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ON THE SPECTRUM OF JACOBI OPERATORS WITH QUASI-PERIODIC ALGEBRO-GEOMETRIC COEFFICIENTS

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Abstract. We characterize the spectrum of one-dimensional Jacobi operators $H = aS^+ + a^-S^- + b$ in $l^2(\mathbb{Z})$ with quasi-periodic complex-valued algebro-geometric coefficients (which satisfy one (and hence infinitely many) equation(s) of the stationary Toda hierarchy) associated with nonsingular hyperelliptic curves. The spectrum of H coincides with the conditional stability set of H and can explicitly be described in terms of the mean value of the Green's function of H.

As a result, the spectrum of H consists of finitely many simple analytic arcs in the complex plane. Crossings as well as confluences of spectral arcs are possible and discussed as well.

1. Introduction

It is well-known from the work of Date and Tanaka [12], [13], Dubrovin, Matveev, and Novikov [17], Flaschka [23], McKean [39], [40], McKean–van Moerbeke [41], van Moerbeke [42], van Moerbeke and Mumford [43], Mumford [44], Novikov, Manakov, Pitaevski, and Zakharov [48], Teschl [54, Chs. 9,13], Toda [55, Ch. 4], [56, Chs. 26-30], that the self-adjoint Jacobi operator

$$H = aS^+ + a^-S^- + b, \quad \mathrm{dom}(H) = \ell^2(\mathbb{Z}), \tag{1.1}$$

in $\ell^2(\mathbb{Z})$ with real-valued periodic, or more generally, algebro-geometric *quasi-periodic* and *real-valued* coefficients a and b (i.e., coefficients that satisfy one (and hence infinitely many) equation(s) of the stationary Toda hierarchy), leads to a finite-gap, or perhaps more appropriately, to a finite-band spectrum $\sigma(H)$ of the form

$$\sigma(H) = \bigcup_{m=1}^{p+1} [E_{2m-2}, E_{2m-1}]. \quad E_0 < E_1 < \cdots < E_{2p+1}. \tag{1.2}$$

Compared to the real-valued case, the corresponding spectral properties of Jacobi operators with *periodic* and *complex-valued* coefficients a and b, to the best of our knowledge, have been studied rather sparingly in the literature. We are only aware of two papers by Naĭman [46], [47], in which it is shown that the spectrum consists of a set of piecewise analytic arcs which may have common endpoints. (This is in anology to the case of (non-self-adjoint) one-dimensional periodic Schrödinger operators, cf. [49], [52].) It seems plausible that the latter case is connected with (complex-valued) stationary solutions of equations of the Toda hierarchy. In particular, the next scenario in line, the determination of the spectrum of H in the case of *quasi-periodic* and *complex-valued* solutions of the stationary Toda hierarchy apparently has never been clarified. The latter spectral problems have been open since the late seventies and it is the purpose of this paper to provide a comprehensive solution of them. For theta function representations of a and b in the complex-valued algebro-geometric case (without addressing spectral theoretic questions) we refer, for instance, to Aptekarev [1], Dubrovin, Krichever, and Novikov [16], Krichever [32]–[35] (cf. also the appendix written by Krichever in [15]).

To describe our results, a bit of preparation is needed. Let

$$G(z,n,n') = (H-z)^{-1}(n,n'), \quad z \in \mathbb{C}\backslash\sigma(H), \ n,n' \in \mathbb{Z}, \tag{1.3}$$

be the Green's function of H (here $\sigma(H)$ denotes the spectrum of H) and denote by $g(z,n)$ the corresponding diagonal Green's function of H defined by

$$g(z,n) = G(z,n,n) = \frac{\prod_{j=1}^{p}[z-\mu_j(n)]}{R_{2p+2}(z)^{1/2}}, \tag{1.4}$$

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z-E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}, \tag{1.5}$$

$$E_m \neq E_{m'} \ \text{for } m \neq m', \quad m,m' = 0,1,\dots,2p+1. \tag{1.6}$$

For any quasi-periodic (in fact, Bohr (uniformly) almost periodic) sequence $f = \{f(k)\}_{k\in\mathbb{Z}}$ the mean value $\langle f\rangle$ of f is defined by

$$\langle f\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{N} f(k). \tag{1.7}$$

Moreover, we introduce the set Σ by

$$\Sigma = \bigg\{\lambda \in \mathbb{C} \,\bigg|\, \mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\lambda,\cdot)-y}{G_{p+1}(\lambda,\cdot)+y}\bigg)\bigg\rangle\bigg) = 0\bigg\}, \tag{1.8}$$

where $y^2 = R_{2p+2}(z)$, and $G_{p+1}(z,n)$ will be defined in (2.21). Here we observe that G_{p+1} is given in terms of the off-diagonal Green's function $G(z,n+1,n)$ by

$$\frac{G_{p+1}(z,n)}{R_{2p+2}(z)^{1/2}} = 1 - 2a(n)G(z,n+1,n) \tag{1.9}$$

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(cf. also (3.13)). In addition, we note that

$$\langle g(z,\cdot)\rangle = \frac{\prod_{j=1}^{p}\big(z-\widetilde\lambda_j\big)}{R_{2p+2}(z)^{1/2}} \tag{1.10}$$

for some constants $\{\widetilde\lambda_j\}_{j=1}^p \subset \mathbb{C}$.

Finally, we denote by $\sigma_{\rm p}(T)$, $\sigma_{\rm r}(T)$, $\sigma_{\rm c}(T)$, $\sigma_{\rm e}(T)$, and $\sigma_{\rm ap}(T)$, the point spectrum (i.e., the set of eigenvalues), the residual spectrum, the continuous spectrum, the essential spectrum (cf. (4.14)), and the approximate point spectrum of a densely defined closed operator T in a complex Hilbert space, respectively.

Our principal new results, to be proved in Section 4, then read as follows:

Theorem 1.1. *Assume that a and b are quasi-periodic* (*complex-valued*) *solutions of the pth stationary Toda equation associated with the hyperelliptic curve $y^2 = R_{2p+2}(z)$ subject to* (1.5) *and* (1.6). *Then the following assertions hold:*
(*i*) *The point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\rm p}(H) = \sigma_{\rm r}(H) = \emptyset, \tag{1.11}$$

$$\sigma(H) = \sigma_{\rm c}(H) = \sigma_{\rm e}(H) = \sigma_{\rm ap}(H). \tag{1.12}$$

(*ii*) *The spectrum of H coincides with Σ and equals the conditional stability set of H,*

$$\sigma(H) = \bigg\{\lambda\in\mathbb{C} \,\bigg|\, \mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\lambda,\cdot)-y}{G_{p+1}(\lambda,\cdot)+y}\bigg)\bigg\rangle\bigg) = 0\bigg\} \tag{1.13}$$

$$= \{\lambda\in\mathbb{C} \,|\, \textit{there exists at least one bounded solution } 0\neq\psi\in\ell^\infty(\mathbb{Z}) \textit{ of } H\psi=\lambda\psi\}. \tag{1.14}$$

(*iii*) $\sigma(H)\subset\mathbb{C}$ *is bounded,*

$$\sigma(H)\subset\{z\in\mathbb{C} \,|\, \mathrm{Re}(z)\in[M_1,M_2],\ \mathrm{Im}(z)\in[M_3,M_4]\}, \tag{1.15}$$

where

$$\begin{aligned}
M_1 &= -2\sup_{n\in\mathbb{Z}}[|\mathrm{Re}(a(n))|] + \inf_{n\in\mathbb{Z}}[\mathrm{Re}(b(n))],\\
M_2 &= 2\sup_{n\in\mathbb{Z}}[|\mathrm{Re}(a(n))|] + \sup_{n\in\mathbb{Z}}[\mathrm{Re}(b(n))],\\
M_3 &= -2\sup_{n\in\mathbb{Z}}[|\mathrm{Im}(a(n))|] + \inf_{n\in\mathbb{Z}}[\mathrm{Im}(b(n))],\\
M_4 &= 2\sup_{n\in\mathbb{Z}}[|\mathrm{Im}(a(n))|] + \sup_{n\in\mathbb{Z}}[\mathrm{Im}(b(n))].
\end{aligned} \tag{1.16}$$

(*iv*) *$\sigma(H)$ consists of finitely many simple analytic arcs. These analytic arcs may only end at the points $\widetilde\lambda_1,\dots,\widetilde\lambda_p$, $E_0,\dots,E_{2p+1}$.*
(*v*) *Each E_m, $m=0,\dots,2p+1$, is met by at least one of these arcs. More precisely, a particular E_{m_0} is hit by precisely $2N_0+1$ analytic arcs, where $N_0\in\{0,\dots,p\}$ denotes the number of $\widetilde\lambda_j$ that coincide with E_{m_0}. Adjacent arcs meet at an angle $2\pi/(2N_0+1)$ at E_{m_0}.* (*Thus, generically, $N_0=0$ and precisely one arc hits E_{m_0}.*)
(*vi*) *Crossings of spectral arcs are permitted and take place precisely when*

$$\mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\widetilde\lambda_{j_0},\cdot)-y}{G_{p+1}(\widetilde\lambda_{j_0},\cdot)+y}\bigg)\bigg\rangle\bigg) = 0 \quad \textit{for some } j_0\in\{1,\dots,p\} \textit{ with } \widetilde\lambda_{j_0}\notin\{E_m\}_{m=0}^{2p+1}. \tag{1.17}$$

In this case $2M_0+2$ *analytic arcs are converging toward* $\widetilde{\lambda}_{j_0}$, *where* $M_0 \in \{1,\dots,p\}$ *denotes the number of* $\widetilde{\lambda}_j$ *that coincide with* $\widetilde{\lambda}_{j_0}$. *Adjacent arcs meet at an angle* $\pi/(M_0+1)$ *at* $\widetilde{\lambda}_{j_0}$. *(Thus, if crossings occur, generically,* $M_0 = 1$ *and two arcs cross at a right angle.)*
(vii) *The resolvent set* $\mathbb{C}\backslash\sigma(H)$ *of* H *is path-connected.*

Naturally, Theorem 1.1 applies to the special case where a and b are periodic complex-valued solutions of the pth stationary Toda equation associated with a nonsingular hyperelliptic curve. Even in this special case, Theorem 1.1 yields new facts which go beyond the previous results by Naĭman [46], [47].

For analogous results in the context of one-dimensional Schrödinger operators with quasi-periodic algebro-geometric KdV potentials we refer to [2].

Theorem 1.1 focuses on stationary quasi-periodic solutions of the Toda hierarchy for the following reasons. First of all, the class of algebro-geometric solutions of the (time-dependent) Toda hierarchy is defined as the class of all solutions of some (and hence infinitely many) equations of the stationary Toda hierarchy. Secondly, time-dependent algebro-geometric solutions of a particular equation of the (time-dependent) Toda hierarchy just represent isospectral deformations (the deformation parameter being the time variable) of fixed stationary algebro-geometric Toda solutions (the latter can be viewed as the initial condition at a fixed time t_0). In the present case of quasi-periodic algebro-geometric solutions of the pth Toda equation, the isospectral manifold of such given solutions is a complex p-dimensional torus, and time-dependent solutions trace out a path in that isospectral torus (cf. the discussions in [24] and [25]).

Finally, we give a brief discussion of the contents of each section. In Section 2 we provide the necessary background material including a quick construction of the Toda hierarchy of nonlinear evolution equations and its Lax pairs using a polynomial recursion formalism. We also discuss the hyperelliptic Riemann surface underlying the stationary Toda hierarchy, the corresponding Baker–Akhiezer function, and the necessary ingredients to describe the analog of the Its–Matveev formula for stationary Toda solutions. Section 3 focuses on the Green's function of the Jacobi operator H, a key ingredient in our characterization of the spectrum $\sigma(H)$ of H in Section 4 (cf. (1.13)). Our principal Section 4 is then devoted to a proof of Theorem 1.1. Appendix A provides the necessary summary of tools needed from elementary algebraic geometry (most notably the theory of compact (hyperelliptic) Riemann surfaces) and sets the stage for some of the notation used in Sections 2–4. Appendix B provides additional insight into one ingredient of the theta function representation of the coefficients a and b.

2. The Toda hierarchy, hyperelliptic curves, and theta function representations of the coefficients a and b

In this section we briefly review the recursive construction of the Toda hierarchy and associated Lax pairs following [7] and especially [25]. Moreover, we discuss the class of algebro-geometric solutions of the Toda hierarchy corresponding to the underlying hyperelliptic curve and recall the analog of the Its–Matveev formula for such solutions. The material in this preparatory section is known and detailed accounts with proofs can be found, for instance, in [7]. For the notation employed in connection with elementary concepts in algebraic geometry (more precisely, the theory of compact Riemann surfaces), we refer to Appendix A.

Throughout this section we assume that

$$a, b \in \ell^\infty(\mathbb{Z}) \tag{2.1}$$

and consider the second-order Jacobi difference expression

$$L = aS^+ + a^- S^- + b, \tag{2.2}$$

where $S^\pm$ denote the shift operators

$$(S^\pm f)(n) = f^\pm(n) = f(n \pm 1), \quad n \in \mathbb{Z}, \ f \in \ell^\infty(\mathbb{Z}). \tag{2.3}$$

To construct the stationary Toda hierarchy we need a second difference expression of order $2p+2$, $p \in \mathbb{N}_0$, defined recursively in the following. We take the quickest route to the construction of P_{2p+2}, and hence to the Toda hierarchy, by starting from the recursion relations (2.4)–(2.6) below.

Define $\{f_j\}_{j \in \mathbb{N}_0}$ and $\{g_j\}_{j \in \mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad g_0 = -c_1, \tag{2.4}$$

$$2f_{j+1} + g_j + g_j^- - 2bf_j = 0, \quad j \in \mathbb{N}_0, \tag{2.5}$$

$$g_{j+1} - g_{j+1}^- + 2\big(a^2 f_j^+ - (a^-)^2 f_j^-\big) - b(g_j - g_j^-) = 0, \quad j \in \mathbb{N}_0. \tag{2.6}$$

Explicitly, one finds

$$\begin{aligned}
&f_0 = 1, \\
&f_1 = b + c_1, \\
&f_2 = a^2 + (a^-)^2 + b^2 + c_1 b + c_2, \\
&f_3 = (a^-)^2(b^- + 2b) + a^2(b^+ + 2b) + b^3 + c_1(a^2 + (a^-)^2 + b^2) + c_2 b + c_3, \\
&\text{etc.}, \\
&g_0 = -c_1, \\
&g_1 = -2a^2 - c_2, \\
&g_2 = -2a^2(b + b^-) - 2c_1 a^2 - c_3, \\
&\text{etc.}
\end{aligned} \tag{2.7}$$

Here $\{c_j\}_{j \in \mathbb{N}}$ denote undetermined summation constants which naturally arise when solving (2.4)–(2.6).

Subsequently, it will be convenient to also introduce the corresponding homogeneous coefficients $\hat{f}_j$ and $\hat{g}_j$, defined by vanishing of the constants c_k, $k \in \mathbb{N}$,

$$\begin{aligned}
&\hat{f}_0 = 1, \quad \hat{f}_j = f_j\big|_{c_k = 0,\, k = 1, \dots, j}, \quad j \in \mathbb{N}, \\
&\hat{g}_j = g_j\big|_{c_k = 0,\, k = 1, \dots, j+1}, \quad j \in \mathbb{N}_0.
\end{aligned} \tag{2.8}$$

Hence,

$$f_j = \sum_{k=0}^{j} c_{j-k} \hat{f}_k, \quad g_j = \sum_{k=1}^{j} c_{j-k} \hat{g}_k - c_{j+1}, \quad j \in \mathbb{N}_0, \tag{2.9}$$

introducing

$$c_0 = 1. \tag{2.10}$$

Next we define difference expressions P_{2p+2} of order $2p+2$ by

$$P_{2p+2} = -L^{p+1} + \sum_{j=0}^{p} \Big(g_j + 2af_j S^+\Big) L^{p-j} + f_{p+1}, \quad p \in \mathbb{N}_0. \tag{2.11}$$

Using the recursion relations (2.4)–(2.6), the commutator of P_{2p+2} and L can be explicitly computed and one obtains

$$\begin{aligned}[P_{2p+2}, L] = &- a\big(g_p^+ + g_p + f_{p+1}^+ + f_{p+1} - 2b^+ f_p^+\big)S^+ \\ &+ 2\big(- b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_p^- + b^2 f_p\big) \\ &- a^-\big(g_p + g_p^- + f_{p+1} + f_{p+1}^- - 2bf_p\big)S^-, \quad p \in \mathbb{N}_0. \end{aligned} \tag{2.12}$$

In particular, (L, P_{2p+2}) represents the celebrated *Lax pair* of the Toda hierarchy. Varying $p \in \mathbb{N}_0$, the stationary Toda hierarchy is then defined in terms of the vanishing of the commutator of P_{2p+2} and L in (2.12) by,

$$[P_{2p+2}, L] = \text{s-Tl}_p(a, b) = 0, \quad p \in \mathbb{N}_0. \tag{2.13}$$

Thus one finds

$$g_p + g_p^- + f_{p+1} + f_{p+1}^- - 2bf_p = 0, \tag{2.14}$$

$$-b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_g^- + b^2 f_p = 0. \tag{2.15}$$

Using (2.5) with $j = p$ one concludes that (2.14) reduces to

$$f_{p+1} = f_{p+1}^-, \tag{2.16}$$

that is, f_{p+1} is a lattice constant. Similarly, one infers by subtracting b times (2.14) from twice (2.15) and using (2.6) with $j = p$, that g_{p+1} is a lattice constant as well, that is,

$$g_{p+1} = g_{p+1}^-. \tag{2.17}$$

Equations (2.16) and (2.17) constitute the pth stationary equation in the Toda hierarchy, which will be denoted by

$$\text{s-Tl}_p(a, b) = 0, \quad p \in \mathbb{N}_0. \tag{2.18}$$

Explicitly,

$$\begin{aligned} \text{s-Tl}_0(a, b) &= \begin{pmatrix} b - b^+ \\ 2\big(a^2 - (a^-)^2\big) \end{pmatrix} = 0, \\ \text{s-Tl}_1(a, b) &= \begin{pmatrix} -(a^+)^2 + (a^-)^2 - (b^+)^2 + b^2 \\ 2a^2(b^+ + b) - 2(a^-)^2(b + b^-) \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} b - b^+ \\ 2\big(a^2 - (a^-)^2\big) \end{pmatrix} = 0, \quad \text{etc.}, \end{aligned} \tag{2.19}$$

represent the first few equations of the stationary Toda hierarchy. By definition, the set of solutions of (2.13), with p ranging in $\mathbb{N}_0$ and $c_k \in \mathbb{C}$, $k \in \mathbb{N}$, represents the class of algebro-geometric Toda solutions.

In the following we will frequently assume that a and b satisfy the pth stationary Toda equation. By this we mean they satisfy one of the pth stationary Toda equations after a particular choice of integration constants $c_k \in \mathbb{C}$, $k = 1, \dots, p$, $p \in \mathbb{N}$, has been made.

Next, we introduce polynomials $F_p(z, n)$ and $G_{p+1}(z, n)$ of degree p and $p + 1$ with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_p(z, n) = \sum_{j=0}^{p} z^j f_{p-j}(n), \tag{2.20}$$

$$G_{p+1}(z,n) = -z^{p+1} + \sum_{j=0}^{p} z^j g_{p-j}(n) + f_{p+1}(n). \tag{2.21}$$

Explicitly, one obtains

$$\begin{aligned}
&F_0 = 1,\\
&F_1 = b + z + c_1,\\
&F_2 = a^2 + (a^-)^2 + b^2 + bz + z^2 + c_1(b+z) + c_2,\\
&G_1 = b - z,\\
&G_2 = (a^-)^2 - a^2 + b^2 - z^2 + c_1(b-z),\\
&G_3 = -a^2 b^+ + (a^-)^2 b^- + 2(a^-)^2 b + b^3 - 2a^2 z - z^3\\
&\qquad + c_1\Big((a^-)^2 - a^2 + b^2 - z^2\Big) + c_2(b-z), \quad \text{etc.}
\end{aligned} \tag{2.22}$$

Next, we study the restriction of the difference expression P_{2p+2} to the two-dimensional kernel (i.e., the formal null space in an algebraic sense as opposed to the functional analytic one) of $(L-z)$. More precisely, let

$$\ker(L-z) = \{\psi \colon \mathbb{Z} \to \mathbb{C} \cup \{\infty\} \mid (L-z)\psi = 0\}. \tag{2.23}$$

Then (2.11) implies

$$P_{2p+2} \mid_{\ker(L-z)} = \big(2aF_p(z)S^+ + G_{p+1}(z)\big)\big|_{\ker(L-z)}. \tag{2.24}$$

Therefore, the Lax relation (2.12) becomes

$$\begin{aligned}
0 = [P_{2p+2}, L] \mid_{\ker(L-z)} &= \Big(a\big(2(z-b^+)F_p^+ - 2(z-b)F_p + G_{p+1}^- - G_{p+1}^+\big)S^+\\
&\quad + \big(2(a^-)^2F_p^- - 2a^2F_p^+ + (z-b)(G_{p+1}^- - G_{p+1})\big)\Big)\Big|_{\ker(L-z)},
\end{aligned} \tag{2.25}$$

or, equivalently,

$$0 = 2(z-b^+)F_p^+ - 2(z-b)F_p + G_{p+1}^- - G_{p+1}^+, \tag{2.26}$$

$$0 = 2(a^-)^2F_p^- - 2a^2F_p^+ + (z-b)(G_{p+1}^- - G_{p+1}). \tag{2.27}$$

Upon summing (2.26) one infers

$$0 = 2(z-b^+)F_p^+ + G_{p+1}^+ + G_{p+1}, \quad p \in \mathbb{N}_0, \tag{2.28}$$

and inserting (2.26) into (2.27) then implies

$$0 = (z-b)^2F_p + (z-b)G_{p+1} + a^2F_p^+ - (a^-)^2F_p^-, \quad p \in \mathbb{N}_0. \tag{2.29}$$

Combining equations (2.27) and (2.28) one concludes that the quantity

$$R_{2p+2}(z) = G_{p+1}(z,n)^2 - 4a(n)^2F_p(z,n)F_p^+(z,n) \tag{2.30}$$

is a lattice constant, and hence depends on z only. Thus, one can write

$$R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \tag{2.31}$$

One can show that equation (2.30) leads to an explicit determination of the integration constants $c_1, \dots, c_p$ in

$$\text{s-Tl}_p(a,b) = 0 \tag{2.32}$$

in terms of the zeros $E_0, \dots, E_{2p+1}$ of the associated polynomial R_{2p+2} in (2.31). In fact, one can prove

$$c_k = c_k(\underline{E}), \quad k = 1, \dots, p, \tag{2.33}$$

where

$$c_k(\underline{E}) = - \sum_{\substack{j_0, \dots, j_{2p+1}=0 \\ j_0 + \cdots + j_{2p+1} = k}}^{k} \frac{(2j_0)! \cdots (2j_{2p+1})!}{2^{2k}(j_0!)^2 \cdots (j_{2p+1}!)^2 (2j_0 - 1) \cdots (2j_{2p+1} - 1)} E_0^{j_0} \cdots E_{2p+1}^{j_{2p+1}},$$
$$k = 1, \dots, p \tag{2.34}$$

are symmetric functions of $\underline{E} = (E_0, \dots, E_{2p+1})$.

Remark 2.1. Since by (2.5), (2.6), (2.20) and (2.21), a enters quadratically in F_p and G_{p+1}, the Toda hierarchy (2.18) is invariant under the substitution

$$a(n) \to a_\epsilon(n) = \epsilon(n) a(n), \quad \epsilon(n) \in \{+1, -1\}, \quad n \in \mathbb{Z}. \tag{2.35}$$

We emphasize that the result (2.24) is valid independently of whether or not P_{2p+2} and L commute. However, the fact that the two difference expressions P_{2p+2} and L commute implies the existence of an algebraic relationship between them. This gives rise to the Burchnall-Chaundy polynomial for the Toda hierarchy first discussed in the discrete context by Naĭman [46], [47].

Theorem 2.2. *Fix $p \in \mathbb{N}_0$ and assume that P_{2p+2} and L commute, $[P_{2p+2}, L] = 0$, or equivalently, suppose* s-Tl$_p(a, b) = 0$. *Then L and P_{2p+2} satisfy an algebraic relationship of the type* (*cf.* (2.31))

$$\begin{aligned} &\mathcal{F}_p(L, P_{2p+2}) = P_{2p+2}^2 - R_{2p+2}(L) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{2.36}$$

The expression $\mathcal{F}_p(L, P_{2p+2})$ is called the Burchnall–Chaundy polynomial of the Lax pair (L, P_{2p+2}). The equation (2.36) naturally leads to the hyperelliptic curve $\mathcal{K}_p$ of (arithmetic) genus $p \in \mathbb{N}_0$, where

$$\begin{aligned} &\mathcal{K}_p \colon \mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \end{aligned} \tag{2.37}$$

The curve $\mathcal{K}_p$ is compactified by joining two points $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, at infinity. For notational simplicity, the resulting curve is still denoted by $\mathcal{K}_p$. Points P on $\mathcal{K}_p \backslash P_{\infty_\pm}$ are represented as pairs $P = (z, y)$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = 0$. The complex structure on $\mathcal{K}_p$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_p$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $p \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_p$ (i.e., by fixing $E_0, \dots, E_{2p+1}$), the integration constants $c_1, \dots, c_p$ in the corresponding stationary s-Tl$_p$ equation are uniquely determined as is clear from (2.33) and (2.34), which establish the integration constants c_k as symmetric functions of $E_0, \dots, E_{2p+1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$. The trivial case $p = 0$, which leads to $a(n)^2 = (E_1 - E_0)^2/16$, $b(n) = (E_0 + E_1)/2$, is of no interest to us in this paper.

In the following, the zeros[1] of the polynomial $F_p(\cdot, n)$ (cf. (2.20)) will play a special role. We denote them by $\{\mu_j(n)\}_{j=1}^p$ and write

$$F_p(z,n) = \prod_{j=1}^{p} [z - \mu_j(n)]. \tag{2.38}$$

The next step is crucial; it permits us to "lift" the zeros μ_j of F_p from $\mathbb{C}$ to the curve $\mathcal{K}_p$. From (2.30) and (2.38) one infers

$$R_{2p+2}(z) - G_{p+1}(z)^2 = 0, \quad z \in \{\mu_j, \mu_k^+\}_{j,k=1,\dots,p}. \tag{2.39}$$

We now introduce $\{\hat{\mu}_j(n)\}_{j=1,\dots,p} \subset \mathcal{K}_p$ by

$$\hat{\mu}_j(n) = \big(\mu_j(n), -G_{p+1}(\mu_j(n), n)\big), \quad j = 1, \dots, p, \ n \in \mathbb{Z}. \tag{2.40}$$

Next, we recall equation (2.30) and define the fundamental meromorphic function $\phi(\cdot, n)$ on $\mathcal{K}_p$ by

$$\begin{aligned} \phi(P,n) &= \frac{-G_{p+1}(z,n) + y}{2a(n)F_p(z,n)} && (2.41)\\ &= \frac{-2a(n)F_p(z,n+1)}{G_{p+1}(z,n) + y}, && (2.42)\\ & P = (z,y) \in \mathcal{K}_p, \ n \in \mathbb{Z} \end{aligned}$$

with divisor $(\phi(\cdot, n))$ of $\phi(\cdot, n)$ given by

$$\big(\phi(\cdot,n)\big) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}(n+1)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}, \tag{2.43}$$

using (2.38). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\} \in \mathrm{Sym}^p(\mathcal{K}_p) \tag{2.44}$$

(cf. the notation introduced in Appendix A). The stationary Baker–Akhiezer function $\psi(\cdot, n, n_0)$ on $\mathcal{K}_p \backslash \{P_{\infty_\pm}\}$ is then defined in terms of $\phi(\cdot, n)$ by

$$\psi(P,n,n_0) = \begin{cases} \prod_{m=n_0}^{n-1} \phi(P,m) & \text{for } n \geq n_0 + 1, \\ 1 & \text{for } n = n_0, \\ \prod_{m=n}^{n_0-1} \phi(P,m)^{-1} & \text{for } n \leq n_0 - 1 \end{cases} \tag{2.45}$$

with divisor $\big(\psi(\cdot, n, n_0)\big)$ of $\psi(P, n, n_0)$ given by

$$\big(\psi(\cdot,n,n_0)\big) = \mathcal{D}_{\underline{\hat{\mu}}(n)} - \mathcal{D}_{\underline{\hat{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{\infty_+}} - \mathcal{D}_{P_{\infty_-}}). \tag{2.46}$$

Basic properties of ϕ and ψ are summarized in the following result $\big($where $W\big(f,g\big)(n) = a(n)\big(f(n)g(n+1) - f(n+1)g(n)\big)$, $n \in \mathbb{Z}$, denotes the Wronskian of two complex-valued sequences f and g, and P^* abbreviates $P^* = (z, -y)$ for $P = (z,y) \in \mathcal{K}_p\big)$.

[1] If $a, b \in \ell^\infty(\mathbb{Z})$, these zeros are the Dirichlet eigenvalues of a bounded operator on $\ell^2(\mathbb{Z})$ associated with the difference expression L and a Dirichlet boundary condition at $n \in \mathbb{Z}$.

Lemma 2.3. *Assume $a, b \in \ell^\infty(\mathbb{Z})$ satisfy the pth stationary Toda equation* (2.18). *Moreover, let $P = (z, y) \in \mathcal{K}_p \backslash \{P_{\infty_\pm}\}$ and $(n, n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$a\phi(P) + a^- \phi^-(P)^{-1} = z - b, \tag{2.47}$$

as well as

$$\phi(P)\phi(P^*) = \frac{F_p^+(z)}{F_p(z)}, \tag{2.48}$$

$$\phi(P) + \phi(P^*) = \frac{-G_{p+1}(z)}{aF_p(z)}, \tag{2.49}$$

$$\phi(P) - \phi(P^*) = \frac{y(P)}{aF_p(z)}. \tag{2.50}$$

Moreover, ψ satisfies

$$\big(L - z(P)\big)\psi(P) = 0, \quad \big(P_{2p+2} - y(P)\big)\psi(P) = 0, \tag{2.51}$$

$$\psi(P, n, n_0)\psi(P^*, n, n_0) = \frac{F_p(z, n)}{F_p(z, n_0)}, \tag{2.52}$$

$$\begin{aligned} &a(n)\big[\psi(P, n, n_0)\psi(P^*, n+1, n_0) + \psi(P^*, n, n_0)\psi(P, n+1, n_0)\big] \\ &\quad = -\frac{G_{p+1}(z, n)}{F_p(z, n_0)}, \end{aligned} \tag{2.53}$$

$$W(\psi(P, \cdot, n_0), \psi(P^*, \cdot, n_0)) = -\frac{y(P)}{F_p(z, n_0)}. \tag{2.54}$$

Combining the polynomial recursion approach with (2.38) readily yields trace formulas for the Toda invariants, which are expressions of a and b in terms of the zeros μ_j of F_p.

Lemma 2.4. *Assume $a, b \in \ell^\infty(\mathbb{Z})$ satisfy the pth stationary Toda equation* (2.18). *Then,*

$$a(n)^2 = \frac{1}{2}\sum_{j=1}^{p} R_{2p+2}^{1/2}(\hat\mu_j(n)) \prod_{\substack{k=1 \\ k \neq j}}^{p} [\mu_j(n) - \mu_k(n)]^{-1} + \frac{1}{4}[b^{(2)}(n) - b(n)^2] > 0, \tag{2.55}$$

$$b(n) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p} \mu_j(n), \tag{2.56}$$

$$b^{(k)}(n) = \frac{1}{2}\sum_{m=0}^{2p+1} E_m^k - \sum_{j=1}^{p} \mu_j(n)^k, \quad k \in \mathbb{N}. \tag{2.57}$$

From this point on we assume that the affine part of $\mathcal{K}_p$ is nonsingular, that is,

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2p+1. \tag{2.58}$$

Since nonspecial divisors play a fundamental role in this context we also recall the following fact.

Lemma 2.5. *Suppose the affine part of $\mathcal{K}_p$ is nonsingular and assume that $a, b \in \ell^\infty(\mathbb{Z})$ satisfy the pth stationary Toda equation* (2.18). *Let $\mathcal{D}_{\underline{\hat\mu}}$, $\underline{\hat\mu} = (\hat\mu_1, \dots, \hat\mu_p)$*

be the Dirichlet divisor of degree p associated with a, b defined according to (2.40)*, that is,*

$$\hat{\mu}_j(n) = \big(\mu_j(n), -G_{p+1}(\mu_j(n), n)\big), \quad j = 1, \dots, p, \; n \in \mathbb{Z}. \tag{2.59}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial for all $n \in \mathbb{Z}$. Moreover, there exists a constant $C_\mu > 0$ such that

$$|\mu_j(n)| \le C_\mu, \quad j = 1, \dots, p, \; n \in \mathbb{Z}. \tag{2.60}$$

We continue with the theta function representation for ψ, a, and b. For general background information and the notation employed we refer to Appendix A.

Let θ denote the Riemann theta function associated with $\mathcal{K}_p$ (whose affine part is assumed to be nonsingular) and a fixed homology basis $\{a_j, b_j\}_{j=1}^p$. Next, choosing a base point $Q_0 \in \mathcal{B}(\mathcal{K}_p)$ in the set of branch points of $\mathcal{K}_p$, we recall that the Abel maps $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ are defined by (A.43) and (A.46), and the Riemann vector $\underline{\Xi}_{Q_0}$ is given by (A.58). Then Abel's theorem (cf. (A.56)) (2.46) yields

$$\begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - \underline{A}_{P_{\infty_-}}(P_{\infty_+})(n - n_0) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - 2\underline{A}_{Q_0}(P_{\infty_+})(n - n_0). \end{aligned} \tag{2.61}$$

Next, let $\omega^{(3)}_{P_{\infty_+}, P_{\infty_-}}$ denote the normalized differential of the third kind defined by

$$\omega^{(3)}_{P_{\infty_+}, P_{\infty_-}} = \frac{1}{y} \prod_{j=1}^{p} (z - \lambda_j) dz \underset{\zeta \to 0}{=} \pm \big(\zeta^{-1} + O(1)\big) d\zeta \text{ as } P \to P_{\infty_\pm}, \tag{2.62}$$
$$\zeta = 1/z,$$

where the constants $\lambda_j \in \mathbb{C}$, $j = 1, \dots, p$, are determined by employing the normalization

$$\int_{a_j} \omega^{(3)}_{P_{\infty+}, P_{\infty-}} = 0, \quad j = 1, \dots, p. \tag{2.63}$$

One then infers

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty+}, P_{\infty-}} \underset{\zeta \to 0}{=} \pm \ln \zeta + e_0^{(3)}(Q_0) + O(\zeta) \text{ as } P \to P_{\infty} \tag{2.64}$$

for some constant $e_0^{(3)}(Q_0) \in \mathbb{C}$. The vector of b-periods of $\omega^{(3)}_{P_{\infty_+}, P_{\infty_-}}/(2\pi i)$ is denoted by

$$\underline{U}_0^{(3)} = \big(U_{0,1}^{(3)}, \dots, U_{0,p}^{(3)}\big), \quad U_{0,j}^{(3)} = \frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{P_{\infty_+}, P_{\infty_-}}, \quad j = 1, \dots, p. \tag{2.65}$$

Since Q_0 is a branch point, $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, one concludes by (A.45) that

$$U_0^{(3)} = \underline{A}_{P_{\infty-}}(P_{\infty+}) = 2\underline{A}_{Q_0}(P_{\infty+}). \tag{2.66}$$

In the following it will be convenient to introduce the abbreviation

$$\begin{aligned} &\underline{z}(P, \underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \mathcal{K}_p, \; \underline{Q} = \{Q_1, \dots, Q_p\} \in \mathrm{Sym}^p(\mathcal{K}_p). \end{aligned} \tag{2.67}$$

We note that $\underline{z}(\cdot, \underline{Q})$ is independent of the choice of base point Q_0.

The zeros and the poles of ψ as recorded in (2.46) suggest consideration of the following expression involving θ-functions (cf. (A.32))

$$\frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(n))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(n_0))\big)}\exp\bigg(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg). \tag{2.68}$$

Here we agree to use the same path of integration from Q_0 to P on $\mathcal{K}_p$ in the Abel map $\underline{\hat{A}}_{Q_0}(P)$ in $\underline{z}(P,\underline{\hat{\mu}}(n))$ and in the integral of $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}$ in the exponent of (2.68). With this convention the expression (2.68) is well-defined on $\mathcal{K}_p$ and one concludes

$$\psi(P,n,n_0)=C(n,n_0)\frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(n))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(n_0))\big)}\exp\bigg((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg). \tag{2.69}$$

To determine $C(n,n_0)$ one can use (2.52) for $P=P_{\infty_+}$ and $P^*=P_{\infty_-}$. Hence,

$$C(n,n_0)^2=\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0-1))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n-1))\big)}. \tag{2.70}$$

Thus, one obtains the following well-known result.

Theorem 2.6. *Suppose that $a,b\in\ell^\infty(\mathbb{Z})$ satisfy the pth stationary Toda equation* (2.18) *on $\mathbb{Z}$. In addition, assume the affine part of $\mathcal{K}_p$ to be nonsingular and let $P\in\mathcal{K}_p\backslash\{P_{\infty_\pm}\}$ and $n,n_0\in\mathbb{Z}$. Then $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial for $n\in\mathbb{Z}$. Moreover,*[2]

$$\psi(P,n,n_0)=C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\bigg((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg), \tag{2.71}$$

where

$$C(n,n_0)=\bigg[\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n_0-1))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n-1))\big)}\bigg]^{1/2}, \tag{2.72}$$

with the linearizing property of the Abel map,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)})=\big(\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)})-2\underline{A}_{Q_0}(P_{\infty_+})(n-n_0)\big)\pmod{L_p}. \tag{2.73}$$

The coefficients a and b are given by

$$a(n)=\tilde{a}\bigg[\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n-1))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n+1))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n))\big)^2}\bigg]^{1/2},\quad n\in\mathbb{Z}, \tag{2.74}$$

$$b(n)=\frac{1}{2}\sum_{m=0}^{2p+1}E_m-\sum_{j=1}^{p}\lambda_j+\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\bigg[\frac{\theta\big(\underline{\omega}+\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n))\big)}{\theta\big(\underline{\omega}+\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(n-1))\big)}\bigg]\bigg|_{\underline{\omega}=0},\quad n\in\mathbb{Z}, \tag{2.75}$$

where the constant $\tilde{a}$ depends only on $\mathcal{K}_p$ and $c_j(p)$ is given by (A.25) *and* (A.28).

Combining (2.73) and (2.75), one observes the remarkable linearity of the theta function with respect to n in formulas (2.74), (2.75). In fact, one can rewrite (2.75) as

$$b(n)=\Lambda_0+\sum_{j=1}^{p}c_j(p)\frac{\partial}{\partial\omega_j}\ln\bigg(\frac{\theta(\underline{\omega}+\underline{A}-\underline{B}n)}{\theta(\underline{\omega}+\underline{C}-\underline{B}n)}\bigg)\bigg|_{\underline{\omega}=0}, \tag{2.76}$$

[2] To avoid multi-valued expressions in formulas such as (2.71), etc., we agree to always choose the same path of integration connecting Q_0 and P and refer to Remark A.4 for additional tacitly assumed conventions.

where

$$\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty+}) + \underline{U}_0^{(3)} n_0 + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}), \tag{2.77}$$

$$\underline{B} = \underline{U}_0^{(3)}, \tag{2.78}$$

$$\underline{C} = \underline{A} + \underline{B}, \tag{2.79}$$

$$\Lambda_0 = \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p} \lambda_j. \tag{2.80}$$

Hence, the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^p$ are uniquely determined by $\mathcal{K}_p$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^p$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(n_0) = (\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)) \in \operatorname{Sym}^p \mathcal{K}_p$ at the point n_0.

Remark 2.7. If one assumes a, b in (2.74) and (2.75) to be quasi-periodic (cf. (3.6) and (3.7)), then there exists a homology basis $\{\tilde{a}_j, \tilde{b}_j\}_{j=1}^p$ on $\mathcal{K}_p$ such that $\underline{\widetilde{B}} = \underline{\widetilde{U}}_0^{(3)}$ satisfies the constraint

$$\underline{\widetilde{B}} = \underline{\widetilde{U}}_0^{(3)} \in \mathbb{R}^p. \tag{2.81}$$

This is studied in detail in Appendix B.

3. The Green's function of H

In this section we focus on the properties of the Green's function of H and derive a variety of results to be used in our principal Section 4.

Introducing

$$\mathrm{G}(P,m,n) = \frac{1}{W\big(\psi(P,\cdot,n_0)\big),\psi(P^*,\cdot,n_0)} \begin{cases} \psi(P^*,m,n_0)\psi(P,n,n_0), & m \le n, \\ \psi(P,m,n_0)\psi(P^*,n,n_0), & m \ge n, \end{cases}$$
$$P \in \mathcal{K}_p \backslash \{P_{\infty_\pm}\},\ n, n_0 \in \mathbb{Z}, \tag{3.1}$$

and

$$g(P,n) = G(P,n,n) = \frac{\psi(P,n,n_0)\psi(P^*,n,n_0)}{W\big(\psi(P,\cdot,n_0),\psi(P^*,\cdot,n_0)\big)}, \tag{3.2}$$

equations (2.52) and (2.54) then imply

$$g(P,n) = -\frac{F_p(z,n)}{y(P)}, \quad P = (z,y) \in \mathcal{K}_p \backslash \{P_{\infty_\pm}\},\ n \in \mathbb{Z}. \tag{3.3}$$

Together with $g(P,n)$ we also introduce its two branches $g_\pm(z,n)$ defined on the upper and lower sheets $\Pi_\pm$ of $\mathcal{K}_p$ (cf. (A.3), (A.4), and (A.14))

$$g_\pm(z,n) = \mp\frac{F_p(z,n)}{R_{2p+2}(z)^{1/2}}, \quad z \in \Pi,\ n \in \mathbb{Z} \tag{3.4}$$

with $\Pi = \mathbb{C}\backslash\mathcal{C}$ the cut plane introduced in (A.4).

For convenience we shall focus on g_- whenever possible and use the simplified notation

$$g(z,n) = g_-(z,n), \quad z \in \Pi,\ n \in \mathbb{Z} \tag{3.5}$$

from now on.

Next we briefly review a few properties of quasi-periodic and almost-periodic discrete functions.

We denote by $\mathrm{QP}(\mathbb{Z})$ and $\mathrm{AP}(\mathbb{Z})$ the sets of quasi-periodic and almost periodic sequences on $\mathbb{Z}$, respectively.

In particular, a sequence f is called quasi-periodic with fundamental periods $(\Omega_1,\dots,\Omega_N)\in(0,\infty)^N$ if the frequencies $2\pi/\Omega_1,\dots,2\pi/\Omega_N$ are linearly independent over $\mathbb{Q}$ and if there exists a continuous function $F\in C(\mathbb{R}^N)$, periodic of period 1 in each of its arguments,

$$F(x_1,\dots,x_j+1,\dots,x_N)=F(x_1,\dots,x_N),\quad x_j\in\mathbb{R},\ j=1,\dots,N, \tag{3.6}$$

such that

$$f(n)=F(\Omega_1^{-1}n,\dots,\Omega_N^{-1}n),\quad n\in\mathbb{Z}. \tag{3.7}$$

Any quasi-periodic sequence on $\mathbb{Z}$ is almost periodic on $\mathbb{Z}$. Moreover, a sequence $f=\{f(k)\}_{k\in\mathbb{Z}}$ is almost periodic on $\mathbb{Z}$ if and only if there exists a Bohr almost periodic function g on $\mathbb{R}$ such that $f(k)=g(k)$ for all $k\in\mathbb{Z}$ (see, e.g., [11, p. 47]).

For any almost periodic sequence $f=\{f(k)\}_{k\in\mathbb{Z}}$, the mean value $\langle f\rangle$ of f, defined by

$$\langle f\rangle=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=n_0-N}^{n_0+N}f(k), \tag{3.8}$$

exists and is independent of $n_0\in\mathbb{Z}$. Moreover, we recall the following facts for almost periodic sequences that can be deduced from corresponding properties of Bohr almost periodic functions, see, for instance, [5, Ch. I], [6, Sects. 39–92], [11, Ch. I], [22, Chs. 1,3,6], [29], [37, Chs. 1,2,6], and [51].

Theorem 3.1. *Assume $f,g\in\mathrm{AP}(\mathbb{Z})$ and $n_0,n\in\mathbb{Z}$. Then the following assertions hold:*
(i) $f\in\ell^\infty(\mathbb{Z})$.
(ii) $\overline{f}$, cf, $c\in\mathbb{C}$, $f(\cdot+n)$, $f(n\,\cdot)$, $n\in\mathbb{Z}$, $|f|^\alpha$, $\alpha\geq 0$ *are all in* $\mathrm{AP}(\mathbb{Z})$.
(iii) $f+g, fg\in\mathrm{AP}(\mathbb{Z})$.
(iv) $1/g\in\mathrm{AP}(\mathbb{Z})$ *if and only if* $1/g\in\ell^\infty(\mathbb{Z})$.
(v) Let G be uniformly continuous on $\mathcal{M}\subseteq\mathbb{R}$ and $f(n)\in\mathcal{M}$ for all $n\in\mathbb{Z}$. Then $G(f)\in\mathrm{AP}(\mathbb{Z})$.
(vi) Let $\langle f\rangle=0$, *then* $\sum_{k=n_0}^n f(k)\underset{|n|\to\infty}{=}o(|n|)$.
(vii) Let $F(n)=\sum_{k=n_0}^n f(k)$. *Then* $F\in\mathrm{AP}(\mathbb{Z})$ *if and only if* $F\in\ell^\infty(\mathbb{Z})$.
(viii) If $0\leq f\in\mathrm{AP}(\mathbb{Z})$, $f\not\equiv 0$, *then* $\langle f\rangle>0$.
(ix) If $1/f\in\ell^\infty(\mathbb{Z})$ *and* $f=|f|\exp(i\varphi)$, *then* $|f|\in\mathrm{AP}(\mathbb{Z})$ *and φ is of the type* $\varphi(n)=cn+\psi(n)$, *where* $c\in\mathbb{R}$ *and* $\psi\in\mathrm{AP}(\mathbb{Z})$ *(and real-valued).*
(x) If $F(n)=\exp\Big(\sum_{k=n_0}^n f(k)\Big)$, *then* $F\in\mathrm{AP}(\mathbb{Z})$ *if and only if* $f(n)=i\beta+\psi(n)$, *where* $\beta\in\mathbb{R}$, $\psi\in\mathrm{AP}(\mathbb{Z})$, *and* $\Psi\in\ell^\infty(\mathbb{Z})$, *where* $\Psi(n)=\sum_{k=n_0}^n\psi(k)$.

For the rest of this paper it will be convenient to introduce the following hypothesis:

Hypothesis 3.2. Assume the affine part of $\mathcal{K}_p$ to be nonsingular. Moreover, suppose that a, $b\in\mathrm{QP}(\mathbb{Z})$ satisfy the pth stationary Toda equation (2.18) on $\mathbb{Z}$.

Next, we note the following result.

Lemma 3.3. *Assume Hypothesis* 3.2. *Then all z-derivatives of $F_p(z,\cdot)$ and $G_{p+1}(z,\cdot)$, $z\in\mathbb{C}$, and $g(z,\cdot)$, $z\in\Pi$, are quasi-periodic. Moreover, taking limits to points on $\mathcal{C}$, the last result extends to either side of cuts in the set $\mathcal{C}\backslash\{E_m\}_{m=0}^{2p+1}$ (cf.* (A.3)) *by continuity with respect to z.*

Proof. Since f_ℓ and g_ℓ are polynomials with respect to a and b, f_ℓ and g_ℓ, $\ell \in \mathbb{N}$, are quasi-periodic by Theorem 3.1. The corresponding assertion for $F_p(z,\cdot)$ then follows from (2.20) and that for $g(z,\cdot)$ follows from (3.4). $\square$

In the following we represent $G_{p+1}(z,n) + G^+_{p+1}(z,n)$ as

$$G_{p+1}(z,n) + G^+_{p+1}(z,n) = -2\prod_{k=1}^{p+1}[z - \nu_k(n)], \quad z \in \mathbb{C}, \ n \in \mathbb{Z}, \tag{3.9}$$

and note that the roots ν_k are bounded,

$$\|\nu_k\|_\infty \leq \widetilde{C}, \quad k = 1, ..., p+1 \tag{3.10}$$

for some constant $\widetilde{C} > 0$, since the coefficients of $G_{p+1}(z,n)$ are defined in terms of bounded coefficients a and b by(2.6). For future purposes we introduce the set

$$\begin{aligned}\Pi_C = \Pi \Big\backslash \Big(&\{z \in \mathbb{C} \,|\, |z| \leq C+1\} \cup \\ &\Big\{ z \in \mathbb{C} \,|\, \min_{m=0,\dots,2p+1}[\mathrm{Re}(E_m)] - 1 \leq \mathrm{Re}(z) \leq \max_{m=0,\dots,2p+1}[\mathrm{Re}(E_m)] + 1, \\ &\min_{m=0,\dots,2p+1}[\mathrm{Im}(E_m)] - 1 \leq \mathrm{Im}(z) \leq \max_{m=0,\dots,2p+1}[\mathrm{Im}(E_m)] + 1 \Big\} \Big),\end{aligned} \tag{3.11}$$

where $C = \max\{C_\mu, \|b\|_\infty, \widetilde{C}\}$ and C_μ is the constant in (2.60). Without loss of generality, we may also assume that Π_C contains no cuts, that is,

$$\Pi_C \cap \mathcal{C} = \emptyset. \tag{3.12}$$

Next, we derive a fundamental equation for the mean value of the diagonal Green's function $g(z,\cdot)$ that will allow us to analyze the spectrum of the Jacobi operator H. First, we note that by (2.49), (2.50), (2.54), and (3.1) one obtains

$$-\frac{\mathrm{G}(P,n,n+1)}{\mathrm{G}(P^*,n,n+1)} = \frac{G_{p+1}(z,n) - y}{G_{p+1}(z,n) + y}, \quad P = (z,y) \in \mathcal{K}_p, \ n \in \mathbb{Z}. \tag{3.13}$$

Differentiating the logarithm of the expression on the right-hand side of (3.13) with respect to z and using (2.30), one infers

$$\frac{1}{2}\frac{d}{dz}\ln\bigg(\frac{G_{p+1}(z,n) - y}{G_{p+1}(z,n) + y}\bigg) = \frac{\frac{R^\bullet_{2p+2}(z)}{2y}G_{p+1}(z,n) - yG^\bullet_{p+1}(z,n)}{-4a(n)^2F_p(z,n)F^+_p(z,n)}, \ z \in \Pi_C. \tag{3.14}$$

Here $\bullet$ abbreviates d/dz. We note that the left-hand side of (3.14) is well-defined since by (2.30), (2.38), and (3.11),

$$\begin{aligned}[G_{p+1}(z,n) - y][G_{p+1}(z,n) + y] &= G_{p+1}(z,n)^2 - R_{2p+2}(z) \\ &= 4a(n)^2F_p(z,n)F^+_p(z,n) \\ &= 4a(n)^2\prod_{j=1}^{p}[z - \mu_j(n)][z - \mu_j(n+1)] \neq 0, \quad z \in \Pi_C.\end{aligned} \tag{3.15}$$

Adding and subtracting $g(z,n)$ on the right-hand side of (3.14) yields

$$\frac{1}{2}\frac{d}{dz}\ln\bigg(\frac{G_{p+1}(z,n) - y}{G_{p+1}(z,n) + y}\bigg) = g(z,n) + \frac{K(z,n)}{y}, \ z \in \Pi_C, \tag{3.16}$$

where

$$K(z,n) = \frac{1}{2}G_{p+1}(z,n)\bigg(\frac{F_p^\bullet(z,n)}{F_p(z,n)} + \frac{(F_p^+)^\bullet(z,n)}{F_p^+(z,n)}\bigg) - G_{p+1}^\bullet(z,n) - F_p(z,n). \quad (3.17)$$

Next we prove that the mean value of $K(z,\cdot)$ equals zero.

Lemma 3.4. *Assume Hypothesis* 3.2. *Then*

$$\langle K(z,\cdot)\rangle = 0, \quad z\in\Pi_C. \quad (3.18)$$

Proof. Let $z\in\Pi_C$. Using (2.28) we rewrite (3.17) as

$$\begin{aligned}
K(z,n) =& \frac{1}{2}G_{p+1}(z,n)\bigg[\frac{d}{dz}\ln\big(G_{p+1}(z,n)+G_{p+1}^-(z,n)\big)\\
&\qquad + \frac{d}{dz}\ln\big(G_{p+1}^+(z,n)+G_{p+1}(z,n)\big)\bigg]\\
&- \frac{d}{dz}G_{p+1}(z,n) + \frac{1}{2}\bigg(\frac{G_{p+1}^-(z,n)}{z-b(n)} - \frac{G_{p+1}(z,n)}{z-b^+(n)}\bigg)\\
=& \frac{1}{2}G_{p+1}(z,n)\bigg[\frac{G_{p+1}^\bullet(z,n)+(G_{p+1}^-)^\bullet(z,n)}{G_{p+1}(z,n)+G_{p+1}^-(z,n)}\\
&\qquad + \frac{(G_{p+1}^+)^\bullet(z,n)+G_{p+1}^\bullet(z,n)}{G_{p+1}^+(z,n)+G_{p+1}(z,n)}\bigg]\\
&- G_{p+1}^\bullet(z,n) + \frac{1}{2}\bigg(\frac{G_{p+1}^-(z,n)}{z-b(n)} - \frac{G_{p+1}(z,n)}{z-b^+(n)}\bigg)\\
=& \frac{1}{2}\bigg[\frac{(G_{p+1}^+)^\bullet(z,n)G_{p+1}(z,n) - G_{p+1}^\bullet(z,n)G_{p+1}^+(z,n)}{G_{p+1}^+(z,n)+G_{p+1}(z,n)}\\
&\quad - \frac{G_{p+1}^\bullet(z,n)G_{p+1}^-(z,n) - (G_{p+1}^-)^\bullet(z,n)G_{p+1}(z,n)}{G_{p+1}(z,n)+G_{p+1}^-(z,n)}\bigg]\\
&+ \frac{1}{2}\bigg(\frac{G_{p+1}^-(z,n)}{z-b(n)} - \frac{G_{p+1}(z,n)}{z-b^+(n)}\bigg), \quad z\in\Pi_C. \quad (3.19)
\end{aligned}$$

Since $K(z,\cdot)$ is a sum of two difference expressions and $G_{p+1}(z,\cdot)$ and $G_{p+1}^\bullet(z,\cdot)$ are bounded for fixed $z\in\Pi_C$, one obtains

$$\langle K(z,\cdot)\rangle = 0, \quad z\in\Pi_C. \quad (3.20)$$

□

Using (3.16) and Lemma 3.4, one derives the following result that will subsequently play a crucial role in this paper.

Lemma 3.5. *Assume Hypothesis* 3.2 *and let* $z, z_0\in\Pi$. *Then*

$$\bigg\langle \ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle = 2\int_{z_0}^{z} dz'\langle g(z',\cdot)\rangle + \bigg\langle \ln\bigg(\frac{G_{p+1}(z_0,\cdot)-y}{G_{p+1}(z_0,\cdot)+y}\bigg)\bigg\rangle, \quad (3.21)$$

where the path connecting z_0 and z is assumed to lie in the cut plane Π. *Moreover, by taking limits to points on $\mathcal{C}$ in* (3.21), *the result* (3.21) *extends to either side of the cuts in the set $\mathcal{C}$ by continuity with respect to z.*

Proof. Let $z, z_0 \in \Pi_C$. Integrating equation (3.16) from z_0 to z along a smooth path in Π_C yields

$$\ln\left(\frac{G_{p+1}(z,n)-y}{G_{p+1}(z,n)+y}\right) - \ln\left(\frac{G_{p+1}(z_0,\cdot)-y}{G_{p+1}(z_0,\cdot)+y}\right) = 2\int_{z_0}^{z} dz'\, g(z',n) + \\ + 2\int_{z_0}^{z} dz'\, \frac{K(z',n)}{y}. \tag{3.22}$$

By Lemma 3.3, $K(z,\cdot)$ is quasi-periodic. Consequently, also

$$\int_{z_0}^{z} dz'\, \frac{K(z',\cdot)}{y}, \quad z \in \Pi_C, \tag{3.23}$$

is a family of uniformly almost periodic functions for z varying in compact subsets of Π_C as discussed in [22, Sect. 2.7]. By Lemma 3.4 one thus obtains

$$\left\langle \left[\int_{z_0}^{z} dz'\, \frac{K(z',\cdot)}{y} \right] \right\rangle = 0. \tag{3.24}$$

Hence, taking mean values in (3.22) (taking into account (3.24)), proves (3.21) for $z \in \Pi_C$. Since f_ℓ, $\ell \in \mathbb{N}_0$, are quasi-periodic by Lemma 3.3 (we recall that $f_0 = 1$), (2.20) and (3.4) yield

$$\int_{z_0}^{z} dz'\, \langle g(z',\cdot)\rangle = \sum_{\ell=0}^{p} \langle f_{p-\ell}\rangle \int_{z_0}^{z} dz'\, \frac{(z')^\ell}{R_{2p+2}(z')^{1/2}}. \tag{3.25}$$

Thus, $\int_{z_0}^{z} dz'\, \langle g(z',\cdot)\rangle$ has an analytic continuation with respect to z to all of Π and consequently, (3.21) for $z \in \Pi_C$ extends by analytic continuation to $z \in \Pi$. By continuity this extends to either side of the cuts in $\mathcal{C}$. Interchanging the role of z and z_0, analytic continuation with respect to z_0 then yields (3.21) for $z, z_0 \in \Pi$. $\square$

Remark 3.6. For $z \in \Pi_C$, the sequence $\ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)$ is quasi-periodic and hence $\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle$ is well-defined. But if one analytically continues $\ln\big(\frac{G_{p+1}(z,n)-y}{G_{p+1}(z,n)+y}\big)$ with respect to z, then $(G_{p+1}(z,n)-y)$ and $(G_{p+1}(z,n)+y)$ may acquire zeros for some $n \in \mathbb{Z}$ and hence $\ln\big(\frac{G_{p+1}(z,n)-y}{G_{p+1}(z,n)+y}\big) \notin \mathrm{QP}(\mathbb{Z})$. Nevertheless, as shown by the right-hand side of (3.21), $\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle$ admits an analytic continuation in z from Π_C to all of Π, and from now on, $\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle$, $z \in \Pi$, always denotes that analytic continuation (cf. also (3.27)).

Next, we will invoke the Baker-Akhiezer function $\psi(P,n,n_0)$ and analyze the expression $\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle$ in more detail.

Theorem 3.7. *Assume Hypothesis* 3.2*, let* $P = (z,y) \in \Pi_\pm$*, and* $n, n_0 \in \mathbb{Z}$*. Moreover, select a homology basis* $\{\tilde a_j, \tilde b_j\}_{j=1}^p$ *on* $\mathcal{K}_p$ *such that* $\underline{\widetilde{B}} = \underline{\widetilde{U}}_0^{(3)}$*, with* $\underline{\widetilde{U}}_0^{(3)}$ *the vector of* $\tilde b$*-periods of the normalized differential of the third kind,* $\widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}$*, satisfies the constraint*

$$\underline{\widetilde{B}} = \underline{\widetilde{U}}_0^{(3)} \in \mathbb{R}^p \tag{3.26}$$

(*cf. Appendix* B)*. Then,*

$$\mathrm{Re}\left(\left\langle \ln\left(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\right)\right\rangle\right) = 2\mathrm{Re}\left(\int_{Q_0}^{P} \widetilde{\omega}^{(3)}_{P_{\infty_+},\infty_-}\right). \tag{3.27}$$

Proof. Using (2.41), (2.45) and (2.46) one obtains the following representation of the Baker-Akhiezer function $\psi(P,n,n_0)$ for $n>n_0$, $n,n_0\in\mathbb{Z}$, $P\in\mathcal{K}_p$,

$$\begin{aligned}\psi(P,n,n_0) &= \prod_{m=n_0}^{n-1}\phi(P,m) = \bigg[\prod_{m=n_0}^{n-1}\frac{y-G_{p+1}(z,m)}{-y-G_{p+1}(z,m)}\,\frac{F_p(z,m+1)}{F_p(z,m)}\bigg]^{1/2}\\ &= \bigg(\frac{F_p(z,n)}{F_p(z,n_0)}\bigg)^{1/2}\bigg[\prod_{m=n_0}^{n-1}\frac{G_{p+1}(z,m)-y}{G_{p+1}(z,m)+y}\bigg]^{1/2}\\ &= \bigg(\frac{F_p(z,n)}{F_p(z,n_0)}\bigg)^{1/2}\\ &\quad\times\exp\bigg(\frac{1}{2}\sum_{m=n_0}^{n-1}\bigg[\ln\bigg(\frac{G_{p+1}(z,m)-y}{G_{p+1}(z,m)+y}\bigg)-\bigg\langle\ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle\bigg]\bigg)\\ &\quad\times\exp\bigg(\frac{1}{2}(n-n_0)\bigg\langle\ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle\bigg),\\ &\qquad P=(z,y)\in\Pi_\pm,\ z\in\Pi_C,\ n,n_0\in\mathbb{Z}.\end{aligned}\tag{3.28}$$

A similar representation can be written for $\psi(P,n,n_0)$ if $n<n_0$, $n,n_0\in\mathbb{Z}$, $P\in\mathcal{K}_p$.

Since $\Big[\ln\Big(\frac{G_{p+1}(z,m)-y}{G_{p+1}(z,m)+y}\Big)-\Big\langle\ln\Big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\Big)\Big\rangle\Big]$ has mean zero,

$$\bigg(\frac{1}{2}\sum_{m=n_0}^{n-1}\bigg[\ln\bigg(\frac{G_{p+1}(z,m)-y}{G_{p+1}(z,m)+y}\bigg)-\bigg\langle\ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle\bigg]\bigg)\underset{|n|\to\infty}{=}o(|n|),\quad z\in\Pi_C,\tag{3.29}$$

by Theorem 3.1 (vi). In addition, the factor $F_p(z,n)/F_p(z,n_0)$ in (3.28) is quasi-periodic and hence bounded on $\mathbb{Z}$.

On the other hand, (2.71) yields

$$\begin{aligned}\psi(P,n,n_0) &= C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\bigg((n-n_0)\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)\\ &= \Theta(P,n,n_0)\exp\bigg((n-n_0)\int_{Q_0}^{P}\widetilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg),\\ &\qquad P\in\mathcal{K}_p\backslash\big(\{P_{\infty_\pm}\}\cup\{\hat{\mu}_j(n_0)\}_{j=1}^p\big).\end{aligned}\tag{3.30}$$

Taking into account (2.67), (2.73), (2.81), (A.32), and the fact that by (2.60) no $\hat{\mu}_j(n)$ can reach $P_{\infty_\pm}$ as n varies in $\mathbb{Z}$, one concludes that

$$\Theta(P,\cdot,n_0)\in\ell^\infty(\mathbb{Z}),\quad P\in\mathcal{K}_p\backslash\{\hat{\mu}_j(n_0)\}_{j=1}^p.\tag{3.31}$$

A comparison of (3.28) and (3.30) then shows that the $o(|n|)$-term in (3.29) must actually be bounded on $\mathbb{Z}$ and hence the left-hand side of (3.29) is almost periodic (in fact, quasi-periodic). In addition, the term

$$\exp\bigg(\frac{1}{2}\sum_{m=n_0}^{n-1}\bigg[\ln\bigg(\frac{G_{p+1}(z,m)-y}{G_{p+1}(z,m)+y}\bigg)-\bigg\langle\ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle\bigg]\bigg),\quad z\in\Pi_C,\tag{3.32}$$

is then almost periodic (in fact, quasi-periodic) by Theorem 3.1 (x). A further comparison of (3.28) and (3.30) then yields (3.27) for $z\in\Pi_C$. Analytic continuation with respect to z then implies (3.27) for $z\in\Pi$. By continuity with respect to z, taking boundary values to either side of the cuts in the set $\mathcal{C}$, this then extends to $z \in \mathcal{C}$ (cf. (A.3), (A.4)) and hence proves (3.27) for $P = (z, y) \in \mathcal{K}_p \backslash \{P_{\infty_\pm}\}$. ☐

4. Spectra of Jacobi operators with quasi-periodic algebro-geometric coefficients

In this section we establish the connection between the algebro-geometric formalism of Section 2 and the spectral theoretic description of Jacobi operators H in $\ell^2(Z)$ with quasi-periodic algebro-geometric coefficients. In particular, we introduce the conditional stability set of H and prove our principal result, the characterization of the spectrum of H. Finally, we provide a qualitative description of the spectrum of H in terms of analytic spectral arcs.

Suppose that a, $b \in \ell^\infty(\mathbb{Z}) \cap \mathrm{QP}(\mathbb{Z})$ satisfy the pth stationary Toda equation (2.18) on $\mathbb{Z}$. The corresponding Jacobi operator H in $\ell^2(\mathbb{Z})$ is then defined by

$$H = aS^+ + a^- S^- + b, \quad \mathrm{dom}(H) = \ell^2(\mathbb{Z}). \tag{4.1}$$

Thus, H is a bounded operator on $\ell^2(\mathbb{Z})$ (it is self-adjoint if and only if a and b are real-valued).

Before we turn to the spectrum of H in the general non-self-adjoint case, we briefly mention the following result on the spectrum of H in the self-adjoint case with quasi-periodic (or almost periodic) real-valued coefficients a and b. We denote by $\sigma(A)$, $\sigma_{\mathrm{e}}(A)$, and $\sigma_{\mathrm{d}}(A)$ the spectrum, essential spectrum, and discrete spectrum of a self-adjoint operator A in a complex Hilbert space, respectively.

Theorem 4.1 (See, e.g., [53] in the continuous context)**.** *Let $a, b \in \mathrm{QP}(\mathbb{Z})$ be real-valued. Define the self-adjoint Jacobi operator H in $\ell^2(\mathbb{Z})$ as in (4.1). Then,*

$$\begin{aligned}
&\sigma(H) = \sigma_{\mathrm{e}}(H) \\
&\quad \subseteq [-2\sup_{n\in\mathbb{Z}}\big(|\mathrm{Re}(a(n))|\big) + \inf_{n\in\mathbb{Z}}\big(\mathrm{Re}(b(n))\big), 2\sup_{n\in\mathbb{Z}}\big(|\mathrm{Re}(a(n))| + \sup_{n\in\mathbb{Z}}\mathrm{Re}(b(n))\big)], \\
&\sigma_{\mathrm{d}}(H) = \emptyset.
\end{aligned}$$

Moreover, $\sigma(H)$ contains no isolated points, that is, $\sigma(H)$ is a perfect set.

In the special periodic case where a, b are real-valued, the spectrum of H is purely absolutely continuous and a finite union of some compact intervals (see, e.g., [12], [13], [23], [42], [54]–[56]).

Next, we turn to the analysis of the generally non-self-adjoint operator H in (4.1). Assuming Hypothesis 3.2 we introduce the set $\Sigma \subset \mathbb{C}$ by

$$\Sigma = \bigg\{\lambda \in \mathbb{C} \,\bigg|\, \mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\lambda,\cdot) - y}{G_{p+1}(\lambda,\cdot) + y}\bigg)\bigg\rangle\bigg) = 0\bigg\}. \tag{4.2}$$

Below we will show that Σ plays the role of the conditional stability set of H, familiar from the spectral theory of one-dimensional periodic differential and difference operators.

Lemma 4.2. *Assume Hypothesis* 3.2*. Then Σ coincides with the conditional stability set of H, that is,*

$$\begin{aligned}
\Sigma = \{\lambda \in \mathbb{C} \,|\, &\textit{there exists at least one bounded solution} \\
&0 \neq \psi \in \ell^\infty(\mathbb{Z}) \textit{ of } H\psi = \lambda\psi\}.
\end{aligned} \tag{4.3}$$

Proof. By (2.71) and (2.72),

$$\psi(P,n,n_0)=C(n,n_0)\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n_0)))}\exp\bigg((n-n_0)\int_{E_0}^{z}\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg), \quad P=(z,y)\in\Pi_{\pm}, \tag{4.4}$$

is a solution of $H\psi=z\psi$ which is bounded on $\mathbb{Z}$ if and only if the exponential function in (4.4) is bounded on $\mathbb{Z}$. By (3.27), the latter holds if and only if

$$\mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle\bigg)=0. \tag{4.5}$$

□

Remark 4.3. At first sight our *a priori* choice of cuts $\mathcal{C}$ for $R_{2p+2}(\cdot)^{1/2}$, as described in Appendix A, might seem unnatural as they completely ignore the actual spectrum of H. However, the spectrum of H is not known from the outset, and in the case of complex-valued potentials, spectral arcs of H may actually cross each other (cf. Theorem 4.7 (iv)) which renders them unsuitable for cuts of $R_{2p+2}(\cdot)^{1/2}$.

Before we state our first principal result on the spectrum of H, we find it convenient to recall a number of basic definitions and well-known facts in connection with the spectral theory of non-self-adjoint operators (we refer to [18, Chs. I, III, IX], [26, Sects. 1, 21–23], [30, Sects. IV.5.6, V.3.2], and [50, p. 178–179] for more details). Let S be a densely defined closed operator in complex separable Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ and by $\ker(T)$ and $\mathrm{ran}(T)$ the kernel (null space) and range of a linear operator T in $\mathcal{H}$. The resolvent set, $\rho(S)$, spectrum, $\sigma(S)$, point spectrum (the set of eigenvalues), $\sigma_{\mathrm{p}}(S)$, continuous spectrum, $\sigma_{\mathrm{c}}(S)$, residual spectrum, $\sigma_{\mathrm{r}}(S)$, field of regularity, $\pi(S)$, approximate point spectrum, $\sigma_{\mathrm{ap}}(S)$, two kinds of essential spectra, $\sigma_{\mathrm{e}}(S)$, and $\widetilde{\sigma}_{\mathrm{e}}(S)$, the numerical range of S, $\Theta(S)$, and the sets $\Delta(S)$ and $\widetilde{\Delta}(S)$ are defined as follows:

$$\rho(S)=\{z\in\mathbb{C}\,|\,(S-zI)^{-1}\in\mathcal{B}(\mathcal{H})\}, \tag{4.6}$$

$$\sigma(S)=\mathbb{C}\backslash\rho(S), \tag{4.7}$$

$$\sigma_{\mathrm{p}}(S)=\{\lambda\in\mathbb{C}\,|\,\ker(S-\lambda I)\neq\{0\}\}, \tag{4.8}$$

$$\sigma_{\mathrm{c}}(S)=\{\lambda\in\mathbb{C}\,|\,\ker(S-\lambda I)=\{0\}\text{ and }\mathrm{ran}(S-\lambda I)\text{ is dense in }\mathcal{H}\text{ but not equal to }\mathcal{H}\}, \tag{4.9}$$

$$\sigma_{\mathrm{r}}(S)=\{\lambda\in\mathbb{C}\,|\,\ker(S-\lambda I)=\{0\}\text{ and }\mathrm{ran}(S-\lambda I)\text{ is not dense in }\mathcal{H}\}, \tag{4.10}$$

$$\pi(S)=\{z\in\mathbb{C}\,|\,\text{there exists }k_z>0\text{ s.t. }\|(S-zI)u\|_{\mathcal{H}}\geq k_z\|u\|_{\mathcal{H}}\text{ for all }u\in\mathrm{dom}(S)\}, \tag{4.11}$$

$$\sigma_{\mathrm{ap}}(S)=\mathbb{C}\backslash\pi(S), \tag{4.12}$$

$$\Delta(S)=\{z\in\mathbb{C}\,|\,\dim(\ker(S-zI))<\infty\text{ and }\mathrm{ran}(S-zI)\text{ is closed}\}, \tag{4.13}$$

$$\sigma_{\mathrm{e}}(S)=\mathbb{C}\backslash\Delta(S), \tag{4.14}$$

$$\widetilde{\Delta}(S)=\{z\in\mathbb{C}\,|\,\dim(\ker(S-zI))<\infty\text{ or }\dim(\ker(S^*-\overline{z}I))<\infty\}, \tag{4.15}$$

$$\widetilde{\sigma}_{\mathrm{e}}(S)=\mathbb{C}\backslash\widetilde{\Delta}(S), \tag{4.16}$$

$$\Theta(S)=\{(f,Sf)\in\mathbb{C}\,|\,f\in\mathrm{dom}(S),\ \|f\|_{\mathcal{H}}=1\}, \tag{4.17}$$

respectively. One then has

$$\sigma(S) = \sigma_{\rm p}(S) \cup \sigma_{\rm c}(S) \cup \sigma_{\rm r}(S) \quad \text{(disjoint union)} \tag{4.18}$$
$$= \sigma_{\rm p}(S) \cup \sigma_{\rm e}(S) \cup \sigma_{\rm r}(S), \tag{4.19}$$
$$\sigma_{\rm c}(S) \subseteq \sigma_{\rm e}(S) \backslash (\sigma_{\rm p}(S) \cup \sigma_{\rm r}(S)), \tag{4.20}$$
$$\sigma_{\rm r}(S) = \sigma_{\rm p}(S^*)^* \backslash \sigma_{\rm p}(S), \tag{4.21}$$
$$\sigma_{\rm ap}(S) = \{\lambda \in \mathbb{C} \,|\, \text{there exists a sequence } \{f_n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(S) \\ \text{with } \|f_n\|_{\mathcal{H}} = 1,\, n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|(S - \lambda I) f_n\|_{\mathcal{H}} = 0\}, \tag{4.22}$$
$$\widetilde{\sigma}_{\rm e}(S) \subseteq \sigma_{\rm e}(S) \subseteq \sigma_{\rm ap}(S) \subseteq \sigma(S) \ \text{ (all four sets are closed)}, \tag{4.23}$$
$$\rho(S) \subseteq \pi(S) \subseteq \Delta(S) \subseteq \widetilde{\Delta}(S) \quad \text{(all four sets are open)}, \tag{4.24}$$
$$\widetilde{\sigma}_{\rm e}(S) \subseteq \overline{\Theta(S)}, \quad \Theta(S) \ \text{ is convex}, \tag{4.25}$$
$$\widetilde{\sigma}_{\rm e}(S) = \sigma_{\rm e}(S) \ \text{ if } S = S^*. \tag{4.26}$$

Here σ^* in the context of (4.21) denotes the complex conjugate of the set $\sigma \subseteq \mathbb{C}$, that is,

$$\sigma^* = \{\overline{\lambda} \in \mathbb{C} \,|\, \lambda \in \sigma\}. \tag{4.27}$$

We note that there are several other versions of the concept of the essential spectrum in the non-self-adjoint context (cf. [18, Ch. IX]) but we will only use the two in (4.14) and in (4.16) in this paper.

We start with the following elementary result.

Lemma 4.4. *Let H be defined as in* (4.1)*. Then,*

$$\sigma_{\rm e}(H) = \widetilde{\sigma}_{\rm e}(H) \subseteq \overline{\Theta(H)}. \tag{4.28}$$

Proof. Since H and H^* are second-order difference operators on $\mathbb{Z}$,

$$\dim(\ker(H - zI)) \leq 2, \quad \dim(\ker(H^* - \overline{z}I)) \leq 2. \tag{4.29}$$

Moreover, we note that S closed and densely defined and $\dim(\ker(S^* - \overline{z}I)) < \infty$ implies that $\mathrm{ran}(S - zI)$ is closed (cf. [18, Theorem I.3.2]). Equations (4.13)–(4.16) and (4.25) then prove (4.28). $\square$

Theorem 4.5. *Assume Hypothesis* 3.2*. Then the point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\rm p}(H) = \sigma_{\rm r}(H) = \emptyset, \tag{4.30}$$
$$\sigma(H) = \sigma_{\rm c}(H) = \sigma_{\rm e}(H) = \sigma_{\rm ap}(H). \tag{4.31}$$

Proof. First we prove the absence of the point spectrum of H. Suppose $z \in \Pi \backslash (\Sigma \cup \{\mu_j(n_0)\}_{j=1}^p)$. Then $\psi(P, \cdot, n_0)$ and $\psi(P^*, \cdot, n_0)$ are linearly independent solutions of $H\psi = z\psi$ which are unbounded at $+\infty$ or $-\infty$. This argument extends to all $z \in \Pi\backslash\Sigma$ by multiplying $\psi(P, \cdot, n_0)$ and $\psi(P^*, \cdot, n_0)$ with an appropriate function of z and n_0 (independent of n). It also extends to either side of the cut $\mathcal{C}\backslash\Sigma$ by continuity with respect to z. On the other hand, any solution $\psi(z, \cdot) \in \ell^2(\mathbb{Z})$ of $H\psi = z\psi$, $z \in \mathbb{C}$, is necessarily bounded (since any sequence in $\ell^2(\mathbb{Z})$ is bounded). Thus,

$$\{\mathbb{C}\backslash\Sigma\} \cap \sigma_{\rm p}(H) = \emptyset. \tag{4.32}$$

Hence, it remains to rule out eigenvalues located in Σ. We consider a fixed $\lambda \in \Sigma$ and note that by (2.52), there exists at least one solution $\psi_1(\lambda, \cdot) \in \ell^\infty(\mathbb{Z})$ of

$H\psi = \lambda\psi$. Actually, a comparison of (3.28) and (4.2) shows that one can choose $\psi_1(\lambda,\cdot)$ such that $|\psi_1(\lambda,\cdot)| \in \mathrm{QP}(\mathbb{Z})$ and hence $\psi_1(\lambda,\cdot) \notin \ell^2(\mathbb{Z})$.

Next, suppose there exists a second solution $\psi_2(\lambda,\cdot) \in \ell^2(\mathbb{Z})$ of $H\psi = \lambda\psi$ which is linearly independent of $\psi_1(\lambda,\cdot)$. Then one concludes that the Wronskian of $\psi_1(\lambda,\cdot)$ and $\psi_2(\lambda,\cdot)$ lies in $\ell^2(\mathbb{Z})$,

$$W(\psi_1(\lambda,\cdot),\psi_2(\lambda,\cdot)) \in \ell^2(\mathbb{Z}). \tag{4.33}$$

However, by hypothesis, $W(\psi_1(\lambda,\cdot),\psi_2(\lambda,\cdot)) = c(\lambda) \neq 0$ is a nonzero constant. This contradiction proves that

$$\Sigma \cap \sigma_{\mathrm{p}}(H) = \emptyset \tag{4.34}$$

and hence $\sigma_{\mathrm{p}}(H) = \emptyset$.

Next, we note that the same argument yields that H^* also has no point spectrum,

$$\sigma_{\mathrm{p}}(H^*) = \emptyset. \tag{4.35}$$

Indeed, if a, $b \in \ell^\infty(\mathbb{Z}) \cap \mathrm{QP}(\mathbb{Z})$ satisfy the pth stationary Toda equation (2.18) on $\mathbb{Z}$, then $\overline{a}$, $\overline{b}$ also satisfy one of the pth stationary Toda equation (2.18) associated with a hyperelliptic curve of genus p with $\{E_m\}_{m=0}^{2p+1}$ replaced by $\{\overline{E}_m\}_{m=0}^{2p+1}$, etc. Since by general principles (cf. (4.27)),

$$\sigma_{\mathrm{r}}(B) \subseteq \sigma_{\mathrm{p}}(B^*)^* \tag{4.36}$$

for any densely defined closed linear operator B in some complex separable Hilbert space (see, e.g., [27, p. 71]), one obtains $\sigma_{\mathrm{r}}(H) = \emptyset$ and hence (4.30). This proves that the spectrum of H is purely continuous, $\sigma(H) = \sigma_{\mathrm{c}}(H)$. The remaining equalities in (4.31) then follow from (4.20) and (4.23). $\square$

The following result is a fundamental one:

Theorem 4.6. *Assume Hypothesis* 3.2. *Then the spectrum of H coincides with Σ and hence equals the conditional stability set of H,*

$$\begin{aligned}\sigma(H) &= \left\{\lambda \in \mathbb{C} \,\middle|\, \mathrm{Re}\left(\left\langle \ln\left(\frac{G_{p+1}(\lambda,\cdot) - y}{G_{p+1}(\lambda,\cdot) + y}\right)\right\rangle\right) = 0\right\} &(4.37)\\ &= \{\lambda \in \mathbb{C} \,|\, \textit{there exists at least one bounded solution} \\ &\qquad\qquad 0 \neq \psi \in \ell^\infty(\mathbb{Z}) \textit{ of } H\psi = \lambda\psi\}. &(4.38)\end{aligned}$$

In particular,

$$\{E_m\}_{m=0}^{2p+1} \subset \sigma(H), \tag{4.39}$$

and $\sigma(H)$ contains no isolated points.

Proof. First we will prove that

$$\sigma(H) \subseteq \Sigma \tag{4.40}$$

by adapting a method due to Chisholm and Everitt [10] (in the context of differential operators). For this purpose we temporarily choose $z \in \Pi\backslash(\Sigma \cup \{\mu_j(n_0)\}_{j=1}^p)$ and construct the resolvent of H as follows. Introducing the two branches $\psi_\pm(P,n,n_0)$ of the Baker–Akhiezer function $\psi(P,n,n_0)$ by

$$\psi_\pm(P,n,n_0) = \psi(P,n,n_0), \quad P = (z,y) \in \Pi_\pm, \; n,n_0 \in \mathbb{Z}, \tag{4.41}$$

we define

$$\hat{\psi}_+(z,n,n_0) = \begin{cases} \psi_+(z,n,n_0) & \text{if } \psi_+(z,\cdot,n_0) \in \ell^2(n_0,\infty), \\ \psi_-(z,n,n_0) & \text{if } \psi_-(z,\cdot,n_0) \in \ell^2(n_0,\infty), \end{cases} \tag{4.42}$$

$$\hat{\psi}_-(z,n,n_0) = \begin{cases} \psi_-(z,n,n_0) & \text{if } \psi_-(z,\cdot,n_0) \in \ell^2(-\infty,n_0), \\ \psi_+(z,n,n_0) & \text{if } \psi_+(z,\cdot,n_0) \in \ell^2(-\infty,n_0), \end{cases} \quad z \in \Pi\backslash\Sigma, \ n, n_0 \in \mathbb{Z}, \tag{4.43}$$

and

$$G(z,n,n') = \frac{1}{W(\hat{\psi}_-(z,n,n_0), \hat{\psi}_+(z,n,n_0))} \begin{cases} \hat{\psi}_-(z,n',n_0)\hat{\psi}_+(z,n,n_0), & n \geq n', \\ \hat{\psi}_-(z,n,n_0)\hat{\psi}_+(z,n',n_0), & n \leq n', \end{cases} \quad z \in \Pi\backslash\Sigma, \ n, n_0 \in \mathbb{Z}. \tag{4.44}$$

Due to the homogeneous nature of G, (4.44) extends to all $z \in \Pi$. Moreover, we extend (4.42)–(4.44) to either side of the cut $\mathcal{C}$ except at possible points in Σ (i.e., to $\mathcal{C}\backslash\Sigma$) by continuity with respect to z, taking limits to $\mathcal{C}\backslash\Sigma$. Next, we introduce the operator $R(z)$ in $\ell^2(\mathbb{Z})$ defined by

$$(R(z)f)(n) = \sum_{n' \in \mathbb{Z}} G(z,n,n')f(n'), \quad f \in \ell_0^\infty(\mathbb{Z}), \ z \in \Pi, \tag{4.45}$$

where $\ell_0^\infty(\mathbb{Z})$ denotes the linear space of compactly supported (i.e., finite) complex-valued sequences, and extend it to $z \in \mathcal{C}\backslash\Sigma$, as discussed in connection with $G(\cdot,n,n')$. The explicit form of $\hat{\psi}_\pm(z,n,n_0)$, inferred from (3.30) by restricting P to $\Pi_\pm$, then yields the estimates

$$|\hat{\psi}_\pm(z,n,n_0)| \leq C_\pm(z,n_0)e^{\mp\kappa(z)n}, \quad z \in \Pi\backslash\Sigma, \ n \in \mathbb{Z} \tag{4.46}$$

for some constants $C_\pm(z,n_0) > 0$, $\kappa(z) > 0$, $z \in \Pi\backslash\Sigma$. One can follow the second part of the proof of Theorem 5.3.2 in [20] line by line and prove that $R(z)$, $z \in \mathbb{C}\backslash\Sigma$, extends from $\ell_0^\infty(\mathbb{Z})$ to a bounded linear operator defined on all of $\ell^2(\mathbb{Z})$. A straightforward computation then proves

$$(H - zI)R(z)f = f, \quad f \in \ell^2(\mathbb{Z}), \ z \in \mathbb{C}\backslash\Sigma \tag{4.47}$$

and hence also

$$R(z)(H - zI)g = g, \quad g \in \ell^2(\mathbb{Z}), \ z \in \mathbb{C}\backslash\Sigma. \tag{4.48}$$

Thus, $R(z) = (H - zI)^{-1}$, $z \in \mathbb{C}\backslash\Sigma$, and hence (4.40) holds.

Next we will prove that

$$\sigma(H) \supseteq \Sigma. \tag{4.49}$$

We will adapt a strategy of proof applied by Eastham in the continuous case of (real-valued) periodic potentials [19] (reproduced in the proof of Theorem 5.3.2 of [20]) to the (complex-valued) quasi-periodic discrete case at hand. Suppose $\lambda \in \Sigma$. By the characterization (4.3) of Σ, there exists a bounded solution $\psi(\lambda,\cdot)$ of $H\psi = \lambda\psi$. A comparison with the Baker-Akhiezer function (3.30) then shows that one can assume, without loss of generality, that

$$|\psi(\lambda,\cdot)| \in \mathrm{QP}(\mathbb{Z}). \tag{4.50}$$

By Theorem 3.1 (i), one obtains

$$\psi(\lambda,\cdot) \in \ell^\infty(\mathbb{Z}). \tag{4.51}$$

Next, we pick $\Omega \in \mathbb{N}$ and consider $g(n)$, $n = 0, 1, \dots, \Omega$, satisfying

$$\begin{aligned} &g(0) = 0, \quad g(\Omega) = 1, \\ &0 \leq g(n) \leq 1, \quad n = 1, \dots, \Omega - 1. \end{aligned} \tag{4.52}$$

Moreover, we introduce the sequence $\{h_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{Z})$ by

$$h_k(n) = \begin{cases} 1, & |n| \leq (k-1)\Omega, \\ g(k\Omega - |n|), & (k-1)\Omega \leq |n| \leq k\Omega, \\ 0, & |n| \geq k\Omega \end{cases} \tag{4.53}$$

and the sequence $\{f_k(\lambda)\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{Z})$ by

$$f_k(\lambda, n) = d_k(\lambda)\psi(\lambda, n)h_k(n), \quad n \in \mathbb{Z},\ d_k(\lambda) > 0,\ k \in \mathbb{N}. \tag{4.54}$$

Here $d_k(\lambda)$ is determined by the normalization requirement

$$\|f_k(\lambda)\|_2 = 1, \quad k \in \mathbb{N}. \tag{4.55}$$

Of course,

$$f_k(\lambda, \cdot) \in \ell^2(\mathbb{Z}), \quad k \in \mathbb{N}, \tag{4.56}$$

since $f_k(\lambda, \cdot)$ is finitely supported. Next, we note that as a consequence of Theorem 3.1 $(viii)$,

$$\sum_{-N}^{N} |\psi(\lambda, n)|^2 \underset{N\to\infty}{=} (2N+1)\big\langle|\psi(\lambda, \cdot)|^2\big\rangle + o(N) \tag{4.57}$$

with

$$\big\langle|\psi(\lambda, \cdot)|^2\big\rangle > 0. \tag{4.58}$$

Thus, one computes

$$\begin{aligned} 1 = \|f_k(\lambda)\|_2^2 &= d_k(\lambda)^2 \sum_{n\in\mathbb{Z}} |\psi(\lambda, n)|^2 h_k(n)^2 \\ &= d_k(\lambda)^2 \sum_{|n|\leq k\Omega} |\psi(\lambda, n)|^2 h_k(n)^2 \geq d_k(\lambda)^2 \sum_{|n|\leq (k-1)\Omega} |\psi(\lambda, n)|^2 \\ &\geq d_k(\lambda)^2 \big[\big\langle|\psi(\lambda, \cdot)|^2\big\rangle (k-1)\Omega + o(k)\big]. \end{aligned} \tag{4.59}$$

Consequently,

$$d_k(\lambda) \underset{k\to\infty}{=} O\big(k^{-1/2}\big). \tag{4.60}$$

Next, one computes

$$\begin{aligned} (H - \lambda I)f_k(\lambda, n) =& d_k(\lambda)\Big[a(n)\psi(\lambda, n)\big[h_k(n+1) - h_k(n)\big] \\ &+ a(n-1)\psi(\lambda, n-1)\big[h_k(n-1) - h_k(n)\big]\Big] \end{aligned} \tag{4.61}$$

and hence

$$\|(H - \lambda I)f_k\|_2 \leq 2d_k(\lambda)\|a\|_\infty \|\psi(\lambda)(h_k^+ - h_k)\|_2, \quad k \in \mathbb{N}. \tag{4.62}$$

Using (4.51) and (4.53) one estimates

$$\begin{aligned} \|\psi(\lambda)\big[h_k^+ - h_k\big]\|_2^2 &= \sum_{(k-1)\Omega\leq|n|\leq k\Omega} |\psi(\lambda, n)|^2 |h_k(n+1) - h_k(n)|^2 \\ &\leq 2\|\psi(\lambda)\|_\infty^2 \big(\Omega + 1\big). \end{aligned} \tag{4.63}$$

Thus, combining (4.60) and (4.62)–(4.63) one infers

$$\lim_{n\to\infty} \|(H - \lambda I)f_k\|_2 = 0 \tag{4.64}$$

and hence $\lambda \in \sigma_{\mathrm{ap}}(H) = \sigma(H)$ by (4.22) and (4.31).

Relation (4.39) follows from (4.3) and the fact that by (2.52) there exists a solution $\psi((E_m, 0), \cdot, n_0) \in \ell^\infty(\mathbb{Z})$ of $H\psi = E_m\psi$ for all $m = 0, \ldots, 2p+1$.

Finally, $\sigma(H)$ contains no isolated points since those would necessarily be essential singularities of the resolvent of H, as H has no eigenvalues by (4.30) (cf. [30, Sect. III.6.5]). An analysis of the Green's function of H reveals at most an algebraic singularity at the points $\{E_m\}_{m=0}^{2p+1}$ and hence excludes the possibility of an essential singularity of $(H-zI)^{-1}$. $\square$

In the special self-adjoint case where a, b are real-valued, the result (4.37) is equivalent to the vanishing of the Lyapunov exponent of H which characterizes the (purely absolutely continous) spectrum of H as discussed by Carmona–Lacroix [9, Chs. IV, VII] (cf. also [8], [14], [28], [31]).

The explicit formula for Σ in (4.2) permits a qualitative description of the spectrum of H as follows. We recall (3.16) and (3.25) and write

$$\frac{1}{2}\frac{d}{dz}\bigg\langle \ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle = \langle g(z,\cdot)\rangle = \frac{\prod_{j=1}^{p}(z-\widetilde{\lambda}_j)}{\Big(\prod_{j=0}^{2p+1}(z-E_m)\Big)^{1/2}}, \quad z\in\Pi, \tag{4.65}$$

for some constants

$$\{\widetilde{\lambda}_j\}_{j=1}^{p}\subset\mathbb{C}. \tag{4.66}$$

As in similar situations before, (4.65) extends to either side of the cuts in $\mathcal{C}$ by continuity with respect to z.

Theorem 4.7. *Assume Hypothesis* 3.2. *Then the spectrum $\sigma(H)$ of H has the following properties:*
(i) *$\sigma(H)\subset\mathbb{C}$ is bounded,*

$$\sigma(H)\subset\{z\in\mathbb{C}\,|\,\mathrm{Re}(z)\in[M_1,M_2],\ \mathrm{Im}(z)\in[M_3,M_4]\}, \tag{4.67}$$

where

$$\begin{aligned}
M_1 &= -2\sup_{n\in\mathbb{Z}}[|\mathrm{Re}(a(n))|] + \inf_{n\in\mathbb{Z}}[\mathrm{Re}(b(n))],\\
M_2 &= 2\sup_{n\in\mathbb{Z}}[|\mathrm{Re}(a(n))|] + \sup_{n\in\mathbb{Z}}[\mathrm{Re}(b(n))],\\
M_3 &= -2\sup_{n\in\mathbb{Z}}[|\mathrm{Im}(a(n))|] + \inf_{n\in\mathbb{Z}}[\mathrm{Im}(b(n))],\\
M_4 &= 2\sup_{n\in\mathbb{Z}}[|\mathrm{Im}(a(n))|] + \sup_{n\in\mathbb{Z}}[\mathrm{Im}(b(n))].
\end{aligned} \tag{4.68}$$

(ii) *$\sigma(H)$ consists of finitely many simple analytic arcs* (*cf. Remark* 4.8)*. These analytic arcs may only end at the points $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_p$, $E_0,\dots,E_{2p+1}$.*
(iii) *Each E_m, $m=0,\dots,2p+1$, is met by at least one of these arcs. More precisely, a particular E_{m_0} is hit by precisely $2N_0+1$ analytic arcs, where $N_0\in\{0,\dots,p\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with E_{m_0}. Adjacent arcs meet at an angle $2\pi/(2N_0+1)$ at E_{m_0}.* (*Thus, generically, $N_0=0$ and precisely one arc hits E_{m_0}.*)
(iv) *Crossings of spectral arcs are permitted. This phenomenon takes place precisely when for a particular $j_0\in\{1,\dots,p\}$, $\widetilde{\lambda}_{j_0}\in\sigma(H)$ such that*

$$\begin{gathered}
\mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\widetilde{\lambda}_{j_0},\cdot)-y}{G_{p+1}(\widetilde{\lambda}_{j_0},\cdot)+y}\bigg)\bigg\rangle\bigg)=0\\
\text{for some } j_0\in\{1,\dots,p\} \text{ with } \widetilde{\lambda}_{j_0}\notin\{E_m\}_{m=0}^{2p+1}.
\end{gathered} \tag{4.69}$$

In this case $2M_0+2$ analytic arcs are converging toward $\widetilde{\lambda}_{j_0}$, where $M_0\in\{1,\dots,p\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with $\widetilde{\lambda}_{j_0}$. Adjacent arcs meet at an angle

$\pi/(M_0+1)$ *at* $\widetilde\lambda_{j_0}$. *(Thus, if crossings occur, generically,* $M_0 = 1$ *and two arcs cross at a right angle.)*
(v) *The resolvent set* $\mathbb{C}\backslash\sigma(H)$ *of* H *is path-connected.*

Proof. Item (i) follows from (4.28) and (4.31) upon noticing that

$$(f, Hf) = 2\sum_{k=-\infty}^{\infty} a(k)\mathrm{Re}[f(k+1)\overline{f(k)}] + (f, \mathrm{Re}(b)f) + i(f, \mathrm{Im}(b)f), \quad f \in \ell^2(\mathbb{Z}). \tag{4.70}$$

To prove (ii) we first introduce the meromorphic differential of the third kind

$$\Omega^{(3)} = \langle g(P,\cdot)\rangle dz = \frac{\langle F_p(z,\cdot)\rangle dz}{y} = \frac{\prod_{j=1}^p \big(z-\widetilde\lambda_j\big)dz}{R_{2p+2}(z)^{1/2}}, \qquad P=(z,y)\in\mathcal{K}_p\backslash\{P_{\infty_\pm}\} \tag{4.71}$$

(cf. (4.66)). Then, by Lemma 3.5,

$$\bigg\langle \ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle = 2\int_{Q_0}^P \Omega^{(3)} + \bigg\langle \ln\bigg(\frac{G_{p+1}(z_0,\cdot)-y}{G_{p+1}(z_0,\cdot)+y}\bigg)\bigg\rangle, \tag{4.72}$$

for some fixed $Q_0 = (z_0, y_0) \in \mathcal{K}_p\backslash\{P_{\infty_\pm}\}$, is holomorphic on $\mathcal{K}_p\backslash\{P_{\infty_\pm}\}$. By (4.65), (4.66), the characterization (4.37) of the spectrum,

$$\sigma(H) = \bigg\{\lambda\in\mathbb{C}\,\bigg|\, \mathrm{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(\lambda,\cdot)-y}{G_{p+1}(\lambda,\cdot)+y}\bigg)\bigg\rangle\bigg)=0\bigg\}, \tag{4.73}$$

and the fact that $\mathrm{Re}\big(\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle\big)$ is a harmonic function on the cut plane Π, the spectrum $\sigma(H)$ of H consists of analytic arcs which may only end at the points $\widetilde\lambda_1,\dots,\widetilde\lambda_p$, $E_0,\dots,E_{2p+1}$. (Since $\sigma(H)$ is independent of the chosen set of cuts, if a spectral arc crosses or runs along a part of one of the cuts in $\mathcal{C}$, one can slightly deform the original set of cuts to extend an analytic arc along or across such an original cut.)

To prove (iii) one first recalls that by Theorem 4.2 the spectrum of H contains no isolated points. On the other hand, since $\{E_m\}_{m=0}^{2p+1}\subset\sigma(H)$ by (4.39), one concludes that at least one spectral arc meets each E_m, $m=0,\dots,2p+1$. Choosing $Q_0=(E_{m_0},0)$ in (4.72) one obtains

$$\begin{aligned}
&\bigg\langle \ln\bigg(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\bigg)\bigg\rangle = 2\int_{E_{m_0}}^z dz'\,\langle g(z',\cdot)\rangle + \bigg\langle \ln\bigg(\frac{G_{p+1}(E_{m_0},\cdot)-y}{G_{p+1}(E_{m_0},\cdot)+y}\bigg)\bigg\rangle \\
&\quad = 2\int_{E_{m_0}}^z dz' \frac{\prod_{j=1}^p\big(z'-\widetilde\lambda_j\big)}{\big(\prod_{m=0}^{2p+1}(z'-E_m)\big)^{1/2}} + \bigg\langle \ln\bigg(\frac{G_{p+1}(E_{m_0},\cdot)-y}{G_{p+1}(E_{m_0},\cdot)+y}\bigg)\bigg\rangle \\
&\quad \underset{z\to E_{m_0}}{=} \int_{E_{m_0}}^z dz'\,(z'-E_{m_0})^{N_0-(1/2)}[C+O(z'-E_{m_0})] \\
&\qquad + \bigg\langle \ln\bigg(\frac{G_{p+1}(E_{m_0},\cdot)-y}{G_{p+1}(E_{m_0},\cdot)+y}\bigg)\bigg\rangle \\
&\quad \underset{z\to E_{m_0}}{=} \frac{(z-E_{m_0})^{N_0+(1/2)}}{N_0+(1/2)}[C+O(z-E_{m_0})] + \bigg\langle \ln\bigg(\frac{G_{p+1}(E_{m_0},\cdot)-y}{G_{p+1}(E_{m_0},\cdot)+y}\bigg)\bigg\rangle
\end{aligned} \tag{4.74}$$

for some $C = |C|e^{i\varphi_0} \in \mathbb{C}\backslash\{0\}$. Using

$$\operatorname{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(E_m,\cdot)-y}{G_{p+1}(E_m,\cdot)+y}\bigg)\bigg\rangle\bigg) = 0, \quad m = 0,\dots,2p+1, \tag{4.75}$$

as a consequence of (4.39), $\operatorname{Re}\big(\big\langle \ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle\big) = 0$ and $z = E_{m_0} + \rho e^{i\varphi}$ imply

$$0 \underset{\rho\downarrow 0}{=} \cos[(N_0+(1/2))\varphi+\varphi_0]\rho^{N_0+(1/2)}[|C|+O(\rho)]. \tag{4.76}$$

This proves the assertions made in item (iii).

In order to prove (iv) it suffices to refer to (4.65) and observe that locally $\frac{1}{2}\frac{d}{dz}\big\langle\ln\big(\frac{G_{p+1}(z,\cdot)-y}{G_{p+1}(z,\cdot)+y}\big)\big\rangle$ behaves like $C_0(z-\widetilde\lambda_{j_0})^{M_0}$ for some $C_0 \in \mathbb{C}\backslash\{0\}$ in a sufficiently small neighborhood of $\widetilde\lambda_{j_0}$.

Finally we will show that all arcs are simple (i.e., do not self-intersect each other). Assume that the spectrum of H contains a simple closed loop γ, $\gamma \subset \sigma(H)$. Then

$$\operatorname{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(z(P),\cdot)-y(P)}{G_{p+1}(z(P),\cdot)+y(P)}\bigg)\bigg\rangle\bigg) = 0, \quad P \in \Gamma, \tag{4.77}$$

where the closed simple curve $\Gamma \subset \mathcal{K}_p$ denotes an appropriate lift of γ to $\mathcal{K}_p$, yields the contradiction

$$\operatorname{Re}\bigg(\bigg\langle \ln\bigg(\frac{G_{p+1}(z(P),\cdot)-y(P)}{G_{p+1}(z(P),\cdot)+y(P)}\bigg)\bigg\rangle\bigg) = 0 \text{ for all } P \text{ in the interior of } \Gamma \tag{4.78}$$

by Corollary 8.2.5 in [3]. Therefore, since there are no closed loops in $\sigma(H)$ and no analytic arc tends to infinity, the resolvent set of H is connected and hence path-connected, proving (v). $\square$

Remark 4.8. Here $\sigma \subset \mathbb{C}$ is called an *arc* if there exists a parameterization $\gamma \in C([0,1])$ such that $\sigma = \{\gamma(t)\,|\, t\in[0,1]\}$. The arc σ is called *simple* if there exists a parameterization γ such that $\gamma\colon [0,1]\to\mathbb{C}$ is injective.

Appendix A. Hyperelliptic curves and their theta functions

We provide a brief summary of some of the fundamental notations needed from the theory of hyperelliptic Riemann surfaces. More details can be found in some of the standard textbooks [21] and [45] as well as in monographs dedicated to integrable systems such as [4, Ch. 2], [24, App. A, B].

Fix $p \in \mathbb{N}$. We intend to describe the hyperelliptic Riemann surface $\mathcal{K}_p$ of genus p of the Toda-type curve (2.36), associated with the polynomial

$$\begin{aligned} &\mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \\ &R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z-E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C}. \end{aligned} \tag{A.1}$$

To simplify the discussion we will assume that the affine part of $\mathcal{K}_p$ is nonsingular, that is, we assume that

$$E_m \neq E_{m'} \text{ for } m \neq m',\ m,m' = 0,\dots,2p+1 \tag{A.2}$$

throughout this appendix. Next we introduce an appropriate set of (nonintersecting) cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$, $j = 1, \dots, p+1$, and denote

$$\mathcal{C} = \bigcup_{j=1}^{p+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Define the cut plane

$$\Pi = \mathbb{C} \backslash \mathcal{C}, \tag{A.4}$$

and introduce the holomorphic function

$$R_{2p+2}(\cdot)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \bigg(\prod_{m=0}^{2p+1} (z - E_m) \bigg)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5). Next we define the set

$$\mathcal{M}_p = \{ (z, \sigma R_{2p+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{1, -1\} \} \cup \{ P_{\infty_+}, P_{\infty_-} \} \tag{A.6}$$

by extending $R_{2p+2}(\cdot)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_p$ is then the set $\mathcal{M}_p$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_p$ crosswise along the cuts. Moreover, we introduce the set of branch points

$$\mathcal{B}(\mathcal{K}_p) = \{ (E_m, 0) \}_{m=0}^{2p+1}. \tag{A.7}$$

Points $P \in \mathcal{K}_p \backslash \{ P_{\infty_\pm} \}$ are denoted by

$$P = (z, \sigma R_{2p+2}(z)^{1/2}) = (z, y), \tag{A.8}$$

where $y(P)$ denotes the meromorphic function on $\mathcal{K}_p$ satisfying $\mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0$ and

$$y(P) \underset{\zeta \to 0}{=} \mp \bigg(1 - \frac{1}{2} \bigg(\sum_{m=0}^{2p+1} E_m \bigg) \zeta + O(\zeta^2) \bigg) \zeta^{-p-1} \text{ as } P \to P_{\infty_\pm},\ \zeta = 1/z. \tag{A.9}$$

In addition, we introduce the holomorphic sheet exchange map (involution)

$$* \colon \mathcal{K}_p \to \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y),\ P_{\infty_\pm} \mapsto P^*_{\infty_\pm} = P_{\infty_\mp} \tag{A.10}$$

and the two meromorphic projection maps

$$\tilde{\pi} \colon \mathcal{K}_p \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto z,\ P_{\infty_\pm} \mapsto \infty \tag{A.11}$$

and

$$y \colon \mathcal{K}_p \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto y,\ P_{\infty_\pm} \mapsto \infty. \tag{A.12}$$

Thus the map $\tilde{\pi}$ has a pole of order 1 at $P_{\infty_\pm}$ and y has a pole of order $p+1$ at $P_{\infty_\pm}$. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P), \quad y(P^*) = -y(P), \quad P \in \mathcal{K}_p. \tag{A.13}$$

As a result, $\mathcal{K}_p$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$ ($\cong \mathbb{C} \cup \{\infty\}$) branched at the $2p+4$ points $\{(E_m, 0)\}_{m=0}^{2p+1}, P_{\infty_\pm}$. $\mathcal{K}_p$ is compact since $\tilde{\pi}$ is open and $\mathbb{CP}^1$ is compact. Therefore, the compact hyperelliptic Riemann surface resulting in this manner has topological genus p.

Next we introduce the upper and lower sheets $\Pi_\pm$ by

$$\Pi_\pm = \{ (z, \pm R_{2p+2}(z)^{1/2}) \in \mathcal{M}_p \mid z \in \Pi \} \tag{A.14}$$

and the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P \mapsto z. \tag{A.15}$$

Let $\{a_j, b_j\}_{j=1}^p$ be a homology basis for $\mathcal{K}_p$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1,\dots,p. \tag{A.16}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1}^p$ we also recall the canonical dissection of $\mathcal{K}_p$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_p$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_p$ given by

$$\partial\widehat{\mathcal{K}}_p = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_p^{-1} b_p^{-1}. \tag{A.17}$$

Let $\mathcal{M}(\mathcal{K}_p)$ and $\mathcal{M}^1(\mathcal{K}_p)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_p$, respectively. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_p)$ at a point $Q \in \mathcal{K}_p$ is defined by

$$\operatorname{res}_Q(\nu) = \frac{1}{2\pi i} \int_{\gamma_Q} \nu, \tag{A.18}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_p)$ are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1,\dots,p. \tag{A.19}$$

If $\omega_{P_1,m}^{(2)}$ is a differential of the second kind on $\mathcal{K}_p$ whose only pole is $P_1 \in \widehat{\mathcal{K}}_p$ with principal part $\zeta^{-m-2}\, d\zeta$, $m \in \mathbb{N}_0$ near P_1 and $\omega_j = (\sum_{k=0}^\infty d_{j,k}(P_1)\zeta^k)\, d\zeta$ near P_1, then

$$\frac{1}{2\pi i} \int_{b_j} \omega_{P_1,m}^{(2)} = \frac{d_{j,m}(P_1)}{m+1}, \quad m = 0,1,\dots. \tag{A.20}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_p$ not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_p)$ is usually normalized by vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad j = 1,\dots,p. \tag{A.21}$$

A normal differential $\omega_{P_1,P_2}^{(3)}$, associated with two distinct points P_1, $P_2 \in \hat{\mathcal{K}}_p$, by definition, has simple poles at P_1 and P_2 with residues $+1$ at P_1 and -1 at P_2 and vanishing a-periods. If $\omega_{P,Q}^{(3)}$ is a normal differential of the third kind associated with P, $Q \in \hat{\mathcal{K}}_p$, holomorphic on $\mathcal{K}_p \backslash \{P,Q\}$, then

$$\int_{b_j} \omega_{P,Q}^{(3)} = 2\pi i \int_P^Q \omega_j, \quad j = 1,\dots,p. \tag{A.22}$$

We shall always assume (without loss of generality) that all poles of $\omega^{(2)}$ and $\omega^{(3)}$ on $\mathcal{K}_p$ lie on $\hat{\mathcal{K}}_p$ (i.e., not on $\partial\hat{\mathcal{K}}_p$).

Using our local charts one infers that dz/y is a holomorphic differential on $\mathcal{K}_p$ with zeros of order $p-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j=1,\dots,p, \tag{A.23}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_p$. Introducing the invertible matrix C in $\mathbb{C}^p$

$$C = \big(C_{j,k}\big)_{j,k=1,\dots,p}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{A.24}$$

$$\underline{c}(k) = (c_1(k),\dots,c_p(k)), \quad c_j(k) = \big(C^{-1}\big)_{j,k}, \quad j,k=1,\dots,p, \tag{A.25}$$

the normalized differentials ω_j for $j=1,\dots,p$,

$$\omega_j = \sum_{\ell=1}^{p} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k=1,\dots,p, \tag{A.26}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_p$.

In the chart $(U_{P_{\infty_\pm}}, \zeta_{P_{\infty_\pm}})$ induced by $1/\tilde{\pi}$ near $P_{\infty_\pm}$ one infers,

$$\begin{aligned}\underline{\omega} = (\omega_1,\dots,\omega_p) &= \mp \sum_{j=1}^{p} \frac{\underline{c}(j)\zeta^{p-j}d\zeta}{\big(\prod_{m=0}^{2p+1}(1-\zeta E_m)\big)^{1/2}} \\ &= \pm\Big(\underline{c}(p) + \zeta\big[\frac{1}{2}\underline{c}(p)\sum_{m=0}^{2p+1} E_m + \underline{c}(p-1)\big] + O(\zeta^2)\Big)d\zeta \text{ as } P \to P_{\infty_\pm}, \\ &\qquad \zeta = 1/z.\end{aligned} \tag{A.27}$$

Given (A.27), one can compute for the vector $\underline{U}_0^{(2)}$ of b-periods of $\omega^{(2)}_{P_{\infty+},0}/(2\pi i)$, the normalized differential of the second kind, which is holomorphic on $\mathcal{K}_p \backslash \{P_\infty\}$,

$$\underline{U}_0^{(2)} = \big(U_{0,1}^{(2)},\dots,U_{0,p}^{(2)}\big), \quad U_{0,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{\infty+},0} = 2c_j(p), \;\; j=1,\dots,p. \tag{A.28}$$

The matrix $\tau = \big(\tau_{j,\ell}\big)_{j,\ell=1}^{p}$ in $\mathbb{C}^{p\times p}$ of b-periods defined by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j,\ell=1,\dots,p \tag{A.29}$$

satisfies

$$\operatorname{Im}(\tau) > 0 \text{ and } \tau_{j,\ell} = \tau_{\ell,j}, \;\; j,\ell=1,\dots,p. \tag{A.30}$$

Associated with the matrix τ one introduces the period lattice

$$L_p = \{\underline{z} \in \mathbb{C}^p \mid \underline{z} = \underline{m} + \underline{n}\tau, \; \underline{m},\underline{n} \in \mathbb{Z}^p\} \tag{A.31}$$

and the Riemann theta function associated with $\mathcal{K}_p$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,p}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^p} \exp\big(2\pi i(\underline{n},\underline{z}) + \pi i(\underline{n},\underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^p, \tag{A.32}$$

where $(\underline{u},\underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^{p} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^p$. It has the following fundamental properties

$$\theta(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n) = \theta(\underline{z}), \tag{A.33}$$

$$\theta(\underline{z}+\underline{m}+\underline{n}\tau) = \exp\big(-2\pi i(\underline{n},\underline{z}) - \pi i(\underline{n},\underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m},\underline{n} \in \mathbb{Z}^p. \tag{A.34}$$

Next we briefly describe some consequences of a change of homology basis. Let

$$\{a_1,\dots,a_p,b_1,\dots,b_p\} \tag{A.35}$$

be a canonical homology basis on $\mathcal{K}_p$ with intersection matrix satisfying (A.16) and let

$$\{a'_1,\dots,a'_p,b'_1,\dots,b'_p\} \tag{A.36}$$

be a homology basis on $\mathcal{K}_p$ related to each other by

$$\begin{pmatrix} \underline{a'}^\top \\ \underline{b'}^\top \end{pmatrix} = X \begin{pmatrix} \underline{a}^\top \\ \underline{b}^\top \end{pmatrix}, \tag{A.37}$$

where

$$\begin{aligned} \underline{a}^\top &= (a_1,\dots,a_p)^\top, \quad \underline{b}^\top = (b_1,\dots,b_p)^\top, \\ \underline{a'}^\top &= (a'_1,\dots,a'_p)^\top, \quad \underline{b'}^\top = (b'_1,\dots,b'_p)^\top, \end{aligned} \tag{A.38}$$

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A.39}$$

with A, B, C, and D being $p \times p$ matrices with integer entries. Then (A.36) is also a canonical homology basis on $\mathcal{K}_p$ with intersection matrix satisfying (A.16) if and only if

$$X \in \mathrm{Sp}(p,\mathbb{Z}), \tag{A.40}$$

where

$$\mathrm{Sp}(p,\mathbb{Z}) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, X \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} X^\top = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix},\ \det(X) = 1 \right\} \tag{A.41}$$

denotes the symplectic modular group (here A, B, C, D in X are again $p\times p$ matrices with integer entries). If $\{\omega_j\}_{j=1}^p$ and $\{\omega'_j\}_{j=1}^p$ are the normalized bases of holomorphic differentials corresponding to the canonical homology bases (A.35) and (A.36), with τ and τ' the associated b and b'-periods of $\underline{\omega} = \omega_1,\dots,\omega_p$ and $\underline{\omega}' = \omega'_1,\dots,\omega'_p$, respectively, then one computes

$$\underline{\omega}' = \underline{\omega}(A + B\tau)^{-1}, \quad \tau' = (C + D\tau)(A + B\tau)^{-1}. \tag{A.42}$$

Fixing a base point $Q_0 \in \mathcal{K}_p \backslash \{P_{\infty_\pm}\}$, one denotes by $J(\mathcal{K}_p) = \mathbb{C}^p/L_p$ the Jacobi variety of $\mathcal{K}_p$, and defines the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_n \to J(\mathcal{K}_p), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_p \right) \pmod{L_p}, \quad P \in \mathcal{K}_p. \tag{A.43}$$

Next, consider the vector $\underline{U}_0^{(3)}$ of b-periods of $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}/(2\pi i)$, the normalized differential of the third kind, holomorphic on $\mathcal{K}_p \backslash \{P_{\infty_\pm}\}$,

$$\underline{U}_0^{(3)} = \big(U_{0,1}^{(3)}, \dots, U_{0,p}^{(3)}\big), \quad U_{0,j}^{(3)} = \frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}}, \ j = 1,\dots,p. \tag{A.44}$$

Using (A.22) one then computes

$$\underline{U}_0^{(3)} = \underline{A}_{P_{\infty-}}(P_{\infty+}) = 2\underline{A}_{Q_0}(P_{\infty+}), \tag{A.45}$$

where Q_0 is chosen to be a branch point of $\mathcal{K}_p$, $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, in the last part of (A.45).

Similarly, one introduces

$$\underline{\alpha}_{Q_0}\colon \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P\in\mathcal{K}_p} \mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.46}$$

where $\operatorname{Div}(\mathcal{K}_p)$ denotes the set of divisors on $\mathcal{K}_p$. Here a map $\mathcal{D}\colon \mathcal{K}_p \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_p$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_p$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.) For subsequent use in Remark A.4 we also introduce

$$\begin{gathered}\underline{\widehat{A}}_{Q_0}\colon \widehat{\mathcal{K}}_p \to \mathbb{C}^p, \\ P \mapsto \underline{\widehat{A}}_{Q_0}(P) = \big(\widehat{A}_{Q_0,1}(P), \dots, \widehat{A}_{Q_0,p}(P)\big) = \bigg(\int_{Q_0}^{P} \omega_1, \dots, \int_{Q_0}^{P} \omega_p\bigg)\end{gathered} \tag{A.47}$$

and

$$\underline{\widehat{\alpha}}_{Q_0}\colon \operatorname{Div}(\widehat{\mathcal{K}}_p) \to \mathbb{C}^p, \quad \mathcal{D} \mapsto \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}) = \sum_{P\in\widehat{\mathcal{K}}_p} \mathcal{D}(P)\underline{\widehat{A}}_{Q_0}(P). \tag{A.48}$$

In connection with divisors on $\mathcal{K}_p$ we will employ the following (additive) notation,

$$\begin{gathered}\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\ \underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_p, \quad Q_0 \in \mathcal{K}_p, \ m \in \mathbb{N},\end{gathered} \tag{A.49}$$

where for any $Q \in \mathcal{K}_p$,

$$\mathcal{D}_Q\colon \mathcal{K}_p \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_p \backslash \{Q\}, \end{cases} \tag{A.50}$$

and $\operatorname{Sym}^m \mathcal{K}_p$ denotes the mth symmetric product of $\mathcal{K}_p$. In particular, $\operatorname{Sym}^m \mathcal{K}_p$ can be identified with the set of nonnegative divisors $0 \le \mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$ of degree $m \in \mathbb{N}$.

For $f \in \mathcal{M}(\mathcal{K}_p)\backslash\{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_p)\backslash\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_p)\backslash\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_p) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0, \quad \deg((\omega)) = 2(p-1), \quad f \in \mathcal{M}(\mathcal{K}_p)\backslash\{0\}, \ \omega \in \mathcal{M}^1(\mathcal{K}_p)\backslash\{0\}, \tag{A.51}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P\in\mathcal{K}_p} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_p) \mid f = 0 \text{ or } (f) \ge \mathcal{D}\}, \quad r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.52}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_p) \mid \omega = 0 \text{ or } (\omega) \ge \mathcal{D}\}, \quad i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}) \tag{A.53}$$

(with $i(\mathcal{D})$ the index of specialty of $\mathcal{D}$), one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $\omega \in \mathcal{M}^1(\mathcal{K}_p)\backslash\{0\}$. Then*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad p \in \mathbb{N}_0. \tag{A.54}$$

The Riemann-Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - p + 1, \quad n \in \mathbb{N}_0. \tag{A.55}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_p)$, $p \in \mathbb{N}$, is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.56}$$

Finally, assume $p \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_p) \to J(\mathcal{K}_p)$ is surjective (Jacobi's inversion theorem).

Theorem A.2. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^p \mathcal{K}_p$, $\underline{Q} = \{Q_1, \dots, Q_p\}$. Then*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.57}$$

if and only if there are s pairs of the type $\{P, P^*\} \subseteq \{Q_1, \dots, Q_p\}$ (this includes, of course, branch points for which $P = P^*$). Obviously, one has $s \leq p/2$.

Next, we denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,p}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{p} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, p. \tag{A.58}$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_p\} \in \operatorname{Sym}^p \mathcal{K}_p$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})\big) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_p\}. \tag{A.59}$$

Remark A.4. In Section 2 we dealt with theta function expressions of the type

$$\psi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\bigg(-c \int_{Q_0}^{P} \Omega^{(3)}\bigg), \quad P \in \mathcal{K}_p, \tag{A.60}$$

where $\mathcal{D}_j \in \operatorname{Sym}^p \mathcal{K}_p$, $j = 1, 2$, are nonspecial positive divisors of degree p, $c \in \mathbb{C}$ is a constant, and $\Omega^{(3)}$ is a normalized differential of the third kind with a prescribed singularity at $P_{\infty_\pm}$. Even though we agree to always choose identical paths of integration from P_0 to P in all Abelian integrals (A.60), this is not sufficient to render ψ single-valued on $\mathcal{K}_p$. To achieve single-valuedness one needs to replace $\mathcal{K}_p$ by its simply connected canonical dissection $\widehat{\mathcal{K}}_p$ and then replace $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ in (A.60) with $\underline{\widehat{A}}_{Q_0}$ and $\underline{\widehat{\alpha}}_{Q_0}$ as introduced in (A.47) and (A.48). In particular, one regards a_j, b_j, $j = 1, \dots, p$, as curves (being a part of $\partial \widehat{\mathcal{K}}_p$, cf. (A.17)) and not as homology classes. Similarly, one then replaces $\underline{\Xi}_{Q_0}$ by $\underline{\widehat{\Xi}}_{Q_0}$ (replacing $\underline{A}_{Q_0}$ by $\underline{\widehat{A}}_{Q_0}$ in (A.58), etc.). Moreover, in connection with ψ, one introduces the vector of b-periods $\underline{U}^{(3)}$ of $\Omega^{(3)}$ by

$$\underline{U}^{(3)} = (U_1^{(3)}, \dots, U_p^{(3)}), \quad U_j^{(3)} = \frac{1}{2\pi i} \int_{b_j} \Omega^{(3)}, \quad j = 1, \dots, p, \tag{A.61}$$

and then renders ψ single-valued on $\widehat{\mathcal{K}}_p$ by requiring

$$\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_2) = c\,\underline{U}^{(3)} \tag{A.62}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = c\,\underline{U}^{(3)} \pmod{L_p}$). Actually, by (A.34),

$$\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_2) - c\,\underline{U}^{(3)} \in \mathbb{Z}^p, \tag{A.63}$$

suffices to guarantee single-valuedness of ψ on $\widehat{\mathcal{K}}_p$. Without the replacement of $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ by $\underline{\widehat{A}}_{Q_0}$ and $\underline{\widehat{\alpha}}_{Q_0}$ in (A.60) and without the assumption (A.62) (or (A.63)), ψ is a multiplicative (multi-valued) function on $\mathcal{K}_p$, and then most effectively discussed by introducing the notion of characters on $\mathcal{K}_p$ (cf. [21, Sect. III.9]). For simplicity, we decided to avoid the latter possibility and throughout this paper will always tacitly assume (A.62) or (A.63).

Appendix B. Restrictions on $\underline{B} = \underline{U}_0^{(3)}$

The purpose of this appendix is to prove the result (2.81), $\underline{B} = \underline{U}_0^{(3)} \in \mathbb{R}^p$, for some choice of homology basis $\{a_j, b_j\}_{j=1}^p$ on $\mathcal{K}_p$ as recorded in Remark 2.7.

To this end we first recall a few notions in connection with periodic meromorphic functions of p complex variables.

Definition B.1. Let $p \in \mathbb{N}$ and $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ be meromorphic (i.e., a ratio of two entire functions of p complex variables). Then,
(i) $\underline{\omega} = (\omega_1, \dots, \omega_p) \in \mathbb{C}^p \backslash \{0\}$ is called a *period* of F if

$$F(\underline{z} + \underline{\omega}) = F(\underline{z}) \tag{B.1}$$

for all $\underline{z} \in \mathbb{C}^p$ for which F is analytic. The set of all periods of F is denoted by $\mathcal{P}_F$.
(ii) F is called *degenerate* if it depends on less than p complex variables; otherwise, F is called *nondegenerate*.

Theorem B.2. *Let $p \in \mathbb{N}$, $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ be meromorphic, and $\mathcal{P}_F$ be the set of all periods of F. Then either*
(i) $\mathcal{P}_F$ has a finite limit point,
or
(ii) $\mathcal{P}_F$ has no finite limit point.
In case (i), $\mathcal{P}_F$ contains infinitesimal periods (i.e., sequences of nonzero periods converging to zero). In addition, in case (i) each period is a limit point of periods and hence $\mathcal{P}_F$ is a perfect set.
Moreover, F is degenerate if and only if F admits infinitesimal periods. In particular, for nondegenerate functions F only alternative (ii) applies.

Next, let $\underline{\omega}_q \in \mathbb{C}^p \backslash \{0\}$, $q = 1, \dots, r$ for some $r \in \mathbb{N}$. Then $\underline{\omega}_1, \dots, \underline{\omega}_r$ are called *linearly independent over $\mathbb{Z}$ (resp. $\mathbb{R}$)* if

$$\begin{aligned} &\nu_1 \underline{\omega}_1 + \dots + \nu_r \underline{\omega}_r = 0, \quad \nu_q \in \mathbb{Z} \text{ (resp., } \nu_q \in \mathbb{R}), \ \ q = 1, \dots, r, \\ &\text{implies } \nu_1 = \dots = \nu_r = 0. \end{aligned} \tag{B.2}$$

Clearly, the maximal number of vectors in $\mathbb{C}^p$ linearly independent over $\mathbb{R}$ equals $2p$.

Theorem B.3. *Let $p \in \mathbb{N}$.*
(i) If $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ is a nondegenerate meromorphic function with periods $\underline{\omega}_q \in \mathbb{C}^p \backslash \{0\}$, $q = 1, \dots, r$, $r \in \mathbb{N}$, linearly independent over $\mathbb{Z}$, then $\underline{\omega}_1, \dots, \underline{\omega}_r$ are also linearly independent over $\mathbb{R}$. In particular, $r \leq 2p$.
(ii) A nondegenerate entire function $F\colon \mathbb{C}^p \to \mathbb{C}$ cannot have more than p periods linearly independent over $\mathbb{Z}$ (or $\mathbb{R}$).

For $p = 1$, $\exp(z)$, $\sin(z)$ are examples of entire functions with precisely one period. Any non-constant doubly periodic meromorphic function of one complex variable is elliptic (and hence indeed has poles).

Definition B.4. Let $p, r \in \mathbb{N}$. A system of periods $\underline{\omega}_q \in \mathbb{C}^p \backslash \{0\}$, $q = 1, \dots, r$, of a nondegenerate meromorphic function $F \colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$, linearly independent over $\mathbb{Z}$, is called *fundamental* or a *basis* of periods for F if every period $\underline{\omega}$ of F is of the form

$$\underline{\omega} = m_1 \underline{\omega}_1 + \dots + m_r \underline{\omega}_r \quad \text{for some } m_q \in \mathbb{Z},\ q = 1, \dots, r. \tag{B.3}$$

The representation of $\underline{\omega}$ in (B.3) is unique since by hypothesis $\underline{\omega}_1, \dots, \underline{\omega}_r$ are linearly independent over $\mathbb{Z}$. In addition, $\mathcal{P}_F$ is countable in this case. (This rules out case (i) in Theorem B.2 since a perfect set is uncountable. Hence, one does not have to assume that F is nondegenerate in Definition B.4.)

This material is standard and can be found, for instance, in [38, Ch. 2].

Next, returning to the Riemann theta function $\theta(\underline{\cdot})$ in (A.32), we introduce the vectors $\{\underline{e}_j\}_{j=1}^p, \{\underline{\tau}_j\}_{j=1}^p \subset \mathbb{C}^p \backslash \{0\}$ by

$$\underline{e}_j = (0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0), \quad \underline{\tau}_j = \underline{e}_j \tau, \quad j = 1, \dots, p. \tag{B.4}$$

Then

$$\{\underline{e}_j\}_{j=1}^p \tag{B.5}$$

is a basis of periods for the entire (nondegenerate) function $\theta(\underline{\cdot}) \colon \mathbb{C}^p \to \mathbb{C}$. Moreover, fixing $k \in \{1, \dots, p\}$, then

$$\{\underline{e}_j, \underline{\tau}_j\}_{j=1}^p \tag{B.6}$$

is a basis of periods for the meromorphic function $\partial_{z_k} \ln \big(\frac{\theta(\underline{\cdot})}{\theta(\underline{\cdot} + \underline{V})}\big) \colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$, $\underline{V} \in \mathbb{C}^p$ (cf. (A.34) and [21, p. 91]).

Next, let $\underline{A} \in \mathbb{C}^p$, $\underline{D} = (D_1, \dots, D_p) \in \mathbb{R}^p$, $D_j \in \mathbb{R} \backslash \{0\}$, $j = 1, \dots, p$, and consider

$$\begin{aligned} f_k \colon \mathbb{R} \to \mathbb{C}, \quad f_k(n) &= \partial_{z_k} \ln \left(\frac{\theta(\underline{A} - \underline{z})}{\theta(\underline{C} - \underline{z})} \right) \bigg|_{\underline{z} = \underline{D} n} \\ &= \partial_{z_k} \ln \left(\frac{\theta(\underline{A} - \underline{z}\, \operatorname{diag}(\underline{D}))}{\theta(\underline{C} - \underline{z}\, \operatorname{diag}(\underline{D}))} \right) \bigg|_{\underline{z} = (n, \dots, n)}. \end{aligned} \tag{B.7}$$

Here $\operatorname{diag}(\underline{D})$ denotes the diagonal matrix

$$\operatorname{diag}(\underline{D}) = \big(D_j \delta_{j,j'}\big)_{j,j'=1}^p. \tag{B.8}$$

Then the quasi-periods D_j^{-1}, $j = 1, \dots, p$, of f_k are in a one-to-one correspondence with the periods of

$$F_k \colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}, \quad F_k(\underline{z}) = \partial_{z_k} \ln \left(\frac{\theta(\underline{A} - \underline{z}\, \operatorname{diag}(\underline{D}))}{\theta(\underline{C} - \underline{z}\, \operatorname{diag}(\underline{D}))} \right) \tag{B.9}$$

of the special type

$$\underline{e}_j \big(\operatorname{diag}(\underline{D}) \big)^{-1} = \big(0, \dots, 0, \underbrace{D_j^{-1}}_{j}, 0, \dots, 0\big). \tag{B.10}$$

Moreover,

$$f_k(n) = F_k(\underline{z})|_{\underline{z} = (n, \dots, n)}, \quad n \in \mathbb{Z}. \tag{B.11}$$

Theorem B.5. *Suppose a and b in (2.75) to be quasi-periodic. Then there exists a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^p$ on $\mathcal{K}_p$ such that the vector $\underline{\widetilde B} = \underline{\widetilde U}_0^{(3)}$ with $\underline{\widetilde U}_0^{(3)}$ the vector of $\tilde b$-periods of the corresponding normalized differential of the third kind, $\widetilde\omega^{(3)}_{P_{\infty+},P_{\infty-}}$, satisfies the constraint*

$$\underline{\widetilde B} = \underline{\widetilde U}_0^{(3)} \in \mathbb{R}^p. \tag{B.12}$$

Proof. By (A.44), the vector of b-periods $\underline{U}_0^{(3)}$ associated with a given homology basis $\{a_j, b_j\}_{j=1}^p$ on $\mathcal{K}_p$ and the normalized differential of the third kind, $\omega^{(3)}_{P_{\infty+},P_{\infty-}}$, is continuous with respect to $E_0, \dots, E_{2p+1}$. Hence, we may assume in the following that

$$B_j \neq 0, \ \ j = 1, \dots, p, \quad \underline{B} = (B_1, \dots, B_p) \tag{B.13}$$

by slightly altering $E_0, \dots, E_{2p+1}$, if necessary. Using (2.76), we may write

$$\begin{aligned} b(n) &= \Lambda_0 - \sum_{j=1}^p c_j(p) \frac{\partial}{\partial \omega_j} \ln\left(\frac{\theta(\underline{\omega} + \underline{A} - \underline{B}n)}{\theta(\underline{\omega} + \underline{C} - \underline{B}n)} \right)\bigg|_{\underline{\omega}=0} \\ &= \Lambda_0 - \sum_{j=1}^p c_j(p) \partial_{z_j} \ln\left(\frac{\theta(\underline{A} - \underline{z})}{\theta(\underline{C} - \underline{z})} \right)\bigg|_{\underline{z}=\underline{B}n}, \end{aligned} \tag{B.14}$$

where by (2.78),

$$\underline{B} = \underline{U}_0^{(3)}. \tag{B.15}$$

Introducing the meromorphic (nondegenerate) function $\mathcal{V}\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ by

$$\mathcal{V}(\underline{z}) = \Lambda_0 - \sum_{j=1}^n c_j(p) \partial_{z_j} \ln\left(\frac{\theta(\underline{A} - \underline{z}\operatorname{diag}(\underline{B}))}{\theta(\underline{C} - \underline{z}\operatorname{diag}(\underline{B}))} \right), \tag{B.16}$$

one observes that

$$b(n) = \mathcal{V}(\underline{z})|_{\underline{z}=(n,\dots,n)}. \tag{B.17}$$

In addition, $\mathcal{V}$ has a basis of periods

$$\Big\{ \underline{e}_j \big(\operatorname{diag}(\underline{B}) \big)^{-1}, \underline{\tau}_j \big(\operatorname{diag}(\underline{B}) \big)^{-1} \Big\}_{j=1}^p \tag{B.18}$$

by (B.6), where

$$\underline{e}_j \big(\operatorname{diag}(\underline{B}) \big)^{-1} = \big(0, \dots, 0, \underbrace{B_j^{-1}}_{j}, 0, \dots, 0 \big), \quad j = 1, \dots, p, \tag{B.19}$$

$$\underline{\tau}_j \big(\operatorname{diag}(\underline{B}) \big)^{-1} = \big(\tau_{j,1} B_1^{-1}, \dots, \tau_{j,p} B_p^{-1} \big), \quad j = 1, \dots, p. \tag{B.20}$$

By hypothesis, b in (B.14) is quasi-periodic and hence has p real (scalar) quasi-periods. The latter are not necessarily linearly independent over $\mathbb{Q}$ from the outset, but by slightly changing the locations of branchpoints $\{E_m\}_{m=0}^{2p+1}$ into, say, $\{\widetilde E_m\}_{m=0}^{2p+1}$, one can assume they are. In particular, since the period vectors in (B.18) are linearly independent and the (scalar) quasi-periods of b are in a one-one correspondence with vector periods of $\mathcal{V}$ of the special form (B.19) (cf. (B.9), (B.10)), there exists a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^p$ on $\mathcal{K}_p$ such that the vector $\underline{\widetilde B} = \underline{\widetilde U}_0^{(3)}$, corresponding to the normalized differential of the third kind, $\widetilde\omega^{(3)}_{P_{\infty+},P_{\infty-}}$ and this particular homology basis, is real-valued. By continuity of $\underline{\widetilde{U}}_0^3$ with respect to $\widetilde{E}_0, \dots, \widetilde{E}_{2p+1}$, this proves (B.12). $\square$

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