2 = \{ \rho = (q,p) \ : \ q \in \mathbb{I} , p \in \mathbb{I} \} .$$

On this phase space, we define two vertical strips $\mathcal{S}_j$ ($j = 1,2$) from the data of four real numbers D_1, $D_2 > 1$ and ℓ_1, $\ell_2 \geq 0$:

$$\mathcal{S}_j = \{ (q,p) \ : \ q \in I_j ,\ p \in \mathbb{I} \} , \qquad \text{with} \quad I_j = \left\{ \frac{\ell_j}{D_j} < q < \frac{\ell_j + 1}{D_j} \right\} \qquad j = 1,2. \tag{3.2}$$

The strips are assumed to be disjoint, which is the case if we impose the conditions:

$$\frac{\ell_1 + 1}{D_1} \leq \frac{\ell_2}{D_2} \quad \text{and} \quad \frac{\ell_2 + 1}{D_2} \leq 1 .$$

The corresponding baker's relation is made of two components B_j, $j = 1,2$ associated with linear symplectic maps defined on the two strips:

$$B_j = \left\{ (\rho';\rho) \ : \ (q',p') = \left(D_j q - \ell_j , \frac{p + \ell_j}{D_j} \right) ,\ \rho = (q,p) \in \mathcal{S}_j \right\} . \tag{3.3}$$

The baker's relation is defined as the graph $B = B_1 \cup B_2$. One clearly notices that each component map is a hyperbolic diffeomorphism, with positive stretching exponent $\log D_1$ (resp. $\log D_2$). At all points where the map is defined, the unstable (stable) direction is the horizontal (vertical) one.

Since the two strips are disjoint, each point $\rho \in \mathbb{T}^2$ has at most one image. In the notations of Proposition 3.1 (taking $q'' = q$, $q' = q'$), each Lagrangian component B_j' can be generated by the function

$$S_j(q,p') = D_j \left(q - \frac{\ell_j}{D_j} \right) \left(p' - \frac{\ell_j}{D_j} \right) \quad \text{defined on the square} \quad \{ (q,p') \in I_j \times I_j \} . \tag{3.4}$$

Let

$$\pi_L , \pi_R \ : \ \mathbb{T}^2 \times \mathbb{T}^2 \ \longrightarrow \ \mathbb{T}^2$$

be the projections on the left and right factors respectively. From the definition (3.3), the set $\pi_R(B)$ is made of points on $\rho \in \mathbb{T}^2$ which have an image through the relation B. Hence, a point $\rho \notin \pi_R(B)$ is said to escape from the torus at time 1. Similarly, a point $\rho \notin \pi_L(B) = \pi_R(B^{-1})$ is said to escape from $\mathbb{T}^2$ at time -1. This "escape" is the reason why we call this relation an "open" relation: the system is not "closed" because it sends particles "to infinity", both in the future and in the past.

We define

$$\Gamma_\pm \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} \pi_R\left(B^{\mp n}\right) \tag{3.5}$$

the set of points which do never escape from $\mathbb{T}^2$ in the past, respectively in the future. One checks that these subsets have the form

$$\Gamma_- = C \times \mathbb{I}\,, \quad \Gamma_+ = \mathbb{I} \times C\,,$$

where $C \subset \mathbb{I}$ is a "cookie-cutter set" in the sense of [14]: if we consider the two contracting maps on $\mathbb{I}$

$$f_j(q) = \frac{q + \ell_j}{D_j}\,, \quad q \in \mathbb{I}\,,$$

this closed set is defined as

$$C = \overline{\bigcup_{n\in\mathbb{N}} \{\, q \in \mathbb{I} \ :\ f_{j_1} \circ \cdots \circ f_{j_n}(q) = q \quad \text{for some sequence } j_m \in \{1,2\}\,\}}\,.$$

The Hausdorff dimension of C (which is equal to its Minkowski and box-counting dimensions) is given by the unique $0 < d < 1$ solving

$$D_1^{-d} + D_2^{-d} = 1\,. \tag{3.6}$$

The trapped set (or set of nonwandering points) is defined as the set of points which never escape from $\mathbb{T}$:

$$K = \Gamma_+ \cap \Gamma_- = C \times C\,, \quad \dim K = 2d\,.$$

The baker's relation is a hyperbolic invertible map on the set K, which is a "fractal repeller". This relation is a model of Smale's horseshoe mechanism.

The simplest case consists in considering a symmetric baker's relation, with $D_1 = D_2 = D$, $\ell = \ell_1 = D - \ell_2 - 1$:

$$\begin{aligned} \frac{\ell}{D} < q < \frac{\ell+1}{D} \quad &\Longrightarrow \quad (q', p') = \left(Dq - \ell, \frac{p+\ell}{D}\right) \\ \frac{\ell}{D} < 1 - q < \frac{\ell+1}{D} \quad &\Longrightarrow \quad (q', p') = \left(D(q-1) + \ell + 1, \frac{p - \ell - 1}{D} + 1\right)\,. \end{aligned} \tag{3.7}$$

Now $C \subset \mathbb{I}$ is a symmetric $1/D-$Cantor set, of dimension $d = \log 2 / \log D$. Notice that if we take $D = 2$, $\ell_1 = 0$, $\ell_2 = 1$, we obtain the usual (closed) baker's map described in the example of Section 3.1, for which the trapped set ($= \mathbb{T}^2$) has dimension 2.

For such a symmetric baker's relation, the analog of the fractal exponent of (2.6) is:

$$\mu_E \quad \longleftrightarrow \quad d = \frac{\log 2}{\log D}\,.$$

4. Quantized maps and relations

Before giving the definition of the quantized Baker's relation, we need to define the quantum Hilbert space corresponding to $\mathbb{T}^2$, as well as the algebra of quantum observables.

4.1. **Quantized tori.** The quantization of tori $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, has a long tradition in mathematical physics [19, 12, 4]. It can be considered as a special case of the Berezin-Toeplitz quantization of compact symplectic Kähler manifolds — see [24] and references given there. Here we will give a self-contained presentation of the simplest case from the point of view of pseudodifferential operators.

We first recall from [13] the quantization of functions $f \in \mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$,

$$\mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n) \stackrel{\rm def}{=} \{f \in \mathcal{C}^\infty(T^*\mathbb{R}^n)\ :\ \forall \alpha, \beta \in \mathbb{N}^n,\ \sup_{(q,p)\in T^*\mathbb{R}^n} |\partial_q^\alpha \partial_p^\beta f(q,p)| < \infty\}\,.$$

To any $f \in \mathcal{S}(T^*\mathbb{R}^n)$ we associate its h-Weyl quantization, that is the operator $f^w(q, hD)$ acting as follows on $\psi \in \mathcal{S}(\mathbb{R}^n)$:

$$[f^w(q,hD)\,\psi](q) \stackrel{\rm def}{=} \frac{1}{(2\pi h)^n} \int\!\!\int f\Big(\frac{q+r}{2}, p\Big)\, e^{\frac{i}{h}\langle q-r, p\rangle}\, \psi(r)\, dr\, dp\,. \tag{4.1}$$

This operator clearly has the mapping properties

$$f^w(q,hD)\ :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)\,,\quad f^w(q,hD)\ :\ \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)\,.$$

It can be shown [13, Lemma 7.8] that $f \mapsto f^w(q, hD)$ can be extended to any $f \in \mathcal{C}^\infty_{\rm b}(T^*\mathbb{R}^n)$, and that the resulting operator has the same mapping properties. Furthermore, $f^w(q, hD)$ is a bounded operator on $L^2(\mathbb{R}^n)$.

We now introduce quantum spaces associated with the torus $\mathbb{T}^{2n}$. For that aim, we fix our notations for the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$:

$$\mathcal{F}_h\psi(p) \stackrel{\rm def}{=} \frac{1}{(2\pi h)^{n/2}} \int \psi(q)\, e^{-\frac{i}{h}\langle q,p\rangle}\, dq\,, \qquad \mathcal{F}_h^* = \mathcal{F}_h^{-1}\,,$$

and as usual in quantum mechanics, $\mathcal{F}_h\psi(p)$ is the "momentum representation" of the state ψ. The quantum spaces of the torus are indexed by a pair of Bloch angles $(\theta_p, \theta_q) \in \mathbb{I}^n \times \mathbb{I}^n$. Given any such pair, the quantum space is made of distributions $\psi \in \mathcal{S}'(\mathbb{R}^n)$ which are both quasiperiodic in position and momentum:

$$\psi(q+\ell) = e^{2\pi i\langle\theta_p,\ell\rangle}\psi(q)\,, \qquad \mathcal{F}_h\psi(p+\ell) = e^{2\pi i\langle\theta_q,\ell\rangle}\mathcal{F}_h\psi(p)\,. \tag{4.2}$$

For simplicity of presentation, we will put $(\theta_q, \theta_p) = (0, 0)$ in the rest of this section. In numerical computations (corresponding to $n = 1$), we will sometimes choose the angles

$(1/2, 1/2)$ for which specific maps have better symmetry properties. Let us denote by $\mathcal{H}_h^n$ the space of distributions satisfying (4.2) with $\theta = (0, 0)$. We have the following elementary

Lemma 4.1. *$\mathcal{H}_h^n \neq \{0\}$ if and only if $h = (2\pi N)^{-1}$ for some positive integer N, in which case $\dim \mathcal{H}_h^n = N^n$ and*

$$\mathcal{H}_h^n = \operatorname{span}\left\{ \frac{1}{\sqrt{N^n}} \sum_{\ell \in \mathbb{Z}^n} \delta(q - \ell - j/N) \ : \ j \in (\mathbb{Z}/N\mathbb{Z})^n \right\} . \tag{4.3}$$

The distributions elements of this basis will be denoted by

$$|Q_j\rangle \, , \quad Q_j = \frac{j}{N} \in \mathbb{I}^n \text{ is the position on which that state is microlocalized.} \tag{4.4}$$

The j-th component of a state $\psi \in \mathcal{H}_h^n$ will be denoted by $\psi(Q_j) = \langle Q_j | \psi \rangle$.

One can check that for such value of h, the Fourier transform $\mathcal{F}_h$ maps $\mathcal{H}_h^n$ to itself. In the above basis, it is represented by the discrete Fourier transform

$$(\mathcal{F}_N)_{j,j'} = \frac{e^{-2i\pi\langle j, j'\rangle/N}}{\sqrt{N^n}} \, , \quad j, j' \in (\mathbb{Z}/N\mathbb{Z})^n \, . \tag{4.5}$$

It is also easy check the following

Lemma 4.2. *Suppose that $f \in \mathcal{C}_{\mathrm{b}}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies $f(q + \ell, p + m) = f(q, p)$ for any $\ell, m \in \mathbb{Z}^m$. Then the operator $f^w(q, hD)$ maps $\mathcal{H}_h^n$ to itself.*

Identifying a function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with a periodic function on $\mathbb{R}^{2n}$, we will write $\operatorname{Op}_h(f)$ for the restriction of $f^w(q, hD)$ on $\mathcal{H}_h^n$,

$$\mathcal{C}^\infty(\mathbb{T}^{2n}) \ni f \ \longmapsto \ \operatorname{Op}_h(f) \in \mathcal{L}(\mathcal{H}_h^n) \, .$$

Lemma 4.3. *There exists a unique (up to a multiplicative constant) Hilbert structure on $\mathcal{H}_h^n$ for which all $\operatorname{Op}_h(f) \, : \, \mathcal{H}_h^n \ \to \ \mathcal{H}_h^n$ with $f \in \mathcal{C}^\infty(\mathbb{T}^{2n}; \mathbb{R})$ are self-adjoint.*

One can choose the constant such that the basis in (4.3) *is orthonormal. This implies that the Fourier transform on $\mathcal{H}_h^n$ (represented by the unitary matrix* (4.5)*) is unitary.*

Proof. Let $\langle \bullet, \bullet \rangle_0$ be the inner product for which the basis in (4.3) is orthonormal. We write the operator $f^w(q, hD)$ on $\mathcal{H}_h^n$ explicitely in that basis using the Fourier expansion of its symbol:

$$f(q, p) = \sum_{\ell, m \in \mathbb{Z}^n} \hat{f}(\ell, m) \, e^{2\pi i(\langle \ell, q\rangle + \langle m, p\rangle)} \, .$$

For that let $L_{\ell,m}(q, p) = \langle \ell, q\rangle + \langle m, p\rangle$, so that

$$f^w(q, hD) = \sum_{\ell, m \in \mathbb{Z}^n} \hat{f}(\ell, m) \, \exp(2\pi i L_{\ell,m}^w(q, hD)) \, .$$

Applying this operator to the distributions (4.4), we get

$$\exp\left(2\pi i L^w_{\ell,m}(q,hD)\right)|Q_j\rangle = \exp\left(\frac{\pi i}{N}(2\langle j,\ell\rangle - \langle m,\ell\rangle)\right)|Q_{j-m}\rangle\,,$$

and consequently,

$$f^w(q,hD)\,|Q_j\rangle = \sum_{m\in\mathbb{Z}^n/(N\mathbb{Z})^n} F_{mj}\,|Q_m\rangle\,,$$

$$F_{mj} = \sum_{\ell,r\in\mathbb{Z}^n} \hat f(\ell, j-m-rN)(-1)^{\langle r,\ell\rangle}\exp\left(\frac{\pi i}{N}\langle j+m,\ell\rangle\right).$$

Since

$$\begin{aligned}\bar F_{jm} &= \sum_{\ell,r\in\mathbb{Z}^n} \hat{\bar f}(-\ell, j-m+rN)(-1)^{\langle r,\ell\rangle}\exp\left(-\frac{\pi i}{N}\langle j+m,\ell\rangle\right)\\ &= \sum_{\ell,r\in\mathbb{Z}^n} \hat{\bar f}(\ell, j-m-rN)(-1)^{\langle r,\ell\rangle}\exp\left(\frac{\pi i}{N}\langle j+m,\ell\rangle\right),\end{aligned}$$

we see that for real f, $f=\bar f$, $F_{jm}=\bar F_{mj}$. This means that $f^w(q,hD)$ is self-adjoint for the inner product $\langle\bullet,\bullet\rangle_0$. We also see that the map $f\mapsto (F_{jm})_{j,m\in(\mathbb{Z}/N\mathbb{Z})^n}$ is onto, from $\mathcal{C}^\infty(\mathbb{T}^{2n};\mathbb{R})$ to the space of Hermitian matrices.

Any other metric on $\mathcal{H}^n_h$ could be written as $\langle u,v\rangle = \langle Bu,v\rangle_0 = \langle u,Bv\rangle_0$. If $\langle f^w u,v\rangle = \langle u,f^w v\rangle$ for all f's, then $Bf^w = f^w B$ for all f's, and hence for all Hermitian matrices. That shows that $B = c\,\mathrm{Id}$, as claimed. □

This choice of normalization $\langle\bullet,\bullet\rangle_0$ can be obtained in a natural way, if we use the following periodization operator to construct $\mathcal{H}^n_h$ from $\mathcal{S}(\mathbb{R}^n)$ [4]:

Lemma 4.4. *For any $h=(2\pi N)^{-1}$, the periodization operator $P_{\mathbb{T}^{2n}}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{H}^n_h$ defined below is surjective:*

$$\forall\psi\in\mathcal{S}(\mathbb{R}^n),\qquad [P_{\mathbb{T}^{2n}}\psi](Q_j) \stackrel{\text{def}}{=} \frac{1}{N^{n/2}}\sum_{\nu\in\mathbb{Z}^n}\psi(Q_j-\nu)\,,\qquad j\in(\mathbb{Z}/N\mathbb{Z})^n\,. \tag{4.6}$$

In the rest of this article we will always assume that $h=(2\pi N)^{-1}$ for some $N\in\mathbb{N}$, so the semiclassical limit corresponds to $N\to\infty$. The scalar product on $\mathcal{H}^n_h$ will be $\langle\bullet,\bullet\rangle_0$ (from now on we will omit the subscript 0, and also often use Dirac's notation $\langle\bullet|\bullet\rangle$ for this product). The associated Hilbert norm will be denoted by $\|\bullet\|$.

4.2. **Lagrangian states.** We want to characterize the semiclassical localization in phase space of sequences of states of the form $\psi=\{\psi_h\in\mathcal{H}^n_h\}_{h\to0}$. In general we will assume that each element of this sequence is normalized, $\|\psi_h\|=1$, but all definitions can be extended to sequences such that the norms satisfy $\|\psi_h\|\asymp 1$ as $h\to0$.

The localization of this sequence is first characterized through its *microsupport*, or *wave front set*, which is the following subset of $\mathbb{T}^{2n}$:

$$\mathrm{WF}_h(\psi) = \complement \left\{ \rho \in \mathbb{T}^{2n} \; : \; \exists \, f \in \mathcal{C}^\infty(\mathbb{T}^{2n}) \, , \; f(\rho) \neq 0 \, , \quad \| \operatorname{Op}_h(f) \psi_h \| = \mathcal{O}(h^\infty) \right\} . \tag{4.7}$$

It is not hard to show [36, Proposition IV-8′] that this definition is equivalent to the following: $\rho \notin \mathrm{WF}_h(\psi)$ if and only if there exists a neighbourhood W_ρ of ρ such that, for any $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ supported in W_ρ, $\| \operatorname{Op}_h(f)\psi_h\| = \mathcal{O}(h^\infty)$. This yields the following

Lemma 4.5. *For any function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $f \equiv 0$ in an open neighbourhood of $\mathrm{WF}_h(\psi)$, we have $\| \operatorname{Op}_h(f)\psi_h\| = \mathcal{O}(h^\infty)$. As a consequence, the microsupport of a normalized sequence $\{\psi_h\}$ cannot be empty.*

Proof. The (compact) support of f can be covered by finitely many W_{ρ_i}, and using a partition of unity associated with these sets we can decompose it as $f = \sum_i f_i$, with $\mathrm{supp}(f_i) \subset W_{\rho_i}$. We get the result by linearity, and using the second definition of $\mathrm{WF}_h(\psi)$. □

We also make the following observation:

Lemma 4.6. *Take $\psi = \{\psi_h \in \mathcal{H}_h^n\}_{h \to 0}$, $\|\psi_h\| = 1$. Considering ψ_h as a N^n-component vector in the basis* (4.3)*, we define $\bar{\psi}_h$ as the vector with complex conjugate components. Then*

$$\mathrm{WF}_h(\bar{\psi}) = \{(q, -p) \; : \; (q,p) \in \mathrm{WF}_h(\psi)\} \, .$$

Proof. We simply observe that (4.1) gives

$$a^w(q, hD) \, \bar{\psi} = \overline{\bar{a}^w(q, -hD) \, \psi} \, .$$

The lemma follows from the definition 4.7. □

Now let $\Lambda \subset \mathbb{T}^{2n}$ be a union of Lagrangian submanifolds of $\mathbb{T}^{2n}$ with piecewise smooth boundaries.

Definition 4.7. A sequence of states $\psi = \{\psi_h \in \mathcal{H}_h^n\}$ is a *Lagrangian state associated to* Λ, which we denote by $\psi \in I(\Lambda)$, if for any $M \in \mathbb{N}$ and any sequence of functions,

$$f_j \in \mathcal{C}^\infty(\mathbb{T}^{2n}) \, , \quad 1 \leq j \leq M \, , \quad f_j \restriction_\Lambda = 0 \, ,$$

we have

$$\| \operatorname{Op}_h(f_M) \circ \cdots \circ \operatorname{Op}_h(f_1) \, \psi_h \| = \mathcal{O}(h^M) \, \|\psi_h\| \, . \tag{4.8}$$

From the definition (4.7) of the microsupport, we obtain the inclusion

$$\psi \in I(\Lambda) \implies \mathrm{WF}_h(\psi) \subset \Lambda \, . \tag{4.9}$$

Indeed, suppose that $\rho \notin \Lambda$. Then there exists $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $f \restriction_\Lambda = 0$ and $f \equiv 1$ in a neighbourhood of ρ. We can also find $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $f = 1$ on a neighbourhood of the support of a, and $a(\rho) \neq 0$. The symbol calculus (see [13, Chapter 7]) shows that for any

M, $\mathrm{Op}_h(a)\,\mathrm{Op}_h(f)^M = \mathrm{Op}_h(a) + \mathcal{O}_M(h^\infty)$. On the other hand $\|\,\mathrm{Op}_h(f)^M\psi_h\| = \mathcal{O}(h^M)$, and as M is arbitrary, $\|\,\mathrm{Op}_h(a)\,\psi_h\| = \mathcal{O}(h^\infty)$, which in view of definition 4.7 gives (4.9).

We stress that the opposite implication in (4.9) is not true in general. To see that consider $\Lambda = \{(0,p)\ :\ p\in\mathbb{I}\}$. Let $\psi_h\in\mathcal{H}_h^1$ be the "torus coherent state at the origin":

$$\psi_h(Q_j) = \Big(\frac{2}{N}\Big)^{1/4}\sum_{r\in\mathbb{Z}}\exp\{-\pi N(Q_j-r)^2\}\,,\quad j=0,\ldots,N-1\,.$$

Then one can check that $\|\psi_h\|\xrightarrow{h\to0}1$, that $\mathrm{WF}_h(\psi)=\{\,(0,0)\,\}\subset\Lambda$. On the other hand,

$$\|\,\mathrm{Op}_h\big(\sin(2\pi q)\big)\psi_h\|\sim\pi\sqrt{2h}\,,$$

which shows that $\psi_h\notin I(\Lambda)$.

In the physics literature, Lagrangian states are usually called WKB states, and are introduced as Ansätze for eigenstates of integrable systems, using Bohr-Sommerfeld quantization formulae [25]. For instance, in the case $n=1$, if Λ is generated by the function $S\in\mathcal{C}^\infty(\mathbb{I})$:

$$\Lambda_S = \{\,(q,-S'(q)),\ q\in\mathbb{I}\,\}\,, \tag{4.10}$$

then for any function $a(q)\in\mathcal{C}^\infty(\mathbb{I})$, the state $\psi_h\in\mathcal{H}_h^1$ defined as

$$\psi_h(Q_j) = \frac{a(Q_j)}{\sqrt{N}}\exp(-2i\pi NS(Q_j))\,,\quad j=0,\ldots,N-1\,, \tag{4.11}$$

is in $I(\Lambda_S)$. In the next proposition, we generalize this construction to any dimension.

Proposition 4.8. *Suppose that $\Lambda\subset\mathbb{T}^{2n}$ is an embedded Lagrangian manifold. Then for any $\rho\in\Lambda$ there exist Lagrangian states $\psi\in I(\Lambda)$, such that $\rho\in\mathrm{WF}_h(\psi)$.*

Proof. We take $\rho_0=(q_0,p_0)\in\Lambda$, and assume that there exists a neighbourhood V of ρ_0 (which projects on $V_q\in\mathbb{R}^n_q$) and a function $S\in\mathcal{C}^\infty(V_q)$, such that $\Lambda\cap V=\{(q;-d_qS(q))\,,\,q\in V_q\}$. This is a particular case of Proposition 3.1. The general case of a generating function $S(q'',p')$ can be transformed to that of $S=S(q)$ using the symplectic rotation $(q',p')\mapsto(-p',q')$. On the quantum mechanical side, this rotation is performed through a partial Fourier transform in the variable q'. Our construction below can be transposed to this general case through this Fourier transform (which acts covariantly on the Weyl quantization).

We also assume that the neighbourhood V is contained in the interior of $\mathbb{I}^{2n}$, and we identify V_q with a subset of $\mathbb{I}^n$. We first construct a Lagrangian state in $L^2(\mathbb{R}^n)$:

$$u_h(q) = a(q)\,e^{-\frac{i}{h}S(q)}\,, \tag{4.12}$$

with a symbol $a\in\mathcal{C}^\infty(\mathbb{R}^n)$ compactly supported inside V_q, and such that $a(q_0)\neq0$. This state admits the norm $\|u_h\|_{L^2}=\|a\|_{L^2}$. For any $f\in\mathcal{C}^\infty(\mathbb{T}^{2n})$, we apply the operator

$f^w(q, hD)$ to that state. Although we could do it directly using (4.1), we prefer to reduce the problem to the case of $S = 0$ by conjugation with the unitary multiplication operator

$$v(q) \longmapsto e^{\frac{i}{h}S^w(q)} v(q)\,, \tag{4.13}$$

where we can assume that $S \in \mathcal{C}_\mathrm{b}^\infty(\mathbb{R}^n)$. We then apply the operator

$$G^w(q, hD) \stackrel{\text{def}}{=} e^{\frac{i}{h}S^w(q)}\, f^w(q, hD)\, e^{-\frac{i}{h}S^w(q)}\,,$$

to the function $a(q)$. The symbol calculus shows that $G(q, p)$ admits an h-expansion, with principal symbol $g(q, p) = f(q, p + d_qS(q))$: if f vanishes on Λ, then g vanishes on $\{(q, 0) : q \in V_q\}$. We get

$$[f^w(q, hD)\, u_h](q) = e^{-\frac{i}{h}S(q)}\, G^w(q, hD)\, a(q) = e^{-\frac{i}{h}S(q)}\, g^w(q, hD)\, a(q) + \mathcal{O}(h)\,,$$

The explicit integral

$$[g^w(q, hD)\, a](q) = \frac{1}{(2\pi h)^n} \int\!\!\int g\Big(\frac{q + r}{2}, p\Big)\, a(r)\, e^{\frac{i}{h}\langle q-r,p\rangle}\, dr\, dp$$

can be evaluated through the stationary phase method. The derivative of the phase vanishes at $r = q$, $p = 0$, so the integral admits the following expansion [21, §7.7] for $q \in V_q$:

$$[g^w(q, hD)\, a](q) = L_0(g\, a)(q) + hL_1(g\, a)(q) + \mathcal{O}(h^2)\,. \tag{4.14}$$

Here each function $L_j(g\, a)$ is obtained by applying a certain differential operator (in (r, p)) on the function $g((q + r)/2, p)\, a(r)$, taking the output at the point $(r = p,\ p = 0)$. The first term is simply $L_0(g\, a)(q) = g(q, 0)\, a(q)$. For q outside V_q, the nonstationary phase estimates show that

$$f^w(q, hD)\, u_h(q) = \mathcal{O}\left(\Big(\frac{h}{h + \mathrm{dist}(q, V_q)}\Big)^\infty\right)\,. \tag{4.15}$$

If $f(\rho_0) \neq 0$, then $L_0(g\, a)$ is nonzero in a neighbourhood W of q_0, and we obtain

$$\|f^w(q, hD)\, u_h\|_{L^2(\mathbb{R}^n)} = \|g^w(q, hD)\, a\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h) \geq \|L_0(g\, a)\|_{L^2(W)} + \mathcal{O}(h)\,. \tag{4.16}$$

The left hand side is thus bounded from below by a positive constant.

On the opposite, if f vanishes on Λ, then at each point $q \in V_q$ we get $L_0(g\, a)(q) = 0$, which implies that $\|f^w(q, hD)\, u_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$. The same procedure can be iterated to show that

$$\|f_M^w(q, hD) \circ \cdots \circ f_1^w(q, hD) u_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^M) \tag{4.17}$$

for any family of functions $f_i \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ vanishing on Λ.

We can now carry over these estimates onto the state $\psi_h = P_{\mathbb{T}^{2n}}\, u_h \in \mathcal{H}_h^n$, where $P_{\mathbb{T}^{2n}}$ is the periodizing operator (4.6). Since $a(q)$ was supported inside $V_q \subset \mathbb{I}^n$, this state admits the following representation, which generalizes (4.10):

$$\psi_h(Q_j) = \frac{a(Q_j)}{N^{n/2}}\ \exp(-2i\pi NS(Q_j))\,, \qquad j \in (\mathbb{Z}/N\mathbb{Z})^n\,. \tag{4.18}$$

The norm of this state is therefore the sum

$$\|\psi_h\|^2 = N^{-n} \sum_{j\in(\mathbb{Z}/N\mathbb{Z})^n} |a(Q_j)|^2 = \int dq\, |a(q)|^2 + \mathcal{O}(h^\infty)\,,$$

where we used the smoothness of $a(q)$. Considering the functions f_i appearing in (4.17), the state

$$u_h^{(M)}(q) \stackrel{\text{def}}{=} h^{-M}\, [f_M^w \circ \cdots \circ f_1^w\, u_h](q)\,, \tag{4.19}$$

is not compactly supported, but it is small for $q \notin V_q$, as in (4.15). As a result, its projection on $\mathcal{H}_h^n$ satisfies

$$\begin{aligned} h^{-M}\ \mathrm{Op}_h(f_M) \circ \cdots \circ \mathrm{Op}_h(f_1)\, \psi_h(Q_j) &= P_{\mathbb{T}^{2n}}\, u_h^{(M)}(Q_j) \\ &= \frac{u_h^{(M)}(Q_j)}{N^{n/2}} + \mathcal{O}(h^\infty), \quad j \in (\mathbb{Z}/N\mathbb{Z})^n\,. \end{aligned} \tag{4.20}$$

This immediately implies that

$$\|\, \mathrm{Op}_h(f_M) \circ \cdots \circ \mathrm{Op}_h(f_1)\psi_h\| = h^M \|P_{\mathbb{T}^{2n}}\, u_h^{(M)}\| = h^M \|u_h^{(M)}\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^\infty) = \mathcal{O}(h^M)\,.$$

These estimates show that the family $\psi \in I(\Lambda)$, and that $\rho_0 \in \mathrm{WF}_h(\psi)$. □

Remark 4.1. The definition of $I(\Lambda)$ mimicks the Hörmander-Melrose definition of Lagrangian distributions [21, Definition 25.1.1] (see [1] for an adaptation to the standard semiclassical setting). The requirement that Λ is Lagrangian reflects the uncertainty principle, in the following sense. A Lagrangian submanifold is the lowest dimensional submanifold for which the conclusion of Proposition 4.8 holds, that is, for any $\rho \in \Lambda$, there exists a state ψ satisfying $\psi \in I(\Lambda)$ and $\rho \in \mathrm{WF}_h(\psi)$.

Indeed, let Λ be an embedded submanifold of $\mathbb{T}^{2n}$. Let us assume that $\psi \in I(\Lambda)$, so (4.8) must hold for any family of functions $f_j\restriction_\Lambda = 0$. From the identity

$$\frac{i}{h}[\mathrm{Op}_h(f_i), \mathrm{Op}_h(f_j)] = \mathrm{Op}_h(\{f_i, f_j\}) + \mathcal{O}(h)\,.$$

we see that $\|\,\mathrm{Op}_h(\{f_i, f_j\})\, \psi_h\| = \mathcal{O}(h)$. As in the proof of (4.9), we can show that if $\{f_i, f_j\}(\rho) \neq 0$ for some $\rho \in \Lambda$, then $\rho \notin \mathrm{WF}_h(\psi)$. Hence, if we want the conclusion of Proposition 4.8 to hold for Λ, then this submanifold must satisfy

$$\forall f_i,\ f_j \in C^\infty(\mathbb{T}^{2n}), \qquad f_i\restriction_\Lambda,\, f_j\restriction_\Lambda = 0 \implies \{f_i, f_j\}\restriction_\Lambda = 0\,.$$

This property means that Λ is *co-isotropic*, and must be of dimension $\geq n$. Lagrangian manifolds are co-isotropic manifolds of minimal dimension.

4.2.1. *Singular Lagrangian states.* We now give an example where Λ is a union of Lagrangians with piecewise smooth boundaries. Let Λ_S be given by (4.10) and ψ_h by (4.11). Let us truncate ψ_h to some proper subinterval $[Q, Q'] \subset \mathbb{I}$, that is, replace the symbol $a(q)$ by the discontinuous function $\tilde{a}(q) = a(q)\chi_{[Q,Q']}(q)$. That gives a state $\tilde{\psi}_h \in \mathcal{H}_h^1$. One could expect $\tilde{\psi}_h$ to be a Lagrangian state in $I(\Lambda_S)$ (as is ψ_h), or rather in $I(\tilde{\Lambda}_S)$, where

$$\tilde{\Lambda}_S \stackrel{\text{def}}{=} \Lambda_S \cap ([Q, Q'] \times \mathbb{I}) . \tag{4.21}$$

This is not the case: one needs to include in the Lagrangian the *singularity set*

$$\Lambda_{\text{sing}} = \{(Q, p) \ : \ p \in \mathbb{I}\} \cup \{(Q', p) \ : \ p \in \mathbb{I}\} ,$$

which is the "periodized" conormal bundle of the boundary $\partial\tilde{\Lambda}_S$. We will indeed prove that $\tilde{\psi}_h \in I(\tilde{\Lambda}_S \cup \Lambda_{\text{sing}})$, which can be considered as a semiclassical, discrete analogue of singular Lagrangian distributions of Guillemin-Uhlmann [17] and Melrose-Uhlmann [31]. We have the following

Lemma 4.9. *Let us truncate the state* (4.18) *to a hypercube D in $\mathbb{I}^n$, $D = \prod_{\ell=1}^n [\alpha_\ell, \beta_\ell]$,*

$$\tilde{\psi}_h(Q_j) = \frac{a(Q_j)\,\chi_D(Q_j)}{N^{n/2}} \exp(-2i\pi N S(Q_j)) , \quad j \in (\mathbb{Z}/N\mathbb{Z})^n , \tag{4.22}$$

where χ_D is the characteristic function of D. Then $\tilde{\psi}_h$ is associated with the singular Lagrangian $\tilde{\Lambda}_S \cup \Lambda_{\text{sing}}$, where $\tilde{\Lambda}_S = \{(q, -d_q S(q)) \, , \ q \in D\}$ and

$$\begin{aligned} \Lambda_{\text{sing}} = \bigcup_{\ell=1}^n \Big(& \big\{(q,p) \ : \ q_\ell = \alpha_\ell, \ p_\ell \in \mathbb{I}, \ q_m \in [\alpha_m, \beta_m], \ p_m = -d_{q_m} S(q), \ m \neq \ell\big\} \\ & \cup \ \big\{(q,p) \ : \ q_\ell = \beta_\ell, \ p_\ell \in \mathbb{I}, \ q_m \in [\alpha_m, \beta_m], \ p_m = -d_{q_m} S(q), \ m \neq \ell\big\}\Big) . \end{aligned} \tag{4.23}$$

Remark. It would be tempting to generalize the lemma to arbitrary sets D with smooth boundaries. However, if $n = 2$, $S = 0$, $a = 1$, and ∂D does not contain a segment with rational slopes then

$$WF_h(\tilde{\psi}_h) = (D \times \{0\}) \cup (\partial D \times \mathbb{I}^2) .$$

The second component being 3-dimensional, this set is certainly not contained in a finite union of Lagrangians.

Proof. As in the proof of Proposition 4.8, we can, by conjugation with the operator (4.13), reduce the proof to the case $S = 0$. We first consider states defined on $\mathbb{R}^n$, localized on the hypercube $D \subset \mathbb{R}^n$:

$$u_h(q) = \chi_D(q)\, a(q) , \quad a \in \mathcal{C}^\infty(\mathbb{R}^n) .$$

We use the following

Lemma 4.10. *The ideal $\mathcal{J}$ of periodic functions vanishing on the singular Lagrangian $\tilde{\Lambda}_0 \cup \Lambda_{\text{sing}}$, where $\tilde{\Lambda}_0$ is as in (4.21), is (infinitely) generated by*

$$\begin{aligned}
&g_j(p,q) \stackrel{\text{def}}{=} \sin\big(\pi(q_j-\alpha_j)\big)\,\sin\big(\pi(q_j-\beta_j)\big)\,\sin(\pi p_j)\,,\\
&g_{ij}(q,p) \stackrel{\text{def}}{=} \sin(\pi p_i)\sin(\pi p_j) \quad i\neq j\,,\ 1\leq i,j\leq n\,,\\
&\phi_j(q,p) = \phi(q_j,p_1,\cdots,p_{j-1},p_{j+1},\cdots,p_n)\,, \qquad \textit{where } \phi(q_j,\bullet)\equiv 0,\ \alpha_j\leq q_j\leq\beta_j,\\
&\psi(q)\,, \quad \textit{where } \phi\in\mathcal{C}^\infty(\mathbb{I}^n) \textit{ vanishes on } D.
\end{aligned}$$

Proof. The lemma becomes easier to prove in a model, $n=2$, case of finding functions vanishing on

$$(\{q_1=p_2=0\}\cup\{q_2=p_1=0\}\cup\{p_1=p_2=0\})\cap\{q_1\geq 0\,,q_2\geq 0\}\,. \tag{4.24}$$

The functions vanishing on the first factor in the intersection are generated by q_1p_1 , q_2p_2, and p_1p_2. Writing an arbitrary function $F(q,p)$ as

$$\begin{aligned}F(q_1,q_2,p_1,p_2) = F_0(p_1,p_2) + q_1F_1(q_1,q_2,p_2) + q_2F_2(q_1,q_2,p_1) +\\ + q_1p_1F_{11}(q_1,q_2,p_1,p_2) + q_2p_2F_{22}(q_1,q_2,p_1,p_2)\,.\end{aligned}$$

We need to find conditions for $q_1F_1(q_1,q_2,p_2)$ and $q_2F_2(q_1,q_2,p_1)$ to vanish on (4.24). We treat the first function by expanding it as

$$F_1(q_1,q_2,p_2) = F_{10}(q_1,q_2) + p_2F_{12}(q_1,p_2) + q_2p_2F_{122}(q_1,q_2,p_2)\,.$$

This forces $F_{10}(q_1,q_2)$ to vanish identically in $\{q_1,q_2\geq 0\}$ and $F_{12}(q_1,p_2)$ to vanish identically in $\{q_1\geq 0\}$.

The function $F_2(q_1,q_2,p_1)$ is treated identically. Hence the functions vanishing on (4.24) are generated by q_1p_1, q_2p_2, p_1p_2, and all the smooth fuctions $\psi(q_1,q_2)$, $\phi_1(q_1,p_2)$, $\phi_2(q_2,p_1)$ vanishing on $\{q_1,q_2\geq 0\}$. The transposition to the torus setting gives the lemma. □

This lemma means that any $F\in\mathcal{J}$ can be decomposed as

$$F = \sum_{j\neq i} f_{ij}g_{ij} + \sum_j (f_{jj}g_j + f_j\phi_j + \psi)\,,$$

where the functions $f_\bullet$ are smooth and either periodic or antiperiodic in each variable, so that $f_\bullet g_\bullet$, are $\mathbb{Z}$-periodic in all variables.

The action of each term $(f\,g)^w(q,hD)$ on $u_h(q)$ can be written

$$(fg)^w\,u_h = ((f\,a)^w\circ g^w + h\,L(f_J,a,g_J))\,\chi_D\,,$$

where $L(f,a,g)$ is a pseudodifferential operator of norm $\mathcal{O}(1)$. Therefore, we are reduced to study the action of the generators $g^w(q,hD)$, $g=g_{ij}$, g_j , ϕ_j,ψ, on the characteristic function $\chi_D(q)$. We first note that $\psi\,\chi_D = \phi_j^w\chi_D\equiv 0$, and consequently there is nothing to prove in this case.

For each $j \in \{1, \dots, n\}$, the generators g_j, contain a factor $\sin(\pi p_j)$. Up to an error $\mathcal{O}(h)$, we first quantize this factor and apply it to χ_D:

$$\sin(\pi h D_j)\chi_D(q) = \frac{1}{2i}\big(\chi_D(q_j + \pi h, q') - \chi_D(q_j - \pi h, q')\big) \stackrel{\text{def}}{=} b_j(q)\,.$$

The function $b_j(q)$ is supported in the strips $S_j = \{\, |q_j - \alpha_j| \le \pi h \,\} \cup \{\, |q_j - \beta_j| \le \pi h \,\}$, where it takes values ± 1. We now apply the remaining factors of g_j. This amounts to multiplying $b_j(q)$ by the product $\sin\big(\pi(q_j - \alpha_j)\big)\sin\big(\pi(q_j - \alpha_j)\big)$. This gives a function taking values $\mathcal{O}(h)$. Taking the error into account, we obtain $\|g_j^w\,\chi_D\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$.

In the case of g_{ij}, $i \neq j$, we apply $\sin(\pi h D_i)$ to $b_j(q)$: the resulting function takes values ± 1 on its support $S_j \cap S_{j'}$, so that $\|g_{ij}^w\,\chi_D\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$.

We have now proved that $\|F^w\, u_h\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ for any $F \in \mathcal{J}$. The procedure can be easily iterated to any finite product of functions $F_i \in \mathcal{J}$, yielding an estimate (4.17).

The proof is completed by the periodization argument as in the proof of Proposition 4.8. The only slight difference lies in the fact that the analogues of the functions $u_h^{(M)}(q)$ of (4.19) may now have discontinuities near ∂D, so that $\|P_{\mathbb{T}^{2n}}\, u_h^{(M)}\| - \|u_h^{(M)}\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h)$ instead of $\mathcal{O}(h^\infty)$. □

4.3. **Quantum relations.** Suppose that $\Lambda \subset \mathbb{T}^{2n} \times \mathbb{T}^{2n}$ is a Lagrangian submanifold. The basic example is given by the twisted graph Γ'_κ of a symplectic diffeomorphism κ on $\mathbb{T}^{2n}$ (see Section 3.2.1):

$$\Gamma'_\kappa = \big\{\, (q', q; p', -p) \;:\; (q', p') = \kappa(q, p),\ (q, p) \in \mathbb{T}^{2n} \,\big\}\ .$$

As we noticed in that section, the choice of change of sign depends on the choice of the splitting of variables (q, p), which is itself related with the choice of a *polarization* in the quantization $a \mapsto \mathrm{Op}_h(a)$ [21, §25.2]. This somewhat cumbersome convention is explained as follows.

Any state $v \in \mathcal{H}_h^n$ is naturally identified to a linear form $f_v \in (\mathcal{H}_h^n)^*$ through $f_v(w) = \langle v, w\rangle$. In our notations*, this scalar product is antilinear in the *first* component. To make the identification linear, we choose instead

$$v \in \mathcal{H}_h^n \Longrightarrow f_v(\cdot) = \langle \bar v, \cdot\rangle\,, \tag{4.25}$$

where states v are written as a vectors in the basis (4.3).

Let $\mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^n \otimes (\mathcal{H}_h^n)^*$ be the space of linear operators on $\mathcal{H}_h^n$. The linear identification (4.25) of $\mathcal{H}_h^n$ with $(\mathcal{H}_h^n)^*$ gives the identification,

$$\mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^{2n}\,, \quad \text{through} \quad (u \otimes v)(w) = u\,\langle \bar v, w\rangle\,, \quad u, v, w \in \mathcal{H}_h^n\,. \tag{4.26}$$

We observe that the norm on $\mathcal{H}_h^{2n}$ is the same as Hilbert-Schmidt norm on $\mathcal{L}(\mathcal{H}_h^n)$:

$$\|T\|_{\mathcal{H}_h^{2n}} = (\mathrm{tr}_{\mathcal{H}_h^n}(T^*T))^{\frac{1}{2}}\,, \tag{4.27}$$

*This is the physicists' convention.

It is related to the operator norm on $\mathcal{L}(\mathcal{H}_h^n)$ as follows:

$$\|T\|_{\mathcal{L}(\mathcal{H}_h^n)} \leq \|T\|_{\mathcal{H}_h^{2n}} \leq N^{n/2} \|T\|_{\mathcal{L}(\mathcal{H}_h^n)} \,. \tag{4.28}$$

Unitary operators have Hilbert-Schmidt norms $N^{n/2}$, and the subunitary operators considered in this paper have norms comparable with $N^{n/2}$.

The identification (4.26) dictates the way an operator of the type $A_1 \otimes A_2$ (with $A_i \in \mathcal{L}(\mathcal{H}_h^n)$) acts on $u \otimes v \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n)$. Indeed, if we take any $w \in \mathcal{H}_h^n$, we have

$$\begin{aligned}
[(A_1 \otimes A_2)(u \otimes v)](w) &= [A_1 u \otimes A_2 v](w) \\
&= A_1 u \, \langle \overline{A_2 v}, w \rangle \\
&= A_1 u \, \langle \bar{v}, A_2' w \rangle \\
&= [(A_1 u \otimes v) \circ A_2'](w) \,.
\end{aligned}$$

Here A_2' is the transposed of the operator A_2, written as a matrix in the basis (4.3). In the case $A_1 = \mathrm{Op}_h(a)$, $A_2 = \mathrm{Op}_h(b)$ for some real functions $a, b \in \mathcal{C}^\infty(\mathbb{T}^{2n})$, one checks that $A_2' = \mathrm{Op}_h(\tilde{b})$, with the same twisted function as in the proof of Lemma 4.6: $\tilde{b}(q,p) = b(q,-p)$. By linearity, for any $C_h \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n)$, we have

$$\mathrm{Op}_h(a \otimes b) \, C_h = \mathrm{Op}_h(a) \circ C_h \circ \mathrm{Op}_h(\tilde{b}) \,. \tag{4.29}$$

The sign change in the tilting $\Gamma \rightsquigarrow \Gamma'$ parallels the transformation $a(\rho')\, b(\rho) \rightsquigarrow a(\rho')\, \tilde{b}(\rho)$.

We are now in position to quantize a symplectic map, more generally a symplectic relation Γ as defined in Section 3.2.

Definition 4.11. A semiclassical sequence $U = \{ U_h \in \mathcal{H}_h^{2n} \}_{h \to 0}$, $\|U_h\|_{\mathcal{H}_h^{2n}} \leq C h^{-K}$, where K is fixed, is a quantum relation associated with the symplectic relation Γ if U is a Lagrangian state in $I(\Gamma')$, in the sense of Definition 4.7.

Explicitly, for any $M \in \mathbb{N}$ and any sequence of functions

$$g_j \in \mathcal{C}^\infty(\mathbb{T}^{2n} \times \mathbb{T}^{2n}) \,, \quad g_j \restriction_{\Gamma'} = 0 \,, \quad 1 \leq j \leq M \,,$$

we must have

$$\| \mathrm{Op}_h(g_M) \circ \cdots \circ \mathrm{Op}_h(g_1) \, U_h \|_{\mathcal{H}_h^{2n}} = \mathcal{O}(h^M) \, \|U_h\|_{\mathcal{H}_h^{2n}} \,. \tag{4.30}$$

If a function $g \in \mathcal{C}^\infty(\mathbb{T}^{4n})$ vanishes on Γ', then the function $\tilde{g}$ defined as $\tilde{g}(q',p';q,p) = g(q',p';q,-p)$ vanishes on Γ. The condition $g_j \restriction_{\Gamma'} = 0$ can thus be written $\tilde{g}_j \restriction_{\Gamma} = 0$.

Remark 4.2. Following the Section 4.2, in the case when Γ' is a singular Lagrangian it is useful to include in the definition sequences U in the (larger) space $I(\Gamma' \cup \Lambda_{\mathrm{sing}})$; the quantum baker's relation we define in next section will belong to such an enlarged space.

Through the identification (4.26), U_h is an operator on $\mathcal{L}(\mathcal{H}_h^n)$. We now show that this operator "classically transports" the microsupport of a sequence $w = \{ w_h \in \mathcal{H}_h^n \}$.

Proposition 4.12. *Take $U = \{ U_h \in \mathcal{H}_h^{2n} \simeq \mathcal{L}(\mathcal{H}_h^n) \}$ a quantum relation $U \in I(\Gamma')$. Then for any normalized sequence $w = \{ w_h \in \mathcal{H}_h^n \}$, $\|w\|_{\mathcal{H}_h^n} \asymp 1$, the microsupport of the image sequence $U(w) = \{ U_h(w_h) \}$ satisfies:*

$$\mathrm{WF}_h(U(w)) \subset \Gamma\big(\mathrm{WF}_h(w)\big) = \big\{ \rho \in \mathbb{T}^{2n} \ : \ \exists\, \mu \in \mathrm{WF}_h(w)\,,\ (\rho,\mu) \in \Gamma \big\}\ .$$

Proof. Assume that $\rho_0' \notin \Gamma(WF_h(w))$, which means that $\Gamma^{-1}(\rho_0') \not\subset \mathrm{WF}_h(w)$. Then there exists a function $f \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $f \equiv 1$ near ρ_0' but with $\mathrm{supp}(f)$ sufficiently small so that $\Gamma^{-1}(\mathrm{supp}(f)) \Subset \complement\, \mathrm{WF}_h(w)$. Consequently, there exists a function $g \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ with $g \equiv 1$ near $\mathrm{WF}_h(w)$ but $g \equiv 0$ on $\Gamma^{-1}(\mathrm{supp}(a))$. The function $f \otimes \tilde{g} \in \mathcal{C}^\infty(\mathbb{T}^{4n})$ then automatically satisfies $f \otimes \tilde{g}\restriction_{\Gamma'} = 0$.

Our aim is to show that $\rho_0' \notin \mathrm{WF}_h(U(w))$. For this, we introduce a function $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ such that $a(\rho_0') > 0$ and $f \equiv 1$ on $\mathrm{supp}(a)$. As in the proof of (4.9) we see that for any $M \in \mathbb{N}$, $\mathrm{Op}_h(a)\,\mathrm{Op}_h(f)^M = \mathrm{Op}_h(a) + \mathcal{O}(h^\infty)$. Hence

$$\begin{aligned} \|\,\mathrm{Op}_h(a) U_h w_h\| &= \|\,\mathrm{Op}_h(a)\,\mathrm{Op}_h(f)^M\, U_h w_h\| + \mathcal{O}(h^\infty) \\ &= \|\,\mathrm{Op}_h(a)\,\mathrm{Op}_h(f)^M\, U_h\ \mathrm{Op}_h(g)^M\, w_h\| \\ &\qquad + \|\,\mathrm{Op}_h(a)\,\mathrm{Op}_h(f)^M\, U_h\,(1 - \mathrm{Op}_h(g)^M)\, w_h\| + \mathcal{O}(h^\infty) \end{aligned}$$

To bound the second term on the right hand side, we notice that the function $(1 - g^M)$ vanishes near $\mathrm{WF}_h(w)$, so from Lemma 4.5 we get $\|(1 - \mathrm{Op}_h(g)^M)\, w_h\| = \mathcal{O}(h^\infty)$. Thus the second term is negligible. The first term on the right hand side is estimated using he identity

$$\mathrm{Op}_h(f)^M\, U_h\ \mathrm{Op}_h(g)^M = \mathrm{Op}_h(f \otimes \tilde{g})^M\, U_h\,.$$

Because $f \otimes \tilde{g}$ vanishes on Γ', the Hilbert-Schmidt norm of that operator is an $\mathcal{O}(h^M)$ by the assumption $U \in I(\Gamma')$. Equation (4.28) implies that its operator norm is an $\mathcal{O}(h^{M-K})$, where K comes from the bound on $\|U_h\|_{\mathcal{H}_h^{2n}}$. Hence

$$\|\,\mathrm{Op}_h(a) U_h\, w_h\| = \mathcal{O}(h^{M-K})\,,$$

for an arbitrary $M \in \mathbb{N}$, which shows that $\rho_0' \notin \mathrm{WF}_h(U(w))$. □

4.4. **Quantized baker's relation.** We explicitly construct quantum relations $B_h \in \mathcal{L}(\mathcal{H}_h^1)$ associated with the "open baker's maps" described in Section 3.3. For simplicity, we will assume that the coefficients D_j and ℓ_j are integers. For such a classical map, we will only consider the subsequence of Planck's constants of the form $h = (2\pi N)^{-1}$ such that $N/D_1 = M_1 \in \mathbb{N}$ and $N/D_2 = M_2 \in \mathbb{N}$ (that is, the lowest common multiple of (D_1, D_2) divides N).

Restricting ourselves to this subsequence, we define the quantizations of the baker's relation (3.3) as the following operators (written as $N \times N$ matrices in the bases (4.3)):

$$B_h = B_N \stackrel{\text{def}}{=} \mathcal{F}_N^* \circ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{F}_{M_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{F}_{M_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B_{1,h} + B_{2,h}. \tag{4.31}$$

The numbers of columns in successive blocks are respectively given by

$$\ell_1 M_1\,,\ M_1\,,\ \ell_2 M_2 - (\ell_1+1)M_1\,,\ M_2\,,\ (D_2 - \ell_2 - 1)M_2\,,$$

and $\mathcal{F}_M$ is the discrete Fourier transform given in (4.5). These matrices obviously generalize the unitary matrices associated with the closed baker's map [2]. Similar matrices can be constructed for any value of Bloch angles θ [39]. Although we restrict our presentation to the periodic boundary conditions $\theta = (0,0)$, our numerical computations (see next section) are performed for the antiperiodic case $\theta = (1/2, 1/2)$, because that choice preserves the parity symmetry of the classical relations (5.1,5.2) at the quantum level, so the desymmetrized matrices have the size $N/2$ [37].

We now check that the matrices (4.31) satisfy the definition 4.11, if we select the appropriate Lagrangian surface on $\mathbb{T}^4$, namely by adjoining a singularity set $\Lambda_{\rm sing}$ to the twisted graph B' (see Remark 4.2). By linearity, we can separately consider the two blocks $B_{j,h}$. Let us treat the left block $B_{1,h}$. Since the classical relation B_1 is generated by the function $S_1(q,p')$ of (3.4), it is natural to express the operator $B_{1,h}$ in the mixed representation (p',q), that is by a matrix from the basis $\{\,|Q_j\rangle\,\}$ to the basis $\{\,|P_k\rangle\,\}$. Since the change of basis matrix, $(|P_k\rangle\langle Q_j|)_{j,k=0,\ldots,N-1}$, equals $\mathcal{F}_N$, the operator $A_{1,h}$ defined as the matrix

$$(\langle Q_k|A_{1,h}|Q_j\rangle)_{j,k=0,\ldots,N-1} \stackrel{\text{def}}{=} (\langle P_k|B_{1,h}|Q_j\rangle)_{j,k=0,\ldots,N-1} = \mathcal{F}_N \circ B_{1,h}$$

is given by the Fourier block $\mathcal{F}_{M_1}$ at the same position as in (4.31), and zeros everywhere else.

The following lemma reduces finding the Lagrangian relation associated to $B_{1,h}$ to finding the Lagrangian associated to $A_{1,h}$.

Lemma 4.13. *Suppose that* $U_h \in \mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^{2n}$ *and that*

$$V_h \stackrel{\text{def}}{=} \mathcal{F}_N \circ U_h \in \mathcal{L}(\mathcal{H}_h^n) \simeq \mathcal{H}_h^{2n}\,.$$

Then

$$U_h \in I(\mathcal{C}') \iff V_h \in I(\mathcal{D}')\,,$$

where

$$\mathcal{D}' = \{(p', q; -q', p) \,:\, (q', q; p', p) \in \mathcal{C}'\}\,,$$

or equivalently $\mathcal{D} = \{(p', -q'; q, p) \,:\, (q', p'; q, p) \in \mathcal{C}\}\,.$

Proof. This is a special case of a composition formula which can be proved by mimicking the semiclassical proof in [1]. Here it follows from the the covariance properties of Weyl quantization with respect to the Fourier transform: for any $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$, $\mathrm{Op}(a)_h \circ \mathcal{F}_h = \mathcal{F}_h \circ \mathrm{Op}_h(b)$ where $b(q,p) = a(p,-q)$. As a result, for any $f \in \mathcal{C}^\infty(\mathbb{T}^{4n})$,

$$\mathrm{Op}_h(f)(\mathcal{F}_h \circ U_h) = \mathcal{F}_h \circ \mathrm{Op}_h(g)(U_h)\,,$$

where $g \in \mathcal{C}^\infty(\mathbb{T}^{4n})$ is given by $g(q',p';q,p) = f(p',-q';q,p)$.

Notice that the transformation $\mathcal{C} \mapsto \mathcal{D}$ is obtained by rotating each plane (q'_i, p'_i) by the angle $-\pi/2$ around the origin. □

We now identify the Lagrangian associated with the operator $A_{1,h}$. If we call $\chi_1(q)$ the characteristic function of the interval $I_1 = [\ell_1/D_1, \ell_1 + 1/D_1]$, then we can write

$$\langle Q_k|A_{1,h}|Q_j\rangle = \langle P_k|B_{1,h}|Q_j\rangle = \sqrt{\frac{D_1}{N}}\,\chi_1\big(\frac{k}{N}\big)\,\chi_1\big(\frac{j}{N}\big)\,e^{-2i\pi N\,S_1(\frac{k}{N},\frac{j}{N})}\,. \tag{4.32}$$

The operator,

$$A_{1,h} \in \mathcal{L}(\mathcal{H}_h^1) \simeq \mathcal{H}_h^2\,,$$

has a structure similar to the state analyzed in Lemma 4.9, after the following identifications:

$$2 \to n, \quad (q',q) \in \mathbb{I}^2 \to q \in \mathbb{I}^n, \quad I_1 \times I_1 \to D, \quad \sqrt{\frac{D_1}{N}} \to \frac{a(q)}{N^{n/2}}, \quad S_1(q',q) \to S(q)\,.$$

This state is therefore associated with the Lagrangian $\Lambda_{S_1} \cup \tilde{\mathcal{D}}_{\rm sing}$, where the first component is

$$\Lambda_{S_1} = \big\{\, \big(q', q;\, -(D_1 q' - \ell_1),\, -(D_1 q - \ell_1)\big) \;:\; q, q' \in I_1 \,\big\}\,.$$

This manifold is the image of the twisted graph B'_1 through the transformation given in Lemma 4.13.

The singular part $\mathcal{D}_{\rm sing}$ can be split into 4 components, each one sitting above a side of the square ∂D. For instance, the side $\{q' \in I_1,\, q = \ell_1/D_1\}$ leads to the Lagrangian component

$$\begin{aligned}\mathcal{D}_{\rm sing,1} &= \Big\{\, \big(q',\, q = \frac{\ell_1}{D_1}\,;\, p' = -\partial_{q'} S_1(q', \ell_1/D_1),\, p\big) \;:\; q' \in I_1,\, p \in \mathbb{I} \,\Big\} \\ &= \Big\{\, \big(q',\, \frac{\ell_1}{D_1}\,;\, 0,\, p\big) \;:\; q' \in I_1,\, p \in \mathbb{I} \,\Big\}\,.\end{aligned}$$

Through the inverse transformation in Lemma 4.13, this yields the Lagrangian

$$\Lambda_{\rm sing,1} = \Big\{\, \big(q' = 0,\, q = \frac{\ell_1}{D_1};\, p',\, p\big) \;:\; p' \in I_1\,, p \in \mathbb{I} \,\Big\}\,,$$

which contains the corresponding side of $\partial B'_1$. The three remaining sides of $I_1 \times I_1$ lead to similar Lagrangians $\Lambda_{\rm sing,i}$, $i = 2,3,4$, each of which containings a side of $\partial B'_1$. The same analysis applies to $B_{2,h}$ and hence we have proved the

Proposition 4.14. *The sequence of matrices B_h given in* (4.31) *quantizes the classical baker's relation $B = B_1 \cup B_2$ of* (3.3)*, in the sense of Definition 4.11 and Remark 4.2:*

$$B_h \in I\left(B' \cup \bigcup_{j=1}^{8} \Lambda_{\mathrm{sing,j}}\right).$$

This quantization of the baker's relation is very close to the "quantum horseshoe" defined by Saraceno-Vallejos in [38]. The operator B_h is subunitary, and its eigenstates can be seen as "metastable states", "decaying states" or "resonances". This subunitarity mirrors the decay of a classical probability density evolved through the open map B (due to the "escape" of particles to infinity). This classical decay can be analyzed in terms of a "conditionally invariant measure" on $\mathbb{T}^2$, which decays according to the classical decay rate $0 < \lambda_{\mathrm{cl}} < 1$ [7].

5. Numerical results for the baker's relation

We numerically computed the spectrum of the quantum Baker relations in the following symmetric cases:

$$\text{the “3-baker”:} \qquad D_1 = D_2 = 3, \qquad \ell_1 = 0,\ \ell_2 = 2 \tag{5.1}$$

$$\text{the “5-baker”:} \qquad D_1 = D_2 = 5, \qquad \ell_1 = 1,\ \ell_2 = 3\,, \tag{5.2}$$

but we will only plot results concerning the first case. These relations have simple trapped sets, which are pure Cantor sets of dimensions

$$2d = 2 \times \frac{\log 2}{\log D}\,.$$

Besides, these relations enjoy the obvious parity symmetry $q \to 1-q, \quad p \to 1-p$. The quantization preserves this symmetry if we choose anti-periodic conditions for the quantum torus, that is Bloch angles $\theta_1 = (1/2, 1/2)$: the matrix B_{h,θ_1} then commutes with the quantum parity operator $\Pi_{kl} = \delta_{k,N-1-l}$, so we can separately diagonalize the even and odd parts

$$B_{h,\mathrm{ev}} = B_h \circ \frac{1+\Pi}{2}\,, \quad B_{h,\mathrm{odd}} = B_{h,\theta_1} \circ \frac{1-\Pi}{2}\,,$$

which are both of rank N/D; they give the full nontrivial spectrum of B_h. We checked that for both relations, the odd spectrum has the same characteristics as the even one, so we only analyze the latter.

We compared the spectra of the matrices $B_{h,\mathrm{ev}}$ along geometric sequences $\{N_k = N_o D^k\}$ or arbitrary sequences. As a first illustration, we show in Fig. 5 the spectrum of the 3-baker $B_{h,\mathrm{ev}}$ along the geometric sequence $N = 81 \times 3^k$ for $k = 1, 2, 3, 4$.

In Fig. 6 (left), we plot the number

$$n(N, r) = \#\{\lambda \in \mathrm{Spec}(B_{h,\mathrm{ev}}) \,:\, |\lambda| \geq r\}$$

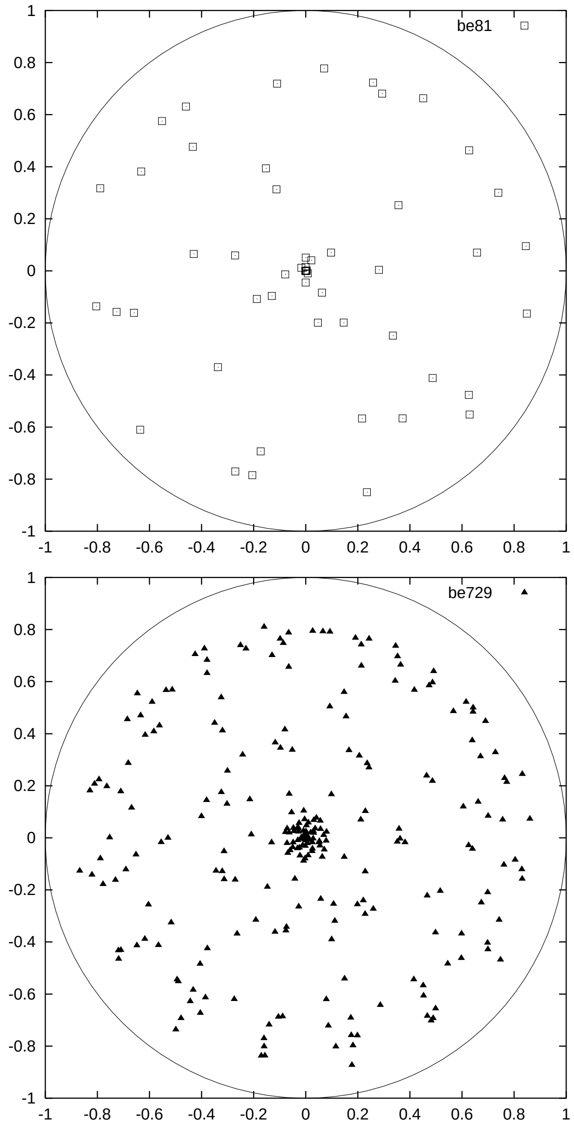

FIGURE 5. Even spectrum of the quantum 3-baker's relation B_h for $N/3 = 81, 243, 729, 2187$.

as a function of $0 < r < 1$, for the 3-baker and N taken along an arbitrary sequence. According to the theory, this number should scale as

$$n(N, r) \sim C(r)\, N^{\frac{\log 2}{\log 3}} \tag{5.3}$$

in the limit $N \to \infty$. To check this conjecture, we rescaled vertical coordinate of the curves in Fig. 6 (left) by a factor $N^{-\log 2/\log 3}$, and plotted the rescaled curves in Fig. 6 (right). These rescaled curves are now fairly superposed on each other, which shows that the theoretical scaling is approximately correct. A similar study was performed in [42] for the open kicked rotator. In this reference, the fractal exponent was related to the *mean dwell time* of the classical dynamics, that is, the average time spent in the cavity before leaving it: according to eq. 7 in that reference, the number of resonances behaves like $\sim N^{d'}$ with $d' = 1 - 1/(\lambda \tau_{\rm dwell})$. For our symmetric baker's relations, the dwell time is $\tau_{\rm dwell} = D/(D-2)$, so that d' is not exactly equal to d. However, one should have $d' \approx d$ for a system with a large dwell time.

The authors of [42] noticed that the profile function $C(r)$ for the kicked rotator can be fitted reasonably well with a prediction of random matrix theory [52]. In our case, the superposition of the rescaled counting functions on Fig. 6 (right) seems too approximate for us to claim such an agreement with random matrix theory, although the data are compatible with it.

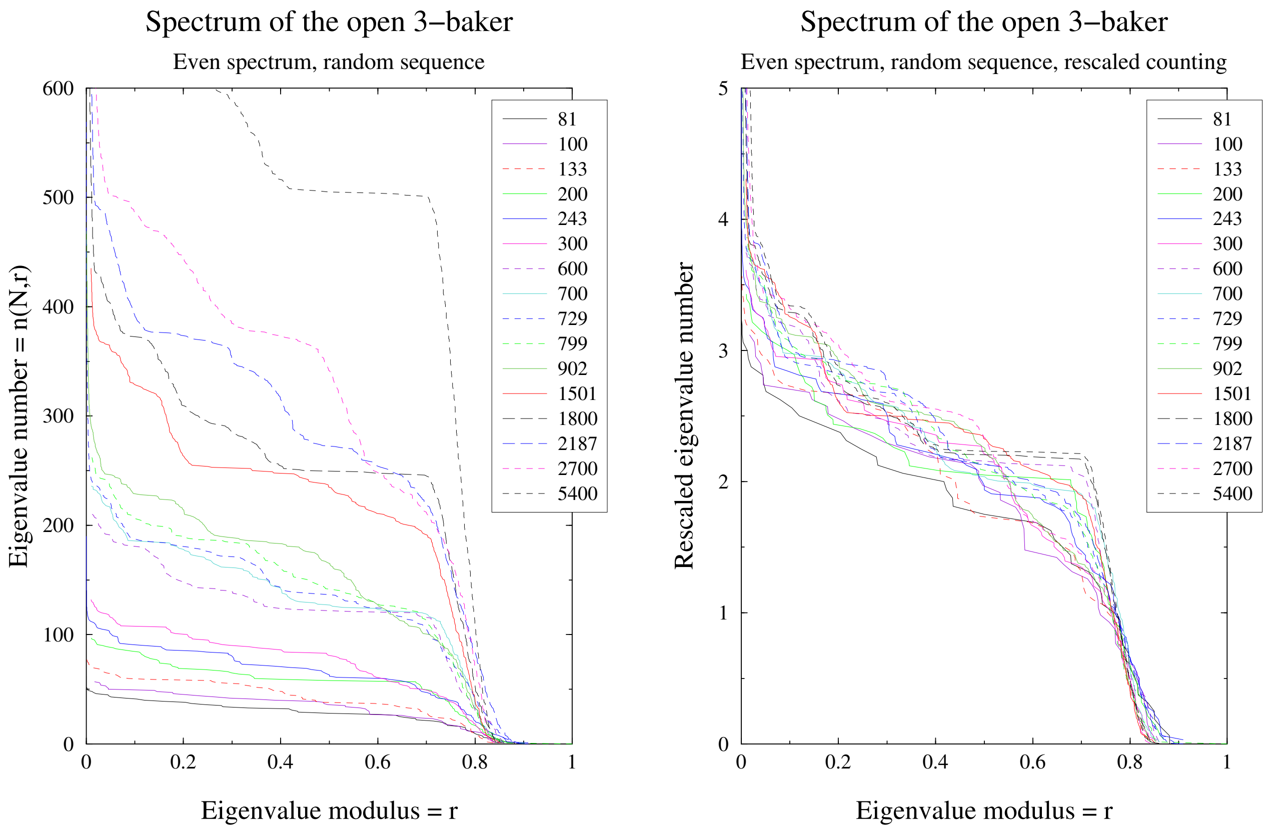

FIGURE 6. On the left, we plot the number $n(N, r)$ of even eigenvalues of modulus $\geq r$ of the 3-baker B_h (the numbers in the box is the rank $N/3$ of the desymmetrized matrices). On the right we have rescaled that number by the factor $N^{-\log 2/\log 3}$.

To check the fractal law (5.3) more precisely, we fix some values for r, and look at the N-dependence of $n(N,r)$, for N taken either along geometric sequences, or along an arbitrary sequence (with always $3|N$). On Figure 7, we plot this dependence in logarithmic scale. For $r = 0.5$, the points seem to converge to the theoretical scaling law quite nicely as $N \to \infty$, even along an "arbitrary" sequence. For smaller r, the convergence is not yet clear, especially for $r = 0.03$. This phenomenon is to be expected: for each N there are exactly $N/3$ nonzero even eigenvalues; the above scaling shows that most of them concentrate near the origin. Therefore, when counting eigenvalues in annuli closer and closer to the origin, one finally starts to count the "bulk" of the $N/3$ eigenvalues, departing from (5.3).

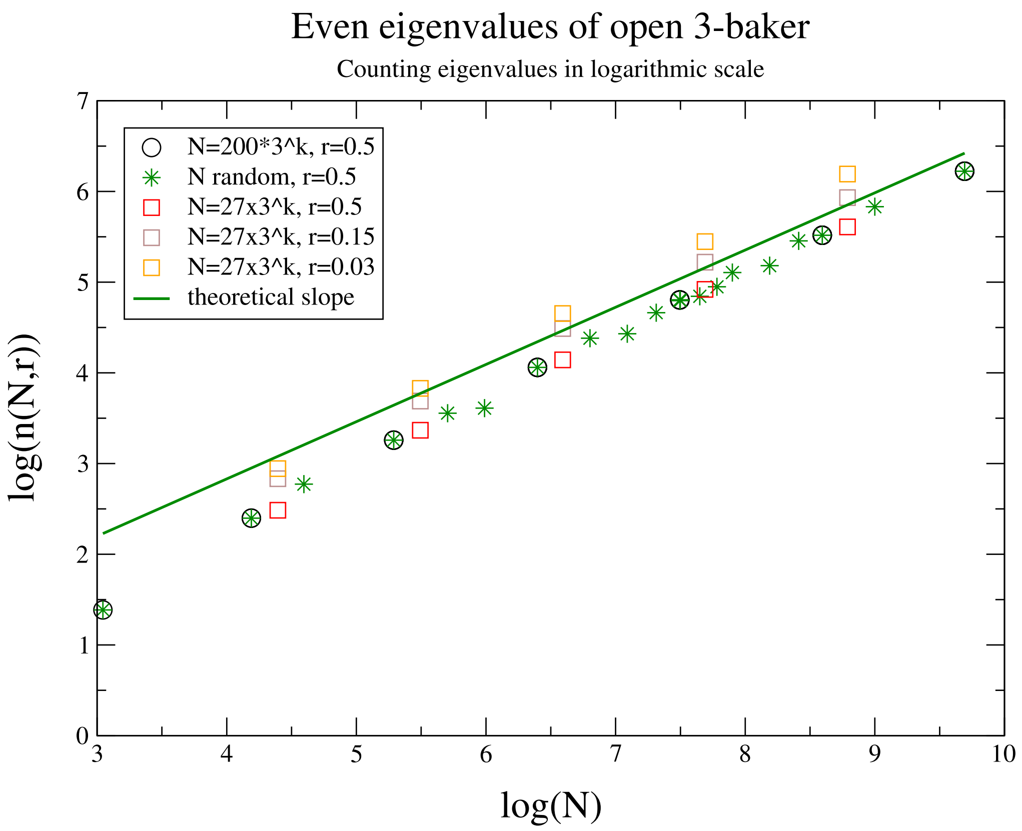

FIGURE 7. Checking the N-dependence of $n(N,r)$ for various values of r, along geometric and arbitrary sequences for N.

6. A toy model

Let us explicitly compute the matrix elements of the two vertical blocks $B_{1,h}$, $B_{2,h}$ in (4.31), for the symmetric example (5.1) of the open 3-baker. Both are matrices $N \times N/3$, which we index by $0 \leq k \leq N-1$, $0 \leq l \leq N/3-1$:

$$
\begin{aligned}
&(B_{1,h})_{k\,l} = \begin{cases} (1-\exp(2i\pi(k-3l)/(3N)))^{-1}(1-\omega_3^k)/N & \text{if } k \neq 3l, \\ 1 & \text{if } k = 3l, \end{cases}, \\
&(B_{2,h})_{k\,l} = \omega_3^{2k}\,(B_{1,h})_{k\,l}\,, \quad \omega_3 = e^{2i\pi/3}\,.
\end{aligned}
\tag{6.1}
$$

On Figure 8 (left) we represent the moduli of these matrix elements. The largest matrix elements are near the "tilted diagonals" $k \approx 3l$, and decay as $1/|k-3l|$ away from them.

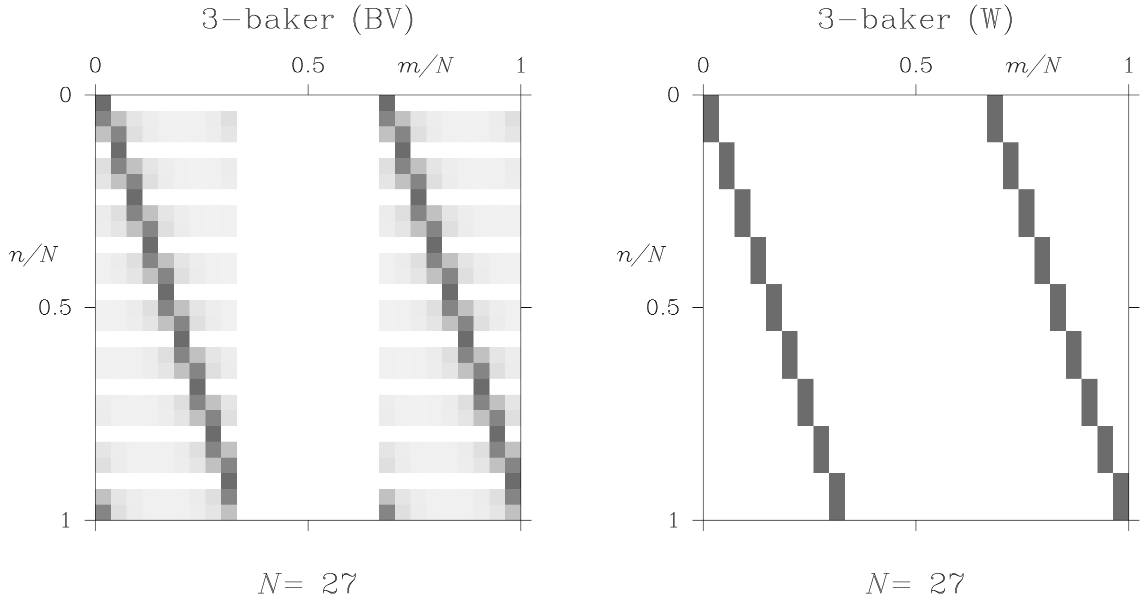

FIGURE 8. Matrices B_{27} (left) and its toy model $\widetilde{B}_{27}$ (right) for the quantum 3-baker. The gray scale corresponds to the modulus of the matrix elements, (white = 0, black= 1).

Being unable to rigorously analyze the spectrum of B_h, we replace it by the following simplified model:

$$
\begin{aligned}
&\widetilde{B}_h = \widetilde{B}_N = \frac{1}{\sqrt{3}}\,[\widetilde{B}_{1,h}, 0, \widetilde{B}_{2,h}]\,, \\
&(\widetilde{B}_{1,h})_{k\,l} = \begin{cases} 1 & \text{if } l = \lfloor k/3 \rfloor \\ 0 & \text{if } l \neq \lfloor k/3 \rfloor \end{cases}, \qquad (\widetilde{B}_{2,h})_{k\,l} = \omega_3^{2k}\,(\widetilde{B}_{1,h})_{k\,l}\,,
\end{aligned}
\tag{6.2}
$$

where $\lfloor x \rfloor$ denotes the integer part of x. For $N = 9$, this gives

$$\widetilde{B}_{N=9} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3^2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega_3 \end{pmatrix}, \ \omega_3 = e^{2\pi i/3}\,. \tag{6.3}$$

The model has been obtained "by hand", by replacing "lower order" terms in the matrix B_h by 0, keeping only nonzero elements on the "tilted diagonals", and replacing $(1 - e^{2\pi i(\pm 1)/3})/(N(1 - e^{2i\pi\pm 1/(3N)}))$ by 1.

The new matrix $\widetilde{B}_h$ retains some qualitative features of B_h but there is no immediate connection between their spectra: the "lower order" terms are not small enough for that, and B_h cannot be considered as a "small perturbation" of $\widetilde{B}_h$. Still, a comparison of density plots of both matrices for $N = 27$ (Fig. 8) gives a visual motivation for the toy model.

The simplicity of the matrices $\widetilde{B}_h$ will allow us to prove the fractal Weyl law which we could numerically observe for B_h (see Section 6.2). It is interesting to notice that the simplified operator $\widetilde{B}_h$ is in fact not associated with the same classical relation as B_h:

Proposition 6.1. *In the notations of Section 4.2, the quantum relation $\{\widetilde{B}_h\}$ is associated with the following singular Lagrangian:*

$$\widetilde{B}_h \in I(\widetilde{B}' \cup \widetilde{\Lambda}_{\rm sing})\,,$$

where

$$\widetilde{B} = \left\{ (q' = 3q, p' = \frac{p+j}{3}; q, p) \ : \ q \in \mathbb{I} \setminus [1/3, 2/3]\,, p \in \mathbb{I}\,, j = 0, 1, 2 \right\},$$
$$\widetilde{\Lambda}_{\rm sing} = \left\{ \left(q' = 0, q\,;\, p',\, p\right), \ : \ q \in \{\, 0, 1/3, 2/3\,\}\,,\, p',\, p \in \mathbb{I} \,\right\}\,.$$

Proof. In place of $\widetilde{B}_h$ we will consider $\widetilde{A}_h = \mathcal{F}_N \circ \widetilde{B}_h$, and apply Lemma 4.13. From the structure of $\widetilde{B}_h$, the operator $\widetilde{A}_h$ can obviously be split into $\widetilde{A}_{1,h} + \widetilde{A}_{2,h}$. We will analyze the first component in detail, the analysis for the second one being similar. The matrix $\langle Q_k | \widetilde{A}_{1,h} | Q_j \rangle$ is nonzero in the vertical strip $I_1 \times \mathbb{I}$, with $I_1 = [0, 1/3)$. It takes the value

$$\langle Q_k | \widetilde{A}_{1,h} | Q_j \rangle = \frac{\chi_1(Q_j)}{\sqrt{3N}} \sum_{\ell=0}^{2} \exp(-2i\pi k(3j+\ell)/N) = \Big(\frac{1}{3} \sum_{\ell=0}^{2} e^{-2i\pi k\ell/N} \Big)\, (\mathcal{F}_{N/3})_{k\,j}\,.$$

Using the notations of Lemma 4.9 and comparing this expression with (4.32), the term between brackets is part of the "symbol" $a(Q_k, Q_j)$. On the opposite, the last factor

$(\mathcal{F}_{N/3})_{k\,j}$, which takes values for $Q_k \in I_1$, $Q_j \in \mathbb{I}$, can be rewritten as $\sqrt{3}Ne^{-2i\pi N S_1(Q_k,Q_j)}$, with $S_1(q',q) = 3q'q$. The latter function generates the Lagrangian

$$\Lambda_{S_1} = \{\, (q',\, q;\, p' = -3q,\, p = -3q'),\ :\ q' \in \mathbb{I},\, q \in I_1 \,\}\ ,$$

which, after inverse transformation through Lemma 4.13, leads to

$$\widetilde{B}'_1 = \{\, (q' = 3q,\, q\,; p',\, p = -3p'),\ :\ q \in I_1,\, p' \in \mathbb{I} \,\}\ .$$

One easily checks that the symbol $a(q',q)$ is continuous across $q' = 1$, so its discontinuities only concern the q variable. The domain supporting this matrix is therefore the strip $D = \mathbb{I} \times I_1$, with boundaries $\partial D = \mathbb{I} \times (\{\, q = 0 \,\} \cup \{\, q = 1/3 \,\})$. To this boundary is associated the Lagrangian

$$\mathcal{D}_{\text{sing},1} = \left\{ \left(q' = 0,\, q = \frac{j}{3};\, p' \in \mathbb{I},\, p \in \mathbb{I}\right),\ j = 0, 1 \right\}\ ,$$

which transforms into a part of $\widetilde{\Lambda}_{\text{sing}}$.

Using the same approach, one easily finds that the second component $\widetilde{B}_{2,h}$ of the quantum relation is associated to $\widetilde{B}'_2$ and the second part of $\widetilde{\Lambda}_{\text{sing}}$. $\square$

We notice that the graph $\widetilde{B}$ can be obtained by adjoining to each point $(\rho';\rho) \in B$ the translated points $(\rho' + (0,1/3);\rho)$ and $(\rho' + (0,2/3);\rho)$. Now a point $\rho \in \mathbb{T}^2$ has either 3 images through $\widetilde{B}$, or none. The relation $\widetilde{B}$ is therefore genuinely different from B. Both these relations share the same forward trapped set $\widetilde{\Gamma}_- = \Gamma_- = C \times \mathbb{I}$ (see §3.3), but the backwards trapped set of $\widetilde{B}$he trapped sets of $\widetilde{B}$ is easily shown to be $\widetilde{\Gamma}_+ = \mathbb{T}^2$, which drastically differs from Γ_+. This asymmetry between $\widetilde{\Gamma}_-$ and $\widetilde{\Gamma}_+$ reflects the fact that, unlike B, the relation $\widetilde{B}$ is not time reversal symmetric.

The fact that $\widetilde{B}_h$ is not associated with the relation B (in the sense of definition 4.11 and Remark 4.2) should not bother us too much though. In the next section, we will give a more "formal" construction of the quantized open 3-baker $\widetilde{B}_h$, in the case where N is a power of 3 (this construction will also hold for any symmetric D-baker, for N a power of D). We will show that this matrix appears naturally as the quantization of the classical relation, if one substitutes the discrete Fourier transform (4.5) by the Walsh Fourier transform.

6.1. **Walsh quantization of the baker's relation.** The *Walsh model* of harmonic analysis has been originally devoted to fast signal processing [26]. It has been used recently in mathematics to obtain simpler (and provable) versions of statements of the usual harmonic analysis — see [33] for an application in scattering theory and for pointers to the recent literature. The major advantage of Walsh harmonic analysis is the possibility to completely localize a wavepacket both in position and momentum: for our problem, this has the effect to avoid diffraction problems due to the discontinuities of the map, which spoil the usual semiclassics [39]. Closer to our context, Meenakshisundaram and Lakshminarayan recently used the Hadamard Fourier transform (which is related with the Walsh transform we give below) to analyze the multifractal structure of some eigenstates of the (unitary) quantum 2-baker B_N [30].

6.1.1. *The quantum torus as a system of quantum Dits.* We first fix the coefficient $D \in \mathbb{N}$ ($D \geq 2$) of the symmetric baker's relation (3.7), and will consider in this section only the inverse Planck's constants of the form $N = D^k$ for some $k \in \mathbb{N}$. In this case, integers $j \in \mathbb{Z}_{D^k} = \{0, \ldots, D^k - 1\}$ are in one-to-one correspondence with the words $\boldsymbol{\epsilon} = \epsilon_1\epsilon_2 \cdots \epsilon_k$ made of symbols (or "Dits") $\epsilon_\ell \in \mathbb{Z}_D$:

$$\mathbb{Z}_{D^k} \ni j = \sum_{\ell=1}^{k} \epsilon_\ell \, D^{k-\ell} \,. \tag{6.4}$$

The natural order for $j \in \mathbb{Z}_{D^k}$ corresponds to the lexicographic order for the symbolic words $\{\boldsymbol{\epsilon} \in (\mathbb{Z}_D)^k\}$. This way, each position eigenstate $|Q_j\rangle$ of the basis (4.3) (the *computational basis* in the quantum computing framework) can be associated with the unique symbolic sequence $\epsilon_1\epsilon_2 \cdots \epsilon_k$ which gives its Dnary expansion

$$Q_j = \frac{j}{N} = 0 \cdot \epsilon_1\epsilon_2 \cdots \epsilon_k \,. \tag{6.5}$$

Let us denote the canonical basis of $\mathbb{C}^D$ by $\{e_0, e_1, \ldots, e_{D-1}\}$. Then, each $|Q_j\rangle$ can be written as

$$|Q_j\rangle = e_{\epsilon_1} \otimes e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_k} \,. \tag{6.6}$$

Following [41], we can denote each $|Q_j\rangle$ by $|\boldsymbol{\epsilon}\rangle = |\epsilon_1\epsilon_2 \cdots \epsilon_k\rangle$ to emphasize the above tensor product decomposition. This way, the quantum space $\mathcal{H}_h^1$ is naturally identified with the *tensor product* of k spaces $\mathbb{C}^D$:

$$\mathcal{H}_h = (\mathbb{C}^D)_1 \otimes (\mathbb{C}^D)_2 \otimes \cdots \otimes (\mathbb{C}^D)_k \,.$$

In the quantum computating framework, each space $(\mathbb{C}^D)_\ell$ is interpreted as a "quantum Dit", or " qu$D$it" [32]; viewed in our toral phase space, the qu$D$it $(\mathbb{C}^D)_\ell$ is associated with the *scale* $D^{-\ell}$ in the position variable, so $(\mathbb{C}^D)_1$ is called the "most significant qu$D$it".

6.1.2. *Walsh Fourier transform.* The discrete Fourier transform of (4.5) (with $n = 1$, $N = D^k$) is the Fourier transform (in the sense of abstract harmonic analysis) on the group $\mathbb{Z}_{D^k}$. The simpler Walsh Fourier transform is the Fourier transform on the (more complicated!) group $(\mathbb{Z}_D)^k$. More explicitly, each row of $\mathcal{F}_{D^k}$ corresponds to the character $j' \mapsto \exp\Big(- 2i\pi jj'/D^k\Big)$ of $\mathbb{Z}_{D^k}$:

$$(\mathcal{F}_{D^k})_{jj'} = D^{-k/2} \exp\Big(- 2i\pi \frac{jj'}{D^k}\Big) = D^{-k/2} \prod_{\ell=1}^{k} \exp\Big(- 2i\pi \frac{\epsilon_\ell(jj')}{D^\ell}\Big) \,, \tag{6.7}$$

Using (6.4), we can easily express each $\epsilon_m(jj')$ in terms of the symbols of j and j':

$$\epsilon_m(jj') = \sum_{\ell+\ell'=k+1-m} \epsilon_\ell(j)\,\epsilon_{\ell'}(j')\,.$$

The Walsh Fourier transform is defined by keeping only the first factor on the right hand side of (6.7): one obtains the matrix

$$(W_k)_{jj'} = D^{-k/2}\,\exp\Big(-2i\pi\frac{\epsilon_1(jj')}{D}\Big) = \prod_{\ell=1}^{k} D^{-1/2}\,\omega_D^{-\epsilon_\ell(j)\epsilon_{k+1-\ell}(j')}\,, \tag{6.8}$$

where $\omega_D = e^{2i\pi/D}$.

Using the tensor product structure of $\mathcal{H}_h^1 \simeq (\mathbb{C}^D)^{\otimes k}$, this definition can be recast as follows.

Lemma 6.2. *The Walsh Fourier transform W_k acts simply on tensor product states $v_1\otimes v_2\otimes\cdots\otimes v_k$:*

$$W_k(v_1\otimes\cdots\otimes v_k) = \mathcal{F}_D v_k\otimes\cdots\otimes\mathcal{F}_D v_1\,,\qquad v_\ell\in\mathbb{C}^D,\ \ell=1,\ldots,k\,.$$

Here $\mathcal{F}_D = W_1$ is the discrete Fourier transform on $\mathbb{C}^D$. This expression shows that W_k is a unitary tranformation on $\mathcal{H}_n^1 \simeq (\mathbb{C}^D)^{\otimes k}$.

Proof. By linearity, we only need to consider the case where each v_ℓ is a basis state e_{ϵ_ℓ}. We then apply the definition (6.8):

$$\begin{aligned}
W_k(e_{\epsilon_1}\otimes\cdots\otimes e_{\epsilon_k}) &= \sum_{\boldsymbol{\epsilon}'\in(\mathbb{Z}_D)^k}\Big(\prod_{\ell=1}^{k} D^{-1/2}\,\omega_D^{-\epsilon'_{k+1-\ell}\,\epsilon_\ell}\Big)\,e_{\epsilon'_1}\otimes\cdots\otimes e_{\epsilon'_k}\\
&= \bigotimes_{\ell=1}^{k}\Bigg(\sum_{\epsilon'_\ell\in\mathbb{Z}_D} D^{-1/2}\,\omega_D^{-\epsilon'_\ell\,\epsilon_{k+1-\ell}}\,e_{\epsilon'_\ell}\Bigg)\\
&= \bigotimes_{\ell=1}^{k}\mathcal{F}_D\,e_{\epsilon_{k+1-\ell}}\,.
\end{aligned}$$

Since W_k is the combination of the unitary operator $\mathcal{F}_D\otimes\cdots\otimes\mathcal{F}_D$ with a (unitary) permutation of the qu$D$its, it is unitary. □

This expression shows that (as opposed to the discrete Fourier transform), the Walsh Fourier transform *does not entangle* the different qu$D$its: a tensor product state is sent to another tensor product state.

Example. To illustrate this simple lemma we take $D = 2$, and consider the following $2^k \times 2^k$ matrix

$$A_{2^k} = \frac{1}{\sqrt{2}}[A_{0,2^k}, A_{1,2^k}], \quad (A_{j,2^k})_{0\leq n\leq 2^k-1, 0\leq m\leq 2^{k-1}-1} = \begin{cases} (-1)^{jn}, & m = \lfloor n/2 \rfloor \\ 0, & m \neq \lfloor n/2 \rfloor . \end{cases} \tag{6.9}$$

For instance when $k = 2$ we get

$$A_{2^2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} .$$

This sequence of matrices has been obtained as the "extreme" possibility among a family of different quantizations of the (closed) 2-baker's map [41][♭]. In a different context, this (unitary) matrix belongs to the family of transfer matrices associated with the binary (or "de Bruijn") graph with 2^k vertices [48].

The transformation A_{2^k} acts as follows on tensor product states

$$v_1 \otimes \cdots \otimes v_k \longmapsto v_2 \otimes \cdots \otimes v_k \otimes F_2 v_1 .$$

From this and the identity $(F_2)^2 = I$ we easily draw $(A_{2^k})^k = F_2^{\otimes k}$ and from $F_2^2 = Id_2$ we get that the matrix A_{2^k} is periodic:

$$(A_{2^k})^{2k} = Id_{2^k} . \tag{6.10}$$

The simple action of A_{2^k} on tensor products also shows that this matrix can be easily expressed in terms of the Walsh Fourier transform (for $D = 2$):

$$A_{2^k} = W_k \begin{pmatrix} W_{k-1} & 0 \\ 0 & W_{k-1} \end{pmatrix} . \tag{6.11}$$

The 2×2 block structure corresponds to the most significant qubit:

$$\begin{pmatrix} W_{k-1} & 0 \\ 0 & W_{k-1} \end{pmatrix} (v_1 \otimes \cdots \otimes v_k) = v_1 \otimes W_{k-1}(v_2 \otimes \cdots \otimes v_k) .$$

The left (right) block contains the indices $\{j < N/2\}$ ($\{j \geq N/2\}$), so the expression (6.11) exactly parallels the one defining the Balazs-Voros quantum baker [2]. Compared to this "usual" quantum baker, A_{2^k} is thus obtained by replacing the discrete Fourier matrices $\mathcal{F}_{2^k}$, $\mathcal{F}_{2^{k-1}}$ by their Walsh analogues W_k, W_{k-1}.

The matrix A_{2^k} is unitary and, as we will see in the next section, our toy model $\widetilde{B}_h$ for the quantum open 3-baker (see Eq. (6.2)) is its subunitary analogue.

[♭] We thank M. Saraceno for pointing out this interpretation to us, as well as the equation (6.10).

6.2. **Resonances for the Walsh quantization of the open Baker relation.** We now set $D = 3$ to concentrate on the open 3-baker (5.1). By analogy with the example in last section, we modify the quantization (4.31,6.1), in the case $N = 3^k$, by replacing the discrete Fourier matrices by their Walsh analogues. The resulting operator exactly coincides with the toy model (6.2) introduced in the beginning of this section:

Lemma 6.3. *In the case $N = 3^k$, the matrix $\widetilde{B}_h$ defined in* (6.2) *can be rewritten in terms of the Walsh Fourier transforms as follows:*

$$\widetilde{B}_h = W_k^* \begin{pmatrix} W_{k-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{k-1} \end{pmatrix} .$$

Proof. The matrix

$$V_k \stackrel{\text{def}}{=} \begin{pmatrix} W_{k-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{k-1} \end{pmatrix} ,$$

acts on the computational basis as:

$$V_k(e_{\epsilon_1} \otimes \cdots \otimes e_{\epsilon_k}) = \begin{cases} e_{\epsilon_1} \otimes W_{k-1}(e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_k}) \,, & \epsilon_1 \neq 1 \\ 0 \,, & \epsilon_1 = 1 \,. \end{cases}$$

Hence

$$W_k^* V_k(e_{\epsilon_1} \otimes \cdots \otimes e_{\epsilon_k}) = \begin{cases} e_{\epsilon_2} \otimes \cdots \otimes e_{\epsilon_k} \otimes \mathcal{F}_3^* e_{\epsilon_1} \,, & \epsilon_1 \neq 1 \\ 0 \,, & \epsilon_1 = 1 \,. \end{cases} \tag{6.12}$$

Since

$$\mathcal{F}_3^* e_0 = \frac{1}{\sqrt{3}}(e_0 + e_1 + e_2) \,, \quad \mathcal{F}_3^* e_2 = \frac{1}{\sqrt{3}}(e_0 + \omega_3^2\, e_1 + \omega_3\, e_2) \,,$$

it follows that the matrix (6.12) is identical with $\widetilde{B}_h$ of (6.2). □

As a byproduct of the above proof, if we define the "truncated" inverse Fourier matrix

$$\widetilde{F}_3^* \stackrel{\text{def}}{=} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \omega_3^2 \\ 1 & 0 & \omega_3 \end{pmatrix} , \tag{6.13}$$

the toy model $\widetilde{B}_h$ acts as follows on tensor products:

$$\widetilde{B}_h(v_1 \otimes \cdots \otimes v_k) = v_2 \otimes v_3 \otimes \cdots \otimes \widetilde{F}_3^* v_1 \,. \tag{6.14}$$

This form is particularly nice to compute the spectrum of $\widetilde{B}_h$. We start by computing the spectrum of its power $(\widetilde{B}_h)^k$, which is enough to obtain the radial distribution of resonances (that is, the distribution of *resonance widths*).

Proposition 6.4. *Let $\lambda_\pm$, $|\lambda_-| < |\lambda_+|$, be the eigenvalues of the matrix*

$$\Omega_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & \omega_3 \end{pmatrix} .$$

The non-zero eigenvalues of $(\widetilde{B}_h)^k$ (for $N = (2\pi h)^{-1} = 3^k$) are given by $\lambda_+^{k-p}\lambda_-^p$, $0 \leq p \leq k$, each occurring with multiplicity $\binom{k}{p}$. From this we get the radial distribution of the eigenvalues of $\widetilde{B}_h$:

$$\forall\, r \in [0,1], \qquad \frac{1}{2^k}\, \# \Big\{ \mathrm{Spec}(\widetilde{B}_h) \setminus D(0,r) \Big\} \xrightarrow{k\to\infty} C(r)\,,$$
$$C(r) = \begin{cases} 1\,, & r < |\det \Omega_3|^{\frac12} \\ 0\,, & r > |\det \Omega_3|^{\frac12}\,, \end{cases} \tag{6.15}$$

which shows that the nontrivial resonances accumulate near the circle of radius $r_0(\widetilde{B}) = |\det \Omega_3|^{\frac12}$.

This proposition gives Theorem 1, where $\widetilde{B}$ is the baker's relation described in Proposition 6.1, $\widetilde{B}_h$ the matrices (6.2), and Planck's constants are taken along the sequence $\{h_k = (2\pi \times 3^k)^{-1},\ k \in \mathbb{N}\}$.

Proof. From the expression (6.14) on tensor products, we see that

$$(\widetilde{B}_h)^k (v_1 \otimes \cdots \otimes v_k) = \widetilde{F}_3^* v_1 \otimes \cdots \otimes \widetilde{F}_3^* v_k\,.$$

That means that $(\widetilde{B}_h)^k = (\widetilde{F}_3^*)^{\otimes k}$, so one eigenbasis is obtained by taking the tensor products of eigenstates of $\widetilde{F}_3^*$, and the eigenvalues of $(\widetilde{B}_h)^k$ are the corresponding products of eigenvalues of $\widetilde{F}_3^*$. The nonzero eigenvalues λ_+, λ_- of $\widetilde{F}_3^*$ are the eigenvalues of Ω_3, so the first part of the proposition follows. To prove the second part, notice that each eigenvalue $\lambda_+^{k-p}\lambda_-^p$ of $\widetilde{B}_h^k$ corresponds to an eigenvalue (possibly in the generalized sense) of modulus $|\lambda_+^{1-p/k}\lambda_-^{p/k}|$ of $\widetilde{B}_h$. Therefore, we are able to count eigenvalues of $\widetilde{B}_h$ (with multiplicities) in a given annulus.

Let $H(t)$ denote the Heaviside function, $H(t) = 0$ for $t < 0$, and $H(t) = 1$, otherwise. Then, for any $0 < r < 1$,

$$\begin{aligned} \# \Big\{ \mathrm{Spec}(\widetilde{B}_h) \setminus D(0,r) \Big\} &= \sum_{p=0}^{k} H(|\lambda_+|^{1-p/k}|\lambda_-|^{p/k} - r) \binom{k}{p} \\ &= \sum_{p=0}^{k} H(-p/k + 1/2 + \rho) \binom{k}{p}, \quad \rho = \frac{\log(|\lambda_-\lambda_+|^{\frac12}/r)}{\log(|\lambda_+|/|\lambda_-|)}\,. \end{aligned}$$

Using Stirling's formula, one easily shows that in the limit $k \to \infty$,

$$\frac{1}{2^k}\sum_{p=0}^{k} H(-p/k+1/2+\rho)\binom{k}{p} \sim \sqrt{\frac{2k}{\pi}}\int_{-\infty}^{\rho} e^{-2kx^2}\,dx \to H(\rho)\,.$$

This expression shows that the distribution of resonances is semiclassically dominated by the degrees $|p-k/2| = \mathcal{O}(k^{1/2})$, and proves the second part of the proposition. $\square$

The explicit eigenvalues are $\lambda_\pm = \frac{1+i\sqrt{3}}{4\sqrt{3}} \pm \sqrt{\frac{11-i3\sqrt{3}}{24}}$, with approximate values

$$\lambda_+ \approx 0.8390 + i0.0942, \quad |\lambda_+| \approx 0.8443, \quad \lambda_- \approx -0.5504 + i4058, \quad |\lambda_-| \approx 0.6838\,.$$

The geometric mean of their moduli is $|\lambda_-\lambda_+|^{1/2} = \sqrt{|\det\Omega_3|} = 3^{-1/4}$. On figure 1 we plot the circles with radii $|\lambda_+|$, $|\lambda_-|$ and $|\lambda_-\lambda_+|^{1/2}$, together with the spectrum of $\widetilde{B}_h$.

We need to analyze the spectrum of $\widetilde{B}_h$ more precisely to show that distribution of resonances is asymptotically uniform with respect to the angular variable.

Proposition 6.5. *Let $h = (2\pi 3^k)^{-1}$. As a set, the nontrivial spectrum of $\widetilde{B}_h$ is given by*

$$\{\lambda_+\} \cup \{\lambda_-\} \cup \bigcup_{\omega^k=1} \{\omega\lambda_+^{1-p/k}\lambda_-^{p/k} \;:\; 1 \leq p \leq k-1\}\,.$$

For each $p \neq 0, k$, the k eigenvalues asymptotically have the same degeneracy $\frac{1}{k}\binom{k}{p}$, which shows that their distribution is uniform in the angular variable. Therefore, for any continuous function $f \in C(D(0,1))$ we have (counting degeneracies in the LHS):

$$\frac{1}{2^k}\sum_{0\neq\lambda\in\mathrm{Spec}(\widetilde{B}_h)} f(\lambda) \xrightarrow{k\to\infty} \int_0^{2\pi} f(|\lambda_-\lambda_+|^{\frac{1}{2}},\theta)\,\frac{d\theta}{2\pi}\,.$$

Proof. To classify the nontrivial spectrum of $\widetilde{B}_h$, we will use the eigenvectors $v_\pm$ of $\widetilde{F}_3^*$ associated with the eigenvalues $\lambda_\pm$. Call

$$\{\,\boldsymbol{\eta} = \eta_1\eta_2\cdots\eta_k \;:\; \eta_\ell \in \{\pm\}\,\} \simeq (\mathbb{Z}_2)^k$$

the set of binary sequences of length k. The number of symbols $\eta_\ell = -$ in the sequence $\boldsymbol{\eta}$ is called the *degree* of $\boldsymbol{\eta}$. The *cyclic shift* τ acts on these sequences as $\tau(\eta_1\cdots\eta_k) = \eta_2\cdots\eta_k\eta_1$. The shift allows us to partition $(\mathbb{Z}_2)^k$ into periodic orbits, each orbit $\mathcal{O} = \{\,\boldsymbol{\eta}, \tau(\boldsymbol{\eta}), \ldots, \tau^{\ell_{\mathcal{O}}-1}(\boldsymbol{\eta})\,\}$ being of (primitive) period $\ell_{\mathcal{O}}$. Since $\tau^k = id$, the primitive period must divide k. We call $\deg(\mathcal{O})$ the common degree of the elements of $\mathcal{O}$ and observe that

$$k \mid \ell_{\mathcal{O}} \deg(\mathcal{O})\,. \tag{6.16}$$

To each sequence $\boldsymbol{\eta}$ we associate the state $|\boldsymbol{\eta}\rangle \stackrel{\text{def}}{=} v_{\eta_1}\otimes v_{\eta_2}\otimes\cdots\otimes v_{\eta_k}$, which is obviously an eigenstate of $(\widetilde{B}_h)^k$, with eigenvalue $\lambda_+^{k-\deg(\boldsymbol{\eta})}\lambda_-^{\deg(\boldsymbol{\eta})}$. These 2^k states form an independent family, which span the nontrivial eigenspaces of $\widetilde{B}_h$. This operator acts very simply on these states:

$$\forall \boldsymbol{\eta} \in (\mathbb{Z}_2)^k, \qquad \widetilde{B}_h|\boldsymbol{\eta}\rangle = \lambda_{\eta_1}\,|\tau(\boldsymbol{\eta})\rangle\,.$$

This equation shows that, for any orbit $\mathcal{O}$, $\widetilde{B}_h$ leaves invariant the $\ell_{\mathcal{O}}$-dimensional subspace $V_{\mathcal{O}} \stackrel{\text{def}}{=} \operatorname{span}\{\,|\boldsymbol{\eta}\rangle\,,\,\boldsymbol{\eta}\in\mathcal{O}\,\}$. To compute the spectrum of $\widetilde{B}_h\restriction_{V_{\mathcal{O}}}$ we first observe that it is contained in the set of k-th roots of $\lambda_+^{k-\deg(\mathcal{O})}\lambda_-^{\deg(\mathcal{O})}$, which in view of (6.16) is equal to

$$S_{\mathcal{O}} \stackrel{\text{def}}{=} \left\{\,\omega_{\ell_{\mathcal{O}}}^{j}\,\lambda_+^{1-\deg(\mathcal{O})/k}\lambda_-^{\deg(\mathcal{O})/k}\,,\ j=0,\dots,\ell_{\mathcal{O}}-1\,\right\}.$$

We claim that $\operatorname{Spec}(\widetilde{B}_h\restriction_{V_{\mathcal{O}}}) = S_{\mathcal{O}}$ (clearly with no degeneracies). In fact, let $\Omega_{\mathcal{O}}\ :\ V_{\mathcal{O}}\to V_{\mathcal{O}}$ be defined by $\Omega_{\mathcal{O}}|\tau^\ell(\boldsymbol{\eta})\rangle = \omega_{\ell_{\mathcal{O}}}^{-\ell}|\tau^\ell(\boldsymbol{\eta})\rangle$, for a choice of $\boldsymbol{\eta}\in\mathcal{O}$. This operator is invertible on $V_{\mathcal{O}}$. By a verification on basis elements,

$$\widetilde{B}_h\restriction_{V_{\mathcal{O}}}\circ\,\Omega_{\mathcal{O}} = \omega_{\ell_{\mathcal{O}}}\,\Omega_{\mathcal{O}}\circ\widetilde{B}_h\restriction_{V_{\mathcal{O}}}\,,$$

and hence if $\lambda\in\operatorname{Spec}(\widetilde{B}_h\restriction_{V_{\mathcal{O}}})$ then $\omega_{\ell_{\mathcal{O}}}^{j}\lambda\in\operatorname{Spec}(\widetilde{B}_h\restriction_{V_{\mathcal{O}}})$ for any j.

Since $\mathcal{O}\neq\mathcal{O}'\Longrightarrow V_{\mathcal{O}}\cap V_{\mathcal{O}'}=\{\,0\,\}$, enumerating the orbits decomposition of $(\mathbb{Z}_2)^k$ yields the full nontrivial spectrum of $\widetilde{B}_h$, with degeneracies.

The degree $p=0$ corresponds to the unique orbit $\mathcal{O}=\{\,\boldsymbol{\eta}=+++\cdots+++\,\}$, so the eigenvalue λ_+ is nondegenerate. Similarly, the degree $p=k$ leads to the simple eigenvalue λ_-.

For any degree $1\leq p\leq k-1$, call $g=\gcd(k,p)$. The sequences of degree p will take all possible periods $\ell_{\boldsymbol{\eta}}=k/\ell$, where $\ell\in\mathbb{N}$, $\ell|g$. We show below that, in the semiclassical limit, the huge majority of the sequences of any degree $p\neq 0,k$ have primitive period k.

Lemma 6.6. *There exists $C>0$, $K>0$ s.t., for any $k\geq K$ and any degree $1\leq p\leq k-1$,*

$$\frac{\#\{\,\boldsymbol{\eta}\in(\mathbb{Z}_2)^k\ :\ \deg(\boldsymbol{\eta})=p\,,\ell_{\boldsymbol{\eta}}<k\,\}}{\#\{\,\boldsymbol{\eta}\in(\mathbb{Z}_2)^k\ :\ \deg(\boldsymbol{\eta})=p\,\}}\leq C\,\frac{\log k}{k}\,.$$

Proof. We still use $g=\gcd(k,p)$.

If $g=1$, then all orbits of degree p are of primitive period k.

If $g>1$, there exists $\ell>1$, $\ell|g$. For any P prime divisor of ℓ, any sequence of primitive period $\ell_{\boldsymbol{\eta}}=k/\ell$ is also of (nonnecessarily primitive) period k/P. Any sequence of degree p and (nonnecessarily primitive) period k/P can be seen as the P repetitions of a sequence of k/P bits, among which p/P take the value $(-)$. Therefore, the number of such sequences is exactly $\binom{k/P}{p/P}$. As a consequence, we have

$$\frac{\#\{\,\boldsymbol{\eta}\in(\mathbb{Z}_2)^k\ :\ \deg(\boldsymbol{\eta})=p\,,\ell_{\boldsymbol{\eta}}<k\,\}}{\#\{\,\boldsymbol{\eta}\in(\mathbb{Z}_2)^k\ :\ \deg(\boldsymbol{\eta})=p\,\}}\leq\sum_{P\text{ prime},\ P|g}\frac{\binom{k/P}{p/P}}{\binom{k}{p}}\,. \tag{6.17}$$

We will now estimate each term of the right hand side. From the symmetry $\binom{k}{p} = \binom{k}{k-p}$, we can assume $p \le k/2$. Expanding the coefficient $\binom{k}{p}$ into

$$\binom{k}{p} = \frac{k(k-1)\cdots(k-p+1)}{p(p-1)\cdots 1},$$

we notice that both the numerator and the denominator contain exactly p/P factors which are multiples of P. Their ratio gives $\binom{k/P}{p/P}$, while the ratio of the remaining factors is

$$\begin{aligned}\frac{\binom{k}{p}}{\binom{k/P}{p/P}} &= \frac{(k-1)\cdots(k-P+1)(k-P-1)\cdots(k-p+1)}{(p-1)\cdots(p-P+1)(p-P-1)\cdots 1}\\ &\ge \frac{k-p+1}{1}\\ &\ge \frac{k}{2}+1\,.\end{aligned}$$

Here we used the fact that each factor $(k-m)/(p-m) > 1$, $0 \le m \le p-2$, and only kept explicit the last factor. The last inequality comes from $p \le k/2$.

We have obtained a uniform upper bound for each term on the right hand side of (6.17). By standard arguments, there exists K, $\widetilde{C} > 0$ s.t. the number of prime factors of any $k \ge K$ is $\le \widetilde{C} \log k$, so the number of terms in the sum is $\le \widetilde{C} \log k$. As a result, (6.17) is bounded from above by $\widetilde{C} \log k/(k+2)$, which proves the lemma. $\square$

This lemma shows that the orbits of period $\ell_{\mathcal{O}} < k$ have a negligible contribution to the asymptotic density of resonances. We can therefore act as if, for any $1 \le p \le k-1$, each orbit of degree p had period k, leading to the k eigenvalues $\left\{ \omega_k^j \, \lambda_+^{1-p/k} \lambda_-^{p/k} ,\ j = 0, \ldots, k-1 \right\}$. In the semiclassical limit, these k eigenvalues are uniformly distributed on the circle of radius $|\lambda_+^{1-p/k} \lambda_-^{p/k}|$, and each of them has multiplicity $\frac{1}{k}\binom{k}{p}$. This shows that the asymptotic resonance distribution is circular-symmetric, with the radial distribution described in Proposition 6.4. $\square$

Remark 6.1. Several features of the (nontrivial) spectrum of $\widetilde{B}_h$ are very different from what one expects for a random subunitary matrix of size $2^k \times 2^k$: the (logarithms of the) resonances form a regular lattice, most eigenvalues are highly degenerate, and the radial density is a delta function at $r_0(\tilde{B})$. Actually, the only generic feature seems to be the global fractal scaling of the Weyl law, and the uniform angular distribution.

Remark 6.2. The radial density of resonances is governed by $r_0(\tilde{B}_h) = \sqrt{|\det \Omega_3|}$, which seems to depend on the subtelties of the quantization. As an example of this fact, in §7 we will consider the open baker's map with $D = 4$, which we call B, obtained by keeping only the second and third strips. It has Lyapounov exponent $\log 4$, and the Cantor set C (see §3.3) has dimension $d = \log 2/\log 4 = 1/2$. The open map B' obtained by removing the first and third strips has the same characteristics. However, if we Walsh-quantize these two maps, the spectra of $\widetilde{B}_h$ and $\widetilde{B}'_h$ are very different. These spectra are obtained from the eigenvalues of different 2×2 blocks of the inverse Fourier matrix $\mathcal{F}_4^*$. The first map leads to the matrix

$$\Omega_4 = \frac{1}{2} \begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix} ,$$

with two nonzero eigenvalues $\lambda_\pm$ of different moduli, so the spectrum of $\widetilde{B}_h$ will satisfy the fractal Weyl law, and be concentrated around the circle of radius

$$r_0(\widetilde{B}) = \sqrt{|\det \Omega_4|} = 2^{-3/4} .$$

In an opposite way, the second map leads to the singular matrix,

$$\Omega'_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The nontrivial spectrum of $\widetilde{B}'_h$ then reduce to a simple eigenvalue $\lambda_+ = 1$. In that case, the Weyl law is singular, corresponding to a vanishing function $C(r)$. This qualitative difference between both spectra cannot be explained from purely classical data.

7. Conductance in the Walsh model

7.1. **Quantum transport.** In this section, we consider open baker's relations, for which the "opening" consists in two disjoint intervals, which are supposed to represent two "leads" connecting a quantum dot to the outside world. We will prove Theorem 2 in this setting: (1.1) in §7.2 and (1.2) in §7.3.

The baker's relations defined in Section 3.3 can all be seen as truncations of invertible maps on $\mathbb{T}^2$. More precisely there exists a closed baker's map, $\kappa : \mathbb{T}^2 \to \mathbb{T}^2$, such that the graph $B = B_1 \cup B_2$ of the open baker's map is

$$B = \Gamma_\kappa \cap \{ (q, p) \, : \, q \in I_1 \cup I_2 = I, \; p \in \mathbb{I} \} .$$

For admissible values of N, one can quantize the closed map κ into a unitary transformation $U_h = U_{\kappa,h}$ on $\mathcal{H}_h^1$ by straightforwardly generalizing the method of [2, 37]. Multiplying this unitary operator by the quantum projector $\Pi_I = \sum_{Q_j \in I} |Q_j\rangle\langle Q_j|$, we obtain the quantum open baker's map

$$B_h = U_{\kappa,h} \, \Pi_I .$$

To obtain an agreement with the notation of Section 2.4.3, we can interpret the set $I = I_1 \cup I_2$ as the "wall" of the quantum dot, while the complementary interval $L \stackrel{\text{def}}{=} \mathbb{I} \setminus I$ represents the "openings" of the dot, connected with "leads". In the previous sections, we studied the resonances, that is, the eigenvalues of $B_h = U_{\kappa,h}\Pi_I$ (or of $\tilde{B}_h$ when choosing the Walsh quantization). Now, we want to study the "transport" through the dot, using the formalism presented in Section 2.4.3. We assume that the opening L splits into two disjoint "leads" $L = L_1 \cup L_2$, and we study the transmission matrix from lead L_1 to lead

L_2. As presented in Section 2.4.3, this matrix is obtained by decomposing the scattering matrix

$$\tilde{S}(\vartheta) = \Pi_L \sum_{n\geq 0} \left(e^{i\vartheta} U_h \Pi_I)\right)^n e^{i\vartheta} U_h \, \Pi_L$$

into 4 blocks, according to the decomposition $\Pi_L = \Pi_{L_1} \oplus \Pi_{L_2}$. The transmission matrix is defined as

$$t(\vartheta) = \sum_{n\geq 1} e^{in\vartheta} \, \Pi_{L_2} \, U_h \, (\Pi_I U_h)^{n-1} \, \Pi_{L_1} \stackrel{\text{def}}{=} \sum_{n\geq 1} e^{in\vartheta} \, t_n \,. \tag{7.1}$$

According to Landauer's theory of coherent transport, each eigenvalue $T_i(\vartheta)$ of the matrix $t^*(\vartheta)t(\vartheta)$ corresponds to a "transmission channel". The dimensionless conductance of the system is then given by the sum over these transmission eigenvalues:

$$g(\vartheta) = \operatorname{tr}\left(t^*(\vartheta)t(\vartheta)\right).$$

A transmission channel is "classical" if the eigenvalue T_i is very close to unity (perfect transmission) or close to zero (perfect reflection). The intermediate values correspond to the "nonclassical channels", the importance of which is reflected in the noise power

$$P(\vartheta) = \operatorname{tr}\Big(t^*t(\vartheta)\big(Id - t^*t(\vartheta)\big)\Big), \quad \text{or the Fano factor } F(\vartheta) = \frac{P(\vartheta)}{g(\vartheta)}. \tag{7.2}$$

In general it may be necessary to perform the "ensemble averaging" [49] (averaging over ϑ) to obtain significant results. However, for the model we will study below, these quantities will depend very little on ϑ, so this averaging will not be necessary. To alleviate notations we will supress the dependence in ϑ in the transmission matrix t .

Our model. In the remainder of this section, we will compute the quantities characterizing the transport through the "dot" when U_h is a Walsh-quantized baker's map similar with the operator (6.9), but with $D = 4$ instead of $D = 2$. We will choose the two leads $L_1 = [0, 1/4]$ and $L_2 = [3/4, 1]$: this way, the projectors Π_{L_i} as well as the projector $\Pi_I = Id - \Pi_{L_1} - \Pi_{L_2}$ can be represented as a tensor product:

$$\begin{aligned}
\Pi_{L_1} &= \pi_0 \otimes Id_4 \otimes \cdots \otimes Id_4 \,, \\
\Pi_{L_2} &= \pi_3 \otimes Id_4 \otimes \cdots \otimes Id_4 \,, \\
\Pi_I &= \pi_I \otimes Id_4 \otimes \cdots \otimes Id_4 \,,
\end{aligned}$$

where $\pi_i = |e_i\rangle \langle e_i|$ is a rank-1 orthogonal projector acting on $\mathbb{C}^4$, and we note $\pi_I = \pi_1 \oplus \pi_2$.

The "inside" propagator for this model, namely $\widetilde{B}_h = U_h \Pi_I$, is the first among the two quantum maps constructed in Remark 6.2: its nontrivial spectrum satisfies the fractal Weyl law with exponent $d = \frac{1}{2}$, and is concentrated near the radius $r_0(\widetilde{B}) = 2^{-3/4}$.

7.2. **Conductance.** We will crucially use the fact that all operators under consideration act nicely on the tensor product structure $\mathcal{H}_h^1 = (\mathbb{C}^4)^{\otimes k}$, that is, they do not entangle the qu*D*its. It is then logical to compute the trace of t^*t in a basis adapted to this tensor product, and we naturally choose the computational (or position) basis. The conductance is then given by

$$\operatorname{tr}(t^*t) = \sum_{Q_j \in L_1} \langle Q_j | t^* t | Q_j \rangle = \sum_{j=0}^{4^{k-1}-1} \| t | Q_j \rangle \|^2 .$$

Let us consider an arbitrary $j = \epsilon_1 \epsilon_2 \cdots \epsilon_k$ with $\epsilon_1 = 0$, that is $0 \le j \le 4^{k-1} - 1$. Using (7.1) we write

$$t|Q_j\rangle = \sum_{n\ge 1} e^{in\vartheta}\, \Pi_{L_2}\, U_h\, (\Pi_I\, U_h)^{n-1} |Q_j\rangle = \sum_{n \ge 1} e^{in\vartheta}\, t_n |Q_j\rangle\,, \tag{7.3}$$

so that

$$\|t|Q_j\rangle\|^2 = \sum_{m,n\ge 0} e^{i(n-m)\vartheta}\, \langle Q_j | t_m^*\, t_n | Q_j \rangle\,.$$

From now on, we replace the notation $|Q_j\rangle$, $j \in [0, 4^{k-1} - 1]$, by the symbolic notation $|\boldsymbol{\epsilon}\rangle$, where the sequence $\boldsymbol{\epsilon} = 0\, \epsilon_2 \cdots \epsilon_k$ corresponds to j.

To understand the action of t_n on $|\boldsymbol{\epsilon}\rangle$ we notice that $\Pi_I\, U_h$ acts on tensor products as

$$\Pi_I\, U_h (v_1 \otimes \cdots \otimes v_k) = \pi_I v_2 \otimes \cdots \otimes v_k \otimes \mathcal{F}_4^* v_1\,.$$

7.2.1. *Classical transmission channels.* If $n < k$, we obtain

$$t_n|\boldsymbol{\epsilon}\rangle = \pi_3 e_{\epsilon_{n+1}} \otimes e_{\epsilon_{n+2}} \otimes \cdots e_k \otimes \mathcal{F}_4^* e_0 \otimes \mathcal{F}_4^* \pi_I e_{\epsilon_2} \otimes \cdots \otimes \mathcal{F}_4^* \pi_I e_{\epsilon_n}\,, \tag{7.4}$$

frow which we draw the

Lemma 7.1. *Consider a sequence $\boldsymbol{\epsilon} = 0\, \epsilon_2 \epsilon_3 \cdots \epsilon_k$, and assume that there exists an index $2 \le \ell \le k$ such that $\epsilon_\ell \in \{0, 3\}$. Let ℓ_0 be the smallest such index. Then*

$$\|t|\boldsymbol{\epsilon}\rangle\| = \begin{cases} 0 & \text{if } \epsilon_{\ell_0} = 0\,, \\ 1 & \text{if } \epsilon_{\ell_0} = 3\,. \end{cases}$$

Proof. For any $1 \le n \le \ell_0 - 2$, $\epsilon_{n+1} \in \{1, 2\}$ by assumption. Hence the first qu*D*it on the right hand side of (7.4) vanishes and $t_n|\boldsymbol{\epsilon}\rangle = 0$. Furthermore, the state $(\Pi_I U_h)^{\ell_0 - 1}|\boldsymbol{\epsilon}\rangle$ admits as first qu*D*it $\pi_I\, e_{\epsilon_{\ell_0}} = 0$, so that $t_n|\boldsymbol{\epsilon}\rangle = 0$ for any $n \ge \ell_0$. The only remaining term in (7.3) is $t_{\ell_0 - 1}|\boldsymbol{\epsilon}\rangle$:

- if $\epsilon_{\ell_0} = 0$, the first qu*D*it of that term is $\pi_3 e_{\epsilon_{\ell_0}} = 0$, so $t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle = t|\boldsymbol{\epsilon}\rangle = 0$.
- if $\epsilon_\ell = 3$, $t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle = e_{\epsilon_3} \otimes e_{\epsilon_{\ell_0+1}} \otimes \cdots \otimes \mathcal{F}_4^* e_0 \otimes \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \mathcal{F}_4^* e_{\epsilon_{\ell_0 - 1}}$. Since $\mathcal{F}_4^*$ is unitary, $\|t_{\ell_0-1}|\boldsymbol{\epsilon}\rangle\| = \|t|\boldsymbol{\epsilon}\rangle\| = 1$.

□

The total number of such "classical channels" discussed in Lemma 7.1 is easy to compute: it is obtained by removing from the set $[0, 4^{k-1} - 1] \equiv \{ \epsilon_2 \cdots \epsilon_k \in (\mathbb{Z}_4)^{k-1} \}$ the sequences $\boldsymbol{\epsilon}$ such that $\epsilon_\ell \in \{1, 2\}$ for all $2 \le \ell \le k$. The number of the latter is 2^{k-1}, so the number of "classical channels" is $4^{k-1} - 2^{k-1}$. Among them, half are fully reflected, $t|\epsilon\rangle = 0$, and half are fully transmitted, $\|t|\epsilon\rangle\| = 1$. Hence the conductance through these states is

$$\mathrm{tr}_{\mathrm{cl}}(t^* t) = \frac{4^{k-1} - 2^{k-1}}{2} .$$

Remark 7.1. Such classical channels are mentioned in the analysis of [49] for the transmission through an open kicked rotator. They sit in the phase space regions above the lead L_1 which are either sent back to L_1, or sent to L_2 through the classical dynamics, in a time smaller than the Ehrenfest time $T_{\mathrm{Ehr}} = \log N/(\log 4) = k$. For our baker's map B, these regions are vertical strips of widths $4^{-\ell}$, $\ell = 2, \ldots, k$ which exit to the lead L_1 or L_2 at time ℓ. The particularity of the Walsh quantization is the exact full transmission (or reflection) of these quantum states.

7.2.2. *Nonclassical transmission channels.* The nonclassical channels are necessarily combinations of the position states $|\boldsymbol{\epsilon}\rangle$ with $\epsilon_\ell \in \{1, 2\}$ for all $2 \le \ell \le k$. The associated positions $4Q_j = 0 \cdot \epsilon_2 \epsilon_3 \cdots \epsilon_k$ lie close to the Cantor set C, such that $\Gamma_- = C \times \mathbb{I}$ is the set of points never escaping through B or $\widetilde{B}$ (see Eq. (3.5)).

For such a state $|\boldsymbol{\epsilon}\rangle$, the term (7.4) vanishes for all $n < k$, due to the first qu*D*it $\pi_I e_{\epsilon_{n+1}} = 0$. That state therefore accomplishes k "unitary bounces" inside the cavity, before it starts to decay out of it.

We first consider the terms $t_{k+m}|\boldsymbol{\epsilon}\rangle$ for $0 \le m < k$:

$$\begin{aligned} t_k|\boldsymbol{\epsilon}\rangle &= (e_3/2) \otimes \mathcal{F}_4^* e_{\epsilon_2} \otimes \mathcal{F}_4^* e_{\epsilon_3} \cdots \ \mathcal{F}_4^* e_{\epsilon_k} , \\ t_{k+m}|\boldsymbol{\epsilon}\rangle &= \pi_3 \mathcal{F}_4^* e_{\epsilon_{m+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m+2}} \cdots \ \mathcal{F}_4^* e_{\epsilon_k} \otimes \mathcal{F}_4^* (e_3/2) \otimes \mathcal{F}_4^* \pi_I \mathcal{F}_4^* e_{\epsilon_2} \cdots \mathcal{F}_4^* \pi_I \mathcal{F}_4^* e_{\epsilon_m} . \end{aligned} \tag{7.5}$$

Now, in the basis $\{e_i\}$ (we only show nonzero entries),

$$\pi_3 \mathcal{F}_4^* = \frac{1}{2} \begin{pmatrix} & & & \\ & & & \\ & & & \\ 1 & -i & -1 & i \end{pmatrix} , \qquad \pi_I \mathcal{F}_4^* = \frac{1}{2} \begin{pmatrix} & & & \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ & & & \end{pmatrix} , \tag{7.6}$$

which shows that

$$\|\pi_3 \mathcal{F}_4^* e_j\|^2 = \frac{1}{4} , \qquad \|\pi_I \mathcal{F}_4^* e_j\|^2 = \frac{1}{2} , \quad j = 0, \cdots, 3 .$$

Thus,

$$\|t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2 = \frac{1}{4 \times 2^m} , \quad 0 \le m \le k-1 . \tag{7.7}$$

Furthermore, for larger times $n = pk + m$, $p > 1$, $m \in [0, k-1]$, the state $t_n|\boldsymbol{\epsilon}\rangle$ is obtained from (7.5) by inserting the operator $(\pi_I \mathcal{F}_4^*)^{p-1}$ in front of each qu*D*it e_{ϵ_ℓ}. Since $\pi_I \mathcal{F}_4^*$ has a spectral radius $|\lambda_+| < 1$, the norms of these states will decay exponentially, with an asymptotic rate $|\lambda_+|$ as $n \to \infty$. This argument gives the following

Lemma 7.2. *For any $0 < \Theta < 1$, there exists $C > 0$ such that, for any $k \geq 1$ and any nonclassical state $|\boldsymbol{\epsilon}\rangle$, we have*

$$\sum_{m > \lfloor \Theta k \rfloor} \|t_{k+m}|\boldsymbol{\epsilon}\rangle\| \leq C\, 2^{-\Theta k/2}\,.$$

Up to exponentially small errors we now need to compute

$$\|\sum_{m=0}^{\lfloor \Theta k \rfloor} t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2\,.$$

The norms (7.7) give the diagonal terms in the expansion of the expression above:

$$\sum_{m=0}^{\lfloor \Theta k \rfloor} \|t_{k+m}|\boldsymbol{\epsilon}\rangle\|^2 = \frac{1}{2} + \mathcal{O}(2^{-\Theta k})\,. \tag{7.8}$$

In the next lemma we will show that the contribution to the conductance of the nondiagonal terms is negligible in the semiclassical limit.

Lemma 7.3. *Let $0 < \Theta \leq 1/5$. There exists $C = C(\Theta) > 0$ such that for any $k \geq 1$,*

$$\frac{\#\left\{\,\text{nonclassical } \boldsymbol{\epsilon},\ \exists m, m' \in [0, \Theta k],\quad \langle \boldsymbol{\epsilon}| t^*_{k+m} t_{k+m'} |\boldsymbol{\epsilon}\rangle \neq 0\,\right\}}{\#\left\{\,\text{nonclassical } \boldsymbol{\epsilon}\,\right\}} \leq C\, 2^{-k/2}\,.$$

In other words, in the semiclassical limit, a "generic" nonclassical state $|\boldsymbol{\epsilon}\rangle$ will satisfy

$$\forall m, m' \in [0, \Theta k], \quad m \neq m' \Longrightarrow \langle \boldsymbol{\epsilon}| t^*_{k+m} t_{k+m'} |\boldsymbol{\epsilon}\rangle = 0\,.$$

Proof. Take an arbitrary nonclassical state $|\boldsymbol{\epsilon}\rangle$, and any $m, m' \in [0, \Theta k]$, $m > m'$. From (7.5), the first $(k - m)$ qu$D$its of the states $t_{k+m}|\boldsymbol{\epsilon}\rangle$ and $t_{k+m'}|\boldsymbol{\epsilon}\rangle)$ are

$$\pi_3 \mathcal{F}_4^* e_{\epsilon_{m+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m+2}} \otimes \cdots \otimes \mathcal{F}_4^* e_{\epsilon_k}\,,$$
$$\pi_3 \mathcal{F}_4^* e_{\epsilon_{m'+1}} \otimes \mathcal{F}_4^* e_{\epsilon_{m'+2}} \otimes \cdots \otimes \mathcal{F}_4^* e_{\epsilon_{k+m'-m}}\,,$$

respectively. Due to the unitarity of $\mathcal{F}_4^*$ and the fact that the e_i form an orthonormal basis of $\mathbb{C}^4$, the two states $t_{k+m}|\boldsymbol{\epsilon}\rangle$, $t_{k+m'}|\boldsymbol{\epsilon}\rangle$ will be orthogonal if the sequences $\epsilon_{m+2}\cdots\epsilon_k$ and $\epsilon_{m'+2}\cdots\epsilon_{k+m'-m}$ are not equal. Since we took $m < \Theta k$, these two sequences are subsequences of length $(k - m - 1) \geq (1 - \Theta)k$ of the sequence $\boldsymbol{\epsilon}$, shifted from one another by $(m - m')$ steps.

If the two subsequences are equal, then all subsequences

$$\epsilon_{k-(p+1)\Delta+1} \cdots \epsilon_{k-p\Delta}\,, \quad p = 0, \cdots, R\,,$$
$$\Delta \stackrel{\text{def}}{=} (m - m')\,, \quad R \stackrel{\text{def}}{=} \Big[\frac{k - m - 1}{\Delta}\Big]\,,$$

have to be equal to them as well. Hence $\boldsymbol{\epsilon}$ contains a subsequence of length $(R+1)\Delta$ which is periodic with period Δ.

Let us count the number of such sequences $\boldsymbol{\epsilon}$. Once we have fixed the Δ bits $\epsilon_{k-\Delta+1}\cdots\epsilon_k$, the remaining free bits are $\epsilon_2\cdots\epsilon_{k-(R+1)\Delta}$. The number $\#(m,m')$ of such sequences is therefore $2^{\Delta}\times 2^{k-(R+1)\Delta-1}$. From the definition of R, we get the inequality $m\leq \Delta+k-(R+1)\Delta-1<m+\Delta$, so $\#(m,m')$ is bounded from above by

$$\#(m,m')<2^{2m-m'}\leq 2^{2\Theta k}\,.$$

Taking into account all possible pairs (m,m'), we get

$$\#\left\{\text{nonclassical }\boldsymbol{\epsilon},\ \exists\, m,m',\ 0\leq m'<m\leq\Theta k,\ \langle\boldsymbol{\epsilon}|t^*_{k+m}t_{k+m'}|\boldsymbol{\epsilon}\rangle\neq 0\right\}\leq(\Theta k)^2\,2^{2\Theta k}\,.$$

Since the total number of nonclassical channels is 2^{k-1} and $2\Theta\leq 2/5$, we have proven the lemma. □

Using Lemma 7.2 and Eq. (7.8), a generic nonclassical sequence $\boldsymbol{\epsilon}$ will satisfy

$$\|t|\boldsymbol{\epsilon}\rangle\|^2=\frac{1}{2}+\mathcal{O}(2^{-\Theta k/2})\,.$$

For a nongeneric nonclassical sequence $\boldsymbol{\epsilon}$, we use the simple bound $\|t|\boldsymbol{\epsilon}\rangle\|^2\leq 1$. As a result, we get the following estimate for the sum over nonclassical channels:

$$\operatorname{tr}_{\mathrm{noncl}}(t^*t)=\sum_{\substack{\text{nonclassical}\\ \text{generic}}}\|t|\boldsymbol{\epsilon}\rangle\|^2+\sum_{\substack{\text{nonclassical}\\ \text{nongeneric}}}\|t|\boldsymbol{\epsilon}\rangle\|^2=\frac{2^{k-1}}{2}\big(1+\mathcal{O}(2^{-\Theta k/2})\big)\,. \tag{7.9}$$

Adding this to the classical conductance, we get the full conductance

$$g(\vartheta)=\operatorname{tr}(t^*t(\vartheta))=\frac{4^{k-1}}{2}+\mathcal{O}(2^{(1-\Theta/2)k})\,, \tag{7.10}$$

the implied constant being independent of $\vartheta\in[0,2\pi)$ and $0<\Theta\leq 1/5$. The number of scattering channels in our model is given by $M(h)=4^{k-1}$, $2\pi h=1/4^k$. Hence we proved (1.1) in Theorem 2.

7.3. **Noise power.** The conductance corresponds to the first moment of the distribution of transmission eigenvalues. It can not distinguish between a purely classical transport ($T_i\in\{\,0,1\,\}$) and a quantum one (some T_i take intermediate values). To do so, we need to compute the second moment of these eigenvalues, that is, the trace

$$\operatorname{tr}((t^*t)^2)=\sum_{Q_j\in L_1}\|t^*t|Q_j\rangle\|^2\,,$$

or equivalently the noise power (7.2). As in the previous section, we split the sum on the right hand side between the classical and nonclassical states $|Q_j\rangle=|\boldsymbol{\epsilon}\rangle$.

Lemma 7.1 shows that half the classical states are in the kernel of t^*t, half in the eigenspace of t^*t associated with the eigenvalue 1 (full transmission). As a consequence, the trace over the classical states takes the value

$$\mathrm{tr}_{\mathrm{cl}}((t^*t)^2) = \frac{4^{k-1} - 2^{k-1}}{2}\,.$$

Obviously, the classical states are noiseless.

The analysis of the nonclassical states is more delicate. According to last section, for any such state $|\boldsymbol{\epsilon}\rangle$ we have (for any $0 < \Theta < 1$)

$$t|\boldsymbol{\epsilon}\rangle = \sum_{m=0}^{\lfloor \Theta k \rfloor} e^{in\vartheta}\, t_{k+m}|\boldsymbol{\epsilon}\rangle + \mathcal{O}(2^{-\Theta k/2})\,.$$

We now apply to each state $t_{k+m}|\boldsymbol{\epsilon}\rangle$, $m \leq \Theta k$ the adjoint operator

$$t^* = \sum_{n\geq 0} e^{-in\vartheta}\, t_n^*\,. \tag{7.11}$$

According to (7.5), the state $t_{k+m}|\boldsymbol{\epsilon}\rangle$ has the form

$$t_{k+m}|\boldsymbol{\epsilon}\rangle = e_3 \otimes \mathcal{F}_4^*\pi_I w_2 \otimes \cdots \otimes \mathcal{F}_4^*\pi_I w_k\,,$$

for some explicit set of qu$D$its, $w_\ell \in \mathbb{C}^4$, $2 \leq \ell \leq k$ (we use the fact that $e_{\epsilon_\ell} = \pi_I e_{\epsilon_\ell}$ for all $\ell \geq 2$). From the expression

$$\Pi_I U_h^*(v_1 \otimes \cdots \otimes v_k) = \pi_I \mathcal{F}_4 v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}\,,$$

we can write the action of $t_n^* = \Pi_{L_1} U_h^* \,(\Pi_I U_h^*)^{n-1}\, \Pi_{L_2}$ on $t_{k+m}|\boldsymbol{\epsilon}\rangle$. If $n < k$, then

$$\begin{aligned} t_n^*\, t_{k+m}|\boldsymbol{\epsilon}\rangle &= \pi_0\pi_I w_{k-n+1} \otimes \pi_I w_{k-n+2} \otimes \cdots \otimes \pi_I w_k \otimes e_3 \otimes \mathcal{F}_4^*\pi_I w_2 \otimes \cdots \otimes \mathcal{F}_4^*\pi_I w_{k-n} \\ &= 0\,. \end{aligned}$$

The first non-trivial case of $n = k$ is given by

$$t_k^* t_{k+m}|\boldsymbol{\epsilon}\rangle = \pi_0 \mathcal{F}_4 e_3 \otimes \pi_I w_2 \otimes \cdots \pi_I w_k\,.$$

More generally, for any $0 \leq m' \leq k-1$, $t_{k+m'}^* t_{k+m}|\boldsymbol{\epsilon}\rangle =$

$$\pi_0\mathcal{F}_4\pi_I w_{k-m'+1} \otimes \pi_I\mathcal{F}_4\pi_I w_{k-m'+2} \otimes \cdots \pi_I\mathcal{F}_4\pi_I w_k \otimes \pi_I\mathcal{F}_4 e_3 \otimes \pi_I w_2 \otimes \cdots \pi_I w_{k-m'}\,.$$

As for the case of the operator t, we see that by increasing m' we increase the number of qu$D$its on which we apply the operator $\pi_I\mathcal{F}_4$. Therefore, for any index m, the norm of $t_{k+m'}^* t_{k+m}|\boldsymbol{\epsilon}\rangle$ will decrease exponentially fast while increasing m'. As in Lemma 7.2, we truncate the expansion (7.11) to the range $m' \leq \Theta k$, and this results in an exponentially small remainder.

We now need to replace the qu$D$its w_ℓ by their explicit values, which depend on the index m. We introduce the following operators on $\mathbb{C}^4$:

$$\mathcal{P}_{\alpha\beta} \stackrel{\text{def}}{=} \pi_\alpha \mathcal{F}_4 \pi_\beta \mathcal{F}_4^*\,, \quad \alpha, \beta \in \{0, I, 3\}\,,$$

and calculate the resulting matrices in the canonical basis:

$$
\mathcal{P}_{03} = \frac{1}{4}\begin{pmatrix} 1 & -i & -1 & i \\ & & & \\ & & & \\ & & & \end{pmatrix}, \qquad \mathcal{P}_{0I} = \frac{1}{4}\begin{pmatrix} 2 & i-1 & 0 & -i-1 \\ & & & \\ & & & \\ & & & \end{pmatrix},
$$
$$
\mathcal{P}_{II} = \frac{1}{4}\begin{pmatrix} & & & \\ -i-1 & 2 & i-1 & 0 \\ 0 & -i-1 & 2 & i-1 \\ & & & \end{pmatrix}, \qquad \mathcal{P}_{I3} = \frac{1}{4}\begin{pmatrix} & & & \\ i & 1 & -i & -1 \\ -1 & i & 1 & -i \\ & & & \end{pmatrix}. \tag{7.12}
$$

The form of

$$
|\psi_{m',m}(\boldsymbol{\epsilon})\rangle \stackrel{\text{def}}{=} t^*_{k+m'} t_{k+m} |\boldsymbol{\epsilon}\rangle
$$

depends on the sign of $m - m'$, and on whether $m, m' = 0$ or not (we will generally not indicate the dependence in $\boldsymbol{\epsilon}$):

- for $m' = m$, we have
$$
\begin{aligned}
|\psi_{0,0}\rangle &= \mathcal{P}_{03} e_0 \otimes e_{\epsilon_2} \otimes \cdots e_{\epsilon_k}\,, \\
|\psi_{m,m}\rangle &= \mathcal{P}_{0I} e_0 \otimes \mathcal{P}_{II} e_{\epsilon_2} \otimes \cdots \mathcal{P}_{II} e_{\epsilon_m} \otimes \mathcal{P}_{I3} e_{\epsilon_{m+1}} \otimes e_{\epsilon_{m+2}} \otimes \cdots e_{\epsilon_k}\,.
\end{aligned}
$$
- for $m = m' + \Delta$, $\Delta > 0$,
$$
\begin{aligned}
|\psi_{0,\Delta}\rangle &= \mathcal{P}_{03} e_{\epsilon_{\Delta+1}} \otimes e_{\epsilon_{\Delta+2}} \otimes \cdots\cdots e_{\epsilon_k} \otimes \pi_I \mathcal{F}_4^* e_0 \otimes \pi_I \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \pi_I \mathcal{F}_4^* e_{\epsilon_\Delta} \\
|\psi_{m',m'+\Delta}\rangle &= \mathcal{P}_{0I} e_{\epsilon_{\Delta+1}} \otimes \mathcal{P}_{II} e_{\epsilon_{\Delta+2}} \otimes \cdots \mathcal{P}_{II} e_{\epsilon_{\Delta+m'}} \otimes \\
&\qquad \otimes \mathcal{P}_{I3} e_{\epsilon_{\Delta+m'+1}} \otimes e_{\epsilon_{\Delta+m'+2}} \otimes \cdots e_{\epsilon_k} \otimes \pi_I \mathcal{F}_4^* e_0 \otimes \pi_I \mathcal{F}_4^* e_{\epsilon_2} \otimes \cdots \pi_I \mathcal{F}_4^* e_{\epsilon_\Delta}
\end{aligned}
$$
- for $m = m' + \Delta$, $\Delta < 0$,
$$
\begin{aligned}
|\psi_{|\Delta|,0}\rangle &= \pi_0 \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+1}} \otimes \pi_I \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+2}} \otimes \cdots \pi_I \mathcal{F}_4 e_{\epsilon_k} \otimes \mathcal{P}_{I3} e_0 \otimes e_{\epsilon_2} \otimes \cdots e_{\epsilon_{k-|\Delta|}}\,, \\
|\psi_{m+|\Delta|,m}\rangle &= \pi_0 \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+1}} \otimes \pi_I \mathcal{F}_4 e_{\epsilon_{k-|\Delta|+2}} \otimes \cdots \pi_I \mathcal{F}_4 e_{\epsilon_k} \otimes \mathcal{P}_{II} e_0 \otimes \\
&\qquad \otimes \mathcal{P}_{II} e_{\epsilon_2} \otimes \cdots \otimes \mathcal{P}_{II} e_{\epsilon_m} \otimes \mathcal{P}_{I3} e_{\epsilon_{m+1}} \otimes e_{\epsilon_{m+2}} \otimes \cdots e_{\epsilon_{k-|\Delta|}}\,.
\end{aligned}
$$

We notice that each of these states contain subfactors $e_{\epsilon_{m+2}} \otimes \cdots \otimes e_{\epsilon_k}$ if $m \geq m'$, and $e_{\epsilon_{m+2}} \otimes \cdots \otimes e_{\epsilon_{k+m-m'}}$ if $m < m'$. Compared to its position in the tensor product expansion of $|\boldsymbol{\epsilon}\rangle$, this subfactor is shifted by $m' - m = -\Delta$ steps.

We now need

Lemma 7.4. *Let $0 < \Theta < 1/6$ and for any pair of indices m, m, denote $\Delta_j = m - m'$. There exists $C = C(\theta) > 0$ such that*

$$
\frac{\#\left\{\text{ nonclass. } \boldsymbol{\epsilon} \,:\, \exists\, m_1, m_1', m_2, m_2' \in [0, \Theta k],\ \Delta_1 \neq \Delta_2,\ \langle \psi_{m_1',m_1} | \psi_{m_2',m_2} \rangle \neq 0 \right\}}{\#\left\{\text{ nonclass. } \boldsymbol{\epsilon}\right\}} \leq C\, 2^{-k/C}\,.
$$

In other words, for a generic nonclassical sequence $\boldsymbol{\epsilon}$, any two states

$$|\psi_{m_1',m_1}(\boldsymbol{\epsilon})\rangle\,,\quad |\psi_{m_2',m_2}(\boldsymbol{\epsilon})\rangle\,,$$

with m_i, $m_i' \leq \Theta k$, $\Theta < 1/6$, will be orthogonal to each other if $\Delta_1 \neq \Delta_2$.

Proof. Denote $M = [\Theta k] + 2$. For any pair $m, m' \leq \Theta k$, the state $|\psi_{m,m'}\rangle$ contains the subfactor $\boldsymbol{\epsilon}' = e_{\epsilon_M} \otimes \cdots \otimes e_{\epsilon_{k-M}}$, shifted by $m - m'$ from its position in $\boldsymbol{\epsilon}$. The number of choices for the subfactors $\boldsymbol{\epsilon}'$ is 2^{k-2M+1}.

Calling $\delta = \Delta_1 - \Delta_2$, the condition $\langle\psi_{m_1',m_1}|\psi_{m_2',m_2}\rangle \neq 0$ implies that $\boldsymbol{\epsilon}'$ satisfies

$$e_{\epsilon_M} \otimes \cdots \otimes e_{\epsilon_{k-M-\delta}} = e_{\epsilon_{M+\delta}} \otimes \cdots \otimes e_{\epsilon_{k-M}}\,.$$

The combinatorial problem we face now is essentially the same as in the proof of Lemma 7.3. The sequence $\boldsymbol{\epsilon}'$ must contain R repetitions of the subsequence

$$e_{\epsilon_M} \otimes \cdots \otimes e_{\epsilon_{M+\delta-1}}\,,\qquad R = \left[\frac{k-2M+1}{\delta}\right]\,.$$

The number of such sequences $\boldsymbol{\epsilon}'$ is bounded from above by $2^\delta \times 2^\delta$ (2^δ for the periodic component and 2^δ for the "tail" of length $k - 2M + 1 - R\delta < \delta$). Since $\delta \leq 2M$,

$$\frac{\#\left\{\boldsymbol{\epsilon}'\,:\, e_{\epsilon_M} \otimes \cdots \otimes e_{\epsilon_{k-M-\delta}} = e_{\epsilon_{M+\delta}} \otimes \cdots \otimes e_{\epsilon_{k-M}}\right\}}{\#\left\{\boldsymbol{\epsilon}'\right\}} \leq 2^{6M-k} = \mathcal{O}(2^{-k(1-6\Theta)})\,.$$

Summing over all possible shifts $\delta \in [0, 2M]$, produces a factor k in the upper bound. Taking any $C < 1 - 6\Theta$ proves the lemma. $\square$

From now on we will assume that $\boldsymbol{\epsilon}$ is a generic sequence in the sense of the above lemma. Namely, if we group the states $|\psi_{m+\Delta,m}(\boldsymbol{\epsilon})\rangle$ into

$$|\Psi_\Delta(\boldsymbol{\epsilon})\rangle \stackrel{\text{def}}{=} \sum_{\substack{0\leq m,m'\leq\Theta k\\ m=m'+\Delta}} |\psi_{m+\Delta,m}(\boldsymbol{\epsilon})\rangle\,,$$

then genericity means that $\langle\Psi_\Delta(\boldsymbol{\epsilon})|\Psi_\Delta(\boldsymbol{\epsilon})\rangle = 0$ for any $\Delta \neq \Delta'$. As a consequence, the square-norm of $t^*t|\boldsymbol{\epsilon}\rangle$ is then given by

$$\|t^*t|\boldsymbol{\epsilon}\rangle\|^2 = \sum_{|\Delta|\leq\Theta k} \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 + \mathcal{O}(2^{-\Theta k/2})\,. \tag{7.13}$$

The remainder comes from the truncation in m, m', and will be proven in the Remark 7.2, after we have computed the norms of the states $|\psi_{m',m}\rangle$. As we will see, no further simplification occurs in this expression, meaning that two different states $|\psi_{m',m}\rangle$ with the same Δ will generally interfere with each other.

Our objective is now to compute each square norm on the right hand side of (7.13). We will naturally use the fact that all states $|\psi_{m',m}\rangle$ are tensor products, so that the overlap between two of them is the product of the overlaps of their tensor factors. Beside the matrices given in (7.6,7.12) the following ones appear in the expressions for the $|\psi_{m'm}\rangle$:

$$\pi_0\mathcal{F}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \\ & & & \end{pmatrix}, \qquad \pi_I\mathcal{F}_4 = \frac{1}{2}\begin{pmatrix} 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \end{pmatrix}. \tag{7.14}$$

We split the lengthy, yet straightforward computation according to the value of Δ.

7.3.1. *Norm of* Ψ_0. We have

$$\|\Psi_0\|^2 = \sum_{m\le\Theta k} \|\psi_{m,m}\|^2 + 2\sum_{0\le m<n\le\Theta k} \mathrm{Re}\langle\psi_{m,m}|\psi_{n,n}\rangle\,. \tag{7.15}$$

The successive diagonal terms take the values

$$\|\psi_{0,0}\|^2 = \|\mathcal{P}_{03}e_0\|^2 = \frac{1}{16}, \quad \text{while for } m\ge 1\,,$$
$$\|\psi_{m,m}\|^2 = \|\mathcal{P}_{0I}e_0\|^2\Big(\prod_{\ell=2}^{m}\|\mathcal{P}_{II}e_{\epsilon_\ell}\|^2\Big)\,\|\mathcal{P}_{I3}e_{\epsilon_{m+1}}\|^2 = \frac{1}{4}\,\Big(\frac{3}{8}\Big)^{m-1}\frac{1}{8}\,.$$

The sum over the diagonal terms is therefore

$$\sum_{m\le\Theta k}\|\psi_{m,m}\|^2 = \frac{1}{16}+\frac{1}{32}\,\frac{1-\big(\frac{3}{8}\big)^{[\Theta k]}}{1-\frac{3}{8}} = \frac{9}{80}+\mathcal{O}\big((3/8)^{\Theta k}\big)\,.$$

The nondiagonal terms are, for any $0<n\le\Theta k$,

$$\langle\psi_{0,0}|\psi_{n,n}\rangle = \langle\mathcal{P}_{03}e_0,\mathcal{P}_{0I}e_0\rangle\,\Big(\prod_{\ell=2}^{n}\langle e_{\epsilon_\ell},\mathcal{P}_{II}e_{\epsilon_\ell}\rangle\Big)\langle e_{\epsilon_{n+1}},\mathcal{P}_{I3}e_{\epsilon_{n+1}}\rangle = \frac{1}{8}\,\Big(\frac{1}{2}\Big)^{n-1}\frac{1}{4}\,,$$

and for $0<m<n\le\Theta k$,

$$\begin{aligned}
&\langle\psi_{m,m}|\psi_{n,n}\rangle = \\
&\quad\|\mathcal{P}_{0I}e_0\|^2\,\Big(\prod_{\ell=2}^{m}\|\mathcal{P}_{II}e_{\epsilon_\ell}\|^2\Big)\,\langle\mathcal{P}_{I3}e_{\epsilon_{m+1}},\mathcal{P}_{II}e_{\epsilon_{m+1}}\rangle\,\Big(\prod_{\ell=m+2}^{n}\langle e_{\epsilon_\ell},\mathcal{P}_{II}e_{\epsilon_\ell}\rangle\Big)\,\langle e_{\epsilon_{n+1}},\mathcal{P}_{I3}e_{\epsilon_{n+1}}\rangle \\
&\quad= \frac{1}{4}\,\Big(\frac{3}{8}\Big)^{m-1}\frac{1\pm i}{16}\,\Big(\frac{1}{2}\Big)^{n-m-1}\frac{1}{4}\,.
\end{aligned}$$

The sign is $+$ for $\epsilon_{m+1}=1$, and $-$ for $\epsilon_{m+1}=2$. Adding up these contributions, we find for the second part of (7.15):

$$2\sum_{0\le m<n\le\Theta k}\mathrm{Re}\langle\psi_{m,m}|\psi_{n,n}\rangle = \frac{3}{20}+\mathcal{O}(2^{-\Theta k})\,.$$

We observe that this contribution is of the same order as the diagonal one. Summing the diagonal and nondiagonal parts yields

$$\|\Psi_0\|^2 = \frac{21}{80} + \mathcal{O}(2^{-\Theta k})\,. \tag{7.16}$$

7.3.2. *Norm of* Ψ_Δ *with* $\Delta > 0$. We notice that all states $|\psi_{m,m+\Delta}\rangle$, $0 \leq m \leq \Theta k - \Delta$ share the same Δ last qu$D$its, which result in a common factor

$$\prod_{\ell=1}^{\Delta} \|\pi_I \mathcal{F}_4^* e_{\epsilon_\ell}\|^2 = \frac{1}{2^\Delta}$$

in $\|\Psi_\Delta\|^2$. To avoid taking this factor into account at all steps, we call $|\psi'_{m,m+\Delta}\rangle$ the state obtained by removing these last Δ qu$D$its from $|\psi_{m,m+\Delta}\rangle$, compute the norms and overlaps of those truncated states, and finally multiply everything by the above factor.

We then compute the square-norm

$$\|\psi'_{0,\Delta}\|^2 = \|\mathcal{P}_{03} e_{\epsilon_{\Delta+1}}\|^2 = \frac{1}{16}\,.$$

For all $m > 0$, the first qu$D$it of $|\psi'_{m,m+\Delta}\rangle$ is $\mathcal{P}_{0I} e_{\epsilon_{\Delta+1}}$. By inspecting the expression (7.12), we see that this qu$D$it vanishes if $\epsilon_{\Delta+1} = 2$, so that

$$\|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{1}{16}\,\frac{1}{2^\Delta}\,, \qquad \text{if } \ 1 \leq \Delta \leq \Theta k\,,\ \epsilon_{\Delta+1} = 2\,. \tag{7.17}$$

In the opposite case $\epsilon_{\Delta+1} = 1$, the states $|\psi'_{m,m+\Delta}\rangle$ are nontrivial: for $0 < m \leq \Theta k - \Delta$,

$$\|\psi'_{m,m+\Delta}\|^2 = \|\mathcal{P}_{0I} e_1\|^2 \big(\prod_{\ell=2}^{m} \|\mathcal{P}_{II} e_{\epsilon_{\Delta+\ell}}\|^2\big)\, \|\mathcal{P}_{I3} e_{\epsilon_{\Delta+m+1}}\|^2 = \frac{1}{8}\,(\frac{3}{8})^{m-1}\,\frac{1}{8}\,.$$

Summing the diagonal contributions to $\|\Psi_\Delta\|^2$, gives

$$\sum_{m=0}^{[\Theta k]-\Delta} \|\psi'_{m,m+\Delta}\|^2 = \frac{7}{80} + \mathcal{O}\big((3/8)^{\Theta k - \Delta}\big)\,. \tag{7.18}$$

For the nondiagonal contributions we compute

$$\begin{aligned}\langle\psi'_{0,\Delta}|\psi'_{m,m+\Delta}\rangle &= \langle\mathcal{P}_{03} e_1, \mathcal{P}_{0I} e_1\rangle\, \big(\prod_{\ell=2}^{m} \langle e_{\epsilon_{\Delta+\ell}}, \mathcal{P}_{II} e_{\epsilon_{\Delta+\ell}}\rangle\big) \langle e_{\epsilon_{\Delta+m+1}}, \mathcal{P}_{I3} e_{\epsilon_{\Delta+m+1}}\rangle \\ &= \frac{-1-i}{16}\,\frac{1}{2^{m-1}}\,\frac{1}{4}\,,\end{aligned}$$

and

$$\begin{aligned}\langle\psi'_{m,m+\Delta}|\psi'_{n,n+\Delta}\rangle &= \|\mathcal{P}_{0I}e_1\|^2\,\Big(\prod_{\ell=2}^{m}\|\mathcal{P}_{II}e_{\epsilon_{\Delta+\ell}}\|^2\Big)\,\langle\mathcal{P}_{I3}e_{\epsilon_{\Delta+m+1}},\mathcal{P}_{II}e_{\epsilon_{\Delta+m+1}}\rangle\times\\ &\qquad\times\Big(\prod_{\ell=2}^{n-m}\langle e_{\epsilon_{\Delta+m+\ell}},\mathcal{P}_{II}e_{\epsilon_{\Delta+m+\ell}}\rangle\Big)\,\langle e_{\epsilon_{\Delta+n+1}},\mathcal{P}_{I3}e_{\epsilon_{\Delta+n+1}}\rangle\\ &= \frac{1}{8}\,(\frac{3}{8})^{m-1}\,\frac{1\pm i}{16}\,\frac{1}{2^{n-m-1}}\,\frac{1}{4}\,.\end{aligned}$$

(the sign $\pm$ in the last line depends on $\epsilon_{\Delta+m+1}$). This gives

$$2\sum_{0\leq m<n\leq\Theta k-\Delta}\mathrm{Re}\langle\psi'_{m,m+\Delta}|\psi'_{n,n+\Delta}\rangle = -\frac{1}{20}+\mathcal{O}(2^{-\Theta k+\Delta})\,.$$

Adding this to the diagonal terms (7.18), restoring the factor $2^{-\Delta}$, and using (7.17) results in the following norm:

$$\|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{1}{2^\Delta}\Big(\frac{3}{80}\,\delta_{\epsilon_{\Delta+1}=1}+\frac{1}{16}\,\delta_{\epsilon_{\Delta+1}=2}\Big)+\mathcal{O}(2^{-\Theta k})\quad\text{if }\ 1\leq\Delta\leq\Theta k\,,\ \epsilon_{\Delta+1}=1\,. \tag{7.19}$$

Hence the norm of $\Psi_\Delta(\boldsymbol{\epsilon})$ explicitly depends on the sequence $\boldsymbol{\epsilon}$.

7.3.3. *Norm of Ψ_Δ with $\Delta<0$.* As in §7.3.2 we notice that all components of Ψ_Δ share the same $|\Delta|$ first qu*D*its, which lead to the following factor:

$$\|\pi_0\mathcal{F}_4e_{\epsilon_{k-|\Delta|+1}}\|^2\Big(\prod_{\ell=2}^{|\Delta|}\|\pi_I\mathcal{F}_4e_{\epsilon_{k-|\Delta|+\ell}}\|^2\Big) = \frac{1}{4}\,\frac{1}{2^{|\Delta|-1}}\,. \tag{7.20}$$

We call $\psi'_{m+|\Delta|,m}$ the states with these $|\Delta|$ qu*D*its removed. They have the norms

$$\|\psi'_{|\Delta|,0}\|^2 = \|\mathcal{P}_{I3}e_0\|^2 = \frac{1}{8}\,,\quad\text{and for } m\geq 1,$$

$$\|\psi'_{m+|\Delta|,m}\|^2 = \|\mathcal{P}_{II}e_0\|^2\,\Big(\prod_{\ell=2}^{m}\|\mathcal{P}_{II}e_{\epsilon_\ell}\|^2\Big)\,\|\mathcal{P}_{I3}e_{\epsilon_{m+1}}\|^2 = \frac{1}{8}\,(\frac{3}{8})^{m-1}\,\frac{1}{8}\,,$$

and hence,

$$\sum_{m=0}^{\lfloor\Theta k\rfloor-|\Delta|}\|\psi'_{m+|\Delta|,m}\|^2 = \frac{3}{20}+\mathcal{O}\big((3/8)^{\Theta k-|\Delta|}\big)\,.$$

The nondiagonal contributions for $0<n\leq\Theta k-|\Delta|$ are

$$\begin{aligned}\langle\psi'_{|\Delta|,0}|\psi'_{n+|\Delta|,n}\rangle &= \langle\mathcal{P}_{I3}e_{\epsilon_0},\mathcal{P}_{II}e_0\rangle\,\Big(\prod_{\ell=2}^{n}\langle e_{\epsilon_\ell},\mathcal{P}_{II}e_{\epsilon_\ell}\rangle\Big)\,\langle e_{\epsilon_{n+1}},\mathcal{P}_{I3}e_{\epsilon_{n+1}}\rangle\\ &= \frac{-1+i}{16}\,\frac{1}{2^{n-1}}\,\frac{1}{4}\,,\end{aligned}$$

while for any pair $1 \leq m < n \leq \Theta k - |\Delta|$,

$$\begin{aligned}
&\langle \psi'_{m+|\Delta|,m} | \psi'_{n+|\Delta|,n} \rangle = \\
&\quad \Big(\prod_{\ell=1}^{m} \|\mathcal{P}_{II} e_{\epsilon_\ell}\|^2 \Big) \langle \mathcal{P}_{I3} e_{\epsilon_{m+1}}, \mathcal{P}_{II} e_{\epsilon_{m+1}} \rangle \Big(\prod_{\ell=2}^{n-m} \langle e_{\epsilon_{m+\ell}}, \mathcal{P}_{II} e_{\epsilon_{m+\ell}} \rangle \Big) \langle e_{\epsilon_{n+1}}, \mathcal{P}_{I3} e_{\epsilon_{n+1}} \rangle \\
&\quad = \frac{1}{8} \big(\frac{3}{8}\big)^{m-1} \frac{1 \pm i}{16} \frac{1}{2^{n-m-1}} \frac{1}{4} .
\end{aligned}$$

These contributions sum up to

$$2 \sum_{0 \leq m < n \leq \Theta k - |\Delta|} \mathrm{Re} \langle \psi'_{m+|\Delta|,m} | \psi'_{n+|\Delta|,n} \rangle = -\frac{1}{20} + \mathcal{O}(2^{-\Theta k + |\Delta|}) .$$

Putting together this with the diagonal contributions and restoring the factor (7.20), yields

$$\|\Psi_\Delta\|^2 = \frac{1}{20} \frac{1}{2^{|\Delta|}} + \mathcal{O}(2^{-\Theta k}) , \quad -\Theta k \leq \Delta \leq -1 . \tag{7.21}$$

As opposed to the case of $\Delta > 0$, this norm does not depend on $\boldsymbol{\epsilon}$.

Remark 7.2. In the course of the computations, we have obtained expressions for the norms of all states $|\psi_{m',m}\rangle$, which allow us to estimate the remainder in (7.13). Indeed, the expressions obtained in the previous subsections are valid for any pair $m, m' < k/2$, and in this case the norms satisfy:

$$\|\psi_{m',m}(\boldsymbol{\epsilon})\| \leq C \left(\frac{3}{8}\right)^{\min(m',m)/2} 2^{-|m'-m|/2} ,$$

for a constant $C > 0$ independent of $\boldsymbol{\epsilon}$. Using this we see that for any nonclassical $\boldsymbol{\epsilon}$,

$$\|t^* t |\boldsymbol{\epsilon}\rangle - \sum_{0 \leq m', m \leq \Theta k} |\psi_{m',m}(\boldsymbol{\epsilon})\rangle\| = \mathcal{O}(2^{-\Theta k/2}) .$$

This proves the estimate (7.13) for a generic nonclassical sequence.

7.3.4. *Summing up.* We can now sum over all shifts Δ, $|\Delta| \leq \Theta k$ for a given nonclassical sequence $\boldsymbol{\epsilon}$. The sum over the shifts $\Delta \leq 0$ is simple, and independent on the sequence $\boldsymbol{\epsilon}$:

$$\sum_{-\Theta k \leq \Delta \leq 0} \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 = \frac{25}{80} + \mathcal{O}(k\, 2^{-\Theta k}) .$$

The sum over the shifts $\Delta > 0$ is slightly more delicate, since the norm of $|\Psi_\Delta(\boldsymbol{\epsilon})\rangle$ depends on $\boldsymbol{\epsilon}$ — see Eqs. (7.17,7.19). However, we notice that if a sequence $\boldsymbol{\epsilon}$ is nonclassical and generic (in the sense of Lemma 7.4), then its "mirror sequence" $\overline{\boldsymbol{\epsilon}}$, defined by $\epsilon'_1 = 0$ and

$$\forall \ell \in [2, k], \quad \overline{\epsilon}_\ell = 1 \Longleftrightarrow \epsilon_\ell = 2 , \quad \overline{\epsilon}_\ell = 2 \Longleftrightarrow \epsilon_\ell = 1$$

is also a nonclassical and generic sequence, different from $\boldsymbol{\epsilon}$. Therefore, we can partition the set of generic nonclassical sequences into pairs of mirror sequences. The sum of $\|\Psi_\Delta(\boldsymbol{\epsilon})\|^2$ $(\Delta > 0)$ over any pair $(\boldsymbol{\epsilon}, \overline{\boldsymbol{\epsilon}})$ is now independent of the pair:

$$\forall \Delta > 0, \qquad \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 + \|\Psi_\Delta(\overline{\boldsymbol{\epsilon}})\|^2 = \frac{1}{2^\Delta}\,\frac{1}{10} + \mathcal{O}(2^{-\Theta k})\,.$$

Notice that this contribution is identical (up to the remainder) with $\|\Psi_{-\Delta}(\boldsymbol{\epsilon})\|^2 + \|\Psi_{-\Delta}(\overline{\boldsymbol{\epsilon}})\|^2$. Therefore there is a sort of symmetry around $\Delta = 0$, which carries the largest contribution. Summing over Δ, we get for any pair:

$$\sum_{-\Theta k \leq \Delta \leq \Theta k} \|\Psi_\Delta(\boldsymbol{\epsilon})\|^2 + \|\Psi_\Delta(\overline{\boldsymbol{\epsilon}})\|^2 = \frac{58}{80} + \mathcal{O}(k\, 2^{-\Theta k})\,.$$

For any pair of generic sequences, this implies

$$\|t^* t|\boldsymbol{\epsilon}\rangle\|^2 + \|t^* t|\overline{\boldsymbol{\epsilon}}\rangle\|^2 = \frac{58}{80} + \mathcal{O}(2^{-\Theta k/2})\,.$$

Using this estimate and Lemma 7.4, we obtain the trace over the nonclassical states:

$$\mathrm{tr}_{\mathrm{noncl}}((t^* t)^2) = 2^{k-2}\left(\frac{58}{80} + \mathcal{O}(e^{-Ck}) + \mathcal{O}(2^{-\Theta k/2})\right).$$

Substracting this expression from the "nonclassical conductance" (7.9), and calling $\widetilde{C} = \min(C, \frac{1}{2}\Theta)$, we find for the noise power

$$P = \mathrm{tr}(t^* t - (t^* t)^2) = \mathrm{tr}_{\mathrm{noncl}}(t^* t - (t^* t)^2) = 2^{k-1}\left(\frac{11}{80} + \mathcal{O}(e^{-\widetilde{C} k})\right). \tag{7.22}$$

This proves (1.2) in Theorem 2. As remarked in §1.1 this number is close to the random-matrix prediction for this quantity, namely 1/8 [23, 49]. This is in contrast with our remark 6.1 that the semiclassical resonance spectrum of the propagator inside the dot, $\widetilde{B}_h = U_h \Pi_I$, is quite different from that of a random subunitary matrix. Somehow, the matrix t, obtained by summing iterates of $\widetilde{B}_h$, has acquired some "randomness", as far as the distribution of its singular values is concerned.

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