On the Mathematical Structure of Quantum Measurement Theory

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Abstract

We show that the key problems of quantum measurement theory, namely the reduction of the wave-packet of a microsystem and the specification of its quantum state by a macroscopic measuring instrument, may be rigorously resolved within the traditional framework of the quantum mechanics of finite conservative systems. The argument is centred on the generic model of a microsystem, S, coupled to a finite macroscopic measuring instrument $\mathcal{I}$, which itself is an N-particle quantum system. The pointer positions of $\mathcal{I}$ correspond to the macrostates of this instrument, as represented by orthogonal subspaces of the Hilbert space of its pure states. These subspaces, or 'phase cells', are the simultaneous eigenspaces of a set of coarse grained intercommuting macro-observables, M, and, crucially, are of astronomically large dimensionalities, which increase exponentially with N. We formulate conditions on the conservative dynamics of the composite $(S+\mathcal{I})$ under which it yields both a reduction of the wave packet describing the state of S and a one-to-one correspondence, following a measurement, between the observed value of M and the resultant eigenstate of S; and we show that these conditions are fulfilled by the finite version of the Coleman-Hepp model.

Key Words: Schroedinger dynamics of microsystem-cum-measuring instrument; macroscopic phase cells as pointer positions; macroscopic decoherence; reduction of wave packet of microsystem.

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1. Introductory Discussion.

The quantum theory of measurement is concerned with the determination of the state of a microsystem, S, such as an atom, by a macroscopic measuring instrument, $\mathcal{I}$. In Von Neumann's [1] *phenomenological* picture, the $S - \mathcal{I}$ coupling leads to two essential effects. Firstly it converts a pure state of S, as given by a linear combination $\sum_{r=1}^{n} c_r u_r$ of its orthonormal eigenstates u_r, into a statistical mixture of these states for which $|c_r|^2$ is the probability of finding this system in the state u_r: this is the phenomenon often termed the 'reduction of the wave packet'. Secondly, it sends a certain set of classical, i.e. intercommuting, macroscopic variables M of $\mathcal{I}$ to values, indicated by pointers, that specify the state u_r of S that is actually realised. The problem of the quantum theory of this process is to characterise the properties of the macroscopic observables M and the dynamics of the composite $S_c = (S+\mathcal{I})$ that lead to these two effects. Our objective here is to treat this problem on the basis of the model for which S_c is a strictly conservative finite quantum system, whose dynamics is governed by its many-particle Schroedinger equation; and our main result is that this model does indeed contain the structures required for the resolution of this problem. This result provides mathematical justification for the heuristic arguments of Van Kampen [2], which led to essentially the same conclusion. It also establishes that there is no need to base quantum measurement theory on the model, advocated by some authors [3-7], in which S_c is a dissipative system, as a result of *either* its interaction with the 'rest of the Universe' [3-6] *or* a certain postulated nonlinear modification of its Schroedinger equation that leads to a classical deterministic evolution of its macroscopic observables [7].

As regards the main requirements of a satisfactory theory of the measurement process, it is clear from the works of Bohr [8], Jauch [9] and Van Kampen [2] that such a theory demands both a characterisation of the macroscopicality of the observables M and an amplification property of the $S - \mathcal{I}$ coupling whereby different microstates of S give rise to macroscopically different states of $\mathcal{I}$. Evidently, this implies that the initial state in which $\mathcal{I}$ is prepared must be unstable against microscopic changes in the state of S. On the other hand, as emphasised by Whitten-Wolfe and Emch [10, 11], the correspondence between the microstate of S and the eventual observed macrostate of $\mathcal{I}$ must be stable against macroscopically small changes in the initial state of this instrument, of the kind that are inevitable in experimental procedures. Thus, the initial state of $\mathcal{I}$ must be *metastable* by virtue of this combination of stability and instability properties.

There are basically two ways of characterising the macroscopicality of the observables M of $\mathcal{I}$. The first is to represent this instrument as a large but finite N- particle system for which N is extremely large, e.g. of the order of 10^{24}. M is then represented according to the scheme of Van Kampen [12] and Emch [13] as an intercommuting set of observables, which typically are coarse-grained extensive conserved variables of parts or of the whole of this instrument. The simultaneous eigenspaces of these observables then correspond to classical 'phase cells' , which represent the possible positions of the pointers of $\mathcal{I}$. Moreover, for suitably coarse-grained macroscopic observables M, the dimensionality of each of these cells is astronomically large [2, 12] since, by Boltzmanns formula, it is just the exponential of the entropy of the macrostate that it represents, and thus it increases exponentially with

N.

The second way of characterising the macroscopicality of M is to idealise the instrument $\mathcal{I}$ as an infinitely extended system of particles, with finite number density, and to take M to be a set of global intensive observables, which necessarily intercommute [14, 10, 11]. This corresponds to the picture employed in the statistical mechanical description of large systems in the thermodynamic limit [15-17], and it has the merit of sharply distinguishing between macroscopically different states, since different values of M correspond to disjoint primary representations of the observables. Moreover, in the treatments of the measurement problem based on this idealisation, the models of Hepp [14] and Whitten-Wolfe and Emch [10, 11] do indeed exhibit the required reduction of the wave-packet and the one-to-one correspondence between the pointer position of $\mathcal{I}$ and the resultant state of S; and these results are stable against all localised perturbations of the initial state of $\mathcal{I}$. On the debit side, however, Hepp's model requires an infinite time for the measurement to be effected (cf. Bell [18]), while although that of Whitten-Wolfe and Emch achieves its measurements in finite times, it does so only by dint of a physically unnatural, globally extended $S - \mathcal{I}$ interaction. In view of these observations, it appears to be worthwhile to explore the mathematical structure of the measuring process on the basis of the model for which $\mathcal{I}$ is large but finite, with the aim of obtaining conditions under which it yields the essential results obtained for the infinite model instrument, but with a finite realistic observational time. Evidently, this requires rigorous control of any approximations that arise as a result of the finiteness of N.

The object of this article is to investigate the mathematical structure of the measurement process by means of a dynamical treatment of the generic model of the composite, S_c, of a microsystem, S, and a macroscopic, but finite, measuring instrument $\mathcal{I}$. Our treatment of this model is designed to obtain conditions on the $S-\mathcal{I}$ coupling that lead to both the reduction of the wave-packet and the required correspondence between the reading of the instruments pointer and the resultant state of S. As in the works [2] , [8-11] and [14], we avoid the assumption of Von Neumann [1] and Wigner [19] that the observation of the pointer of $\mathcal{I}$ requires another measuring instrument, $\mathcal{I}_2$, which in turn requires yet another instrument, and so on, in such a way that the whole process involves an infinite regression ending up in the observer's brain! Instead, we assume that the measurement process ends with the reading of the pointers that evaluate the macrovariables M of $\mathcal{I}$. This carries the implicit assumption that the dynamics of these variables is sufficiently robust to ensure that the act of reading the pointers has negligible effect on their positions. In this sense, the macroscopic variables of $\mathcal{I}$ behaves radically differently from the observables of S, since the states of 'small' quantum microsystems are susceptible to drastic changes as a result of microscopic disturbances. A further crucial property of $\mathcal{I}$ is that, as pointed out above, the dimensionality of each of its macroscopic phase cells is of astronomically large dimensionality, which increases exponentially with N; and, by a similar argument, the same is true for its density of energy eigenstates We shall see that the enormous phase cells of the finite instrument $\mathcal{I}$ play the essential role of the disjoint representation spaces of the infinite one and consequently that the finite model possesses all the positive properties of the infinite one, with the bonus that it achieves its measurements in finite, realistic times.

Furthermore, the enormity of its density of energy eigenstates ensures that the periods of its Poincare recurrences are astronomically long. We can therefore discount these recurrences by restricting our treatment of the dynamics to finite intervals of much shorter duration.

Turning now to the formulation of the measurement process, we assume, in a standard way, that the observables of S and the macroscopic ones, M, of $\mathcal{I}$, on which measurements are performed, generate $W^{\star}$-algebras $\mathcal{A}$ and $\mathcal{M}$, respectively, the latter being abelian. The process is expressed in terms of the state on the algebra $\mathcal{A}\otimes\mathcal{M}$ that results from the evolution of S_c from an initial state obtained by independent preparations of S and $\mathcal{I}$. The resultant evolved state of S_c then determines the expectation values of the observables, A, of S and their conditional expectation values, $E(A|\mathcal{M})$, given the M's. Thus it determines the probabilistic state, ρ, of S *before* those macroscopic variables are measured and its *subsequent* state, as given by the form of $E(.|\mathcal{M})$, following the measurement.

On relating the state ρ and the conditional expectation functional $E(.|\mathcal{M})$ to the $S-\mathcal{I}$ interaction, we find that the mathematical model yields two classes of effective instruments $\mathcal{I}$, though from the empirical standpoint these classes are essentially equivalent. The first class of instruments comprises those for which the wave packet of S is reduced according to Von Neumann's prescription and the correspondence between the observed value of M and the microstate of S is strictly one-to-one. The second class of instruments comprises those for which this result arises with overwhelming probability, for large N, rather than with absolute certainty. Thus, in this case, if the result of a measurement is interpreted on the basis of an assumption of a perfect correspondence between the microstate of S and the macrostate of $\mathcal{I}$, then there is a miniscule possibility that the pointer position will correspond to a state of S quite different from (in general orthogonal to) the indicated one. We term the instruments of the first class *ideal* and those of the second class *normal.* As support for this classification of instruments, we show that, in the case of a finite version of the Coleman-Hepp model [14], the instrument is generically normal, though it is ideal for certain special values of its parameters. Furthermore, in the former case, the odds against the pointer indicating the 'wrong' state of S increase exponentially with N.

We present our mathematical treatment of the measurement process as follows. In Section 2, we formulate the generic model of the composite quantum system S_c, employing the phase cell representation of Van Kampen [12] and Emch [13] for the description of the macroscopic observables of $\mathcal{I}$. In particular, we formulate the time-dependent expectation values of the observables, A, of S and of the macroscopic ones, M, of $\mathcal{I}$, as well as the conditional expectation value of A, given $\mathcal{M}$, subject to the assumption that S and $\mathcal{I}$ are independently prepared and then coupled together at time $t = 0$. In Section 3, we formulate the conditions on the dynamics of the model under which the measuring instrument $\mathcal{I}$ is ideal or normal, in the sense described above. In Section 4, we show that the general scheme of Sections 2 and 3 is fully realized by the finite version of the Coleman-Hepp model [14]. There we show that the instrument $\mathcal{I}$ for this model is generically normal, though for certain special values of its parameters it is ideal. Moreover, we show that these results are stable under localised perturbations of the initial state of $\mathcal{I}$, and even under global ones that correspond to small changes in the values of intensive thermodynamical variables (e.g.

temperature, polarisation) of that state. Here we take the generic prevalence of normality of $\mathcal{I}$, for this model, to be an indication that real quantum measuring instruments are generally normal rather than ideal. We conclude, in Section 5, with a brief resume of the picture presented here.and a suggestion about a possible further development in the physics of the quantum measutrement process.

2. The Generic Model.

We assume that the algebras of observables, $\mathcal{A}$ and $\mathcal{B}$, of the microsystem S and the instrument $\mathcal{I}$, are those of the bounded operators in separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Correspondingly, the states of these systems are represented by the density matrices in the respective spaces. The density matrices for the pure states are then the one-dimensional projectors. For simplicity we assume that $\mathcal{H}$ is of finite dimensionality n.

We assume that the coupled composite $S_c := (S+\mathcal{I})$ is a conservative system, whose Hamiltonian operator H_c, in $\mathcal{H}\otimes\mathcal{K}$, takes the form

$$H_c = H\otimes I_{\mathcal{K}} + I_{\mathcal{H}}\otimes K + V, \tag{2.1}$$

where H and K are the Hamiltonians of S and $\mathcal{I}$, respectively, and V is the $S-\mathcal{I}$ interaction. Thus, the dynamics of S_c is given by the one- parameter group U_c of unitary transformations of $\mathcal{H}\otimes\mathcal{K}$ generated by iH_c, i.e.

$$U_c(t) = \exp(iH_c t)\ \forall\ t\in\mathbf{R}. \tag{2.2}$$

We assume that the the systems S and $\mathcal{I}$ are prepared, independently of one another, in their initial states represented by density matrices ω and Ω, respectively, and then coupled together at time $t=0$. Thus the initial state of the composite S_c is given by the density matrix $\omega\otimes\Omega$ in $\mathcal{H}_c := \mathcal{H}\otimes\mathcal{K}$. Further, we assume that the initial state of S is pure, and thus that ω is the projection operator $P(\psi)$ for a vector ψ in $\mathcal{H}$. The initial state of S_c is then

$$\Phi = P(\psi)\otimes\Omega. \tag{2.3}$$

Since $\mathcal{H}$ is n-dimensional, we may take as its basis a complete orthonormal set of eigenvectors, $(u_1,.\ .,u_n)$, of H . Hence, the initial state vector ψ of S is given by a linear combination of these vectors, i.e.

$$\psi = \sum_{r=1}^{n} c_r u_r, \tag{2.4}$$

where

$$\sum_{r=1}^{n} |c_r|^2 = 1; \tag{2.5}$$

while the action of H on u_r is given by the equation

$$Hu_r = \epsilon_r u_r, \tag{2.6}$$

where ϵ_r is the corresponding eigenvalue of this operator.

We assume that the instrument $\mathcal{I}$ is designed to perform measurements of the first kind (cf. Jauch [9]), whereby the $S-\mathcal{I}$ coupling does not induce transitions between the eigenstates $\{u_r\}$ of S. This signifies that the interaction V takes the form

$$V=\sum\nolimits_{r=1}^{n}P(u_r)\otimes V_r,$$

where $P(u_r)$ is the projection operator for u_r and the V_r's are observables of $\mathcal{I}$. Hence, by Eq. (2.1), the Hamiltonian of the composite system S_c is

$$H_c=\sum\nolimits_{r=1}^{n}P(u_r)\otimes K_r, \tag{2.7}$$

where

$$K_r=K+V_r+\epsilon_r I_{\mathcal{K}}. \tag{2.8}$$

Consequently, by Eqs. (2.2) and (2.7), the dynamical group U_c is given by the formula

$$U_c(t)=\exp(iH_ct)=\sum\nolimits_{r=1}^{n}P(u_r)\otimes U_r(t), \tag{2.9}$$

where

$$U_r(t)=\exp(iK_rt). \tag{2.10}$$

Consequently, since the evolute at time t (≥ 0) of the initial state Φ of S_c is $U_c^{\star}(t)\Phi U_c(t):=\Phi(t)$, it follows from Eqs.(2.3), (2.4) and (2.10) that

$$\Phi(t)=\sum\nolimits_{r,s=1}^{n}\bar{c}_rc_sP_{r,s}\otimes\Omega_{r,s}(t), \tag{2.11}$$

where $P_{r,s}$ is the operator in $\mathcal{H}$ defined by the equation

$$P_{r,s}f=(u_s,f)u_r\ \forall\ f\in\mathcal{H} \tag{2.12}$$

and

$$\Omega_{r,s}(t)=U_r^{\star}(t)\Omega U_s(t). \tag{2.13}$$

2.1. The Macroscopic Observables of $\mathcal{I}$. We assume that these conform to the following scheme, due to Van Kampen [12] and Emch [13].

(1) They are intercommuting observables, which typically are coarse grained extensive conserved variables of parts or of the whole of the system $\mathcal{I}$. The algebra, $\mathcal{M}$, of these observables is therefore an abelian subalgebra of the full algebra, $\mathcal{B}$, of bounded observables of $\mathcal{I}$. For simplicity, we assume that $\mathcal{M}$ is finitely generated and thus that it consists of the linear combinations of a finite set of orthogonal projectors $\{\Pi_\alpha|\alpha=1,2.\,.,\nu\}$ that span the space $\mathcal{K}$. It follows from these specifications that

$$\Pi_\alpha\Pi_\beta=\Pi_\alpha\delta_{\alpha\beta}, \tag{2.14}$$

$$\sum\nolimits_{\alpha=1}^{\nu}\Pi_\alpha = I_{\mathcal{K}} \tag{2.15}$$

and that any element, M, of $\mathcal{M}$ takes the form

$$M = \sum\nolimits_{\alpha=1}^{\nu} M_\alpha \Pi_\alpha, \tag{2.16}$$

where the M_α's are constants.The subspaces $\{\mathcal{K}_\alpha := \Pi_\alpha\mathcal{K}\}$ of $\mathcal{K}$ correspond to classical phase cells. Each such cell then represents a macrostate of $\mathcal{I}$, and is identified by the position of a pointer (or set of pointers) in a measurement process.

(2) As noted in Section 1, the dimensionality of each cell $\mathcal{K}_\alpha$ is astronomically large, since it is given essentially by the exponential of the entropy function of the macro-observables and thus grows exponentially with N. The largeness of the phase cells is closely connected to the robustness of the macroscopic measurement.

Note here that these properties of $\mathcal{M}$ are just general ones of macroscopic observables and do not depend on $\mathcal{I}$ being a measuring instrument for the system S. The coordination of these properties with those of S that are pertinent to the measuring process will be treated in Section 3.

2.2. Expectation and Conditional Expectation Values of Observables. The observables of S_c with which we shall be concerned are just the self-adjoint elements of $\mathcal{A}\otimes\mathcal{M}$. Their expectation values for the time- dependent state $\Phi(t)$ are given by the formula

$$E\big(A\otimes M\big) = \mathrm{Tr}\big(\Phi(t)[A\otimes M]\big)\ \forall\ A\in\mathcal{A},\ M\in\mathcal{M}, \tag{2.17}$$

In particular, the expectation values of the observables of S and the macroscopic ones of $\mathcal{I}$ are given by the equations

$$E(A) = E(A\otimes I_{\mathcal{K}}) \tag{2.18}$$

and

$$E(M) = E(I_{\mathcal{H}}\otimes M), \tag{2.19}$$

respectively. Further, in view of the abelian character of $\mathcal{M}$, the expectation functional E is compatible with a unique conditional expectation functional on $\mathcal{A}$ with respect to $\mathcal{M}$, as the following argument shows. Such a conditional expectation is a linear mapping $E(.|\mathcal{M})$ of $\mathcal{A}$ into $\mathcal{M}$ that preserves positivity and normalisation and satisfies the condition

$$E\big(E(A|\mathcal{M})M\big) = E(A\otimes M)\ \forall\ A\in\mathcal{A},\ M\in\mathcal{M}. \tag{2.20}$$

Therefore since, by linearity and Eq. (2.16), $E(.|\mathcal{M})$ must take the form

$$E(A|\mathcal{M}) = \sum\nolimits_\alpha \omega_\alpha(A)\Pi_\alpha, \tag{2.21}$$

where the ω_α's are linear functionals on $\mathcal{A}$, it follows from Eq. (2.20) that

$$\omega_\alpha(A)E(\Pi_\alpha) = E(A\otimes\Pi_\alpha)$$

and consequently, by Eq. (2.21),

$$E(A|\mathcal{M}) = \sum\nolimits_{\alpha}' E(A\otimes\Pi_\alpha)\Pi_\alpha/E(\Pi_\alpha)\ \forall\ A\in\mathcal{A}, \tag{2.22}$$

where the prime over Σ indicates that summation is confined to the α's for which $E(\Pi_\alpha)$ does not vanish. In view of Eq. (2.15), this formula for $E(.|\mathcal{M})$ meets the requirements of positivity and normalisation.

2.3. Properties of the Expectation Functional E**.** By Eqs. (2.11)-(2.13), (2.16) and (2.17),

$$E\big(A\otimes M\big) = \sum\nolimits_{r,s=1}^{n}\sum\nolimits_{\alpha=1}^{\nu}\overline{c}_r c_s(u_r, Au_s)M_\alpha F_{r,s;\alpha}, \tag{2.23}$$

where

$$F_{r,s;\alpha} = \mathrm{Tr}\big(\Omega_{r,s}(t)\Pi_\alpha\big). \tag{2.24}$$

Key properties of $F_{r,s:\alpha}$, which follows from Eqns. (2.13), (2.15) and (2.24) are that

$$\sum\nolimits_{\alpha=1}^{\nu}F_{r,r;\alpha} = 1, \tag{2.25}$$

$$1\geq F_{r,r;\alpha}\geq 0 \tag{2.26}$$

and

$$F_{r,s:\alpha} = \overline{F}_{s,r:\alpha}, \tag{2.27}$$

where the bar over F on the r.h.s. indicates complex conjugation. It also follows from those formulae that, for $z_1,.\ .,z_n\in\mathbf{C}$, the sesquilinear form $\sum_{r,s=1}^{n}\overline{z}_r z_s F_{r,s;\alpha}$ is positive, and hence

$$F_{r,r;\alpha}F_{s,s;\alpha}\geq|F_{r,s;\alpha}|^2. \tag{2.28}$$

3. The Measurement Process

As noted in Section 2, a pointer reading of $\mathcal{I}$ serves to identify the phase cells $\mathcal{K}_\alpha$ that represents its macrostate. Eq. (2.22) therefore signifies that the expectation values of the observables of S following that measurement is

$$E(A|\mathcal{K}_\alpha) := E(A\otimes\Pi_\alpha)/E(\Pi_\alpha). \tag{3.1}$$

Now, in order that the pointer reading specifies the eigenstate of S, we require a one- to-one correspondence between the phase cells $\mathcal{K}_\alpha$ and the eigenstates u_r of S. Accordingly, we assume that, for an instrument designed to identify the microstate of S, the number of these phase cells is just the number of the eigenstates u_r of S, namely n.

3.1. The Ideal Instruments. We term the instrument $\mathcal{I}$ *ideal* if there is a one-to-one correspondence between the pointer reading α and the eigenstate u_r of S, on a realistic observational time scale. Thus $\mathcal{I}$ is ideal if, for times t greater than some critical value, τ, and less, in order of magnitude, than the Poincare recurrence times, (I.1) the time-dependent state $\Omega_{r,r}(t)$ of $\mathcal{I}$, that arises in conjunction with the state u_r of S in the formula Eqs. (2.13), lies in one of the subspaces $\mathcal{K}_\alpha$ of $\mathcal{K}$;

(I.2) the correspondence between r and α here is one-to-one, i.e. $\alpha = a(r)$, where a is an invertible transformation of the point set $\{1, 2, . ., n\}$; and

(I.3) this correspondence is stable with respect to perturbations of the initial state Ω of $\mathcal{I}$ that are localised, in the sense that each of them leaves this state unchanged outside some region contained in a ball of volume $O(1)$ with respect to N.

The conditions (I.1) and (I.2) signify that, for times t in the range specified there,

$$\mathrm{Tr}\big(\Omega_{r,r}(t)\Pi_\alpha\big) = \delta_{a(r),\alpha},$$

i.e., by Eq. (2.24),

$$F_{r,r:\alpha} = \delta_{a(r),\alpha}. \tag{3.2}$$

Moreover, it follows from Eqs. (2.25) and (2.26), together with the invertibility of the function a that Eq. (3.2) not only implies but is actually equivalent to the condition

$$F_{r,r;a(r)} = 1. \tag{3.2$'$}$$

Further, by Eqs. (2.28) and (3.2) and the invertibility of a,

$$F_{r,s;\alpha} = 0 \text{ for } r{\neq}s. \tag{3.3}$$

Consequently, by Eqs. (2.23), (3.2) and (3.3),

$$E(A{\otimes}M) = \sum\nolimits_{r=1}^{n} w_{a(r))} M_{a(r)}(u_r, Au_r), \tag{3.4}$$

where

$$w_{a(r)} = |c_r|^2. \tag{3.5}$$

Hence, by Eqs. (3.1) and (3.4) and the invertibility of a,

$$E(\Pi_\alpha) = w_\alpha, \tag{3.6}$$

$$E(A) = \sum\nolimits_{r=1}^{n} w_{a(r)}(u_r, Au_r) \tag{3.7}$$

and

$$E(A|\mathcal{K}_{a(r)}) = (u_r, Au_r). \tag{3.8}$$

Eqs. (3.6) and (3.7) signify that, *before* the pointer position is read, w_α is the probability that the reading is α and the state of S is given by the density matrix

$$\rho = \sum\nolimits_{r=1}^{n} w_{a(r)} P(u_r),$$

i.e., by Eq. (3.5),

$$\rho = \sum\nolimits_{r=1}^{n} |c_r|^2 P(u_r). \tag{3.9}$$

Thus we have a reduction of the wave packet, as given by the transition from the pure state ψ $(= \sum\nolimits_{r=1}^{n})$ to this mixed state ρ.

According to the standard probabilisitic interpretation of quantum mechanics, Eq. (3.9) specifies the state of S just prior to the reading of the pointers, whereas Eq. (3.8) serves to specify its state following that reading. Thus, by Eq. (3.9), $|c_r|^2$ is the probability that the pointer reading will yield the result that u_r is the state of S; while Eq. (3.8) signifies that, following a reading that yields the result that $\alpha = a(r)$, the state of S is u_r. In the standard picture of quantum theory, there is no causality principle that determines which of the states u_r will be found.

Comments. (1) As shown above, the property (3.2) ensures that $\mathcal{I}$ enjoys all the essential properties of a measuring instrument since it implies both the reduction of the wave-packet and the one-to-one correspondence between the pointer position and the microstate of S. On the other hand, the property (3.3), which ensures the reduction of the wave-packet, does not imply Eq. (3.2) and therefore does not, of itself, imply that $\mathcal{I}$ serves as a measuring instrument

(2) The property (3.3) signifies that the $S - \mathcal{I}$ coupling removes the interference between the different components u_r of the pure state ψ and thus represents a *complete decoherence* effect. To see how this is related to the structure of a typical phase cell $\mathcal{K}_\alpha$, we introduce a complete orthonormal basis $\{\theta_{\alpha,\lambda}\}$ of this cell, where the index λ runs from 1 to $\dim(\mathcal{K}_\alpha)$, the dimensionality of $\mathcal{K}_\alpha$. We then infer from Eqs. (2.13) and (2.23) that

$$F_{r,s;\alpha} = \sum\nolimits_{\lambda=1}^{\dim(\mathcal{K}_\alpha)} \big(U_r(t)\theta_{\alpha,\lambda}, \Omega U_s(t)\theta_{\alpha,\lambda}\big),$$

Hence, as iK_r is the generator of U_r, this equation signifies that the decoherence arises from the aggregated destructive interference of the evolutes of the vectors $\theta_{\alpha,\lambda}$ generated by the different Hamiltonians K_r and K_s. This picture of decoherence corresponds to that assumed by Van Kampen [2].

3.2. Normal Measuring Instruments. We term the instrument $\mathcal{I}$ *normal** if the following conditions are fulfilled.

(N.1)) A weaker form of the ideality condition (3.2), or equivalently (3.2)$'$, prevails, to the effect that the difference between the two sides of the latter formula is negligibly small, i.e., noting Eq. (2.25), that

$$0 < 1 - F_{r,r;a(r)} < \eta(N), \tag{3.10}$$

where, for large N, $\eta(N)$ is miniscule by comparison with unity: in the case of the finite version of the Coleman-Hepp model treated in Section 4, it is $\exp(-cN)$, where c is a fixed

* We conjecture that the behaviour of real instruments is generally normal in the sense specified here and thus that the use of this adjective is approriate. Some support for this conjecture is provided by the results of Section 4 for the Coleman- Hepp model.

positive constant of the order of unity. We note that, by Eq. (2.25) and the positivity of Π_α, the condition (3.10) is equivalent to the inequality

$$0 < \sum\nolimits_{r \neq a^{-1}(\alpha)} F_{r,r;\alpha} < \eta(N). \tag{3.10$'$}$$

Further, it follows from Eqs. (2.28), (3.10) and (3.10)$'$ that

$$|F_{r,s;\alpha}| < \eta(N)^{1/2} \text{ for } r{\neq}s, \tag{3.11}$$

which is evidently a *decoherence condition*, being a slightly weakened version of the complete one given by Eq. (3.3).

(N.2) This condition (N.1) is stable under localised modifications of the initial state Ω of $\mathcal{I}$. This stability condition may even be strengthened to include global perturbations of Ω corresponding to small changes in its intensive thermodynamic parameters (cf. the treatment of the Coleman-Hepp model in Section 4).

It follows now from Eq. (3.11) that the replacement of the ideal condition (3.2) by the normal one (3.10) leads to modifications of the order $\eta(N)^{1/2}$ to the formula (3.4) and its consequences. In particular, it implies that a pointer reading α signifies that it is overwhelmingly probable, but not absolutely certain, that the state of S is $u_{a^{-1}(\alpha)}$, as the following argument shows. Suppose that the initial state of S is u_r. Then, by Eq. (2.11), the state of S_c at time t is $P(u_r){\otimes}\Omega_{r,r}(t)$; and by Eqs. (3.10)$'$, there is a probability of the order of $\eta(N)$ that the pointer reading is given by a value α, different from $a(r)$, of the indicator parameter of $\mathcal{I}$. In the freak case that this possibility is realised, this would mean that the state u_r of S led to a pointer reading $\alpha{\neq}a(r)$. Hence, in this case, any inference to the effect that a pointer reading α signified that the state of S was $u_{a^{-1}(\alpha)}$ would be invalid.

Comments. The scheme proposed here admits two kinds of measuring instruments, namely the ideal and the normal ones. The former fulfill perfectly the demands for the reduction of the wave-packet and the one-to-one correspondence between the pointer reading of the measuring instrument and the eigenstate of the observed microsystem. On the other hand, in the case of a normal instrument, there is just a minuscule possibility that the pointer reading might correspond to the 'wrong' eigenstate of the microsystem. However, as the odds against such an eventuality are overwhelming, the distinction between the two kinds of instruments is essentially mathematical rather than observational.

4. The Finite Coleman-Hepp Model.

This model is a caricature of an electron that interacts with a finite spin chain that serves to measure the electronic spin [14]. In order to fit this model into the scheme of the previous Sections, we regard the electron, $\mathcal{P}$, as the composite of two entities, namely its spin, $\mathcal{P}_1$, and its orbital component, $\mathcal{P}_2$. We then take the system S to be just $\mathcal{P}_1$ and the instrument $\mathcal{I}$ to be the composite of $\mathcal{P}_2$ and the chain $\mathcal{C}$. Thus, we build the model of $S_c = (S + \mathcal{I})$ from its components in the following way.

4.1. The System $S = \mathcal{P}_1$. This is just a single Pauli spin. Thus, its state space is $\mathcal{H} = \mathbf{C}^2$ and its three-component spin observable is given by the Pauli matrices (s_x, s_y, s_z). We denote by $u_\pm$ the eigenvectors of s_z whose eigenvalues are ± 1, respectively. These vectors then form a basis in $\mathcal{H}$. We denote their projection operators by $P_\pm$, respectively.

4.2. The System $\mathcal{I} = (\mathcal{P}_2 + \mathcal{C})$. We assume that $\mathcal{P}$ moves along, or parallel to, the axis Ox and thus that the state space of $\mathcal{P}_2$ is $\tilde{\mathcal{K}} := L^2(\mathbf{R})$. We assume that $\mathcal{C}$ is a chain of Pauli spins located at the sites $(1, 2, . ., 2L+1)$, of Ox, where L is a positive integer. Thus, the state space of $\mathcal{C}$ is $\hat{\mathcal{K}} := (\mathbf{C}^2)^{(2L+1)}$, and therefore that of $\mathcal{I}$ is $\mathcal{K} = \tilde{\mathcal{K}} \otimes \hat{\mathcal{K}}$.

The spin at the site n of $\mathcal{C}$ are represented by Pauli matrices $(\sigma_{n,x}, \sigma_{n,y}, \sigma_{n,z})$ that act on the n'th $\mathbf{C}^2$ component of $\hat{\mathcal{K}}$. Thus, they may be canonically identified with operators in $\hat{\mathcal{K}}$ that satisfy the standard Pauli relations

$$\sigma_{n,x}^2 = \sigma_{n,y}^2 = \sigma_{n,z}^2 = \hat{I};\ \sigma_{n,x}\sigma_{n,y} = i\sigma_{n,z},\ \text{etc}, \tag{4.1}$$

together with the condition that the spins at different sites intercommute.

We assume that $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{C}$ are independently prepared before being coupled together at time $t = 0$. Further, we assume that the initial states of $\mathcal{P}_1$ and $\mathcal{P}_2$ are the pure ones, represented by vectors ψ and ϕ in $\mathcal{H}$ and $\tilde{\mathcal{K}}$, respectively, while that of $\mathcal{C}$ is given by a density matrix $\hat{\Omega}$, in $\hat{\mathcal{K}}$, whose form will be specified below, by Eqs. (4.3) and (4.4).Thus, the initial state of $\mathcal{I}$ is

$$\Omega = P(\phi) \otimes \hat{\Omega}, \tag{4.2}$$

where $P(\phi)$ is the projection operator for ϕ. We assume that ϕ has support in a finite interval $[c, d]$ and that $\hat{\Omega}$ takes the form

$$\hat{\Omega} = \otimes_{n=1}^{2L+1} \hat{\omega}_n, \tag{4.3}$$

where $\hat{\omega}_n$, the initial state of the n'th spin of $\mathcal{C}$, is give by the formula

$$\hat{\omega}_n = \frac{1}{2}(I_n + m\sigma_{n,z}), \tag{4.4}$$

where $0 < m \leq 1$. Thus, asuming that there are no interactions between the spins of $\mathcal{C}$, $\hat{\Omega}$ is the equilibrium state obtained by subjecting this chain to a certain temperature-dependent magnetic field, directed along Oz. m is then the magnitude of the resultant polarisation of this chain. One sees immediately from Eqs. (4.3) and (4.4) that $\hat{\Omega}$ is a pure state if $m = 1$: otherwise it is mixed.

4.3. The Dynamics. Following Hepp [14], we assume that the Hamiltonian for the composite system S_c is

$$H_C = I_{\mathcal{H}} \otimes p \otimes I_{\hat{\mathcal{K}}} + P_- \otimes \sum_{n=1}^{2L+1} V(x - n) \otimes \sigma_{n,x}, \tag{4.5}$$

where p and V are the differential and multiplicative operators in $L^2(\mathbf{R})$ $(= \tilde{\mathcal{K}})$ that transform $f(x)$ to $-i\hbar df(x)/dx$ and $V(x)f(x)$, respectively, and V is a bounded, real valued function on $\mathbf{R}$ with support in a finite interval $[a,b]$. Thus, in the notation of Eq. (2.8), but with r taking just the values $\pm$,

$$K_{+}=p\otimes I_{\hat{\mathcal{K}}} \text{ and } K_{-}=p\otimes I_{\hat{\mathcal{K}}}+\sum\nolimits_{n=1}^{2L+1}V(x-n)\otimes\sigma_{n,x}. \tag{4.6}$$

The assumption here that the Hamiltonian for the free orbital motion of $\mathcal{P}$ is linear rather than quadratic in p serves to simplify the model by avoiding dispersion of the 'electronic wave packet' .

The unitary groups $U_{\pm}$ generated by $iK_{\pm}$ are given by the formula

$$U_{\pm}(t)=\exp(iK_{\pm}t) \tag{4.7}$$

and the evolutes of Ω due to the actions of $U_{\pm}(t)$ are

$$\Omega_{\pm}(t):=U_{\pm}^{\star}(t)\Omega U_{\pm}(t). \tag{4.8}$$

These states are evidently the versions, for this model, of $\Omega_{r,r}(t)$, as defined by Eq. (2.13), with the double suffix (r,r) represented by $+$ or $-$. It follows now from Eqs. (4.2) and (4.6)-(4.8) that

$$\Omega_{+}(t)=P(\phi_{t})\otimes\hat{\Omega}, \tag{4.9}$$

where

$$\phi_{t}(x)=\phi(x+t). \tag{4.10}$$

As for Ω_{-} it is convenient to formulate its evolution in interaction representation, in terms of the unitary operator

$$W(t):=U_{-}(t)\exp(-i[p\otimes I_{\hat{\mathcal{K}}}]t). \tag{4.11}$$

Thus, by Eqs. (4.2), (4.8) and (4.11),

$$\Omega_{-}(t)=\exp(-i[p\otimes I_{\hat{\mathcal{K}}}]t)\big(W^{\star}(t)[P(\phi)\otimes\hat{\Omega}]W(t)\big)\exp(i[p\otimes I_{\hat{\mathcal{K}}}]t). \tag{4.12}$$

By Eqs. (4.6), (4.7) and (4.11), $W(t)$ satisfies the Dyson integral equation

$$W(t)=I_{\mathcal{K}}+i\int_{0}^{t}ds\sum\nolimits_{n=1}^{2L+1}[V(x+s-n)\otimes\sigma_{n,x}]W(s),$$

whose solution is

$$W(t)=\exp\big(i\sum\nolimits_{n=1}^{2L+1}[F_{n,t}(x)\otimes\sigma_{n,x}]\big), \tag{4.13}$$

where

$$F_{n,t}(x)=\int_{0}^{t}dsV(x+s-n). \tag{4.14}$$

Further, since the supports of V and ϕ are $[a,b]$ and $[c,d]$, respectively, it follows from these last two equations that we may replace $F_{n,t}(x)$ by $\int_{\mathbf{R}}dxV(x)$ when employing Eq. (4.13) in the formula (4.12), provided that

$$d\leq a+1 \text{ and } t\geq\tau:=2L+1-b-c. \tag{4.15}$$

Thus, in this case, $W(t)$ may be replaced there by $I_{\tilde{\mathcal{K}}}\otimes Z$, where

$$Z=\exp(iJ\sum\nolimits_{n=1}^{2L+1}\sigma_{n,x})\equiv\otimes_{n=1}^{2L+1}\exp(iJ\sigma_{n,x}) \tag{4.16}$$

and

$$J=\int_{\mathbf{R}}dxV(x). \tag{4.17}$$

Consequently, under the conditions (4.15), Eq. (4.12) reduces to the form

$$\Omega_{-}(t)=P(\phi_t)\otimes Z^{\star}\hat{\Omega}Z,$$

where ϕ_t is given by Eq. (4.10). On combining this equation with Eq. (4.9), we see that

$$\Omega_{\pm}(t)=P(\phi_t)\otimes\hat{\Omega}_{\pm}, \qquad (4.18),$$

where $\hat{\Omega}_{\pm}$ are the *time-independent* states given by the formulae

$$\hat{\Omega}_{+}=\hat{\Omega} \text{ and } \hat{\Omega}_{-}=Z^{\star}\hat{\Omega}Z. \tag{4.19}$$

Thus, under the conditions (4.15), the chain $\mathcal{C}$ takes up the steady states $\hat{\Omega}_{\pm}$ corresponding to the states $u_{\pm}$ of S. It should be noted that the critical time τ, specified in Eq. (4.15), is essentially the time required for the particle $\mathcal{P}$ to travel from end to end of the chain $\mathcal{C}$. It is therefore a reasonable macroscopic observational time.

Further, by Eqs. (4.1)-(4.4), (4.16) and (4.19), the explict forms of the states $\hat{\Omega}_{\pm}$ are given by the equations

$$\hat{\Omega}_{+}=2^{-(2L+1)}\otimes_{n=1}^{2L+1}(I_n+m\sigma_{n,z}) \tag{4.20}$$

and

$$\hat{\Omega}_{-}=2^{-(2L+1)}\otimes_{n=1}^{2L+1}\big(I_n+m\sigma_{n,z}\cos(2J)+im\sigma_{n,y}\sin(2J)\big). \tag{4.21}$$

4.4. The Macroscopic Phase Cells of $\mathcal{I}$. We take these to be the subspaces $\mathcal{K}_{\pm}$ of $\mathcal{K}$ corresponding to positive and negative polarizations, respectively, of the chain $\mathcal{C}$ along the Oz-direction. To formulate these subspaces precisely, we first note that the eigenvalues of the total spin of $\mathcal{C}$ along that direction, namely $\Sigma_z:=\sum_{n=1}^{(2L+1)}\sigma_z$, are the odd numbers between $-(2L+1)$ and $(2L+1)$. We define $\hat{\mathcal{K}}_{+}$ (resp. $\hat{\mathcal{K}}_{-}$) to be the subspace of $\hat{\mathcal{K}}$ spanned by the eigenvectors of Σ_z with positive (resp. negative) eigenvalues. Thus, denoting by $\hat{\Psi}$ the simultaneous eigenvector of the $\sigma_{n,z}$'s with eigenvalues all equal to -1, $\hat{\mathcal{K}}_{\pm}$ are the subspaces of $\hat{\mathcal{K}}$ generated by application to $\hat{\Psi}$ of the monomials of order greater than L and less than $(L+1)$, respectively, in the different $\sigma_{n,x}$'s (or $\sigma_{n,y}$'s). We denote their projection operators by $\hat{\Pi}_{\pm}$, respectively. We then define the phase cells $\mathcal{K}_{\pm}$ to be the subspaces $\tilde{\mathcal{K}}\otimes\hat{\mathcal{K}}_{\pm}$ of $\mathcal{K}$, and denote their respective projection operators by $\Pi_{\pm}$ $(=I_{\tilde{\mathcal{K}}}\otimes\hat{\Pi}_{\pm})$.

Evidently, the formulation of the subspaces $\mathcal{K}_{\pm}$ of $\mathcal{K}$ here corresponds to that of the previous Sections, with α taking the values $+$ and $-$, and fulfills the conditions of Eqs.

(2.14) and (2.15). In the treatment that follows, we shall take the correspondence between the phase cells of $\mathcal{I}$ and the eigenstates of S to be the mapping $r \rightarrow a(r)$ of Section 3, with $a(\pm) = \pm$. Thus, the phase cells $\mathcal{K}_{\pm}$ are the indicators for the vector states $u_{\pm}$, respectively.

4.5. Ideality and Normality Conditions for $\mathcal{I}$. It follows now the definition of $\Pi_{\pm} := I_{\tilde{\mathcal{K}}} \otimes \hat{\Pi}_{\pm}$ that, on translating the ideality and normality conditions $(3.2)'$ and $(3.10)'$, respectively, into the specifications for this model and using Eqs. (2.23)-(2.26), the former condition reduces to the equation

$$\mathrm{Tr}(\hat{\Omega}_{+}\hat{\Pi}_{-}) = \mathrm{Tr}(\hat{\Omega}_{-}\hat{\Pi}_{+}) = 0 \tag{4.22)}$$

and the latter to

$$0 < Max\big[\mathrm{Tr}(\hat{\Omega}_{+}\hat{\Pi}_{-}),\ \mathrm{Tr}(\hat{\Omega}_{-}\hat{\Pi}_{+})\big] < \hat{\eta}(L), \tag{4.23}$$

where

$$\hat{\eta}(L) := \eta(2L+1). \tag{4.24}$$

4.6. Resultant Properties of $\mathcal{I}$. The following proposition establish that $\mathcal{I}$ is an ideal measuring instrument for certain special values of the parameters of the model S_c and is a normal one for a wide range of those parameters. Further, in the latter case, $\hat{\eta}(L)$ is exponentially small, i.e. of the order of $\exp(-cL)$, with c a positive constant of the order of unity.

Proposition 4.1. *Assuming the conditions of Eq. (4.15), $\mathcal{I}$ has the following properties.*

(a) If $J = \pi/2$ and $m = 1$, then $\mathcal{I}$ is an ideal instrument, with critical time τ. However, although this implies that it satisfies the local stability condition (I.3), it is transformed to a normal instrument by small perturbations of the global polarization m.

(b) If $J \in (\pi/4, \pi/2)$ and $m \in (-1, 0)$, then $\mathcal{I}$ is a normal instrument, again with critical time τ and with $\hat{\eta}(L) = \exp(-cL)$, where c is a numerical constant of the order of unity: specifically $c = -(1/2)\ln(1 - m^2\cos^2(2J))$. Moreover, in this case, the instrument is stable both under small perturbations of the global polarisation, m, and under local modifications of state.

It follows from our specifications that this proposition is a consequence of the following ones, which we shall prove below.

Proposition 4.2. *Assuming the conditions of Eq. (4.15), the model possesses the following properties.*

(i) If $J = \pi/2$ and $m = 1$, it satifisfies the ideality condition (4.22), with critical time τ.

(ii) If $J \in (\pi/4, \pi/2)$ and $m \in (0, 1)$, it fulfills the normality condition (4.23), with critical time τ and $\hat{\eta}(L) = \big(1 - m^2\cos^2(2J)\big)^{L/2}$.

Proposition 4.3. *Assuming the conditions (4.15),* *(i) the results of Prop. 4.1 are stable under any modification of the initial state $\hat{\Omega}$ of $\mathcal{C}$ that is confined to some segment of this chain whose length is $O(1)$ with respect to the large length L;*

(ii) under the conditions of Prop. 4.2 (i), any small perturbations of the global polarization m change $\mathcal{I}$ from an ideal instrument to a normal one; and

(iii) under the conditions of Prop. 4.2 (ii), the normality of the instrument is stable under small perturbations of the global polarization m.

Proof of of Proposition 4.2. Let $v_{n,\pm}$ denote the eigenstate of $\sigma_{n,z}$ whose eigenvalue is ± 1. Then, by definition of $\hat{\Pi}_+$ (resp. $\hat{\Pi}_-$), the eigenstates of this projector are the tensor products of n v_-'s and $(2L+1-n)$ v_+'s (resp. n v_+'s and $(2L+1-n)$ v_-'s) with n running from 0 to L. Hence, by Eqs. (4.20) and (4.21),

$$\mathrm{Tr}(\Omega_+\hat{\Pi}_-) = 2^{-(2L+1)}\sum\nolimits_{n=0}^{L}(1+m)^n(1-m)^{2L+1-n}(2L+1)!/n!(2L+1-n)! \quad (4.25)$$

and

$$\mathrm{Tr}(\hat{\Omega}_-\hat{\Pi}_+) =$$

$$2^{-(2L+1)}\sum\nolimits_{n=0}^{L}(1-(m)\cos(2J))^n(1+(m)\cos(2J))^{2L+1-n}(2L+1)!/n!(2L+1-n)! \quad (4.26)$$

It follows immediately from these equations that, in the case where $m=1$ and $J=\pi/2$, the r.h.s.'s of these last two equations vanish. This completes the proof of Part (i) of the proposition.

In order to prove Part (ii), we assume that $J\in(\pi/4,\pi/2)$ and $0<m<1$. In this case, the summands on the r.h.s's of Eqs. (4.25) and (4.26) are positive for all $n\in[0,L]$, and they take their largest values at $n=L$, since $(2L+1)!/n!(2L+1-n)!$, $(1+m)^n(1-m)^{2L+1-m}$ and $(1-(m)\cos(2J))^n(1+(m)\cos(2J))^{2L+1-n}$ are all maximized at this value of n. Hence

$$0 < \mathrm{Tr}(\hat{\Omega}_+\hat{\Pi}_-)\leq 2^{-(2L+1)}(1+m)^L(1-m)^{L+1}(2L+1)!/(L!)^2. \quad (4.27)$$

and

$$0 < \mathrm{Tr}(\hat{\Omega}_-\hat{\Pi}_+)\leq 2^{-(2L+1)}(1-(m)\cos(2J))^L(1+(m)\cos(2J))^{L+1}(2L+1)!/(L!)^2. \quad (4.28)$$

Further since, by Sterlings formula,

$$\ln\big[(2L+1)!/(L!)^2\big] = \ln(L) + O(1),$$

it follows from Eqs. (4.27) and (4.28) that

$$\ln\big[\mathrm{Tr}(\hat{\Omega}_+\hat{\Pi}_-\big]\leq L\ln(1-m^2) + \ln(L) + O(1)$$

and

$$\ln\big[\mathrm{Tr}(\hat{\Omega}_-\Pi_+\big]\leq L\ln(1-m^2\cos^2(2J)) + \ln(L) + O(1)$$

Consequently, for sufficiently large L, the r.h.s.'s of these inequalities are both majorized by $(L/2)\ln(1-m^2\cos^2(2J))$, and consequently, in view of the first parts of the inequalities (4.27) and (4.28), the normality condition (4.23) is fulfilled with $\hat{\eta}(L) = \exp(-cL)$ and $c = -(1/2)\ln\bigl(1-m^2\cos^2(2J)\bigr)$.

Proof of Proposition 4.3. First consider the question of stability against global perturbations of the initial state corresponding to small changes in the polarisation m which leave this parameter in the range $(0,1]$. In fact, it follows immediately from Prop. 4.2 that the normality condition (4.23) is stable under such perturbations, while the ideality condition (4.22) is changed to that of normality. This establishes Parts (ii) and (iii) of Prop. 4.3.

In order to prove Part (i), we introduce an arbitrary subset, K, of $[1, 2L+1]]$ whose total number of sites, $|K|$, is $O(1)$ with respect to the 'large' length L; and we denote by K^c the complementary subset $[1, 2L+1]]\backslash K$. Correspondingly we denote by $\check{\mathcal{K}}$ and $\check{\mathcal{K}}^c$ the representation spaces for the spins in K and K^c. It follows from this definition and that of $\hat{\mathcal{K}}$ that this latter space is the tensor product $\check{\mathcal{K}}\otimes\check{\mathcal{K}}^c$.

We now let $\hat{\Omega}_1$ be an arbitrary state of $\mathcal{C}$ that coincides with $\hat{\Omega}$ in K^c. Thus

$$\mathrm{Tr}_{\check{\mathcal{K}}}(\hat{\Omega}_1) = \mathrm{Tr}_{\check{\mathcal{K}}}(\hat{\Omega}). \tag{4.30}$$

By Eq. (4.19), the evolutes $\hat{\Omega}_{1,\pm}$ of $\hat{\Omega}_1$ that stem from the coupling of $\mathcal{I}$ to the states $u_\pm$ of S are given by the formulae

$$\hat{\Omega}_{1,+} = \hat{\Omega}_1 \text{ and } \hat{\Omega}_{1,-} = Z^\star\hat{\Omega}_1 Z. \tag{4.31}$$

We now need to show that the states $\hat{\Omega}_{1,\pm}$ satisfy the same condition (4.22) or (4.23)) as $\hat{\Omega}_\pm$, and with the same value of $\hat{\eta}(L)$, according to whether the assumptions of Prop. 4.2(i) or Prop. 4.2(ii) prevail. To this end, we note that, by Eqs. (4.3), (4.4), (4.30) and (4.31),

$$Tr_{\check{\mathcal{K}}}\hat{\Omega}_{1,\pm} = Tr_{\check{\mathcal{K}}}\hat{\Omega}_\pm := \check{\Omega}^c_\pm; \tag{4.32}$$

and further, by Eqs. (4.16), (4.19) and (4.32), the states $\check{\Omega}_{pm}$ are given by the following canonical analogues of Eqs. (4.20) and (4.21).

$$\check{\Omega}^c_+ = 2^{-(2L+1)}\otimes_{n\in K^c}(I_n + m\sigma_{n,z}) \tag{4.33}$$

and

$$\check{\Omega}^c_- = 2^{-(2L+1)}\otimes_{n\in K^c}\bigl(I_n + m\sigma_{n,z}\cos(2J) + im\sigma_{n,y}\sin(2J)\bigr). \tag{4.34}$$

We now denote by $\check{P}^c_\pm$ be the projection operators for the simultaneous eigenvectors of $\{\sigma_{n,z}|n\in K^c\}$ with eigenvalues all equal to ± 1, respectively; and by $\check{\Pi}^c_+$ (resp. $\check{\Pi}^c_-$) the projection operator of the subspace of $\check{\mathcal{K}}^c$ for which $\sum_{n\in K^c}\sigma_{n,z}\leq -|K|$ (resp.$\geq|K|$). It follows from these definitions that

$$I_{\check{\mathcal{K}}}\otimes\check{P}^c_\pm < \hat{\Pi}_\pm < I_{\check{\mathcal{K}}}\otimes\check{\Pi}^c_\pm. \tag{4.35}$$

Hence, by Eqs. (4.32) and (4.35),

$$\mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{\pm}\check{P}^c_{\mp}) \leq \mathrm{Tr}(\hat{\Omega}_{1\pm}\hat{\Pi}_{\mp}) \leq \mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{\pm}\check{\Pi}^c_{\mp}). \tag{4.36}$$

Further, by Eqs.(4.33) and (4.34) and the definitions of $\check{P}^c_{\pm}$ and $\check{\Pi}^c_{\pm}$,

$$\mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{+}\check{P}^c_{-}) = 2^{-(2L+1)}(1-m)^{2L+1-|K|}, \tag{4.37}$$

$$\mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{-}\check{P}^c_{+}) = 2^{-(2L+1)}(1+(m)\cos(2J))^{2L+1-|K|}, \tag{4.38}$$

$$\mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{+}\check{\Pi}^c_{-}) = \sum_{n=0}^{L}(1+m)^n(1-m)^{2L+1-n-|K|}(2L+1-n-|K|)!/n!(2L+1-n-|K|)! \tag{4.39}$$

and

$$\mathrm{Tr}_{\check{\mathcal{K}}^c}(\check{\Omega}^c_{-}\check{\Pi}^c_{+}) = \sum_{n=0}^{L}$$
$$(1-(m)\cos(2J))^n(1+(m)\cos(2J)))^{2L+1-n-|K|}(2L+1-n-|K|)!/n!(2L+1-n-|K|)!. \tag{4.40}$$

Now in the ideal case where $m=1$ and $J=\pi/2$, the r.h.s.'s Eqs. (4.37)-(4.40) all vanish. Therefore, in this case, the two-sided inequalities (4.36) signify that $\mathrm{Tr}(\hat{\Omega}_{1,\pm}\hat{\Pi}_{\mp})$ vanishes, i.e. that the locally modified state Ω_1 satifies the ideality conditions (4.22).

In the normal case, where $m\in(0,1)$ and $J\in(\pi/4,\pi/2)$, we see immediately from Eqs. (4.36)-(4.38) that the quantities $\mathrm{Tr}(\hat{\Omega}_{\pm}\hat{\Pi}_{\mp})$ are strictly positive. In order to obtain upper bounds for them, we treat Eqs. (4.39) and (4.40) by the method employed for the derivation of the estimates (4.28) and (4.29) from (4.25) and (4.26) in the proof of Prop. 4.2. Thus, taking account of the demand that $|K|=O(1)$ with respect to L, we obtain precisely the same estimates for $\mathrm{Tr}(\hat{\Omega}_{1,\pm}\hat{\Pi}_{\mp})$ as those given by Eqs. (4.28) and (4.29) for $\mathrm{Tr}(\hat{\Omega}_{\pm}\hat{\Pi}_{\mp})$. This signifies that $\mathcal{I}$ remains a normal instrument, with unchanged value of $\hat{\eta}(L)$, when the initial state of $\mathcal{C}$ is changed from $\hat{\Omega}$ to $\hat{\Omega}_1$. In other words, the operation of the instrument $\mathcal{I}$ is stable under local modifications of the initial state of the chain $\mathcal{C}$.

5. Concluding Remarks

We have shown that the general scheme of Sections 2 and 3 is fully realized by the model of Section 4. Since that is a Hamiltonian model for the composite S_c of microsystem and measuring instrument, this signifies that the traditional quantum mechanics of finite conservative systems provides a perfectly adequate framework for the quantum theory of measurement. This theory therefore requires no extraneous elements, such as the interaction of S_c with the 'rest of the Universe' or a nonlinear modification of its Schroedinger dynamics, as has been proposed by some authors [4-7]. Furthermore, the treatment of the model of Section provides a clear illustration of the mathematical dichotomy of ideal and normal measuring instruments. It also establishes that, from an empirical standpoint, there is effectively no distinction between these two classes of instruments, since the odds against the indication by a normal instrument, of a 'wrong' state of a the microsystem are truly astronomical, being of the order of $\exp(cL)$ to one, where c is of the order of unity.

This last observation implies that the instrument of the Coleman-Hepp model would work perfectly well if it the chain $\mathcal{C}$ were merely mesoscopic rather than macroscopic, e.g. with the chain $\mathcal{C}$ composed of, say, 10^5 spins. This raises the question, that can be addressed both experimentally and by the study of other models, of whether real quantum measuring instruments of mesoscopic size can be devised.

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