Well-Posedness for Semi-Relativistic Hartree Equations of Critical Type

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Abstract

We prove local and global well-posedness for semi-relativistic, nonlinear Schrödinger equations $i\partial_t u = \sqrt{-\Delta + m^2} u + F(u)$ with initial data in $H^s(\mathbb{R}^3)$, $s \geq 1/2$. Here $F(u)$ is a critical Hartree nonlinearity that corresponds to Coulomb or Yukawa type self-interactions. For focusing $F(u)$, which arise in the quantum theory of boson stars, we derive a sufficient condition for global-in-time existence in terms of a solitary wave ground state. Our proof of well-posedness does not rely on Strichartz type estimates, and it enables us to add external potentials of a general class.

1 Introduction

In this paper we study the Cauchy problem for nonlinear Schrödinger equations with kinetic energy part originating from special relativity. That is, we consider the initial value problem for

$$i\partial_t u = \sqrt{-\Delta + m^2}\, u + F(u), \qquad (t,x) \in \mathbb{R}^{1+3}, \tag{1.1}$$

where $u(t,x)$ is complex-valued, $m \geq 0$ denotes a given mass parameter, and $F(u)$ is some nonlinearity. Here the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space.

Such "semi-relativistic" equations have (though not Lorentz covariant in general) interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons. Equation (1.1) arises, for instance, as an effective description of *boson stars*, see, e. g., [ES05, LY87], where $F(u)$ is a focusing Hartree nonlinearity given by

$$F(u) = (\frac{\lambda}{|x|} * |u|^2)u, \tag{1.2}$$

with some constant $\lambda < 0$ and $*$ as convolution. Motivated by this physical example with focusing self-interaction of Coulomb type, we address the Cauchy problem for equation (1.1) and a class of Hartree nonlinearities including (1.2). In fact, we shall prove well-posedness for initial data $u(0,x) = u_0(x)$ in $H^s = H^s(\mathbb{R}^3)$, $s \geq 1/2$; see Theorems 1–3 below.

Let us briefly point out a decisive feature of the example cited in (1.2) above. Apart from its physical relevance, the nonlinearity given by (1.2) leads to an L^2-*critical* equation as indicated by the fact that the coupling constant λ has to be dimensionless. In consequence of this, L^2-smallness of the initial datum enters as a sufficient condition for global-in-time solutions. More precisely, we derive for $u_0 \in H^s$, $s \geq 1/2$, the following criterion implying global well-posedness

$$|\lambda| \int_{\mathbb{R}^3} |u_0(x)|^2 \, dx < \int_{\mathbb{R}^3} |Q(x)|^2 \, dx, \tag{1.3}$$

irrespectively of the parameter $m \geq 0$ in (1.1). Here $Q \in H^{1/2}$ is a positive solution (ground state) for the nonlinear equation

$$\sqrt{-\Delta}\, Q - \big(\frac{1}{|x|} * |Q|^2\big) Q = -Q, \tag{1.4}$$

see Theorem 2 below. Note that $u(t,x) = e^{it} Q(x)$ is a solitary wave solution for (1.1) with $m = 0$ and nonlinearity given by (1.2) for $\lambda = -1$. Furthermore, criterion (1.3) can be linked with established results as follows. First, it is reminiscent to a well-known condition derived in [Wei83] for global well-posedess of nonrelativistic Schrödinger equations with focusing, local nonlinearity; see also [NO92] for Hartree nonlinearities. Second, criterion (1.3) is in accordance with a sufficient stability condition proved in [LY87] for the related time-independent problem (i. e., a static boson star); see [FL04] for a more details concerning known results on Hartree equations.

We now give an outline of our methods. The proof of well-posedness presented below does *not* rely on Strichartz (i. e., space-time) estimates for the propagator

$$e^{-it\sqrt{-\Delta + m^2}}, \tag{1.5}$$

but it employs sharp estimates to derive local Lipschitz continuity of L^2-critical nonlinearities of Hartree type. Local well-posedness then follows by standard methods for abstract evolution equations. Furthermore, global well-posedness is derived by means of a-priori estimates and conservation of charge and energy whose proof requires a regularization method.

This paper is organized as follows.

- In Section 2 we introduce a class of critical Hartree nonlinearities including (1.2). First, we state Theorems 1 and 2 that establish local and global well-posedness in energy space $H^{1/2}$ for this class of nonlinearities. In Theorem 3 we extend these results to H^s, for every $s \geq 1/2$. Finally, external potentials are included, i. e., we consider

$$i\partial_t u = \big(\sqrt{-\Delta + m^2} + V\big) u + F(u), \tag{1.6}$$

  where $V : \mathbb{R}^3 \to \mathbb{R}$ is given. In Theorem 4 we state local and global well-posedness for (1.6) with initial datum $u(0,x) = u_0(x)$ in the appropriate energy space. The assumption imposed on V is considerably weak and implies that $\sqrt{-\Delta + m^2} + V$ defines a self-adjoint operator via its form sum.

- The main results (i. e., Theorems 1–4) are proved in Section 3.

- Appendix A contains useful facts about fractional derivatives, a discussion of ground states, and some details of the proofs.

Notation

Throughout this text, the symbol $*$ stands for convolution on $\mathbb{R}^3$, i. e.,

$$(f * g)(x) := \int_{\mathbb{R}^3} f(x-y) g(y)\, dy,$$

and $L^p(\mathbb{R}^3)$, with norm $\|\cdot\|_p$ and $1 \leq p \leq \infty$, denotes the usual Lebesgue L^p-space of complex-valued functions on $\mathbb{R}^3$. Moreover, $L^2(\mathbb{R}^3)$ is associated with the scalar product defined by

$$\langle u, v \rangle := \int_{\mathbb{R}^3} \overline{u}(x) v(x)\, dx.$$

For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we introduce fractional Sobolev spaces (see, e. g., [BL76]) with their corresponding norms according to

$$H^{s,p}(\mathbb{R}^3) := \big\{ u \in \mathcal{S}'(\mathbb{R}^3) : \|u\|_{H^{s,p}} := \|\mathcal{F}^{-1}[(1+\xi^2)^{s/2} \mathcal{F} u]\|_p < \infty \big\},$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathcal{S}'(\mathbb{R}^3)$ (space of tempered distributions). In our analysis, the Sobolev spaces

$$H^s(\mathbb{R}^3) := H^{s,2}(\mathbb{R}^3),$$

with norms $\|\cdot\|_{H^s} := \|\cdot\|_{H^{s,2}}$, will play an important role.

In addition to the common L^p-spaces, we also make use of local L^p-space, $L^p_{\mathrm{loc}}(\mathbb{R}^3)$, with $1 \leq p \leq \infty$, and weak (or Lorentz) spaces, $L^p_w(\mathbb{R}^3)$, with $1 < p < \infty$ and corresponding norms given by

$$\|u\|_{p,w} := \sup_{\Omega} |\Omega|^{-1/p'} \int_{\Omega} |u(x)|\, dx,$$

where $1/p + 1/p' = 1$ and Ω denotes an arbitrary measurable set with Lebesgue measure $|\Omega| < \infty$; see, e. g., [LL01] for this definition of L^p_w-norms. Note that $L^p(\mathbb{R}^3) \subsetneq L^p_w(\mathbb{R}^3)$, for $1 < p < \infty$.

The symbol $\Delta = \sum_{i=1}^3 \partial^2_{x_i}$ stands for the usual Laplacian on $\mathbb{R}^3$, and $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space. Note that $\sqrt{-\Delta + m^2} f \in L^2(\mathbb{R}^3)$ whenever $f \in H^{1/2}(\mathbb{R}^3)$. Besides the operator $\sqrt{-\Delta + m^2}$, we also employ Riesz and Bessel potentials of order $s \in \mathbb{R}$, which we denote by $(-\Delta)^{s/2}$ and $(1 - \Delta)^{s/2}$, respectively; see also Appendix A.

Except for theorems and lemmas, we often use the abbreviations $L^p = L^p(\mathbb{R}^3)$, $L^p_w = L^p_w(\mathbb{R}^3)$, and $H^s = H^s(\mathbb{R}^3)$. In what follows, $a \lesssim b$ always denotes an inequality $a \leq cb$, where c is an appropriate positive constant that can depend on fixed parameters.

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2 Main Results

In this section we prove local and global well-posedness for the following initial value problem

$$\begin{cases} i\partial_t u = \sqrt{-\Delta + m^2}\, u + \big(\dfrac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\big) u, \\ u(0,x) = u_0(x), \quad u : [0,T) \times \mathbb{R}^3 \to \mathbb{C}, \end{cases} \tag{2.1}$$

where $m \geq 0$, $\lambda \in \mathbb{R}$, and $\mu \geq 0$ are given parameters. Note that $|\lambda|$ could be absorbed in the normalization of $u(t,x)$, but we shall keep λ explicit in the following.

Our particular choice of the Hartree type nonlinearities in (2.1) is motivated by the fact that (2.1) can be rewritten as the following system of equations

$$\begin{cases} i\partial_t u = \sqrt{-\Delta + m^2}\, u + \Psi u, \\ (\mu^2 - \Delta)\Psi = 4\pi\lambda |u|^2, \quad u(0,x) = u_0(x), \end{cases} \tag{2.2}$$

where $\Psi = \Psi(t,x)$ is real-valued. This reformulation stems from the observation that $\frac{e^{-\mu|x|}}{4\pi|x|}$ is the Green's function of $(\mu^2 - \Delta)$ in $\mathbb{R}^3$; see Appendix A.1 for more details. System (2.2) now reveals the physical intuition behind (2.1), i. e., the function $u(t,x)$ corresponds to a "positive energy wave" with instantaneous self-interaction that is either of Coulomb or Yukawa type depending on whether $\mu = 0$ or $\mu > 0$, respectively. To prove well-posedness we shall, however, use formulation (2.1) instead, and we refer to facts from potential theory only when estimating the nonlinearity.

2.1 Local Well-Posedness

Let us begin with well-posedness in energy space, i. e., we assume that $u_0 \in H^{1/2}$ holds in (2.1). The following Theorem 1 establishes local well-posedness in the strong sense, i. e., we have existence and uniqueness of solutions, their continuous dependence on initial data, and the blow-up alternative. The precise statements is as follows.

Theorem 1. *The initial value problem (2.1) is locally well-posed in $H^{1/2}(\mathbb{R}^3)$. This means that, for every $u_0 \in H^{1/2}(\mathbb{R}^3)$, there exist a unique solution*

$$u \in C^0\big([0,T); H^{1/2}(\mathbb{R}^3)\big) \cap C^1\big([0,T); H^{-1/2}(\mathbb{R}^3)\big),$$

and it depends continuously on u_0. Here $T \in (0,\infty]$ is the maximal time of existence, where we have that either $T = \infty$ or $T < \infty$ and $\lim_{t\uparrow T} \|u(t)\|_{H^{1/2}} = \infty$ holds.

Remark. Continuous dependence means that the map $u_0 \mapsto u \in C^0(I; H^{1/2})$ is continuous for every compact interval $I \subset [0,T)$.

2.2 Global Well-Posedness

The local-in-time solutions derived in Theorem 1 extend to all times, by virtue of Theorem 2 below, provided that either $\lambda \geq 0$ holds (corresponding to a repulsive nonlinearity) or $\lambda < 0$ and the initial datum is sufficiently small.

Theorem 2. *The solution of (2.1) derived in Theorem 1 is global in time, i. e., we have that $T = \infty$ holds, provided that one of the following conditions is met.*

i) $\lambda \geq 0$.

ii) $\lambda < 0$ *and* $|\lambda| \|u_0\|_2^2 < \|Q\|_2^2$, *where* $Q \in H^{1/2}(\mathbb{R}^3)$ *is a strictly positive solution (ground state) of*

$$\sqrt{-\Delta}\, Q - \big(\frac{1}{|x|} * |Q|^2\big) Q = -Q. \tag{2.3}$$

Moreover, we have the estimate $\|Q\|_2^2 > 4/\pi$.

Remarks. 1) Due to the scaling behavior of (2.3), the function $Q_a(x) = a^{3/2} Q(ax)$, with $a > 0$, yields another ground state with $\|Q_a\|_2 = \|Q\|_2$ that satisfies

$$\sqrt{-\Delta}\, Q_a - \big(\frac{1}{|x|} * |Q_a|^2\big) Q_a = -a Q_a. \tag{2.4}$$

We refer to Appendix A.2 for a discussion of $Q \in H^{1/2}$ concerning its existence, regularity, and uniqueness.

2) Condition ii) resembles a well-known criterion derived in [Wei83] for global-in-time existence for L^2-critical nonlinear (nonrelativistic) Schrödinger equations. But in contrast to the cited work and its situation, it is not clear yet whether condition ii) is sharp for (2.1) when $m = 0$, $\lambda < 0$, and $\mu = 0$, such that there are solutions with $|\lambda| \|u_0\|_2^2 \geq \|Q\|_2^2$ that actually blow up within finite time, i. e., we have that $T < \infty$ and $\lim_{t\uparrow T} \|u(t)\|_{H^{1/2}} = \infty$.

2.3 Higher Regularity

We now turn to well-posedness of (2.1) in H^s, for $s \geq 1/2$, which is settled by the following result.

Theorem 3. *For every $s \geq 1/2$, the conclusions of Theorems 1 and 2 hold, where $H^{1/2}(\mathbb{R}^3)$ and $H^{-1/2}(\mathbb{R}^3)$ in Theorem 1 are replaced by $H^s(\mathbb{R}^3)$ and $H^{s-1}(\mathbb{R}^3)$, respectively.*

Remark. For $s = 1$, this result is needed in [ES05] for a rigorous derivation of (2.1) with Coulomb type self-interaction (i. e., $\mu = 0$) from many-body quantum mechanics.

2.4 External Potentials

Now we consider the following extension of (2.1) that arises by adding an external potential:

$$\begin{cases} i\partial_t u = \big(\sqrt{-\Delta + m^2} + V\big)u + \big(\dfrac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\big)u, \\ u(0,x) = u_0(x), \quad u : [0,T) \times \mathbb{R}^3 \to \mathbb{C}, \end{cases} \tag{2.5}$$

where $m \geq 0$, $\lambda \in \mathbb{R}$, $\mu \geq 0$ are given parameters, and $V : \mathbb{R}^3 \to \mathbb{R}$ denotes a preassigned function that meets the following condition.

Assumption 1. *Suppose that $V = V_+ + V_-$ holds with real-valued, measurable functions V_+ and V_- having the following properties.*

i) $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $V_+ \geq 0$.

ii) V_- *is* $\sqrt{-\Delta}$*-form bounded with relative bound less than* 1*, i. e., there exist constants* $0 \leq a < 1$ *and* $0 \leq b < \infty$*, such that*

$$|\langle u, V_- u\rangle| \leq a\langle u, \sqrt{-\Delta}\, u\rangle + b\langle u, u\rangle$$

holds for all $u \in H^{1/2}(\mathbb{R}^3)$.

We mention that Assumption 1 implies that $\sqrt{-\Delta + m^2} + V$ leads to a self-adjoint operator on L^2 via its form sum. Furthermore, the energy space given by

$$X := \big\{u \in H^{1/2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)\,|u(x)|^2\,dx < \infty\big\} \tag{2.6}$$

is complete with norm $\|\cdot\|_X$, and its dual space is denoted by X^*. We refer to Section 3.4 for more details on $\sqrt{-\Delta + m^2} + V$ and X.

After this preparing discussion, the extension of Theorems 1 and 2 for the initial value problem (2.5) can be now stated as follows.

Theorem 4. *Suppose that V satisfies Assumption 1. Then (2.5) is locally well-posed in the following sense. For every $u_0 \in X$, there exists a unique solution*

$$u \in C^0([0,T); X) \cap C^1([0,T); X^*),$$

and it depends continuously on u_0. Here $T \in (0,\infty]$ is the maximal time of existence such that either $T = \infty$ or $T < \infty$ and $\lim_{t\uparrow T} \|u(t)\|_X = \infty$ holds. Moreover, we have that $T = \infty$ holds, if one of the following conditions is satisfied.

i) $\lambda \geq 0$.

ii) $\lambda < 0$ *and* $|\lambda|\|u_0\|_2^2 < (1-a)\|Q\|_2^2$*, where* Q *is the ground state mentioned in Theorem 2 and* $0 \leq a < 1$ *denotes the relative bound introduced in Assumption 1.*

Remarks. 1) To meet Assumption 1 for V_+, we can choose, for example, $V_+(x) = |x|^\beta$, with $\beta \geq 0$; or even super-polynomial growth such as $V_+(x) = e^x$. Note that Assumption 1 for V_- is satisfied (by virtue of Sobolev inequalities), if

$$|V_-(x)| \leq \frac{c}{|x|^{1-\varepsilon}} + d$$

holds for some $0 < \varepsilon \leq 1$ and constants $0 \leq c, d < \infty$. In fact, we can even admit $\varepsilon = 0$ provided that $c < 2/\pi$ holds, as can be seen from inequality (3.6) below.

2) Since we avoid Strichartz estimates in our well-posedness proof below, we only need that V_+ belongs to L^1_{loc}. In contrast to this, compare, for instance, the conditions on V in [YZ04] for deriving Strichartz type estimates for $e^{-it(-\Delta+V)}$ in order to prove local well-posedness for (nonrelativistic) nonlinear Schrödinger equations with external potentials.

3 Proof of the Main Results

In this section we prove Theorems 1–4. Although Theorem 4 generalizes Theorems 1 and 2, we postpone the proof of Theorem 4 to the final part of this section.

3.1 Proof of Theorem 1 (Local Well-Posedness)

Let $u_0 \in H^{1/2}$ be fixed. In view of (2.1) we put

$$A := \sqrt{-\Delta + m^2} \quad \text{and} \quad F(u) := \big(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\big)u, \tag{3.1}$$

and we consider the integral equation

$$u(t) = e^{-itA}u_0 - i\int_0^t e^{-i(t-\tau)A}F(u(\tau))\,d\tau. \tag{3.2}$$

Here $u(t)$ is supposed to belong to the Banach space

$$Y_T := C^0\big([0,T); H^{1/2}(\mathbb{R}^3)\big), \tag{3.3}$$

with some $T > 0$ and corresponding norm $\|u\|_{Y_T} := \sup_{t\in[0,T)} \|u(t)\|_{H^{1/2}}$. The proof of Theorem 1 is now organized in two steps as follows.

Step 1: Estimating the Nonlinearity

We show that the nonlinearity $F(u)$ is locally Lipschitz continuous from $H^{1/2}$ into itself. This is main point of our argument for local well-posedness and it reads as follows.

Lemma 1. *For $\mu \geq 0$, the map $J(u) := \big(\frac{e^{-\mu|x|}}{|x|} * |u|^2\big)u$ is locally Lipschitz continuous from $H^{1/2}(\mathbb{R}^3)$ into itself with*

$$\|J(u) - J(v)\|_{H^{1/2}} \lesssim (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}})\|u - v\|_{H^{1/2}},$$

for all $u, v \in H^{1/2}(\mathbb{R}^3)$.

Proof of Lemma 1. We prove the claim for $\mu = 0$ and $\mu > 0$ in a common way, so let $\mu \geq 0$ be fixed. For $s \in \mathbb{R}$, it is convenient to introduce

$$\mathcal{D}^s := (\mu^2 - \Delta)^{s/2}.$$

Note that due to the equivalence

$$\|u\|_2 + \|\mathcal{D}^{1/2}u\|_2 \lesssim \|u\|_{H^{1/2}} \lesssim \|u\|_2 + \|\mathcal{D}^{1/2}u\|_2,$$

it is sufficient to estimate the quantities

$$I := \|J(u) - J(v)\|_2 \quad \text{and} \quad II := \|\mathcal{D}^{1/2}[J(u) - J(v)]\|_2,$$

where I is needed only if $\mu = 0$. Using now the identity

$$J(u) - J(v) = \frac{1}{2}\Big[\big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\big)(u + v) + \big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\big)(u - v)\Big]$$

together with Hölder's inequality (which we tacitly apply from now on), we find that

$$
\begin{aligned}
I &\lesssim \Big\| \Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big)(u+v) \Big\|_2 + \Big\| \Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \Big)(u-v) \Big\|_2 \\
&\lesssim \Big\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big\|_6 \|u+v\|_3 + \Big\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \Big\|_\infty \|u-v\|_2 .
\end{aligned} \tag{3.4}
$$

Observing that $e^{-\mu|x|}|x|^{-1} \in L^3_w$ holds, the first term of right-hand side of (3.4) can be bounded by means of the weak Young inequality (see, e. g., [LL01]) as follows

$$
\Big\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big\|_6 \lesssim \Big\| \frac{e^{-\mu|x|}}{|x|} \Big\|_{3,w} \| |u|^2 - |v|^2 \|_{6/5} \lesssim \|u+v\|_3 \|u-v\|_2 . \tag{3.5}
$$

The second term occurring in (3.4) can be estimated by noting that

$$
\begin{aligned}
\Big\| \frac{e^{-\mu|x|}}{|x|} * |u|^2 \Big\|_\infty &= \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-\mu|x-y|}|u(x)|^2}{|x-y|} \, dx \leq \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x-y|} \, dx \\
&\leq \frac{\pi}{2} \int_{\mathbb{R}^3} \overline{u}(x) \sqrt{-\Delta}\, u(x) \, dx = \frac{\pi}{2} \|(-\Delta)^{1/4} u\|_2^2 ,
\end{aligned} \tag{3.6}
$$

which follows from the operator inequality $|x-y|^{-1} \leq (\pi/2)(-\Delta_{x-y})^{1/2}$ (see, e. g., [Kat80, Section V.5.4]) and translational invariance, i. e., we use that $\Delta_{x-y} = \Delta_x$ holds for all $y \in \mathbb{R}^3$. Combining now (3.5) and (3.6) we find that

$$
\begin{aligned}
I &\lesssim \|u+v\|_3^2 \|u-v\|_2 + (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2) \|u-v\|_2 \\
&\lesssim (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2) \|u-v\|_{H^{1/2}} ,
\end{aligned}
$$

where we make use of the Sobolev inequality $\|u\|_3 \lesssim \|u\|_{H^{1/2}}$ in $\mathbb{R}^3$.

It remains to estimate II. To do so, we appeal to the generalized (or fractional) Leibniz rule (see Appendix A.1) leading to

$$
\begin{aligned}
II &\lesssim \Big\| \mathcal{D}^{1/2} \Big[\Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big)(u+v) \Big] \Big\|_2 \\
&\quad + \Big\| \mathcal{D}^{1/2} \Big[\Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \Big)(u-v) \Big] \Big\|_2 \\
&\lesssim \Big\| \mathcal{D}^{1/2} \Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big) \Big\|_6 \|u+v\|_3 \\
&\quad + \Big\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2) \Big\|_\infty \|\mathcal{D}^{1/2}(u+v)\|_2 \\
&\quad + \Big\| \mathcal{D}^{1/2} \Big[\Big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \Big] \Big\|_6 \|u-v\|_3 \\
&\quad + \Big\| \frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2) \Big\|_\infty \|\mathcal{D}^{1/2}(u-v)\|_2 .
\end{aligned} \tag{3.7}
$$

By referring to Appendix A.1, we notice that $\frac{e^{-\mu|x|}}{4\pi|x|} * f$ can be expressed as $\mathcal{D}^{-2} f = (\mu^2 - \Delta)^{-1} f$ in $\mathbb{R}^3$ (here $f \in \mathcal{S}(\mathbb{R}^3)$ is initially assumed, but our arguments follow by density). Thus, the first term of the right-hand side of (3.7) is found to be

$$
\begin{aligned}
\big\|\mathcal{D}^{1/2}\big(\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\big)\big\|_6 &\lesssim \|\mathcal{D}^{1/2-2}(|u|^2-|v|^2)\|_6 \\
&\lesssim \|\mathcal{D}^{-3/2}(|u|^2-|v|^2)\|_6 \\
&\lesssim \big\|G^{\mu}_{3/2} * (|u|^2-|v|^2)\big\|_6 \\
&\lesssim \big\|G^{\mu}_{3/2}\big\|_{2,w}\||u|^2-|v|^2\|_{3/2} \\
&\lesssim \|u+v\|_3\|u-v\|_3, \qquad (3.8)
\end{aligned}
$$

where we use weak Young's inequality together with the fact that $\mathcal{D}^{-3/2}f$ corresponds to $G^{\mu}_{3/2} * f$ with some $G^{\mu}_{3/2} \in L^2_w(\mathbb{R}^3)$; see (A.1). The $\|\cdot\|_\infty$-part of the second term occurring in (3.7) can be estimated by using the Cauchy-Schwarz inequality and (3.6) once again:

$$
\begin{aligned}
\big\|\frac{e^{-\mu|x|}}{|x|} * (|u|^2-|v|^2)\big\|_\infty &\le \big\|\frac{1}{|x|} * (|u|^2-|v|^2)\big\|_\infty \\
&\lesssim \sup_{y\in\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\frac{|u(x)|^2-|v(x)|^2}{|x-y|}\,dx\Big| \\
&\lesssim \sup_{y\in\mathbb{R}^3}\Big|\langle (u(x)+v(x)), \frac{1}{|x-y|}(u(x)-v(x))\rangle\Big| \\
&\lesssim \sup_{y\in\mathbb{R}^3}\big\langle (u(x)+v(x)), \frac{1}{|x-y|}(u(x)+v(x))\big\rangle^{1/2} \\
&\quad \cdot \sup_{y\in\mathbb{R}^3}\big\langle (u(x)-v(x)), \frac{1}{|x-y|}(u(x)-v(x))\big\rangle^{1/2} \\
&\lesssim \|(-\Delta)^{1/4}(u+v)\|_2\|(-\Delta)^{1/4}(u-v)\|_2 \\
&\lesssim (\|u\|_{H^{1/2}} + \|v\|_{H^{1/2}})\|u-v\|_{H^{1/2}} \qquad (3.9)
\end{aligned}
$$

The remaining terms in (3.7) deserve no further comment, since they can be estimated in a similar fashion to all estimates derived so far. Thus, we conclude that

$$
\|J(u)-J(v)\|_{H^{1/2}} \lesssim I + II \lesssim (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}})\|u-v\|_{H^{1/2}}
$$

and the proof of Lemma 1 is now complete. □

Remarks. 1) The proof of Lemma 1 relies on (3.6) in a crucial way. Employing just the Sobolev embedding $H^{1/2} \subset L^2 \cap L^3$ (in $\mathbb{R}^3$) together with the (non weak) Young inequality is not sufficient to conclude that $\|\frac{e^{-\mu|x|}}{|x|} * |u|^2\|_\infty < \infty$ whenever $u \in H^{1/2}$.

2) The proof of Lemma 1 fails for "super-critical" Hartree nonlinearities $J(u) = (|x|^{-\alpha} * |u|^2)u$, where $1 < \alpha < 3$. Thus, the choice $\alpha = 1$ represents a borderline case when deriving local Lipschitz continuity in energy space $H^{1/2}$.

Step 2: Conclusion

Returning to the proof of Theorem 1, we note that A defined in (3.1) gives rise to a self-adjoint operator L^2 with domain H^1. Moreover, its extension to $H^{1/2}$, which we denote by $A : H^{1/2} \to H^{-1/2}$, generates a C^0-group of isometries, $\{e^{-itA}\}_{t\in\mathbb{R}}$, acting on $H^{1/2}$. Local well-posedness in the sense of Theorem 1 now follows by standard methods for evolution equations with locally Lipschitz nonlinearities. That is, existence and uniqueness of a solution $u \in Y_T$ for the integral equation (3.2) is deduced by a fixed point argument, for $T > 0$ sufficiently small. The equivalence of the integral formulation (3.2) and the initial value problem (2.1), with $u_0 \in H^{1/2}$, as well as the blow-up alternative can also be deduced by standard arguments; see, e. g., [Paz83] for a general theory on nonlinear evolution equations. Finally, note that $u \in C^1([0,T); H^{-1})$ follows by equation (2.1) itself. The proof of Theorem 1 is now accomplished.

3.2 Proof of Theorem 2 (Global Well-Posedness)

The first step taken in the proof of Theorem 2 settles conservation of energy and charge that are given by

$$E[u] := \frac{1}{2} \int_{\mathbb{R}^3} \overline{u}(x) \sqrt{-\Delta + m^2}\, u(x)\, dx + \frac{1}{4} \int_{\mathbb{R}^3} \big(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2 \big)(x)\, |u(x)|^2\, dx, \tag{3.10}$$

$$N[u] := \int_{\mathbb{R}^3} |u(x)|^2\, dx, \tag{3.11}$$

respectively. After deriving the corresponding conservation laws (where proving energy conservation requires a regularization), we discuss how to obtain a-priori bounds on the energy norm of the solution.

Step 1: Conservation Laws

Lemma 2. *The local-in-time solutions of Theorem 1 obey*

$$E[u(t)] = E[u_0] \quad \textit{and} \quad N[u(t)] = N[u_0]$$

for all $t \in [0, T)$.

Proof of Lemma 2. Let u be a local-in-time solution derived in Theorem 1, and let T be its maximal time of existence. Since $u(t) \in H^{1/2}$ holds, we can multiply (2.1) by $i\bar{u}(t)$ and integrate over $\mathbb{R}^3$. Taking then real parts yields

$$\frac{d}{dt} N[u(t)] = 0 \quad \text{for} \quad t \in [0, T), \tag{3.12}$$

which shows conservation of charge.

At a formal level, conservation of energy follows by multiplying (2.1) with $\dot{\bar{u}}(t) \in H^{-1/2}$ and integrating over space, but the paring of two elements of $H^{-1/2}$ is not well-defined. Thus, we have to introduce a regularization procedure as follows; see also, e. g., [Caz03, GV00] for other regularization methods for nonlinear (nonrelativistic) Schrödinger equations. Let us define the family of operators

$$\mathcal{M}_\varepsilon := (\varepsilon A + 1)^{-1}, \quad \text{for} \quad \varepsilon > 0, \tag{3.13}$$

where the operator $A = \sqrt{-\Delta + m^2} \geq 0$ is taken from (3.1). Consider the sequences of embedded spaces

$$\ldots H^{3/2} \hookrightarrow H^{1/2} \hookrightarrow H^{-1/2} \hookrightarrow H^{-3/2} \ldots \tag{3.14}$$

It is easy to see (by using functional calculus) that the following properties hold.

a) For $\varepsilon > 0$ and $s \in \mathbb{R}$, we have that $\mathcal{M}_\varepsilon$ is a bounded map from H^s into H^{s+1}.

b) $\|\mathcal{M}_\varepsilon u\|_{H^s} \leq \|u\|_{H^s}$ whenever $u \in H^s$ and $s \in \mathbb{R}$.

c) For $u \in H^s$ and $s \in \mathbb{R}$, we have that $\mathcal{M}_\varepsilon u \to u$ strongly in H^s as $\varepsilon \downarrow 0$.

We shall use tacitly properties a) – c) in the following analysis.

By means of $\mathcal{M}_\varepsilon$ and noting that $E \in C^1(H^{1/2};\mathbb{R})$, we can compute in a well-defined way for $t_1, t_2 \in [0,T)$ as follows

$$\begin{aligned}
E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)] &= \int_{t_1}^{t_2} \langle E'(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon \dot{u}\rangle_{H^{-1/2},H^{1/2}}\, dt \\
&= \int_{t_1}^{t_2} \mathrm{Re}\,\langle A\mathcal{M}_\varepsilon u + F(\mathcal{M}_\varepsilon u), -i\mathcal{M}_\varepsilon(Au + F(u))\rangle\, dt \\
&= \int_{t_1}^{t_2} \mathrm{Im}\,\Big[\langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon A u\rangle + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon A u\rangle \\
&\quad + \langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon F(u)\rangle + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big]\, dt \\
&=: \int_{t_1}^{t_2} f_\varepsilon(t)\, dt, \qquad (3.15)
\end{aligned}$$

where we write $u = u(t)$ for brevity and recall the definition of F from (3.1). We observe that the first term in $f_\varepsilon(t)$ is the "most singular" part, i.e., if $\varepsilon = 0$ we would have pairing of two $H^{-1/2}$-elements. But for $\varepsilon > 0$ we can use the obvious fact that $\mathcal{M}_\varepsilon A = A\mathcal{M}_\varepsilon$ holds and conclude that

$$\mathrm{Im}\,\langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon A u\rangle = \mathrm{Im}\,\langle A\mathcal{M}_\varepsilon u, A\mathcal{M}_\varepsilon u\rangle = 0.$$

Notice that this manipulation is well-defined, because $A\mathcal{M}_\varepsilon u$ and $\mathcal{M}_\varepsilon A u$ are in $H^{1/2}$ whenever $u \in H^{1/2}$. After some simple calculations, we find $f_\varepsilon(t)$ to be of the form

$$\begin{aligned}
f_\varepsilon(t) &= \mathrm{Im}\,\Big[\langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon A u\rangle + \langle A\mathcal{M}_\varepsilon u, \mathcal{M}_\varepsilon F(u)\rangle + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big] \\
&= \mathrm{Im}\,\Big[\langle A^{1/2} F(\mathcal{M}_\varepsilon u), A^{1/2}\mathcal{M}_\varepsilon u\rangle + \langle A^{1/2}\mathcal{M}_\varepsilon u, A^{1/2}\mathcal{M}_\varepsilon F(u)\rangle \\
&\qquad + \langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle\Big],
\end{aligned}$$

Since $\mathcal{M}_\varepsilon u \to u$ strongly in $H^{1/2}$ as $\varepsilon \downarrow 0$, we can infer, by Lemma 1, that

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} f_\varepsilon(t) &= \mathrm{Im}\,\big[\langle A^{1/2}F(u), A^{1/2}u\rangle + \langle A^{1/2}u, A^{1/2}F(u)\rangle + \langle F(u), F(u)\rangle\big] \\
&= \mathrm{Im}\,(\text{Real Number}) = 0.
\end{aligned}$$

To interchange the ε-limit with the t-integration in (3.15), we appeal to the dominated convergence theorem. That is, we seek for a uniform bound on $f_\varepsilon(t)$. In fact, by using the Cauchy-Schwarz inequality and Lemma 1 again we find the following estimate

$$\begin{aligned}
|f_\varepsilon(t)| &\lesssim |\langle A^{1/2}F(\mathcal{M}_\varepsilon u), A^{1/2}\mathcal{M}_\varepsilon u\rangle| + |\langle A^{1/2}\mathcal{M}_\varepsilon u, A^{1/2}\mathcal{M}_\varepsilon F(u)\rangle| \\
&\quad + |\langle F(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon F(u)\rangle| \\
&\lesssim \|A^{1/2}F(\mathcal{M}_\varepsilon u)\|_2 \|A^{1/2}\mathcal{M}_\varepsilon u\|_2 + \|A^{1/2}\mathcal{M}_\varepsilon u\|_2 \|A^{1/2}\mathcal{M}_\varepsilon F(u)\|_2 \\
&\quad + \|F(\mathcal{M}_\varepsilon u)\|_2 \|\mathcal{M}_\varepsilon F(u)\|_2 \\
&\lesssim \|u\|_{H^{1/2}}^4 + \|u\|_{H^{1/2}}^6,
\end{aligned}$$

for all $\varepsilon > 0$. Putting now all together leads to conservation of energy, i.e., we find for all $t_1, t_2 \in [0,T)$ that

$$\begin{aligned}
E[u(t_2)] - E[u(t_1)] &= \lim_{\varepsilon\downarrow 0}\big(E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)]\big) \\
&= \lim_{\varepsilon\downarrow 0}\int_{t_1}^{t_2} f_\varepsilon(t)\, dt = \int_{t_1}^{t_2} \lim_{\varepsilon\downarrow 0} f_\varepsilon(t)\, dt = 0.
\end{aligned}$$

This completes the proof of Lemma 2. ∎

Step 2: A-Priori Bounds

To fill the last gap towards the global well-posedness result of Theorem 2, we now discuss how to obtain a-priori bounds on the energy norm. By the blow-up alternative of Theorem 1, global-in-time existence follows from an a-priori bound of the form

$$\|u(t)\|_{H^{1/2}} \leq C(u_0). \tag{3.16}$$

First, let us assume that $\lambda \geq 0$ holds. Then, for all $t \in [0, T)$, we find from Lemma 2 and (3.10) that

$$\|(-\Delta)^{1/4} u(t)\|_2 \lesssim E[u(t)] = E[u_0].$$

This implies together with charge conservation derived in Lemma 2, i. e.,

$$\|u(t)\|_2^2 = N[u(t)] = N[u_0] \tag{3.17}$$

an a-priori estimate (3.16). Therefore condition i) in Theorem 2 is sufficient for global existence.

For $\lambda < 0$, we can proceed as follows.

$$\begin{aligned}
E[u] &= \frac{1}{2}\|(-\Delta + m^2)^{1/4} u\|_2^2 - \frac{|\lambda|}{4} \int_{\mathbb{R}^3} (\frac{e^{-\mu|x|}}{|x|} * |u|^2)(x)\, |u(x)|^2 \, dx \\
&\geq \frac{1}{2}\|(-\Delta + m^2)^{1/4} u\|_2^2 - \frac{|\lambda|}{4} \int_{\mathbb{R}^3} (\frac{1}{|x|} * |u|^2)(x)\, |u(x)|^2 \, dx \\
&\geq \frac{1}{2}\|(-\Delta)^{1/4} u\|_2^2 - \frac{|\lambda|}{4K}\|(-\Delta)^{1/4} u\|_2^2 \|u\|_2^2 \\
&= \left(\frac{1}{2} - \frac{|\lambda|}{4K}\|u\|_2^2\right)\|(-\Delta)^{1/4} u\|_2^2,
\end{aligned} \tag{3.18}$$

where $K > 0$ is a best constant taken from Appendix A.2. Thus, energy conservation leads to an a-priori bound on the $H^{1/2}$-norm of the solution, if

$$|\lambda| \|u_0\|_2^2 < 2K \tag{3.19}$$

holds. In fact, the constant K satisfies

$$K = \frac{\|Q\|_2^2}{2} > \frac{2}{\pi},$$

where $Q(x)$ is a strictly positive (ground state) solution of

$$\sqrt{-\Delta}\, Q - (\frac{1}{|x|} * |Q|^2) Q = -Q; \tag{3.20}$$

see Appendix A.2. Going back to (3.19), we find that

$$|\lambda| \|u_0\|_2^2 < \|Q\|_2^2 \tag{3.21}$$

is sufficient for global existence. The proof of Theorem 2 is now complete.

3.3 Proof of Theorem 3 (Higher Regularity)

To prove Theorem 3, we need the following generalization of Lemma 1, whose proof is a careful but straightforward generalization of the proof of Lemma 1. We defer the details to Appendix A.3.

Lemma 3. *For $\mu \geq 0$ and $s \geq 1/2$, the map $J(u) := (\frac{e^{-\mu|x|}}{|x|} * |u|^2)u$ is locally Lipschitz continuous from $H^s(\mathbb{R}^3)$ into itself with*

$$\|J(u) - J(v)\|_{H^s} \lesssim (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|u - v\|_{H^s}$$

for all $u, v \in H^s(\mathbb{R}^3)$. Moreover, we have that

$$\|J(u)\|_{H^s} \lesssim \|u\|_{H^r}^2 \|u\|_{H^s}$$

holds for all $u \in H^s(\mathbb{R}^3)$, where $r = \max\{s-1, 1/2\}$.

Local well-posedness of (2.5) in H^s, for $s > 1/2$, can be shown now as follows. We note that $\{e^{-itA}\}_{t \in \mathbb{R}}$, with $A = \sqrt{-\Delta + m^2}$, is a C^0-group of isometries on H^s. Moreover, since the nonlinearity defined in (3.1), is locally Lipschitz continuous from H^s into itself, local well-posedness in H^s follows similarly as explained in the proof of Theorem 1 for $H^{1/2}$. To show global well-posedness in H^s, we prove by induction and Lemma 3 that an a-priori bound on the $H^{1/2}$-norm of solution implies uniform bounds on the H^s-norm on any compact interval $[0, T_*] \subset [0, T)$. This claim follows from (3.2) and the second inequality stated in Lemma 3 by noting that

$$\begin{aligned}
\|u(t)\|_{H^s} &\leq \|e^{-itA}u_0\|_{H^s} + \int_0^t \|e^{-i(t-\tau)A}F(u(\tau))\|_{H^s}\, d\tau \\
&\leq \|u_0\|_{H^s} + \int_0^t \|F(u(\tau))\|_{H^s}\, d\tau \\
&\lesssim C_1 + C_2 \int_0^t \|u(\tau)\|_{H^s}\, d\tau,
\end{aligned}$$

holds, provided that $\|u(t)\|_{H^r} \lesssim 1$ for $r = \max\{s-1, 1/2\} < s$. Invoking Gronwall's inequality we conclude that

$$\|u(t)\|_{H^s} \lesssim e^{C_2 T_*}, \quad \text{for} \quad t \in [0, T_*] \subset [0, T).$$

Induction now implies that an a-priori bound on $\|u(t)\|_{H^{1/2}}$ guarantees uniform bounds $\|u(t)\|_{H^s}$ on any compact interval $I \subset [0, T)$. Thus, the maximal time of existence of an H^s-valued solution coincides with the maximal time of existence when viewed as an $H^{1/2}$-valued solution. Therefore sufficient conditions for global existence for $H^{1/2}$-valued solutions imply global-in-time H^s-valued solutions. This completes the proof of Theorem 3.

3.4 Proof of Theorem 4 (External Potentials)

Let $V = V_+ + V_-$ satisfy Assumption 1 in Section 2. We introduce the quadratic form

$$\mathcal{Q}(u, v) := \langle u, \sqrt{-\Delta + m^2}\, v\rangle + \langle u, V_- v\rangle + \langle u, V_+ v\rangle, \tag{3.22}$$

which is well-defined on the set (energy space)

$$X := \big\{u \in L^2(\mathbb{R}^3) : \mathcal{Q}(u, u) < \infty\big\}. \tag{3.23}$$

Note that Assumption 1 also guarantees that $C_0^\infty(\mathbb{R}^3) \subset X$. It easy to show that our assumption on V implies that the quadratic form (3.22) is bounded from below, i. e., we have $\mathcal{Q}(u, u) \geq -M\langle u, u\rangle$ holds for all $u \in X$ and some constant $M \geq 0$. After adding M to $\mathcal{Q}$, we assume from now on (and without loss of generality) that

$$\mathcal{Q}(u, u) \geq 0 \tag{3.24}$$

holds for all $u \in X$. Since $\mathcal{Q}(\cdot, \cdot)$ is closed (it is a sum of closed forms), the energy space X equipped with its norm

$$\|u\|_X := \sqrt{\langle u, u\rangle + \mathcal{Q}(u, u)} \tag{3.25}$$

is complete, and we have the equivalence

$$\|u\|_{H^{1/2}} + \|V_+^{1/2}u\|_2 \lesssim \|u\|_X \lesssim \|u\|_{H^{1/2}} + \|V_+^{1/2}u\|_2. \tag{3.26}$$

Furthermore, there exists a nonnegative, self-adjoint operator

$$A : D(A) \subset L^2 \to L^2 \tag{3.27}$$

with $X = D(A^{1/2})$, such that

$$\langle u, Av \rangle = \mathcal{Q}(u, v) \tag{3.28}$$

holds for all $u \in X$ and $v \in D(A)$; see, e. g., [Kat80]. This operator can be extended to a bounded operator, still denoted by $A : X \to X^*$, where X^* is the dual space of X.

To prove now the assertion about local well-posedness in Theorem 4, we have to generalize Lemma 1 to the following statement.

Lemma 4. *Suppose $\mu \geq 0$ and let V satisfy Assumption 1. Then the map $J(u) := (\frac{e^{-\mu|x|}}{|x|} * |u|^2)u$ is locally Lipschitz continuous from X into itself with*

$$\|J(u) - J(v)\|_X \lesssim (\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X$$

for all $u, v \in X$.

Proof of Lemma 4. By (3.26), it suffices to estimate $\|J(u) - J(v)\|_{H^{1/2}}$ and $\|V_+^{1/2}[J(u) - J(v)]\|_2$ separately. By Lemma 1, we know that

$$\begin{aligned}\|J(u) - J(v)\|_{H^{1/2}} &\lesssim (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2)\|u - v\|_{H^{1/2}} \\ &\lesssim (\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X.\end{aligned}$$

It remains to estimate $\|V_+^{1/2}[J(u) - J(v)]\|_2$, which can be achieved by recalling (3.9) and proceeding as follows.

$$\begin{aligned}\|V_+^{1/2}[J(u) - J(v)]\|_2 &\lesssim \Big\|V_+^{1/2}\big[\big(\frac{e^{-\mu|x|}}{x} * (|u|^2 - |v|^2)\big)(u+v)\big]\Big\|_2 \\ &\quad + \Big\|V_+^{1/2}\big[\big(\frac{e^{-\mu|x|}}{x} * (|u|^2 + |v|^2)\big)(u-v)\big]\Big\|_2 \\ &\lesssim \Big\|\frac{e^{-\mu|x|}}{|x|} * (|u|^2 - |v|^2)\Big\|_\infty \|V_+^{1/2}(u+v)\|_2 \\ &\quad + \Big\|\frac{e^{-\mu|x|}}{|x|} * (|u|^2 + |v|^2)\Big\|_\infty \|V_+^{1/2}(u-v)\|_2 \\ &\lesssim \|u+v\|_{H^{1/2}}\|u-v\|_{H^{1/2}}\|V_+^{1/2}(u+v)\|_2 \\ &\quad + (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2)\|V_+^{1/2}(u-v)\|_2 \\ &\lesssim \big(\|u\|_X^2 + \|v\|_X^2)\|u - v\|_X.\end{aligned}$$

This completes the proof of Lemma 4. ∎

Returning to the proof of Theorem 4, we simply note that $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a C^0-group of isometries on X, where $A = \sqrt{-\Delta + m^2} + V$ is defined in the form sense (see above). By Lemma 4, the nonlinearity is locally Lipschitz on X. Thus, local well-posedness now follows in the same way as for Theorem 1.

To establish global well-posedness we have to prove conservation of charge, $N[u]$, and energy, $E[u]$, which is for (2.5) given by

$$\begin{aligned} E[u] := \frac{1}{2}\int_{\mathbb{R}^3} \overline{u}(x)\sqrt{-\Delta+m^2}\, u(x)\, dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u(x)|^2\, dx \\ + \frac{1}{4}\int_{\mathbb{R}^3} \big(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\big)(x)\, |u(x)|^2\, dx. \end{aligned} \tag{3.29}$$

As done in Section 3.2, we have to employ a regularization method using the class of operators

$$\mathcal{M}_\varepsilon := (\varepsilon A + 1)^{-1}, \quad \text{for} \quad \varepsilon > 0, \tag{3.30}$$

where we assume without loss of generality that $A \geq 0$ holds. The mapping $\mathcal{M}_\varepsilon$ acts on the sequence of embedded spaces

$$\ldots X^{+2} \hookrightarrow X^{+1} \hookrightarrow X^{-1} \hookrightarrow X^{-2} \ldots, \tag{3.31}$$

with corresponding norms given by $\|u\|_{X^s} := \|(1+A)^{s/2}u\|_2$. Note that $X = X^{+1}$ (with equivalent norms) and that its dual space obeys $X^* = X^{-1}$. By using functional calculus, it is easy to show that $\mathcal{M}_\varepsilon$ exhibits properties that are analog to a) – c) in Section 3.2.

The rest of the argument for proving conservation of energy carries over from Section 3.2 without major modifications. Finally, we mention that deriving a-priori bounds on $\|u(t)\|_X$ leads to a similar discussion as presented in Section 3.2, while noting that we have to take care that V_- has a relative $(-\Delta)^{1/2}$-form bound, $0 \leq a < 1$, introduced in Assumption 1. This completes the proof of Theorem 4.

A Appendix

A.1 Fractional Calculus

The following result (generalized Leibniz rule) is proved in [GK96] for Riesz and Bessel potentials of order $s \in \mathbb{R}$, which are denoted by $(-\Delta)^{s/2}$ and $(1-\Delta)^{s/2}$, respectively. But as a direct consequence of the Milhin multiplier theorem [BL76], the cited result holds for $\mathcal{D}^s := (\mu^2 - \Delta)^{s/2}$, where $\mu \geq 0$ is a fixed constant.

Lemma (Generalized Leibniz Rule). *Suppose that $1 < p < \infty$, $s \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, and $1/p_i + 1/q_i = 1/p$ with $i = 1, 2$, $1 < q_i \leq \infty$, $1 < p_i \leq \infty$. Then*

$$\|\mathcal{D}^s(fg)\|_p \leq c(\|\mathcal{D}^{s+\alpha} f\|_{p_1} \|\mathcal{D}^{-\alpha} g\|_{q_1} + \|\mathcal{D}^{-\beta} f\|_{p_2} \|\mathcal{D}^{s+\beta} g\|_{q_2}),$$

where the constant c depends on all of the parameters but not on f and g.

A second fact we use in the proof of our main result is as follows. For $0 < \alpha < 3$ and $\mu \geq 0$, the potential operator $\mathcal{D}^{-\alpha} = (\mu^2 - \Delta)^{-\alpha/2}$ corresponds to $f \mapsto G^\mu_\alpha * f$, with $f \in \mathcal{S}(\mathbb{R}^3)$, and we have that

$$G^\mu_\alpha \in L^{3/(3-\alpha)}_w(\mathbb{R}^3). \tag{A.1}$$

To see this, we refer to the inequality and the exact formula

$$0 \leq G^\mu_\alpha(x) \leq G^0_\alpha(x) = \frac{c_\alpha}{|x|^{3-\alpha}}, \quad \text{for} \quad \mu \geq 0 \quad \text{and} \quad 0 < \alpha < 3, \tag{A.2}$$

with some constant c_α; these facts can be derived from [Ste70, Section V.3.1]. Now (A.1) follows from $|x|^{-\sigma} \in L^{3/\sigma}_w(\mathbb{R}^3)$ whenever $0 < \sigma < 3$. Another observation used in Section 2 is the well-known explicit formula

$$G^\mu_2(x) = \frac{e^{-\mu|x|}}{4\pi|x|}. \tag{A.3}$$

That is, $(\mu^2 - \Delta)$ in $\mathbb{R}^3$ has the Green's function $\frac{e^{-\mu|x|}}{4\pi|x|}$ with vanishing boundary conditions.

A.2 Ground States

We consider the functional (see also [LY87])

$$K[u] := \frac{\|(-\Delta)^{1/4}u\|_2^2\|u\|_2^2}{\int_{\mathbb{R}^3}(|x|^{-1} * |u|^2)(x)\,|u(x)|^2\,dx}, \tag{A.4}$$

which is well-defined for all $u \in H^{1/2}$ with $u \not\equiv 0$. Note that by using (3.6) we can estimate the denominator in $K[u]$ as follows.

$$\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |u|^2\Big)(x)\,|u(x)|^2\,dx \leq \Big\|\frac{1}{|x|} * |u|^2\Big\|_\infty \|u\|_2^2 \leq \frac{\pi}{2}\|(-\Delta)^{1/4}u\|_2^2\|u\|_2^2, \tag{A.5}$$

which leads to the bound

$$\frac{2}{\pi} \leq K[u] < \infty. \tag{A.6}$$

Indeed, we will see that the estimate from below is a strict inequality. With respect to the related variational problem

$$K := \inf\big\{K[u] : u \in H^{1/2}(\mathbb{R}^3),\ u \not\equiv 0\big\} \tag{A.7}$$

we can state the following result.

Lemma (Ground States). *There exists a minimizer, $Q \in H^{1/2}(\mathbb{R}^3)$, for (A.7), and we have the following properties.*

i) $Q(x)$ is a smooth function that can be chosen to be real-valued, strictly positive, and spherically symmetric with respect to the origin. It satisfies

$$\sqrt{-\Delta}\,Q - \Big(\frac{1}{|x|} * |Q|^2\Big)Q = -Q, \tag{A.8}$$

and it is nonincreasing, i. e., we have that $Q(x) \geq Q(y)$ whenever $|x| \leq |y|$.

ii) $Q(x)$ is unique in the following sense. For every $\widetilde{Q}(x)$ minimizing (A.7), there exist constants $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\beta > 0$, and $y \in \mathbb{R}^3$, such that $\widetilde{Q}(x) = \alpha Q(\beta(x+y))$ holds.

iii) The infimum satisfies $K = \|Q\|_2^2/2$ and $K > 2/\pi$.

Sketch of Proof. We present the main ideas for the proof of the preceding lemma. That (A.7) is attained at some real-valued, radial, nonnegative and nonincreasing function $Q(x) \geq 0$ can be proved by direct methods of variational calculus and rearrangement inequalities; see also [Wei83] for a similar variational problem for nonrelativistic Schrödinger equations with local nonlinearities. Furthermore, any minimizer, $Q \in H^{1/2}$, has to satisfy the corresponding Euler-Lagrange equation that reads

$$\sqrt{-\Delta}\,Q - \Big(\frac{1}{|x|} * |Q|^2\Big)Q = -Q, \tag{A.9}$$

after a suitable rescaling $Q(x) \mapsto aQ(bx)$ with some $a, b > 0$.

Let us make some comments about the properties of Q. Using an bootstrap argument and Lemma 3 for the nonlinearity, it follows that Q belongs to H^s, for all $s \geq 1/2$. Hence it is a smooth function. To see that $Q(x) \geq 0$ is strictly positive, i. e., $Q(x) > 0$, we rewrite equation (A.9) such that

$$Q = \big(\sqrt{-\Delta} + 1\big)^{-1} W, \tag{A.10}$$

where $W := (|x|^{-1} * |Q|^2)Q$. By functional calculus, we have

$$\big(\sqrt{-\Delta} + 1\big)^{-1} = \int_0^\infty e^{-t} e^{-t\sqrt{-\Delta}}\,dt. \tag{A.11}$$

Next, we notice that $e^{-t\sqrt{-\Delta}}$, for $t > 0$, is positivity improving; see, e. g., [RS78, Section XIII.12]. This means that if $f \geq 0$ with $f \not\equiv 0$ then $e^{-t\sqrt{-\Delta}} f > 0$ almost everywhere. Hence $(\sqrt{-\Delta}+1)^{-1}$ is also positivity improving, by (A.11), and we conclude that $Q(x) > 0$ holds almost everywhere, thanks to (A.10) and $W \geq 0$. Moreover, we know that $Q(x)$ is a nonincreasing, continuous function. Therefore $Q(x) > 0$ holds in the strong sense, i. e., for every $x \in \mathbb{R}^3$.

The statement about the uniqueness described in ii) follows by adapting a trick invented in [Lie77] for showing uniqueness of positive solutions, $\phi_\omega(x) > 0$, for the nonrelativistic equation

$$-\Delta\phi_\omega - \big(\frac{1}{|x|} * |\phi_\omega|^2\big)\phi_\omega = -\omega\phi_\omega,$$

where $\omega > 0$; details on this adaption will be elaborated in future work.

Finally, to see that iii) holds, we consider the variational problem

$$I_N := \inf\big\{E[u] : u \in H^{1/2}(\mathbb{R}^3),\ \|u\|_2^2 = N\big\}, \tag{A.12}$$

where $N > 0$ is a given parameter and

$$E[u] = \frac{1}{2}\|(-\Delta)^{1/4}u\|_2^2 - \frac{1}{4}\int_{\mathbb{R}^3}\big(\frac{1}{|x|} * |u|^2\big)\,|u(x)|^2\,dx.$$

Due to the scaling behavior $E[\alpha^{3/2}u(\alpha\cdot)] = \alpha E[u]$, we have that either $I_N = 0$ or $I_N = -\infty$ holds. By noting that

$$E[u] \geq \big(\frac{1}{2} - \frac{N}{4K}\big)\|(-\Delta)^{1/4}u\|_2^2,$$

and the fact that equality holds if and only if u minimizes $K[u]$, we find that $I_N = 0$ holds if and only if $N \leq N_c := 2K$. Moreover, $I_N = 0$ is attained if and only if $N = N_c$. Let $\tilde{Q}$ be such a minimizer with $\|\tilde{Q}\|_2^2 = N_c$. Thanks to the proof of part i), we can assume without loss of generality that $\tilde{Q}$ is real-valued, radial, and strictly positive. Calculating the Euler-Lagrange equation for (A.12), with $N = N_c$, yields

$$\sqrt{-\Delta}\,\tilde{Q} - \big(\frac{1}{|x|} * |\tilde{Q}|^2\big)\tilde{Q} = -\theta\tilde{Q},$$

for some multiplier θ, where it is easy to show that $\theta > 0$ holds. Putting now $Q(x) = \theta^{-3/2}\tilde{Q}(\theta^{-1}x)$, which conserves the L^2-norm, leads to the unique ground state $Q(x)$ introduced in i). Thus, we have that

$$K = \|\tilde{Q}\|_2^2/2 = \|Q\|_2^2/2.$$

To prove that $K > 2/\pi$ holds, let us assume $K = \pi/2$. This implies that the first inequality in (A.5) is an equality for $u(x) = Q(x) > 0$. But this leads to $(|x|^{-1} * |Q|^2)(x) = \text{const.}$, which is impossible. $\square$

A.3 Proof of Lemma 3

Proof of Lemma 3. We only show the second inequality derived in Lemma 3, since the first one can be proved in a similar way.

Let $\mu \geq 0$ and $s \geq 1/2$. We put $\mathcal{D}^\alpha := (\mu^2 - \Delta)^{\alpha/2}$ for $\alpha \in \mathbb{R}$. By the generalized Leibniz rule and (3.6), we have that

$$\begin{aligned}\|\mathcal{D}^s J(u)\|_2 &\lesssim \|\mathcal{D}^s[(\mathcal{D}^{-2}|u|^2)u]\|_2\\ &\lesssim \|\mathcal{D}^{s-2}|u|^2\|_{p_1}\|u\|_{q_1} + \|\mathcal{D}^{-2}|u|^2\|_\infty\|\mathcal{D}^s u\|_2\\ &\lesssim \|\mathcal{D}^{s-2}|u|^2\|_{p_1}\|u\|_{q_1} + \|u\|_{H^{1/2}}^2\|u\|_{H^s},\end{aligned} \tag{A.13}$$

where $1/p_1 + 1/q_1 = 1/2$ with $1 < p_1, q_1 \leq \infty$. The first term of the right-hand side of (A.13) can be controlled as follows, where we introduce $r = \max\{s-1, 1/2\}$.

i) For $1/2 \le s < 3/2$, we choose $p_1 = 3/s$ and $q_1 = 6/(3-2s)$ which leads to

$$\begin{aligned}\|\mathcal{D}^{s-2}|u|^2\|_{3/s}\|u\|_{6/(3-2s)} &\lesssim \|G^{\mu}_{2-s}\|_{3/(1+s),w}\||u|^2\|_{3/2}\|u\|_{H^s}\\ &\lesssim \|u\|^2_{H^{1/2}}\|u\|_{H^s} \lesssim \|u\|^2_{H^r}\|u\|_{H^s},\end{aligned}$$

where we use the weak Young inequality, Sobolev's inequality $\|u\|_{6/(3-2s)} \lesssim \|u\|_{H^s}$ in $\mathbb{R}^3$, and (A.1) once again.

ii) For $s \ge 3/2$, we put $p_1 = 6$ and $q_1 = 3$ and find

$$\begin{aligned}\|\mathcal{D}^{s-2}|u|^2\|_6\|u\|_3 &\lesssim \|\mathcal{D}^{s-3/2}|u|^2\|_2\|u\|_3 \lesssim \|\mathcal{D}^{s-3/2}u\|_6\|u\|_3^2\\ &\lesssim \|\mathcal{D}^{s-1}u\|_2\|u\|_3^2 \lesssim \|u\|_{H^r}\|u\|_{H^s},\end{aligned}$$

while using twice Sobolev's inequality $\|f\|_6 \lesssim \|\mathcal{D}^{1/2}f\|_2$ in $\mathbb{R}^3$.

Putting now all together, we conclude that

$$\begin{aligned}\|J(u)\|_{H^s} &\lesssim \|J(u)\|_2 + \|\mathcal{D}^s J(u)\|_2\\ &\lesssim \|u\|^2_{H^{1/2}}\|u\|_2 + \|u\|^2_{H^r}\|u\|_{H^s} \lesssim \|u\|^2_{H^r}\|u\|_{H^s}.\end{aligned}$$

□

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