# Sufficient conditions for the anti-Zeno effect

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**Abstract**

The ideal anti-Zeno effect means that a perpetual observation leads to an immediate disappearance of the unstable system. We present a straightforward way to derive sufficient conditions under which such a situation occurs expressed in terms of the decaying states and spectral properties of the Hamiltonian. They show, in particular, that the gap between Zeno and anti-Zeno effects is in fact very narrow.
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The Zeno effect which means that an unstable system will never decay if we monitor its decay perpetually is known for decades. For the first time it was formulated explicitly in this context by Beskow and Nilsson [3] and soon after a mathematical analysis [5, 14] revealed sufficient conditions under which it exists; it became truly popular after the authors of [20] coined its present name. Recently the effect attracted a new wave of mathematical [8, 9, 19, 22] and physical [11, 10, 12, 13, 16, 18] interest; in the mentioned papers one can find a more complete bibliography.

Although the opposite situation, in which a frequent measurement can on the contrary speed up the decay, or ideally to lead to an immediate disappearance of the unstable system, was also noticed early [6], it attracted attention only recently – see, e.g., [1, 2, 17, 21] and also [22] and references therein. As in the case of the Zeno effect, the problem can be tackled from two points of view. The more practical one concerns the increase of the measured lifetime in case when the measurement are performed with a certain frequency. On the other hand, theoretically one can ask what happens if the period between two successive measurements tend to zero. The distinction between the two is important, because in typical examples, where the spectral distribution differs from the one giving exponential decay by an energy cut-off, the decay law unperturbed by the measurements oscillates around an exponential function. In such a case the mean value of energy is finite and the system exhibits Zeno effect for continual observation but at finite measurement frequencies it switches between Zeno-type and anti-Zeno-type behaviors. We will thus use the label anti-Zeno only for the infinite-frequency limit having in mind that validity of such idealizations "is the heart and soul of theoretical physics and has the same fundamental significance as the reproducibility of experimental data" as Bratelli and Robinson once put it [4].

Conditions under which the anti-Zeno effect occurs were discussed in the above mentioned papers. In particular, a necessary and sufficient condition formulated in a probabilistic language was derived in [2] and reproduced in the review paper [22]. Our aim in the present letter is to present an alternative simple derivation a sufficient condition given purely in spectral terms – see relations (10), (11) below. Moreover, our argument will work also in the situation when the unstable system has internal degrees of freedom and more than a simple Fourier transform is needed to express the decay law – see (14) and (15). The same approach give us also a fresh look at the Zeno effect and shows that the gap between it and the anti-Zeno effect is in fact very narrow.

We will work within the general framework of quantum kinematics of a decaying system [7], in other words, we base our discussion on three objects: a Hilbert space $\mathcal{H}$ corresponding to an isolated system describing the unstable system together with its decay products, a projection $P$ specifying a subspace of $\mathcal{H}$ referring to the unstable system alone, and a unitary evolution $e^{-iHt}$ on $\mathcal{H}$ corresponding to a self-adjoint total Hamiltonian $H$. We exclude the trivial situation, of course, assuming that the subspace $P\mathcal{H}$ is *not* invariant under $e^{-iHt}$ for $t>0$.

For simplicity we will consider pure states only. If the system is prepared at the initial instant $t=0$ in a state $\psi \in P\mathcal{H}$, its *decay law* unperturbed by measurements, or non-decay probability at a later time $t$, is

$$P(t) = \|P\, e^{-iHt}\psi\|^2, \tag{1}$$

in particular, $P(t) = |(\psi, e^{-iHt}\psi)|^2$ if $\dim P = 1$ (this quantity has always a time argument so it cannot be confused with the projection $P$). In the general case the decay law should be labelled by the initial state, $P_\psi(t)$, but we will avoid it if there no danger of misunderstanding. If we perform non-decay measurements at times $t/n$, $2t/n$ $\ldots$, $t$, all with the positive outcome, the resulting non-decay probability is

$$M_n(t) = P_\psi(t/n) P_{\psi_1}(t/n) \cdots P_{\psi_{n-1}}(t/n)\,, \tag{2}$$

where $\psi_{j+1}$ is the normalized projection of $e^{-iHt/n}\psi_j$ on $P\mathcal{H}$ and $\psi_0 := \psi$, in particular,

$$M_n(t) = (P_\psi(t/n))^n \tag{3}$$

if $\dim P = 1$. Combining the last relation with the fact that $\lim_{n\to\infty}(f(t/n)^n = \exp\{-\dot f(0+)t\}$ whenever $f(0) = 1$ and the one-sided derivative $\dot f(0+)$ exists we see that the Zeno effect, i.e. $M(t) := \lim_{n\to\infty} M_n(t) = 1$ for all $t > 0$, and its anti-Zeno counterpart, i.e. $M(t) = 0$ for any $t > 0$, require that $\dot P(0+)$ is zero and negative infinite, respectively. The same is true if $\dim P > 1$ provided the derivative $\dot P_\psi(0+)$ has such a property for *any* $\psi \in P\mathcal{H}$.

It is thus crucial to estimate the quantity $1 - P(t)$, or more explicitly $(\psi, P\psi) - (\psi, e^{iHt}Pe^{-iHt}\psi)$, to find its behavior for small values of $t$. It is easy to see that we can rewrite it in the form

$$1 - P(t) = 2\,\mathrm{Re}\,(\psi, P(I - e^{-iHt})\psi) - \|P(I - e^{-iHt})\psi\|^2 \tag{4}$$

The left-hand side of (4) can be expressed as

$$4\int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2}\, d\|E_\lambda^H \psi\|^2 - 4\left\| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin\frac{\lambda t}{2}\, dPE_\lambda^H \psi \right\|^2 \tag{5}$$

if we use the spectral representation of $e^{-iHt}$ in terms of the spectral measure $E_H$ of the Hamiltonian $H$ (generated by a non-decreasing projection-valued function $\lambda \mapsto E_\lambda^H := E_H((-\infty, \lambda]))$. By Schwarz inequality the quantity (5) is non-negative; our aim is to find tighter upper and lower bounds for it.

Let us begin with the case $\dim P = 1$ when (5) becomes

$$4\int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2}\, d\omega(\lambda) - 4\left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin\frac{\lambda t}{2}\, d\omega(\lambda) \right|^2 \tag{6}$$

with $d\omega(\lambda) := d(\psi, E_\lambda^H \psi)$. In most decay models $\psi$ belongs to the absolutely continuous spectral subspace of $H$ in which case one has $d\omega(\lambda) = \omega(\lambda)d\lambda$ with some $\omega \in L^1$, however, we will not need this assumption. Using the spectral-measure normalization, $\int_{-\infty}^{\infty} d\omega(\lambda) = 1$, we can rewrite the difference (6) as

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sin^2\frac{\lambda t}{2}+\sin^2\frac{\mu t}{2}\right)d\omega(\lambda)d\omega(\mu)$$
$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\frac{(\lambda-\mu)t}{2}\,\sin\frac{\lambda t}{2}\,\sin\frac{\mu t}{2}\,d\omega(\lambda)d\omega(\mu)$$

or

$$1-P(t) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{(\lambda-\mu)t}{2}\,d\omega(\lambda)d\omega(\mu) \tag{7}$$

Take $\alpha \in (0,2]$. Using the inequalities $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda-\mu|^\alpha \leq 2^\alpha(|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the expression (7) that

$$\begin{aligned}\frac{1-P(t)}{t^\alpha} &\leq 2^{1-\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda-\mu|^\alpha d\omega(\lambda)d\omega(\mu)\\ &\leq 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|\lambda|^\alpha+|\mu|^\alpha)d\omega(\lambda)d\omega(\mu)\\ &\leq 4\langle|H|^\alpha\rangle_\psi.\end{aligned}$$

This means that $1-P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in \mathrm{Dom}\,(|H|^\alpha)$. If this is true for some $\alpha > 1$ we have naturally Zeno effect, although this requirement is slightly stronger than the other known sufficient conditions; recall that $\dot{P}_\psi(0+) = 0$ holds whenever $\langle|H|\rangle_\psi$ is finite as it is known for longtime, see [15], [7, Thm. 1.3.1], and also [14].

On the other hand, by negation we infer that $\psi \notin \mathrm{Dom}\,(|H|)$ is a *necessary condition* for the (one-dimensional) anti-Zeno effect. Notice that in case of the absolutely continuous spectrum this necessary condition follows also from the Lipschitz regularity, since $P(t) = |\hat{\omega}(t)|^2$ and $\hat{\omega}$ is bounded and uniformly $\alpha$-Lipschitz *iff* $\int_{\mathbb{R}} \omega(\lambda)(1+|\lambda|^\alpha)\,d\lambda < \infty$.

Some may believe that by this the problem is closed – it is a common mistake that only states from the domain of the Hamiltonian make physical sense. In reality one can never test experimentally that a given does *not* belong to $D(H)$, see [7, Sec. I.6]. To find a *sufficient condition* let us observe that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive $C$ independent of $t$ such that

$$\left|\sin\frac{(\lambda-\mu)t}{2}\right| \geq C|\lambda-\mu|t\,; \tag{8}$$

one can make the constant explicit but it is not necessary. Consequently, we have the estimate

$$1 - P(t) \geq 2C^2 t^2 \int_{-1/t}^{1/t} d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\mu)(\lambda-\mu)^2$$

which in turn implies

$$\frac{1-P(t)}{t} \geq 4C^2 t \left\{ \int_{-1/t}^{1/t} \lambda^2\, d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\lambda) - \left( \int_{-1/t}^{1/t} \lambda\, d\omega(\lambda) \right)^2 \right\}. \tag{9}$$

The anti-Zeno effect occurs if the the right-hand side diverges as $t \to 0$ which is true if

$$\int_{-N}^{N} \lambda^2\, d\omega(\lambda) \int_{-N}^{N} d\omega(\lambda) - \left( \int_{-N}^{N} \lambda\, d\omega(\lambda) \right)^2 \geq cN^\alpha \tag{10}$$

holds for any $N$ and some $c > 0$, $\alpha > 1$, or slightly more generally, if the inverse of the right-hand-side expression in (10) behaves like $o(N)$ as $N \to \infty$.

The obtained sufficient condition can be also written in a slightly more compact form if we introduce the operators $H_N^\beta := H^\beta E_H(\Delta_N)$ with the spectral cut-off to the interval $\Delta_N := (-N, N)$, in particular, we denote $I_N := E_H(-N, N)$. In this notation, the inequality (10) becomes

$$\langle H_N^2 \rangle_\psi \langle I_N \rangle_\psi - \langle H_N \rangle_\psi^2 \geq cN^\alpha \,. \tag{11}$$

Let us stress that the condition which we have derived does *not* require the Hamiltonian $H$ to be unbounded from below as it is the case with the exponential decay law; to satisfy (10) it is enough that the spectral distribution has a slow decay in one direction only.

As an *example*, consider a Hamiltonian bounded from below and $\psi$ from its absolutely continuous spectral subspace such that the corresponding distribution function behaves as $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1, 2)$. While $\int_{-N}^{N} \omega(\lambda)\, d\lambda$ tends to one as $\lambda \to +\infty$, the other two integrals diverge giving

$$cN^{2-\beta} - c^2 N^{4-2\beta}$$

as the asymptotic behavior of the left-hand side, where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so the anti-Zeno effect occurs. The above argument shows that in the same situation $\beta > 2$ leads to Zeno effect; this shows that the exponential decay law indeed walks a thin rope between the Scylla of æternal preservation and Charybda of immediate destruction. Of course, the exponential decay appears only if the spectrum of $H$ is the whole real line. For a *semibounded* $H$ with the asymptotic behavior $\omega(\lambda) \approx c\lambda^{-1}$ the reduced evolution $(\psi, e^{-iHt}\psi)$ typically exhibits rapid oscillations around $t = 0$ which may obscure existence of the Zeno limit – cf. [7], Rem. 2.4.9.

Let us show next how the situation looks like when the unstable system is allowed to have internal degrees of freedom, $\dim P > 1$. One might expect that the sufficient condition is given by (11) again but this guess is wrong. To find the answer, we denote by $\{\chi_j\}$ an orthonormal basis in $P\mathcal{H}$; it allows to write the second term in (5) as

$$-4 \sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2}\, d(\chi_m, E_\lambda^H \psi) \right|^2 .$$

We also expand the initial state vector $\psi$ as $\psi = \sum_j c_j \chi_j$ with $\sum_j |c_j|^2 = 1$ and denote $d\omega_{jk}(\lambda) := d(\chi_j, E_\lambda^H \chi_k)$, which is naturally a real-valued measure symmetric with respect to interchange of the indices. Since the other measure appearing in (5) can be written as

$$d\|E_\lambda^H \psi\|^2 = \sum_{jk} \bar{c}_j c_k d\omega_{jk}(\lambda) ,$$

we can cast the quantity of interest into the form

$$\begin{aligned} 1 - P(t) = 4 \sum_{jk} \bar{c}_j c_k \Big\{ & \int_{-\infty}^{\infty} \sin^2 \frac{\lambda t}{2}\, d\omega_{jk}(\lambda) \\ & - \sum_m \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin \frac{\lambda t}{2}\, d\omega_{jm}(\lambda) \int_{-\infty}^{\infty} e^{i\mu t/2} \sin \frac{\mu t}{2}\, d\omega_{km}(\mu) \Big\}. \end{aligned} \tag{12}$$

If $\dim P = \infty$ one has to check, of course, the convergence of the series used in the argument and correctness of interchanging of the summation and integration which is easily done by means of Parseval relation.

In the next step we employ normalization of the spectral measure which gives $\int_{-\infty}^{\infty} d\omega_{jk}(\lambda) = \delta_{jk}$. It is then a straightforward exercise to rewrite the curly bracket and to show that the right-hand side of (12) can be rewritten as

$$1 - P(t) = 2 \sum_{jkm} \bar{c}_j c_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2}\, d\omega_{jm}(\lambda) d\omega_{km}(\mu) . \tag{13}$$

Returning to the projection-valued measures we can write the right-hand side of (13) also concisely as

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{(\lambda-\mu)t}{2}\,(\psi,dE_\lambda^H P dE_\mu^H\psi)\,.$$

To get a lower bound to the left-hand side of (13) we employ again the inequality (8) which gives

$$\begin{aligned}
1-P(t) &\geq 2C^2t^2\int_{-1/t}^{1/t}\int_{-1/t}^{1/t}(\lambda-\mu)^2\,(\psi,dE_\lambda^H P dE_\mu^H\psi) \\
&= 4C^2t^2\Big\{\int_{-1/t}^{1/t}\int_{-1/t}^{1/t}\lambda^2\,(\psi,dE_\lambda^H P dE_\mu^H\psi) \\
&\quad -\int_{-1/t}^{1/t}\int_{-1/t}^{1/t}\lambda\mu\,(\psi,dE_\lambda^H P dE_\mu^H\psi)\Big\} \\
&= 4C^2t^2\left\{(\psi,H_{1/t}^2 P I_{1/t}\psi)-\|PH_{1/t}\psi\|^2\right\}\,,
\end{aligned}$$

where the symbol $H_b$ denotes again the cut-off Hamiltonian, $HE_H(\Delta_b)$ with $\Delta_b:=(-b,b)$. Hence a sufficient condition for the anti-Zeno effect to occur in this more general situation is, for instance,

$$\langle H_N^2 P I_N\rangle_\psi-\|PH_N\psi\|^2\geq cN^\alpha \tag{14}$$

for some $c>0$ and $\alpha>1$, both independent of $\psi$, as a proper generalization of the one-dimensional condition (11) – notice that the second term at the left-hand side of (14) can be also written as $\langle H_N P H_N\rangle_\psi$ – or slightly more generally

$$\left(\langle H_N^2 P I_N\rangle_\psi-\|PH_N\psi\|^2\right)^{-1}=o(N) \tag{15}$$

as $N\to\infty$ uniformly w.r.t. $\psi\in P\mathcal{H}$. The meaning of these conditions is similar as before: the energy distribution, now for any possible state of the unstable system, must be sufficiently spread to ensure that the initial decay rate of such a $\psi$ is infinite.

Let us add some comments. First we observe that in the case $\dim P>1$ a system subject to a perpetual observation can exhibit also a more complicated behaviour. A simple example is a combination of Zeno and anti-Zeno effect. Take $\{\mathcal{H}_j,P_j,H_j\}$, $j=1,2$, with $\dim P_j=1$ such that $\dot{P}_1(0+)=0$ and $\dot{P}_2(0+)=-\infty$, and consider the combined system described by the triple

$\{\mathcal{H}_1 \oplus \mathcal{H}_2, P_1 \oplus P_2, H_1 \oplus H_2\}$. If the initial state of such an unstable system is represented by a non-trivial linear combination $\psi = c_1\psi_1 + c_2\psi_2$ and we monitor it continuously, the second component of $\psi$ disappears immediately while the first one will survive forever. Of course, one can imagine various more complicated combinations, especially in the case when $\dim P$ is infinite.

Another comment concerns physical relevance of the conditions discussed here. The point is that they involve asymptotic behavior of spectral distributions at high energies, i.e. something which cannot be in principle verified experimentally. A similar question arises also in connection with the Zeno effect, of course, but there at least sometimes we can be sure that the needed moment of the spectral distribution exists, in particular, if the experiment involves a filtering into a finite energy window, while checking the divergence of such integrals is *always* out of experimental reach as we have remarked above.

What one can verify, however, is whether an energy distribution of a state coincides with with the one leading to the anti-Zeno effect, for instance, decreasing as $c\lambda^{-\beta}$ with some $\beta \in (1,2)$ over a wide range of energies up to some value $\lambda_c$. If this is the case the difference of the theoretical and actual $P(t)$ will be small and the difference in the initial behavior will be significant at the time scale characterized by $\hbar\lambda_c^{-1}$, hence with the measurement frequencies small enough at this scale one should be able to observe a significant reduction of the measured lifetime, demonstrating the anti-Zeno effect practically.

In conclusion, we have derived sufficient conditions under which unstable systems exhibit anti-Zeno effect using nothing else than properties of spectral distribution of the decaying states. The conditions impose restrictions neither on the lower bound of the spectrum of the corresponding Hamiltonian nor on the dimension of the unstable system subspace.

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