Oriented percolation in one-dimensional $\beta/|x-y|^2$, $\beta > 1$ random-cluster model

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Abstract

We consider the one-dimensional long-range Fortuin–Kasteleyn random-cluster model, generated by the edge occupation probabilities

$$p_{\langle x,y\rangle} = \begin{cases} p & \text{if } |x-y| = 1, \\ 1-\exp\left\{-\beta/|x-y|^2\right\} & \text{otherwise,} \end{cases}$$

and weighting factor $\kappa \geq 1$. We prove the occurrence of oriented percolation when $\beta > 1$ and $\kappa \geq 1$, provided p is chosen sufficiently close to 1. We also show that the oriented truncated connectivity $\tau'(x,y)$ satisfies

$$\tau'(x,y) \leq C|x-y|^{-\theta}$$

with $\theta = \min(2(\beta\eta - 1), 2)$ where $\eta = \eta(p) \nearrow 1$ as $p \nearrow 1$.

1 Introduction

In this work we establish the existence of oriented percolation for the "critical" $1/r^2$ one-dimensional long range Fortuin–Kasteleyn (FK) random cluster model. For motivation, related questions and open problems see comments at the end of this section.

Preliminaries. Consider the infinite complete graph with set of vertices $\mathscr{V} = \mathbb{Z}$, and set of edges $\mathbb{B} = \{\langle x,y\rangle,\ x \neq y,\ x,y \in \mathbb{Z}\}$. The one-dimensional long-range FK random cluster model with parameter $\kappa > 0$ on $\Omega = \{0,1\}^{\mathbb{B}}$, is a probability measure defined on $\sigma(\Omega)$, the canonical σ-algebra on Ω, as follows:

Finite volume FK measure. Given $I \subset \mathbb{Z}$, consider $\mathbb{B}(I) = \{\langle x,y\rangle \in \mathbb{B}\colon x,y \in I\}$ and $\Omega_I = \{0,1\}^{\mathbb{B}\setminus\mathbb{B}(I^c)}$, where $I^c = \mathbb{Z}\setminus I$. Assume that $|I| < \infty$, and fix an "external" configuration $\tilde{\omega} \in$

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$\{0,1\}^{\mathbb{B}(I^c)}$, which plays the role of boundary conditions. The finite volume FK-measure with boundary conditions $\tilde\omega$ is the probability measure $\mu^{\tilde\omega}_{\kappa,I}$ on $\Omega^{\tilde\omega}_I = \{\omega \in \Omega:\ \omega_b = \tilde\omega_b \text{ for } b \in \mathbb{B}(I^c)\}$ defined for any $A \in \sigma(\Omega^{\tilde\omega}_I)$ by

$$\mu^{\tilde\omega}_{\kappa,I}(A) = \frac{\displaystyle\int_{A\cap\Omega^{\tilde\omega}_I} \kappa^{c_I(\omega)}\mu\,(d\omega)}{\displaystyle\int_{\Omega^{\tilde\omega}_I} \kappa^{c_I(\omega)}\mu\,(d\omega)}\,, \tag{1.1}$$

where $c_I(\omega)$ is the number of disjoint connected components (clusters) that intersect I, in the graph determined by $\omega \in \Omega^{\tilde\omega}_I$ (an edge b is present iff $\omega_b = 1$) and $\mu = \prod_{\langle x,y\rangle\in\mathbb{B}} \mu_{\langle x,y\rangle}$ is the Bernoulli product measure on Ω:

$$\mu_{\langle x,y\rangle}(\omega_{\langle x,y\rangle} = 1) = \begin{cases} p & \text{if } |x-y| = 1, \\ 1 - \exp\left\{-\dfrac{\beta}{|x-y|^2}\right\} & \text{otherwise,} \end{cases} \tag{1.2}$$

and

$$\mu_{\langle x,y\rangle}(\omega_{\langle x,y\rangle} = 0) = 1 - \mu_{\langle x,y\rangle}(\omega_{\langle x,y\rangle} = 1),$$

where $0 < p < 1$ and $\beta > 0$ are fixed parameters. Further we write $p_{\langle x,y\rangle} \equiv \mu_{\langle x,y\rangle}(\omega_{\langle x,y\rangle} = 1)$ and $q_{\langle x,y\rangle} = 1 - p_{\langle x,y\rangle}$; for $b = \langle x,y\rangle$ we will write p_b instead of $p_{\langle x,y\rangle}$, and say that b "is open" if $\omega_b = 1$. Throughout the paper we will assume $\kappa \geq 1$.

Two external configurations $\tilde\omega_I$ play particular role when considering the infinite volume limit:

$\tilde\omega_b = 0$ for all $b \in \mathbb{B}(I^c)$, and the corresponding measure $\mu^{\tilde\omega}_{\kappa,I} \equiv \mu^f_{\kappa,I}$ is called "free" measure;

and

$\tilde\omega_b = 1$ for all $b \in \mathbb{B}(I^c)$, and the corresponding measure $\mu^{\tilde\omega}_{\kappa,I} \equiv \mu^w_{\kappa,I}$ is called "wired" measure.

Infinite volume limit. On Ω we consider the usual partial order: $\omega \leq \omega'$ if $\omega_b \leq \omega'_b$ for each $b \in \mathbb{B}$. By the FKG inequality (see [F], [ACCN]),

$$\mu^f_{\kappa,I}(g) \leq \mu^f_{\kappa,I'}(g) \leq \mu^{\tilde\omega}_{\kappa,I'}(g) \leq \mu^w_{\kappa,I'}(g) \leq \mu^w_{\kappa,I}(g)$$

holds for any finite intervals $I \subset I'$, any external configuration $\tilde\omega$ and any non-decreasing function $g : \Omega \to \mathbb{R}$. So, as $I \nearrow \mathbb{Z}$ the limit measures μ^f_κ and μ^w_κ exist. Moreover, $\mu^f_\kappa = \mu^w_\kappa$ if and only if there is a unique accumulation point of the set $\mu^{\tilde\omega}_{\kappa,I}$ as $I \nearrow \mathbb{Z}$. If $\kappa = 1$ we have $\mu^f_\kappa = \mu^w_\kappa = \mu$. Since $p_{\langle x,y\rangle} \equiv p_{|x-y|}$ the measures μ^f_κ and μ^w_κ are translation invariant; both these measures are ergodic. Using usual compactness arguments, the existence of limiting measures is obtained for $\kappa > 0$. Another standard approach to infinite volume random cluster measures is through a suitable DLR condition. These two notions do not coincide in general. (See e.g. [G].)

Fix $\omega \in \Omega$. An alternating sequence of vertices and edges $x = x_1, b_1, x_2, \ldots, x_{n-1}, b_{n-1}, x_n = y$, $n \geq 1$, is called a path connecting x to y, and we say that the path is open if $\omega_{b_i} \equiv \omega_{\langle x_i,x_{i+1}\rangle} = 1$, $1 \leq i \leq n-1$. We say that $C \subset \mathbb{Z}$ is connected if for any two distinct vertices x, y in C there exists an open path π connecting them. A maximal connected set is called *an open cluster*, and by $C_x(\omega)$ we denote the cluster of $x \in \mathbb{Z}$ (we write $C_x(\omega) = \{x\}$ if $\omega_{\langle x,y\rangle} = 0$, $\forall y \in \mathbb{Z} \setminus \{x\}$). A path π connecting x to y, $x < y$, is called *oriented*, if $x_1 = x < x_2 < \cdots < x_{n-1} < x_n = y$, and we write $x \rightsquigarrow y$ when there is an open oriented path connecting x to y. Analogously we define $C_x^+ = \{y \colon x \rightsquigarrow y\}$ and define the event

$$[x \rightsquigarrow \infty] = [|C_x^+| = \infty].$$

We are ready to state our main results.

Theorem 1.1 *For any $\beta > 1$, $\kappa \geq 1$, there exist $0 < p_0 < 1$ such that, if $p > p_0$, then*

$$\sup_{L\geq 1} \mu^w_{\kappa,L}(0 \not\rightsquigarrow \infty) \leq \epsilon, \tag{1.3}$$

with $\epsilon = \epsilon(p) \searrow 0$ as $p \nearrow 1$.

Theorem 1.2 *Under the assumptions of Theorem 1.1 and for $x \in \mathbb{N}$ we have*

$$\sup_{L\geq 1} \mu^w_{\kappa,L}(0 \rightsquigarrow x, 0 \not\rightsquigarrow \infty) \leq C\, x^{-\theta} \tag{1.4}$$

where $\theta = \min(2(\beta\eta - 1), 2)$, $\eta = \eta(p) \nearrow 1$ as $p \nearrow 1$ and $C < \infty$.

Motivation. The questions that we address in the present work were raised in 1986 by Newman and Schulman in [NS], where they have established the existence of *oriented long-range percolation* in one dimension for independent edge percolation models with $p_{\langle x,y\rangle} = f(|x - y|)$ such that $\lim_{x\to\infty} x^s\, f(x) > 0$ holds for some $1 < s < 2$, and also showed existence of *non-oriented* percolation for the critical case $s = 2$. The question whether oriented percolation occurs or not in the critical case $s = 2$ remained unanswered. Theorem 1.1 above gives an affirmative answer to it. Moreover, (1.2) satisfies $\lim_{x\to\infty} x^2\, f(x) = \beta$ and our proofs can be easily adapted to include edge percolation models with any $f(x)$ satisfying $\lim_{x\to\infty} x^2\, f(x) > 1$.

This type of questions have several sources of motivation and we mention a few of them, which are particularly related to our work. Consider the following physical problem: take the one-dimensional Ising model with couplings decaying as r^{-2}, r being the distance between vertices, at inverse temperature $\beta > 1$; this is the model studied by Fröhlich and Spencer ([FS1]), for which phase transition was established. Take now the finite box $\Lambda_L = [-L, L]$ and assume the Dobrushin boundary conditions, i.e. all spins in $(-\infty, -L]$ will be taken as $+1$, and all spins in $[L, +\infty)$ will be taken as -1. What can we say about the behaviour of interface when $L \to \infty$? This question can be rephrased in the language of FK representation in more general form as the question about behavior of connected components of each boundary, conditioned not to touch each other, in the corresponding dependent percolation model (recall that by a random coloring of the clusters the FK model gives origin to a spin system which interpolates the independent percolation model ($\kappa = 1$), the Ising model ($\kappa = 2$) and the q–states Potts model ($\kappa = q > 2$, integer), for details see [FK, ACCN]). Though we still do not understand fully the corresponding interface problem, our results shed some light on it and suggest possible ways to rigorous answers. Namely, the presence of the infinite oriented connected component (which has positive density, as the proof below shows), strongly supports the fact, predicted by theoretical physics arguments, that with probability 1/2 the connected cluster of one of the sides $(-\infty, -L]$ or $[L, +\infty)$, will take over most of the "middle" space, and therefore there is no "usual" interface. Another interesting corollary of Theorem 1.1 is that in the case of one dimensional Ising model with couplings decaying as r^{-2} and inverse temperature $\beta > 1$, there is a phase transition even with "one-sided" (+)-boundary conditions, i.e. when the thermodynamical limit is taken with (+)-boundary conditions on the left side of the interval, and free boundary conditions on the right side.

On the other hand these questions are closely related to another class of fascinating problems on the multiplicity of Gibbs states for Markov chains with infinite connections. Here orientation naturally occurs in form of (discrete) time parameter. Recently Johansson and Öberg [JO] showed that if g is a regular specification and $\{\mathrm{var}_k(g)\}_{k=1}^{+\infty}$ is in ℓ^2, then g admits a unique Gibbs measure. This tells in particular, that there are no multiple limiting measures for chains with connections decaying as r^{-2}, which, as our Theorem states, stays in sharp contrast to analogous percolation and Ising models. The understanding of Markov chains with infinite connections in the non-uniqueness regime is still very poor, and it is known as a notoriously difficult problem. We have strong evidence (see [BHS], which also use multi-scale arguments) that multi-scale analysis techniques analogous to ones developed in this work could be turned into robust tool to study this question.

Finally we should mention another interesting set of problems. Newman and Spohn in an unpublished work from 1998 [NSp] proved the Shiba relation for the spin-boson Hamiltonian. They also considered the related one-dimensional Ising models (with zero field and above the critical temperature) with power law or more generally sub-exponentially decaying pair interactions, and raised the question on the validity of the analogue of Shiba relation. In the Ising case, such analogue is that the two point correlation function should decay asymptotically as the interaction with a prefactor given by the magnetic susceptibility squared. Our Theorem 1.2 gives a partial answer to this question.

Heuristics of the proof. The proof relies on Fröhlich-Spencer multi-scale analysis ideas ([FS], [FS1]), and we will use the version developed in [KMP] and [M]. In the next few paragraphs we outline the scheme of the proof, and comment on some key ideas, avoiding most of consuming technical points. Our goal here is only to give a very schematic and approximate picture, postponing precise formulations (which tend to be quite involved) to later in the text.

The goal. We look for an event of positive probability, whose occurrence implies not only the existence of an infinite open component $\mathscr{C}$, but also guarantees the presence of an infinite oriented open path π (lying in $\mathscr{C}$). Essentially, we will construct such event, and show that it has positive probability. Our key estimate will be

$$\mu(0 \not\rightarrow [L, +\infty)) \leq L^{-\delta}$$

for some $\delta > 0$.

Scales. We choose super-exponentially fast growing scales. Given $1 < \alpha < 2$, $l_0 = 1$ and l_1, an integer sufficiently large, let

$$l_k = \lfloor l_{k-1}^{\alpha-1} \rfloor l_{k-1}, \qquad k = 2, 3, \dots , \tag{1.5}$$

where $\lfloor z \rfloor$ denotes the integer part of $z \in \mathbb{R}$. We will use the so called dynamical blocking argument, and the sizes of blocks will depend on the configuration, but still each block $I^{(k)}$ of k-th level will be approximately of size l_k. More precisely $l_k - 2l_k^{\alpha'/\alpha} - 6l_{k-1} \le |I^{(k)}| \le 3l_k + 6l_{k-1}$.

Defected and good blocks. Further we will use the following recursive definition of "defected" block. Fix $1 < \alpha' < \alpha$, specified later.

1) We say that the 0-level block $[i, i+1]$ is *defected* if the corresponding nearest neighbor edge $\langle i, i+1 \rangle$ is closed;

2) A 1-level block $I^{(1)} = [s, s']$ is *defected* if either it contains two or more defected 0-blocks, or it contains only one defected 0-block $[i, i+1]$ but there is no open edge $\langle a, a' \rangle$, with $a \le i,\ i+1 \le a'$, $[a, a'] \subseteq [s, s']$, and $a' - a \le l_1^{\alpha'/\alpha}$. Otherwise, $I^{(1)}$ is called *good.*

If a 1-level block $[s, s']$ is good, then either it contains an open nearest neighbor path going from s to s' (in the case of no defected 0-level blocks), or it contains an open nearest neighbor path going from s to a, followed by an open edge $\langle a, a' \rangle$ as above, which is followed by an open nearest neighbor path from a' to s' (case of single defected 0-level block). Such a path from s to s' we will call *pedestal*, and denote by Υ. Part of the cluster between a and a' is disregarded for the future construction since it may create difficulties constructing an oriented open path (see Fig.2).

3) A k-level block $I^{(k)} = [s, s']$ is *defected* if either it contains two or more defected $(k-1)$-blocks, or it contains only one defected $(k-1)$-block $[i, i']$ but there is no open edge $\langle a, a' \rangle$, of the length at most $l_k^{\alpha'/\alpha}$, with $a \le i,\ i' \le a', a \in \Upsilon,\ a' \in \Upsilon'$, for some $(k-1)$-level pedestals $\Upsilon,\ \Upsilon'$. Otherwise $I^{(k)}$ is called *good.*

Thus, if a k-level block $[s, s']$ is good, then it contains an oriented open path going from s to s': in the case of no defected $(k-1)$-level blocks, this path can be obtained by concatenating pedestals of the $(k-1)$-level good blocks which constitute the given k-block; in the case of single defected $(k-1)$-level block, and using the above notation, a similar concatenation yields an oriented open path going from s to a, which is followed by an open edge $\langle a, a' \rangle$, and then followed by another concatenation of pedestals of $(k-1)$-level good blocks, from a' to s'. In both cases, such path from s to s' will be called *pedestal*, and denoted by Υ. Part of the cluster between a and a' is again disregarded in the future construction, since we have little control on oriented connectivity in this segment.

Strategy. Being "defected" doesn't necessarily imply that there is no oriented open path connecting endpoints of the block. Nevertheless, in order to avoid substantial technical difficulties, we will follow two rules that simplify our construction:

a) once a block is defected, we will assume the worst possible situation which it may bring, namely it will be considered as if all edges within this block are closed.

b) once we have high concentration of defected $(k-1)$-level blocks within a k-level block I, $k \ge 1$, we will not try to find connections within the k-block helping to fix its connectivity, but rather will "push the problem to the next level", and try to "jump over" this troubled block I by a longer edge of length at most $l_{k+1}^{\alpha'/\alpha}$, which starts at the pedestal of some good block to the left of I, and ends similarly on the right of I.

Estimates. Assume that $\kappa \ge 1$ is fixed. We begin by choosing p:

$$p \ge 1 - \kappa^{-2}(\ln 2)\, l_1^{-\delta-1} \tag{1.6}$$

for some $\delta > 0$, chosen later (see 2.18).

For $k \geq 1$, let $I^{(k)}$ be a k-level block of length l_k, which consists of $N_k = l_k/l_{k-1} = \lfloor l_{k-1}^{\alpha-1} \rfloor$ $(k-1)$-level blocks $\{I_j^{(k-1)}\}_{j=1}^{N_k}$ of length l_{k-1}. Assume that for any $I_j^{(k-1)}$, $1 \leq j \leq N_k$ we have the following estimate

$$\mu(I_j^{(k-1)} \text{ is defected}) \leq l_{k-1}^{-\delta}, \quad 1 \leq j \leq N_k.$$

Fix $J \subseteq \{1, \ldots, N_k\}$. Under above assumption, we shall see that if $|J| \geq 2$, then

$$\mu\left((k-1)\text{-level blocks } I_j^{(k-1)}, j \in J, \text{within } I^{(k)} \text{ are defected}\right) \leq l_k^{-\delta}.$$

When $|J| = 1$ we will assume for the moment that the defected $(k-1)$-level block is not located too close to the boundary of $I^{(k)}$, namely at least in distance $l_k^{\alpha'/\alpha}$. Otherwise we will need to make a sequence of local adjustments of blocks in order to achieve this property (see Sect. 2).

In this case let a and a' be the end-vertices of the unique defected block $I_i^{(k-1)}$. By our construction, there exists an oriented path starting from the left boundary of $I^{(k)}$ and ending at the vertex a and another oriented path starting from vertex a' and going to the right boundary of $I^{(k)}$. Both these path are obtained by concatenating pedestals of all good $(k-1)$-blocks on the left side of the defected block $I_i^{(k-1)}$ and, respectively, on the right side. We denote these new left and right pedestals by Υ and Υ', respectively.

The probability that there is no open edge $\langle x, y \rangle$ with $x \leq a, a' \leq y$, $x \in \Upsilon$, $y \in \Upsilon'$ and $y - x \leq l_k^{\alpha'/\alpha}$:

$$\prod_{\substack{x,y: x \leq a < a' \leq y: \\ y - x < l_k^{\alpha'/\alpha} \\ x \in \Upsilon, y \in \Upsilon'}} q_{\langle x,y \rangle} = \exp\Big\{ - \sum_{\substack{x,y: x \leq a < a' \leq y: \\ y - x < l_k^{\alpha'/\alpha} \\ x \in \Upsilon, y \in \Upsilon'}} \frac{\beta}{|x-y|^2} \Big\} \leq l_{k-1}^{-\beta(\alpha'-1)},$$

where the last inequality requires some work, and in order to obtain it we will need to use certain geometric properties of pedestals Υ and Υ', which propagate inductively from each level to the next one. Namely we will show that, typically, the pedestals are relatively dense sets (see Sect.2, (2.7), (2.8)). Using that, and the inclusion

$$\big[I^{(k)} \text{ has a unique defected } (k-1)\text{-level block } [a, a'] \text{ but remains defected } \big] \subseteq$$

$$\Big\{ \big[I^{(k)} \text{ has unique defected } (k-1)\text{-level block } [a, a']\big] \cap$$

$$\big[\text{ there is no open edge } \langle x, y \rangle \text{ with } x \leq a, a' \leq y,\ x \in \Upsilon,\ y \in \Upsilon' \text{ and } y - x \leq l_k^{\alpha'/\alpha} \ \big] \Big\}$$

we will show that

$$\mu\big[I^{(k)} \text{ has a unique defected } (k-1)\text{-level block } [a, a'] \text{ but remains defected } \big] \leq l_k^{-\delta},$$

provided

$$\delta < \beta(1-\eta) - 1 \tag{1.7}$$

for some $\eta = b(\alpha, \alpha', l_1) \to 0$ as $l_1 \to \infty$ and $\alpha' \to \alpha$ (see Proposition 2.1 for details). Notice that, for any $\beta > 1$ one can choose $1 < \alpha < 2$, l_1 sufficiently large and α' close to α so that (1.7) hold. In equation (1.7) we have used that in case $|J| = 1$ (only one defected block is present), we have

$$l_{k-1}^{\alpha-1-\delta} \leq l_k^{-\delta'} \tag{1.8}$$

with $\delta' = (\delta - \alpha + 1)/\alpha \geq (\alpha - 1)/(2 - \alpha) > 0$.

Consequences and conclusions. Once we have good estimates on the probability of a k-level block ($k \geq 1$) being good, we define for fixed $L > 0$:

$$\widehat{M}_L = \min\{k : \; L \leq l_k\}.$$

If the $\widehat{M}_L$-level block $I^{(\widehat{M}_L)}$ contains the interval $[0, L]$ and is good, it implies the existence of an open oriented path from the left end-vertex of $I^{(\widehat{M}_L)}$ to its right end-vertex that uses only edges of length no more than $l_{\widehat{M}_L}^{\alpha'/\alpha}$. When $L \geq l_{\widehat{M}_L}^{\alpha'/\alpha} + 2$, this implies that there exists an open oriented path connecting $(-\infty, 0]$ to $[L, +\infty)$, having at least one vertex in $[0, L]$. Such path does not need to start at the origin 0. In order to obtain the desired lower bound for the event $[0 \rightsquigarrow [L, +\infty)]$ we still need some extra work, and will use the estimate (1.7) for $k = 1, \ldots, \widehat{M}_L - 1$, applied simultaneously to the intersection of certain events which guarantee the existence of the oriented open path staring from 0. The case $L < l_{\widehat{M}_L}^{\alpha'/\alpha} + 2$ is treated in a similar way (see Sect. 4).

Difficulties. To carry on this scheme we have to go through several "unpleasant" and rather involved points. Since we use a dynamical blocking argument (dynamic renormalization), the definitions of blocks of higher level depend not only on the size and location of lower level blocks, but also on their "status" (defected or good), and therefore, on the status of edges within those blocks. This requires a rather tight bookkeeping of what has happened on lower scales and we do renormalization along what we call "itineraries".

The second difficulty is in the representation of the initial measure, which should be sensitive to the above mentioned dynamic renormalization. We opt to write it as a convex combination of a family of probability measures associated with the above mentioned itineraries.

Once this is achieved, all necessary estimates follow along the scheme of [FS1] and [KMP]

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2 Spatial blocks (Dynamic Renormalization)

Notation. For $L \in \mathbb{N}$ we denote $I(L) = [-L, L]$. The scales $\{l_k\}_{k\in\mathbb{N}}$ we set in the following way: let $l_0 = 1$, l_1 is an integer large enough, defined later (see (2.17) at the end of this Section), and $1 < \alpha < 2$. Then

$$l_k = \lfloor l_{k-1}^{\alpha-1} \rfloor l_{k-1}, \quad k \geq 2. \tag{2.1}$$

Further we denote $x_j^{(k)} = j l_k,\ j \in \mathbb{Z}$.

For $1 < \alpha' < \alpha$ we also define

$$\widehat{M}_L = \min\{k : L \leq l_k\}, \tag{2.2}$$

and

$$M_L = \begin{cases} \widehat{M}_L - 1 & \text{if } \ L \leq l_{\widehat{M}_L}^{\alpha'/\alpha} + 2, \\ \\ \widehat{M}_L & \text{if } \ l_{\widehat{M}_L}^{\alpha'/\alpha} + 2 < L. \end{cases}$$

The difference between both cases $L > l_{\widehat{M}_L}^{\alpha'/\alpha} + 2$ and $L \leq l_{\widehat{M}_L}^{\alpha'/\alpha} + 2$ will appear while bounding the probabilities of certain events, and is irrelevant for the time being. Through this Section and the next one we will assume that

$$L > l_{\widehat{M}_L}^{\alpha'/\alpha} + 2,$$

returning to the other case in Sect. 4.

For any fixed $L > 0$ we will consider the measure $\mu^w_{I(L)}$, corresponding to the "wired" boundary conditions. Since under $\mu^w_{I(L)}$ all the edges $b = \langle x, y\rangle$ with end-vertices $x, y \in [-L,\ L]^c$ are open, it will be convenient to embed Ω_L into $\{\omega \in \Omega : \omega_{\langle x,y\rangle} = 1 \text{ if } x, y \in [-L,\ L]^c\} \subset \Omega$, which we denote by Ω^w_L. Moreover, in order to make formal manipulations and proceed with the inductive estimates, we will be looking at a larger interval $I(l_{M_L}) \supseteq I(L)$, and formally attributing state $G(\cdot)$ or $B(\cdot)$ for *all* sub-boxes of k-level in $I(l_{M_L})$, $\ 0 \leq k \leq M_L - 1$.

Throughout the text $\mathbb{I}_A$ indicates the indicator function of an event A, i.e. $\mathbb{I}_A(\omega) = 1$ or 0 according to $\omega \in A$ or not.

Decomposition of events. 0-Level. We set $I_i^{(0)} = [i, i+1]$, call them 0-level blocks, and for i such that $I_i^{(0)} \cap I(l_{M_L}) \neq \emptyset$ define events on Ω_L:

$$G(I_i^{(0)}) = \{\omega : \omega_{<i,i+1>} = 1\}, \qquad B(I_i^{(0)}) = \{\omega : \omega_{<i,i+1>} = 0\},$$

(recall that $G(I_i^{(0)}) = \Omega_L$ for $I_i^{(0)}$ such that $I_i^{(0)} \cap \big(I(l_{M_L}) \setminus I(L)\big) \neq \emptyset$).

1-Level. Consider the intervals $\widetilde{I}_j^1 = [jl_1, (j+1)l_1]$ and for each j such that $\widetilde{I}_j^1 \cap I(l_{M_L}) \neq \emptyset$ define a partition of Ω_L, into the following events:

$$
\begin{aligned}
G(\widetilde{I}_j^{(1)}) &= \bigcap_{i=jl_1}^{(j+1)l_1-1} G(I_i^{(0)}) \\
H(\widetilde{I}_j^{(1)}) &= \bigcup_{i=jl_1}^{(j+1)l_1-1} \left[B(I_i^{(0)}) \cap \bigcap_{\substack{s=jl_1 \\ s\neq i}}^{(j+1)l_1-1} G(I_s^{(0)}) \right] \\
B(\widetilde{I}_j^{(1)}) &= \Omega_L \Big\backslash \left(G(\widetilde{I}_j^{(1)}) \cup H(\widetilde{I}_j^{(1)}) \right). \qquad (2.3)
\end{aligned}
$$

Denote $H_i(\widetilde{I}_j^{(1)}) = B(I_i^{(0)}) \cap \bigcap_{\substack{s=jl_1 \\ s\neq i}}^{(j+1)l_1-1} G(I_s^{(0)})$, and the $I_i^{(0)}$ for which $B(I_i^{(0)})$ occurs are called defected 0-level blocks.

Adjustment. Given ω, we consider all j for which $\omega \in H(\widetilde{I}_j^{(1)})$ and such that the (unique) i_j, $I_{i_j}^{(0)} \subset \widetilde{I}_j^{(1)}$, for which $\omega \in B(I_{i_j}^{(0)})$ verifies $jl_1 \leq i_j \leq jl_1 + \lfloor l_1^{\alpha'/\alpha} \rfloor - 1$ (respectively $(j+1)l_1 - \lfloor l_1^{\alpha'/\alpha} \rfloor \leq i_j \leq (j+1)l_1 - 1$). If this set is not empty, take arbitrarily one of such j and check if $\widetilde{I}_{j-1}^{(1)}$ (respectively $\widetilde{I}_{j+1}^{(1)}$) has a 0-level defect in the sub-interval $[jl_1 - 2\lfloor l_1^{\alpha'/\alpha} \rfloor, jl_1]$ (respectively $[(j+1)l_1, (j+1)l_1 + 2\lfloor l_1^{\alpha'/\alpha} \rfloor - 1]$).

1) If yes, then we consider a new interval $I_{j-1}^{(1)} = \widetilde{I}_{j-1}^{(1)} \cup \widetilde{I}_j^{(1)}$ (respectively $I_j^{(1)} = \widetilde{I}_j^{(1)} \cup \widetilde{I}_{j+1}^{(1)}$) and say that the event $B(I_{j-1}^{(1)})$ (resp. $B(I_j^{(1)})$) occurs. (This is motivated by the fact that for the chosen ω in the new interval there will be at least two 0-level defected blocks.)

2) If not, then we consider two new intervals $I_{j-1}^{(1)} = [(j-1)l_1, jl_1 - \lfloor l_1^{\alpha'/\alpha} \rfloor]$ and $I_j^{(1)} = [jl_1 - \lfloor l_1^{\alpha'/\alpha} \rfloor, (j+1)l_1]$ (respectively $I_j^{(1)} = [jl_1, (j+1)l_1 + \lfloor l_1^{\alpha'/\alpha} \rfloor]$ and $I_{j+1}^{(1)} = [(j+1)l_1 + \lfloor l_1^{\alpha'/\alpha} \rfloor, (j+2)l_1]$) and say that $G(I_{j-1}^{(1)})$, $H(I_{j-1}^{(1)})$ or $B(I_{j-1}^{(1)})$ occurs if the event $H(I_j^{(1)})$ and, respectively $G(\widetilde{I}_{j-1}^{(1)})$, $H(\widetilde{I}_{j-1}^{(1)})$ or $B(\widetilde{I}_{j-1}^{(1)})$, do occur (respectively $G(I_{j+1}^{(1)})$, $H(I_{j+1}^{(1)})$ or $B(I_{j+1}^{(1)})$ occurs if the event $H(I_j^{(1)})$ and, respectively $G(\widetilde{I}_{j+1}^{(1)})$, $H(\widetilde{I}_{j+1}^{(1)})$ or $B(\widetilde{I}_{j+1}^{(1)})$ do occur). (In this case the adjustment moves the boundary "away" from the unique defected block in $I_j^{(1)}$, but doesn't change the number of defected 0-level blocks in adjusted intervals.)

We set $I_j^{(1)} = \widetilde{I}_j^{(1)}$ if $\widetilde{I}_j^{(1)}$ was not involved in the previous adjustment.

If after this adjustment there remain intervals $I_j^{(1)}$ for which $H(I_j^{(1)})$ occurs and its defected interval $I_{i_j}^{(0)}$ stays within distance $\lfloor l_1^{\alpha'/\alpha} \rfloor$ from the left or the right boundary of $I_j^{(1)}$, we repeat the above procedure (applying it, if needed, already to intervals adjusted in the previous step). After finitely many steps of such adjustment procedure there are left no intervals $I_j^{(1)}$, for which the event $H(I_j^{(1)})$ occurs, and its defected 0-level block stays within distance $\lfloor l_1^{\alpha'/\alpha} \rfloor$ from the boundary, and the adjustment procedure is stopped.

Remark. Observe that the adjustment procedure has the *abelian* property, i.e. the final partition does not depend on the order in which we do adjustments and in which order we pick the blocks that still need to be adjusted (in case we have more than one). In this sense the procedure is "well defined". It also has *a locality* property, i.e. the final modification of each initial 1-level block depends on the values of the configuration in the nearest neighbor and, at most, in the next nearest neighbor blocks only.

Once the procedure is completed, the obtained intervals are called 1-level blocks, and we renumerate them from left to right $I_j^{(1)}$, $j \geq 1$. Notice also, that $l_1 - 2\lfloor l_1^{\alpha'/\alpha} \rfloor \leq |I_j^{(1)}| \leq 3l_1$, and $I(L) \subseteq \cup_j I_j^{(1)}$. For 1-level blocks $I_\cdot^{(1)}$ which do not intersect $I(L)$ we always have $G(I_\cdot^{(1)}) = \Omega_L$.

In other words, for each fixed ω, its values on the nearest neighbor edges determine a partition of $I(l_{M_L})$ into 1-level blocks $\{I_j^{(1)}\}_{j\geq 1} \equiv \{I_j^{(1)}(\omega)\}_{j\geq 1}$. We also say that the collection $\{\hat{I}_j^{(1)}\}_{j\geq 1}$ of 1-level blocks is **admissible** (from $I^{(0)}$), if there exists ω for which $\hat{I}_j^{(1)} = I_j^{(1)}(\omega)$, $j \geq 1$. When ambiguity doesn't arise, we will be omitting ω.

Since the adjustments at 1-level are completely determined by the values of configuration on nearest neighbor edges, it implies, in particular, that the final state of the adjustment is $\sigma(\Omega_L)$-measurable. We denote by

$$\mathscr{I}^{(1)} \equiv \mathscr{I}^{(1)}(I^{(0)}) = \{I^{(1)} = \{I_j^{(1)}\}_j\}$$

the collection of all admissible sequences of intervals, which are possible choices (outcomes) for the final state of the adjustment starting from $I^{(0)} = \{I_j^{(0)}\}_{j\in\mathbb{Z}}$. The procedure also determines where the defected 0-level blocks are located within each 1-block, and we denote by $D_{I_j^{(1)}}$ the set of indices of the defected 0-blocks within $I_j^{(1)}$, and $D(I^{(1)}) = \{D_{I_j^{(1)}}\}_j$. The elements $I^{(1)} \in \mathscr{I}^{(1)}$ we call first step of the *itinerary*, and for fixed $I^{(1)} = \{I_j^{(1)}\}_j \in \mathscr{I}^{(1)}$ we will denote

$$\mathscr{E}(I^{(1)}) = \{\omega : \; I^{(1)}(\omega) = I^{(1)}\},$$

i.e., the set of configurations for which the 1-level adjustment leads to $I^{(1)}$. The fact that $\omega, \omega' \in \mathscr{E}(I^{(1)})$ doesn't imply that $D_{I_j^{(1)}}(\omega) = D_{I_j^{(1)}}(\omega')$.

Pedestals. Fix $I^{(1)}$ and consider all 1-level blocks $I_j^{(1)}$ for which $H(I_j^{(1)}) \cap \mathscr{E}(I^{(1)})$ occurs. For such 1-blocks, the set of vertices $x \in I_j^{(1)}$ to the left (resp. right) of the 0-level defected block of $I_j^{(1)}$ will be called left 1-level pedestal of $I_j^{(1)}$ (resp. right 1-level pedestal) and denoted by $\Upsilon_{\mathscr{L}}(I_j^{(1)})$ (resp. $\Upsilon_{\mathscr{R}}(I_j^{(1)})$). Notice that the vertices in each of these 1-level pedestals are connected by open nearest neighbor edges. Define

$$\begin{aligned} G^+(I_j^{(1)}) &= \mathscr{E}(I^{(1)}) \cap H(I_j^{(1)}) \cap [\omega : \exists x \in \Upsilon_{\mathscr{L}}(I_j^{(1)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(1)}),\ y - x \leq \lfloor l_1^{\alpha'/\alpha} \rfloor : \omega_{<x,y>} = 1], \\ B^+(I_j^{(1)}) &= \big(\mathscr{E}(I^{(1)}) \cap H(I_j^{(1)})\big) \setminus G^+(I_j^{(1)}). \end{aligned} \tag{2.4}$$

If $G^+(I_j^{(1)})$ occurs, and the pair (x,y) such that $x \in \Upsilon_{\mathscr{L}}(I_j^{(1)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(1)}),\ y - x \leq \lfloor l_1^{\alpha'/\alpha} \rfloor :$ $\omega_{<x,y>} = 1$ is not unique, we choose one in arbitrary way, and, once the pair (x,y) is chosen, the interval $[x+1, y-1]$ will be called defected part of $I_j^{(1)}$, and denoted by $\mathscr{D}(I_j^{(1)})$. In this case we define $\Upsilon(I_j^{(1)}) = \Big(\Upsilon_{\mathscr{L}}(I_j^{(1)}) \cup \Upsilon_{\mathscr{R}}(I_j^{(1)})\Big) \setminus \mathscr{D}(I_j^{(1)})$.

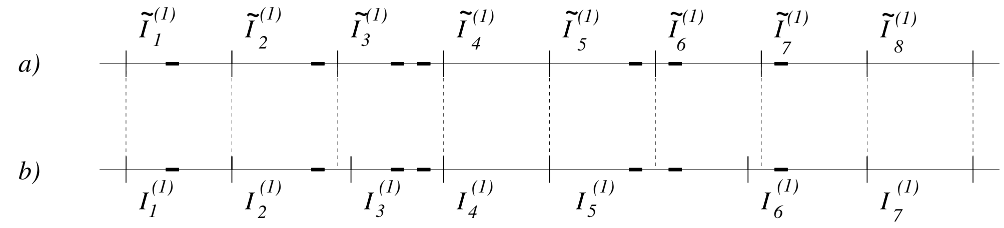

Figure 1: Adjustments: part *a*) shows the deterministic 1-blocks $\widetilde{I}^{(1)}_j$, bold-face segments show location of the defects. Part *b*) shows how these blocks were adjusted. $\widetilde{I}^{(1)}_5$ and $\widetilde{I}^{(1)}_6$, merge into a single 1-block $I^{(1)}_5$.

On the other hand if $G(I^{(1)}_j)$ occurs, it means that there are no closed nearest neighbor edges in $I^{(1)}_j$, and we define the pedestal $\Upsilon(I^{(1)}_j) = I^{(1)}_j$.

This completes the first step, associating with each 1-level block one of the events G, G^+, B or B^+. Since for the next step of the procedure it is irrelevant which particular sub-type G or G^+ (resp. B or B^+) of the event occurs, without creating ambiguities, we will simply denote their occurrence by the letter G (resp. B).

k**-level.** Let $k \geq 2$ and assume we have completed the $(k-1)$-level construction. This means that the following objects are defined:

i) Partitions. For any $r = 0,\, 1, \ldots, k-1$ there is a well defined collection of (finite) sequences of r-blocks $I^{(r)} = \{I^{(r)}_j\}_j = \{[v_j, v'_j]\}_j$, such that

$$1) \quad v'_j = v_{j+1} \quad \text{and} \quad \bigcup\nolimits_j I^{(r)}_j = I(l_{M_L}),$$

$$2) \quad \text{there exists } I^{(r-1)}, \text{ such that } I^{(r)} \text{ is admissible from } I^{(r-1)}.$$

The collection of all such sequences we denote by $\mathscr{I}^{(r)}$, and $\mathscr{I}^{(r)},\ 0 < r \leq k-1$ has the following structure

$$\mathscr{I}^{(r)} = \bigcup_{I^{(r-1)} \in \mathscr{I}^{(r-1)}} \mathscr{I}^{(r)}(I^{(r-1)}),$$

where $\mathscr{I}^{(r)}(I^{(r-1)})$ is the collection of all sequences of r-blocks, which are admissible from $I^{(r-1)}$.

ii) Itineraries. The sequence $J^{(s)} = \{I^{(0)}, I^{(1)}, \ldots, I^{(s)}\},\ 0 \leq s \leq k-1$, such that $I^{(m)} \in \mathscr{I}^{(m)}(I^{(m-1)}),\ 0 < m \leq s$, is called *an itinerary* up to the s-level, and each element $I^{(m)},\ m \leq s$ is called m-th step of the itinerary. We denote by $\mathscr{J}^{(r)}$ the set of all itineraries up to the r-level. Given the itinerary $J^{(k-1)} = \{I^{(0)}, I^{(1)}, \ldots, I^{(k-2)}, I^{(k-1)}\}$ we define

$$\mathscr{E}(J^{(k-1)}) = \mathscr{E}(J^{(k-2)}) \cap \{\omega : I^{(k-1)} \in \mathscr{I}^{(k-1)}(I^{(k-2)})\}.$$

Observe that $\mathscr{E}(J^{(k-1)})$ depends only on the status of edges which have their end-vertices in the same block $I^{(k-2)}_j$, $j \geq 1$, and therefore $\mathscr{E}(J^{(k-1)})$ is $\sigma(\Omega_L)$-measurable.

iii) G, B and Pedestals. Fix $I^{(k-1)} = \{I^{(k-1)}_j\}_j$. With each $(k-1)$-block $I^{(k-1)}_j$ from $I^{(k-1)} \in \mathscr{I}^{(k-1)}$ there are associated events $G(I^{(k-1)}_j)$ and $B(I^{(k-1)}_j)$, and sets $\Upsilon(I^{(k-1)}_j)$, called pedestals, in case if $G(I_j^{(k-1)})$ occurs. Moreover, the occurrence of $G(I_j^{(k-1)})$ implies that $\Upsilon(I_j^{(k-1)})$ is indeed an open oriented path from the left boundary of $I_j^{(k-1)}$ to its right boundary [1]. The intervals $I_i^{(k-1)}$ for which $B(I_i^{(k-1)})$ occurs are called defected (or defects).

Inductive step: Construction of k-blocks. Fix $J^{(k-1)} = \{I^{(0)}, I^{(1)}, \ldots, I^{(k-1)}\} \in \mathscr{J}^{(k-1)}$. For each $z \in \mathbb{Z}$ we set $j_z^k = \min\{j : z \in I_j^{(k-1)}\}$, $\widehat{j}_i^k = j^k_{x_i^{(k)}}$ (see notation after (2.1)), and define the intervals

$$\widetilde{I}_i^{(k)} = \bigcup_{\widehat{j}_i^k+1}^{\widehat{j}_{i+1}^k} I_j^{(k-1)}$$

as well as the events

$$
\begin{aligned}
G(\widetilde{I}_j^{(k)}) &= \mathscr{E}(J^{(k-1)}) \bigcap_{\widehat{j}_i^k+1}^{\widehat{j}_{i+1}^k} G(I_i^{(k-1)}), \\
H(\widetilde{I}_j^{(k)}) &= \bigcup_{i=\widehat{j}_i^k+1}^{\widehat{j}_{i+1}^k} \left[\mathscr{E}(J^{(k-1)}) \cap B(I_i^{(k-1)}) \cap \bigcap_{j \neq i} G(I_j^{(k-1)}) \right], \\
B(\widetilde{I}_j^{(k)}) &= \Omega_L \Big\backslash \left(G(\widetilde{I}_j^{(k)}) \cup H(\widetilde{I}_j^{(k)}) \right). \qquad (2.5)
\end{aligned}
$$

Adjustment of intervals and events at k-th step. Given $\omega \in \Omega_L$ consider all s for which $\mathscr{E}(J^{(k-1)}) \cap H(\widetilde{I}_s^{(k)})$ occurs and the defected interval $I_{i_j}^{(k-1)} \subset \widetilde{I}_s^{(k)}$ satisfies $\widehat{j}_s^k + 1 \le i_j \le j^k_{x_s^{(k)} + \lfloor l_k^{\alpha'/\alpha} \rfloor}$ (respectively $j^k_{x_{s+1}^{(k)} - \lfloor l_k^{\alpha'/\alpha} \rfloor} \le i_j \le \widehat{j}_{s+1}^k$). If this set is not empty, take arbitrarily one of such $\widetilde{I}_s^{(k)}$ and check if $\widetilde{I}_{s-1}^{(k)}$ (respectively $\widetilde{I}_{s+1}^{(k)}$) has a defected block $I_r^{(k-1)}$ with $j^k_{x_s^{(k)} - 2\lfloor l_{k-1}^{\alpha'/\alpha} \rfloor} \le r \le \widehat{j}_s^k$ (resp. $\widehat{j}_{s+1}^k + 1 \le r \le j^k_{x_{s+1}^{(k)} + 2\lfloor l_{k-1}^{\alpha'/\alpha} \rfloor}$).

1) If yes, then we consider a new interval $I_{s-1}^{(k)} = \widetilde{I}_{s-1}^{(k)} \cup \widetilde{I}_s^{(k)}$ (respectively $I_s^{(k)} = \widetilde{I}_s^{(k)} \cup \widetilde{I}_{s+1}^{(k)}$) and say that the event $B(I_{s-1}^{(k)})$ (resp. $B(I_s^{(k)})$) occurs.

2) If not, then we consider two new intervals

$$I_{s-1}^{(k)} = \bigcup_{\widehat{j}_{s-1}^k+1}^{j^k_{x_s^{(k)} - l_{k-1}^{\alpha'/\alpha}}} I_j^{(k-1)} \qquad I_s^{(k)} = \bigcup_{j^k_{x_s^{(k)} - l_{k-1}^{\alpha'/\alpha}}+1}^{\widehat{j}_{s+1}^k} I_j^{(k-1)}$$

$$(I_s^{(k)} = \bigcup_{\widehat{j}_s^k+1}^{j^k_{x_{s+1}^{(k)} + l_{k-1}^{\alpha'/\alpha}}} I_j^{(k-1)} \qquad I_{s+1}^{(k)} = \bigcup_{j^k_{x_{s+1}^{(k)} + l_{k-1}^{\alpha'/\alpha}}+1}^{\widehat{j}_{s+2}^k} I_j^{(k-1)}$$

[1] We are identifying an oriented path with the set of vertices it visits.

respectively), and say that $G(I_{s-1}^{(k)})$, $H(I_{s-1}^{(k)})$ or $B(I_{s-1}^{(k)})$ occurs if respectively $G(\widetilde{I}_{s-1}^{(k)})$, $H(\widetilde{I}_{s-1}^{(k)})$ or $B(\widetilde{I}_{s-1}^{(k)})$, and the event $H(I_s^{(k)})$ do occur (respectively $G(I_{s+1}^{(k)})$, $H(I_{s+1}^{(k)})$ or $B(I_{s+1}^{(k)})$ occurs if respectively $G(\widetilde{I}_{s+1}^{(k)})$, $H(\widetilde{I}_{s+1}^{(k)})$ or $B(\widetilde{I}_{s+1}^{(k)})$, and the event $H(I_s^{(k)})$ do occur). We set $I_s^{(k)} = \widetilde{I}_s^{(k)}$ if $\widetilde{I}_s^{(k)}$ was not involved in the adjustment.

If after this adjustment there remain intervals $I_s^{(k)}$ for which $H(I_s^{(k)})$ occurs and its defected interval $I_i^{(k-1)}$ stays within distance $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from one of the endpoints of $I_s^{(k)}$, we repeat the above procedure (using, if needed, the intervals $I_\cdot^{(k)}$ already previously adjusted).

After finitely many steps of such adjustment there are left no intervals $I_j^{(k)}$, for which the event $H(I_j^{(1)})$ occurs, and the $k-1$-level defected block is close to any of the boundaries by distance $\lfloor l_k^{\alpha'/\alpha} \rfloor$.

Once adjustments are completed, the obtained intervals are called k-level blocks, and we renumerate all k-level blocks from left to right $I_j^{(k)}$, $j \geq 1$. A sequence of k-level blocks $\{I_j^{(k)}\}_j$ which can be obtained after adjustments, starting from $I^{(k-1)}$, is said to be admissible from $I^{(k-1)}$. The set of all such sequences is denoted by $\mathscr{I}^k(I^{(k-1)})$, and $\mathscr{I}^k = \bigcup_{I^{(k-1)}} \mathscr{I}^k(I^{(k-1)})$. The procedure also determines where the defected $(k-1)$-level blocks are located within each k-block, and we denote by $D_{I_j^{(k)}}$ the set of indices of the defected $(k-1)$-blocks within $I_j^{(k)}$, and $D(I^{(k)}) = \{D_{I_j^{(k)}}\}_j$. If it doesn't create ambiguities we will use notation $I^{(k)} = (I^{(k)}, D(I^{(k)}))$. Notice that

$$l_k - (2\lfloor l_k^{\alpha'/\alpha} \rfloor + 6l_{k-1}) < |I_j^{(k)}| \leq 3l_k + 6l_{k-1}, \tag{2.6}$$

and, when it will not create ambiguities, we will write $I_j^{(k)} = \bigcup_{s_0(j)}^{s_1(j)} I_s^{(k-1)}$

k-Pedestals. Finally, consider all (adjusted) k-level blocks $I_j^{(k)}$ for which $H(I_j^{(k)})$ occurs. For such k-block its unique $(k-1)$-sub-block $I_r^{(k-1)} \subset I_j^{(k)}$ for which the event $B(I_r^{(k-1)})$ occurs is called k-level defect. Define

$$\Upsilon_{\mathscr{L}}(I_j^{(k)}) = \bigcup_{s_0(j)}^{r-1} \Upsilon(I_s^{(k-1)}), \qquad \Upsilon_{\mathscr{R}}(I_j^{(k)}) = \bigcup_{r+1}^{s_1(j)} \Upsilon(I_s^{(k-1)}),$$

and call them left and right pedestals of $I_j^{(k)}$. Moreover, on the event $\bigcap_{s_0(j)}^{r-1} G(I_s^{(k-1)}) \cap \bigcap_{r+1}^{s_1(j)} G(I_s^{(k-1)})$ there exists an open oriented path connecting the left boundary of $I_j^{(k)}$ to the right boundary of $I_{r-1}^{(k-1)}$ and an open oriented path connecting the left boundary of $I_{r+1}^{(k-1)}$ to the right boundary of $I_j^{(k)}$. These paths are obtained by concatenation of corresponding $\Upsilon(I_s^{(k-1)})$, $s_0(j) \leq s \leq r-1$ and $r+1 \leq s \leq s_1(j)$.

Define

$$G^+(I_j^{(k)}) = \mathscr{E}(J^{(k)}) \cap H(I_j^{(k)}) \cap [\omega : \exists x \in \Upsilon_{\mathscr{L}}(I_j^{(k)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(k)}),\ y - x \leq \lfloor l_k^{\alpha'/\alpha} \rfloor : \omega_{<x,y>} = 1],$$

$$B^+(I_j^{(k)}) = \mathscr{E}(J^{(k)}) \cap H(I_j^{(k)}) \setminus G^+(I_j^{(k)}).$$

On $G^+(I_j^{(k)})$, if the pair (x, y) such that $x \in \Upsilon_{\mathscr{L}}(I_j^{(k)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(k)})$, $y - x \leq \lfloor l_k^{\alpha'/\alpha} \rfloor$, and $'\omega_{<x,y>} = 1$ is not unique, we choose one in arbitrary way, and, once (x, y) is chosen,

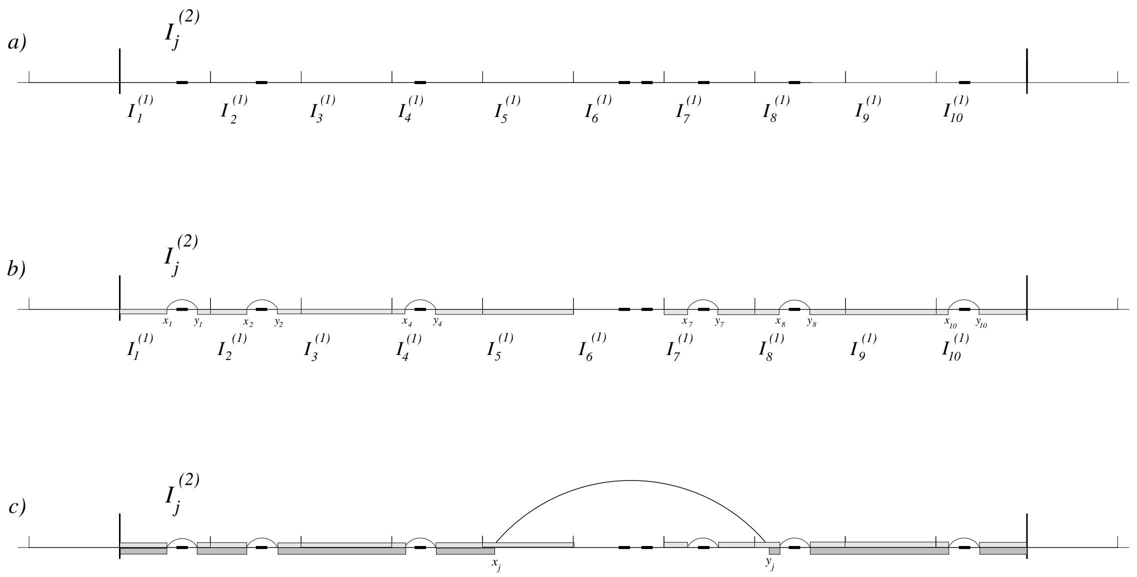

Figure 2: Pedestals and defects: part a) shows the deterministic 1-blocks $I_i^{(1)}$, $1 \leq i \leq 10$, located in the 2-block $I_j^{(2)}$; bold-face segments show location of the 0-level defects. Part b) shows construction of 1-level pedestals, marked by light-gray strips. The block $I_6^{(1)}$ is a defected 1-block. The segments (x_i, y_i) are "enlarged" defects in $I_i^{(1)}$. Part c) shows creation of 2-level pedestals, marked by dark-gray strips, concatenated by long range bonds. The segment (x_j, y_j) is enlarged defect for $I_j^{(2)}$.

the interval $[x+1, y-1]$ will be called defected part of $I_j^{(k)}$ and denoted by $\mathscr{D}(I_j^{(k)})$. Define $\Upsilon(I_j^{(k)}) = \Big(\Upsilon_{\mathscr{L}}(I_j^{(k)}) \cup \Upsilon_{\mathscr{R}}(I_j^{(k)})\Big) \setminus \mathscr{D}(I_j^{(k)})$.

On the other hand, if $G(I_j^{(k)})$ occurs it means that there are no defected block of previous level in $I_j^{(k)}$, and in this case $\Upsilon(I_j^{(k)}) = \bigcup_{s_0(j)}^{s_1(j)} \Upsilon(I_s^{(k-1)})$.

This completes the k-th step, associating with each itinerary $J^{(k-1)}$ its "continuation", the next k-step, i.e. a sequence of k-blocks $I^{(k)} = \{I_j^{(k)}\}_j$, re-numerated from left to right; moreover with each k-block we associate one of the events G, G^+, B or B^+. Since for the next step of the procedure it is irrelevant which particular sub-type G or G^+ (resp. B or B^+) of the event occurs, without creating ambiguities, we will simply denote their occurrence by letter G (resp. B).

Structure of pedestals. We will end this section by mentioning some (elementary) geometric properties of pedestals. They will be used in the next two sections to estimate the probabilities of events such as "jump over defected blocks". Namely, if a k-level block $I = [s, s']$ has only one $(k-1)$-level defected sub-block, here denoted by $[a, a']$, and there are corresponding left and right pedestals, spanning from s to a and from a' to s', respectively, we have to show that the sets

$$\Upsilon \cap [a - \lfloor l_k^{\alpha'/\alpha} \rfloor, a] \quad \text{and} \quad \Upsilon' \cap [a', a' + \lfloor l_k^{\alpha'/\alpha} \rfloor]$$

contain of order $Cl_k^{\alpha'/\alpha}$ vertices, and these vertices are distributed regularly enough.

We recall briefly some consequences of definition of pedestals:

i) If a 1-level block is good we construct a pedestal (within this 1-level block) which contains two nearest neighbor paths concatenated by an edge of length at most $\lfloor l_1^{\alpha'/\alpha} \rfloor$ (this is in the case that there is present one 0-level defected block). By the construction above, a defected 0-level block is always kept away from the boundary of the 1-level block. Nevertheless the concatenating edge may start (or end) at the start- or end- vertex of 1-block. This implies that for two consecutive good 1-level blocks their defected parts may follow each other. Nevertheless this is not true for any three or more consecutive good 1-level blocks. With a little work this implies that for any $\epsilon_0 > 0$ there exists a constant $C \equiv C(\epsilon_0)$, such that

$$|\Upsilon \cap [v,\, v']| \geq C|v' - v| \tag{2.7}$$

for any interval $[v,\, v'] \subseteq \mathrm{env}(\Upsilon)$, such that $|v' - v| \geq l_1^{\frac{\alpha'}{\alpha}+\epsilon_0}$. ($env(\Upsilon)$ denotes the smallest interval in $\mathbb{Z}$ containing Υ.)

ii) Analogously to the previous case, a k-level pedestal (within a k-level block) contains, in "worst case scenario" two paths (lower level pedestals) concatenated by an edge of length at most $\lfloor l_k^{\alpha'/\alpha} \rfloor$. Observe, that even when the defected $(k-1)$-level block is away from the boundary of the k-level block, the concatenating edge may start (or end) at the start- or end- vertex of the k-block. This implies that for two consecutive good k-level blocks their defected parts may follow each other, but again, this is not true for any three or more consecutive good k-level blocks. Taking into account the structure of defects in the lower level pedestals the estimate (2.7) propagates to pedestals of k-level:

$$|\Upsilon \cap [v,\, v']| \geq C|v' - v| \tag{2.8}$$

for any interval $[v,\, v'] \subseteq \mathrm{env}(\Upsilon)$, such that $|v' - v| \geq l_k^{\frac{\alpha'}{\alpha}+\epsilon_0}$.

We will end this section with the estimate which will be important for the inductive step in the next section, and we also fix parameters of the model. By $H_z(I_j^{(k)}) \subset H(I_j^{(k)})$ we denote the event, for which unique defected $k-1$-level block has index z.

Lemma 2.1 *There exists $\eta \equiv \eta(\alpha, \alpha', l_1)$, such that*

$$P[\exists x \in \Upsilon_{\mathscr{L}}(I_j^{(k)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(k)}),\ y - x \leq \lfloor l_k^{\alpha'/\alpha} \rfloor : \omega_{<x,y>} = 1 \big|\, H_z(I_j^{(k)})] \geq 1 - l_{k-1}^{-\beta(1-\eta)(\alpha'-1)}, \tag{2.9}$$

and $\eta \searrow 0$ when $l_1 \nearrow +\infty$.

Proof. We will use repeatedly the following upper and lower bounds: If I and I' are two intervals, and $d = \mathrm{dist}(I, I')$, then there exist constants C_1, C_2, such that

$$C_1\ J(I, I') \leq \sum_{\substack{x \in I \cap \mathbb{Z} \\ y \in I' \cap \mathbb{Z}}} \frac{1}{|x-y|^2} \leq C_2\ J(I, I'), \tag{2.10}$$

where

$$J(I, I') = \int_{I \times I'} dx\, dy \frac{1}{|x-y|^2} = \ln \frac{(|I| + d)(|I'| + d)}{d\,(|I| + |I'| + d)}. \tag{2.11}$$

We shall need also the inequality

$$J(I,I') \leq 2\frac{|I'|}{|I''|}J\left(I,I''\right) \tag{2.12}$$

which holds for every I, I' and I'' such that $I' \subset I''$ and $d' = \mathrm{dist}(I,I'') \geq |I'|$ in view of the first mean value theorem.

If $H_z(I_j^{(k)})$ occurs, we have the left k-level pedestal $\Upsilon_{\mathscr{L}}(I_j^{(k)})$ spanning from the left boundary of $I_j^{(k)}$ to the left boundary of $I_z^{(k-1)}$, and the right k-level pedestal $\Upsilon_{\mathscr{R}}(I_j^{(k)})$, spanning from the right boundary of $I_z^{(k-1)}$ to the right boundary $I_j^{(k)}$. Take two segments S_z^L and S_z^R, such that $|S_z^L| = |S_z^R| = \lfloor l_k^{\alpha'/\alpha} \rfloor$, and lying immediately to the left and to the right from $I_z^{(k-1)}$. Denote

$$\begin{aligned}\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}) &= \Upsilon_{\mathscr{L}}(I_j^{(k)}) \cap S_z^L,\\ \widehat{\Upsilon}_{\mathscr{R}}(I_j^{(k)}) &= \Upsilon_{\mathscr{R}}(I_j^{(k)}) \cap S_z^R.\end{aligned}$$

Then

$$\begin{aligned}&P[\text{all edges } \langle x,y\rangle,\ x \in \widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}),\ y \in \widehat{\Upsilon}_{\mathscr{R}}(I_j^{(k)}) \text{ are closed}]\\ &= \prod_{\substack{x\in S_z^L\\ y\in S_z^R}} q_{\langle x,y\rangle} \prod_{\substack{x\in S_z^L\setminus\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)})\\ y\in S_z^R}} q_{\langle x,y\rangle}^{-1} \prod_{\substack{x\in S_z^L\\ y\in S_z^R\setminus\widehat{\Upsilon}_{\mathscr{R}}(I_j^{(k)})}} q_{\langle x,y\rangle}^{-1}.\end{aligned}$$

Applying (2.10) to S_z^L and S_z^R we immediately get that bound for the first term:

$$\prod_{\substack{x\in S_z^L\\ y\in S_z^R}} q_{\langle x,y\rangle} = \exp\Big\{-\sum_{\substack{x\in S_z^L\\ y\in S_z^R}} \frac{\beta}{|x-y|^2}\Big\} \leq l_{k-1}^{-\beta(\alpha'-1)(1-b)}, \tag{2.13}$$

where $b \equiv b(\alpha', l_1)$ and $b \searrow 0$ when $l_1 \nearrow +\infty$. On the other hand denoting by $\mathscr{D}_n(S_z^L \setminus \widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}))$, $0 \leq n \leq k-2$ (resp. $\mathscr{D}_n(S_z^R \setminus \widehat{\Upsilon}_{\mathscr{R}}(I_j^{(k)}))$) the set of vertices which belong to all n-level defects contained in the segment S_z^L (resp. S_z^R), we get

$$\prod_{\substack{x\in S_z^L\setminus\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)})\\ y\in S_z^R}} q_{\langle x,y\rangle} = \prod_{n=0}^{k-2} \prod_{\substack{x\in \mathscr{D}_n(S_z^L\setminus\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}))\\ y\in S_z^R}} q_{\langle x,y\rangle} = \exp\Big\{-\sum_{n=0}^{k-2} \sum_{\substack{x\in \mathscr{D}_n(S_z^L\setminus\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}))\\ y\in S_z^R}} \frac{\beta}{|x-y|^2}\Big\}.$$

Once again, applying (2.10) for each $0 \leq n \leq k-2$ and taking into account the structure of n-level pedestals together with (2.12), we have

$$\begin{aligned}\sum_{n=0}^{k-2} \sum_{\substack{x\in \mathscr{D}_n(S_z^L\setminus\widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}))\\ y\in S_z^R}} \frac{\beta}{|x-y|^2} &\leq C_2 \sum_{n=0}^{k-2}\sum_{\nu} J\left(I'_\nu, I^R\right)\\ &\leq C \sum_{n=0}^{k-2} \frac{l_{n+1}^{\alpha'/\alpha}}{l_{n+1}} \sum_{\nu} J\left(I''_\nu, I^R\right)\\ &\leq C' l_1^{\alpha'/\alpha-1} J\left(I^L, I^R\right)\end{aligned}$$

where $I^{L(R)}$, I'_ν and I''_ν are intervals in $\mathbb{R}$ such that $I^{L(R)} \cap \mathbb{Z} = S_z^{L(R)}$, $\bigcup_\nu (I'_\nu \cap \mathbb{Z}) = \mathscr{D}_n(S_z^L \setminus \widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}))$ and the sum $\sum_\nu$ is taken over all indices ν of $(n+1)$-level blocks $I_\nu^{(n+1)} = I''_\nu \cap \mathbb{Z}$, where n-level defects are located. From this we can easily get that

$$\prod_{\substack{x \in S_z^L \setminus \widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}) \\ y \in S_z^R}} q_{\langle x,y \rangle} \geq l_{k-1}^{-\beta(\alpha'-1)b'}, \tag{2.14}$$

where $b' \equiv b'(\alpha, \alpha', l_1)$ and $b' \searrow 0$ when $l_1 \nearrow +\infty$. Analogous bound holds for the third term. Finally, since

$$\begin{aligned} & [\omega : \exists x \in \Upsilon_{\mathscr{L}}(I_j^{(k)}), y \in \Upsilon_{\mathscr{R}}(I_j^{(k)}),\ y - x \leq \lfloor l_k^{\alpha'/\alpha} \rfloor : \omega_{<x,y>} = 1] \\ \subseteq\ & [\text{all edges } \langle x, y\rangle,\ x \in \widehat{\Upsilon}_{\mathscr{L}}(I_j^{(k)}),\ y \in \widehat{\Upsilon}_{\mathscr{R}}(I_j^{(k)}) \text{ are closed}], \end{aligned} \tag{2.15}$$

the statement of the proposition follows from (2.13) and (2.14). □

Fixing the parameters. For the fixed $\beta > 1$, which is the first main parameter of the model we choose the pair $\alpha,\ \alpha'$ with $1 < \alpha' < \alpha < 2$ such that

$$\beta(\alpha' - 1) - \frac{2(\alpha-1)^2}{2-\alpha} > \alpha - 1, \tag{2.16}$$

i.e. $\beta(\alpha' - 1) > \alpha\,(\alpha - 1)\,/(2 - \alpha)$. Now we fix $l_1 > 1$ so large that the parameter $\eta = \eta(\alpha, \alpha', l_1)$ in (2.9) becomes so close to zero, that

$$\beta(1-\eta)(\alpha' - 1) - \frac{2(\alpha-1)^2}{2-\alpha} > \alpha - 1. \tag{2.17}$$

Finally we fix

$$\delta > \frac{2(\alpha-1)}{2-\alpha} \tag{2.18}$$

such that

$$\beta(1-\eta)(\alpha' - 1) - \delta(\alpha - 1) > \alpha - 1. \tag{2.19}$$

Inequalities (2.9), (2.18) and (2.19) will be crucial for the inductive estimates.

3 Decomposition of the product measure

During this section we will repeatedly use the following identity:
Let $N > 0$, $c_N = 2^{1/N} - 1$, and $\{\gamma_i\}_1^N$ and $\{\Delta_i\}_1^N$ any two sets of random variables. Then

$$\prod_{j=1}^{N}(\gamma_i + \Delta_i) = \sum_{\substack{J \subset \{1,\dots,N\}: \\ J \neq \emptyset}} c_N^{|J|} \left(\prod_{j=1}^{N} \gamma_i + c_N^{-|J|} \prod_{k \in J^c} \gamma_k \prod_{j \in J} \Delta_j \right), \tag{3.1}$$

with

$$\sum_{\substack{J \subset \{1,\dots,N\}: \\ J \neq \emptyset}} c_N^{|J|} = 1, \tag{3.2}$$

and

$$c_N^{-1} \leq N / \ln 2. \tag{3.3}$$

We set $\Lambda_L^k = \{x \in l_k\mathbb{Z} : [x, x + l_k] \cap I(L) \neq \emptyset\}$, and recall that we are working on the space Ω_L^w.

Step 1. By $\mu^w_{L,I^{(0)}}$ we will denote the marginal of μ_L^w with the variables at the nearest neighbor edges which lie in $I(L)$ integrated out. We have:

$$\begin{aligned}
\mu_L^w &= \prod_{z \in I(L)} \left(p\,\mathbb{I}_{G(I_z^{(0)})} + (1-p)\mathbb{I}_{B(I_z^{(0)})} \right) \mu^w_{L,I^{(0)}} \\
&= \prod_{x \in \Lambda_L^1} \Big(\prod_{z \in \widetilde{I}_x^{(1)}} (p\,\mathbb{I}_{G(I_z^{(0)})} + (1-p)\mathbb{I}_{B(I_z^{(0)})}) \Big) \mu^w_{L,I^{(0)}} \\
&\overset{(3.1)}{=} \prod_{x \in \Lambda_L^1} \left(\sum_{\substack{D_x \subseteq \widetilde{I}_x^{(1)} \\ |D_x| \geq 1}} (c_{l_1})^{|D_x|} \Big(p^{l_1} \prod_{z \in \widetilde{I}_x^{(1)}} \mathbb{I}_{G(I_z^{(0)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)}) \prod_{z \in \widetilde{I}_x^{(1)} \setminus D_x} \mathbb{I}_{G(I_z^{(0)})} \prod_{y \in D_x} \mathbb{I}_{B(I_y^{(0)})} \Big) \right) \mu^w_{L,I^{(0)}} \\
&= \sum_{\{D_x\}_{x \in \Lambda_L^1}} (c_{l_1})^{\sum_x |D_x|} \prod_{x \in \Lambda_L^1} \left(p^{l_1} \mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)}) \mathbb{I}_{D_x(\widetilde{I}_x^{(1)})} \right) \mu^w_{L,I^{(0)}}
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})} &= \prod_{z \in \widetilde{I}_x^{(1)} \setminus D_x} \mathbb{I}_{G(I_z^{(0)})} \prod_{y \in D_x} \mathbb{I}_{B(I_y^{(0)})}, \\
\zeta_{D_x}(\widetilde{I}_x^{(1)}) &= c_{l_1}^{-|D_x|} p^{l_1 - |D_x|} (1-p)^{|D_x|},
\end{aligned}$$

the sum $\sum_{\{D_x\}_{x\in\Lambda_L^1}}$ after the last equality is taken over all collections of defects D_x, such that $|D_x|\ge 1$, and

$$\sum_{\{D_x\}_{x\in\Lambda_L^1}} (c_{l_1})^{\sum_x |D_x|} = 1.$$

Notation. We should call attention of the reader to the fact that D_x stand for the set of indices of blocks which will be chosen as defected inside $\widetilde{I}_x$, while $D_x(\widetilde{I}_x)$ stays for the corresponding event. This will be used for all levels.

Now we choose p such that:

$$p \ge 1-\kappa^{-2}(\ln 2)\, l_1^{-\delta-1}. \tag{3.5}$$

This immediately implies that

$$\zeta_{D_x}(\widetilde{I}_x^{(1)}) \le (l_1/\ln 2)^{|D_x|}(\kappa^{-2}(\ln 2)\, l_1^{-\delta-1})^{|D_x|} = (\kappa^{-2} l_1^{-\delta})^{|D_x|}. \tag{3.6}$$

Remark. The appearance of a factor κ^{-2} in (3.5), thus providing the estimate (3.6), plays an important role in "suppressing" factor $\kappa^{\mathscr{C}_L(\omega)}$ in the definition of the finite volume FK measure. Observe, that if $\omega^{(0)}(L)$ denotes restriction of ω to only nearest neighbor edges in L, and

$$\mathscr{C}_L^{(0)}(\omega) \;:=\; \text{number of open connected components in } L \text{ under } \omega^{(0)}(L),$$

then, using the fact that our system is one-dimensional, we get:

$$\mathscr{C}_L(\omega) \le \mathscr{C}_L^{(0)}(\omega) \le 2\sum_{x\in\Lambda_L^1} |D_x|. \tag{3.7}$$

We will return again to this point after (3.33).

Next, we modify each term

$$\prod_{x\in\Lambda_L^1}\left(p^{l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\right) \tag{3.8}$$

of the above sum (3.4) following the strategy of decomposition of events given in the preceding section. Namely, given the collection $\{D_x\}_{x\in\Lambda_L^1}$ of indices where defected 0-level blocks are located, we can define

$$\begin{aligned} D_x(\widetilde{I}_x^{(1)}) \;&=\; \bigcap_{z\in\widetilde{I}_x^{(1)}\setminus D_x} G(I_z^{(0)}) \cap \bigcap_{y\in D_x} B(I_y^{(0)}), \\ H_v(\widetilde{I}_x^{(1)}) \;&=\; \bigcap_{z\in\widetilde{I}_x^{(1)}\setminus v} G(I_z^{(0)}) \cap B(I_v^{(0)}) \end{aligned}$$

Thus $D_x(\widetilde{I}_x^{(1)}) = H_v(\widetilde{I}_x^{(1)})$ if $D_x=\{v\}$, and $D_x(\widetilde{I}_x^{(1)}) = B(\widetilde{I}_x^{(1)})$ if $|D_x|>1$. The collection $\{D_x\}_{x\in\Lambda_L^1}$ uniquely defines the location of 1-blocks of the first step of the itinerary within the interval $I(l_{M_L})$.

We recall that during the arrangement procedure the initial blocks may remain unchanged, or they may have their boundaries being shifted, or several blocks (at most three) may "merge" into one. In the next paragraphs we will explain how with each step of the arrangement procedure to associate a modification of the product (3.8), in such a way, that it leads to the expression (3.12). (See also comments below (3.12).)

Modifications of factors in (3.8):

i) The initial 1-level block $\widetilde{I}_x^{(1)}$ is **not modified**, i.e. remains unchanged after all arrangements are completed, and after re-indexing of the final partition it will be indexed as $I_s^{(1)}$, for some $s > 0$. In this case the corresponding factor $p^{l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}$ in the product (3.8) remains unchanged, only re-written as $p^{l_1}\mathbb{I}_{G(I_s^{(1)})} + \zeta_{D_s}(I_s^{(1)})\mathbb{I}_{D_s(I_s^{(1)})}$.

ii) Boundary shift case. We start with the case when only one boundary of $\widetilde{I}_x^{(1)}$ is shifted. Assume that $|D_x| = 1$, the defected 0-level block is too close to its right boundary, and $\widetilde{I}_{x+1}^{(1)}$ has no defects close to its left boundary, so that the shift of boundary between $\widetilde{I}_x^{(1)}$ and $\widetilde{I}_{x+1}^{(1)}$ is permitted (see "case 2" of 1-level adjustment in Sect. 2). This implies that:

$$
\begin{aligned}
&\big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + \zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})\mathbb{I}_{D(\widetilde{I}_{x+1}^{(1)})}\big)\\
=\;& p^{\lfloor l_1^{\alpha'/\alpha}\rfloor}\prod_{z=(x+1)l_1}^{(x+1)l_1+\lfloor l_1^{\alpha'/\alpha}\rfloor}\mathbb{I}_{G(I_z^{(0)})}\big(p^{l_1-\lfloor l_1^{\alpha'/\alpha}\rfloor}\mathbb{I}_{G(I_{x+1}'^{(1)})} + \zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})p^{-\lfloor l_1^{\alpha'/\alpha}\rfloor}\mathbb{I}_{D_{x+1}(I_{x+1}'^{(1)})}\big),
\end{aligned}
$$

where $I_{x+1}'^{(1)} = I_{x+1}^{(1)} \setminus [(x+1)l_1,\, (x+1)l_1 + \lfloor l_1^{\alpha'/\alpha}\rfloor]$, and we rewrite two factors corresponding to these neighboring blocks $\widetilde{I}_x^{(1)}$ and $\widetilde{I}_{x+1}^{(1)}$ with altered boundary in the following way:

$$
\begin{aligned}
&\big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\big)\big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + \zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})\mathbb{I}_{D_{x+1}(\widetilde{I}_{x+1}^{(1)})}\big)\\
=\;&\big(p^{l_1+\lfloor l_1^{\alpha'/\alpha}\rfloor}\mathbb{I}_{G(I_x''^{(1)})} + \zeta_{D_x''}(I_x''^{(1)})\mathbb{I}_{D_x''(I_x''^{(1)})}\big)\big(p^{l_1-\lfloor l_1^{\alpha'/\alpha}\rfloor}\mathbb{I}_{G(I_{x+1}'^{(1)})} + \zeta_{D_{x+1}'}(I_{x+1}'^{(1)})\mathbb{I}_{D_{x+1}'(I_{x+1}'^{(1)})}\big)\\
=\;&\big(p^{|I_x''^{(1)}|}\mathbb{I}_{G(I_x''^{(1)})} + \zeta_{D_x''}(I_x''^{(1)})\mathbb{I}_{D_x''(I_x''^{(1)})}\big)\big(p^{|I_{x+1}'^{(1)}|}\mathbb{I}_{G(I_{x+1}'^{(1)})} + \zeta_{D_{x+1}'}(I_{x+1}'^{(1)})\mathbb{I}_{D_{x+1}'(I_{x+1}'^{(1)})}\big),
\end{aligned}
\tag{3.9}
$$

where $I_x''^{(1)} = I_x^{(1)} \cup [(x+1)l_1,\, (x+1)l_1 + \lfloor l_1^{\alpha'/\alpha}\rfloor]$, and

$$
\zeta_{D_x''}(I_x''^{(1)}) = p^{\lfloor l_1^{\alpha'/\alpha}\rfloor}\zeta_{D_x}(\widetilde{I}_x^{(1)}), \qquad \zeta_{D_{x+1}'}(I_{x+1}'^{(1)}) = p^{-\lfloor l_1^{\alpha'/\alpha}\rfloor}\zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)}).
$$

Finally we replace the two factors in (3.8) corresponding to the neighboring blocks $\widetilde{I}_x^{(1)}$ and $\widetilde{I}_{x+1}^{(1)}$ (first line of (3.9)) by the product of modified factors (the last line of (3.9)).

If the resulting blocks $I_x''^{(1)}$ and $I_{x+1}'^{(1)}$ undergo further adjustments, we keep modifying the corresponding factors, taking into account all previously performed modifications.

iii) Two blocks merge case. If $|D_x| = 1$ but D_{x+1} is such that the intervals $\widetilde{I}_x^{(1)}$ and $\widetilde{I}_{x+1}^{(1)}$ have to be taken as one (see "case 1" of 1-level adjustment in Sect. 2), we modify the product of corresponding factors in the following way:

$$
\begin{aligned}
&\big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\big)\big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + \zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})\mathbb{I}_{D_{x+1}(\widetilde{I}_{x+1}^{(1)})}\big)\\
= \;& c_2\Big(p^{2l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + c_2^{-1}p^{l_1}\zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})}\Big)\\
+ \;& c_2\Big(p^{2l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + c_2^{-1}p^{l_1}\zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})\mathbb{I}_{G(\widetilde{I}_x^{(1)})}\mathbb{I}_{D_{x+1}(\widetilde{I}_{x+1}^{(1)})}\Big)\\
+ \;& c_2^2\Big(p^{2l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})}\mathbb{I}_{G(\widetilde{I}_{x+1}^{(1)})} + c_2^{-2}\zeta_{D_x}(\widetilde{I}_x^{(1)})\zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\mathbb{I}_{D_{x+1}(\widetilde{I}_{x+1}^{(1)})}\Big) =
\end{aligned}
$$

$$
\begin{aligned}
= \;& c_2\Big(p^{2l_1}\mathbb{I}_{G(I_x'^{(1)})} + \zeta_{D_x'}(I_x'^{(1)})\mathbb{I}_{D_x'(I_x'^{(1)})}\Big)\\
+ \;& c_2\Big(p^{2l_1}\mathbb{I}_{G(I_x'^{(1)})} + \zeta_{D_x''}(I_x'^{(1)})\mathbb{I}_{D_x''(I_x'^{(1)})}\Big)\\
+ \;& c_2^2\Big(p^{2l_1}\mathbb{I}_{G(I_x'^{(1)})} + \zeta_{D_x'''}(I_x'^{(1)})\mathbb{I}_{D_x'''(I_x'^{(1)})}\Big),
\end{aligned}
\tag{3.10}
$$

where $I_x'^{(1)} = \widetilde{I}_x^{(1)} \cup \widetilde{I}_{x+1}^{(1)}$,

$$
\begin{aligned}
D_x'(I_x'^{(1)}) &= D_x(\widetilde{I}_x^{(1)}) \cap G(\widetilde{I}_{x+1}^{(1)}),\\
D_x''(I_x'^{(1)}) &= G(\widetilde{I}_x^{(1)}) \cap D_{x+1}(\widetilde{I}_{x+1}^{(1)}),\\
D_x'''(I_x'^{(1)}) &= D_x(\widetilde{I}_x^{(1)}) \cap D_{x+1}(\widetilde{I}_{x+1}^{(1)}),
\end{aligned}
$$

and

$$
\zeta_{D_x'} = c_2^{-1}p^{l_1}\zeta_{D_x}(\widetilde{I}_x^{(1)}), \quad \zeta_{D_x''} = c_2^{-1}p^{l_1}\zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)}), \quad \zeta_{D_x'''} = c_2^{-2}\zeta_{D_x}(\widetilde{I}_x^{(1)})\zeta_{D_{x+1}}(\widetilde{I}_{x+1}^{(1)}).
$$

In this case the product of two factors corresponding to blocks $\widetilde{I}_x^{(1)}$ and $\widetilde{I}_{x+1}^{(1)}$ in the product (3.8) will be replaced by a sum of three terms from the last equality in (3.10), and thus, since $2c_2 + c_2^2 = 1$, the whole product in (3.8) will be written as the convex sum of three terms, each of which is of the form of a product, but now with one factor less than the original product in (3.8).

If after this step the adjustment procedure continues, we will continue to modify the already obtained products, corresponding to the step of the itinerary, following the above rules, and taking into account all modifications of the original product (3.8), which were performed up to this last step.

Finally, taking into account (3.9) and (3.10), the last equality in (3.4) can be continued:

$$\sum_{\{D_x\}_{x\in\Lambda_L^1}} (c_{l_1})^{\sum_x |D_x|} \prod_{x\in\Lambda_L^1} \Big(p^{l_1}\mathbb{I}_{G(\widetilde{I}_x^{(1)})} + \zeta_{D_x}(\widetilde{I}_x^{(1)})\mathbb{I}_{D_x(\widetilde{I}_x^{(1)})}\Big)\mu^w_{L,\,I^{(0)}}$$

$$= \sum_{\{D_s\}_s} c(\{D_s\}) \prod_i \Big(p^{|I_i^{(1)}|}\mathbb{I}_{G(I_i^{(1)})} + \zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}\Big)\mu^w_{L,\,I^{(0)}}, \tag{3.11}$$

where the sum $\sum_{\{D_s\}_s}$ is taken over all possible choices of defected sets such that $|D_s| \geq 1$, for all values of s, and the constants $c(\{D_s\})$ are such that $\sum_{\{D_s\}_s} c(\{D_s\}) = 1$. We also define the events G_z^+ and B_z^+ as in (2.4), with the event H replaced by H_z.

Remark. To simplify the notation we will write $\zeta_D(I_i^{(1)})$ and $\mathbb{I}_{D(I_i^{(1)})}$ instead of $\zeta_{D_i}(I_i^{(1)})$ and $\mathbb{I}_{D_i(I_i^{(1)})}$, when it is clear about which collection of defected blocks $\{D_s\}_s$ we are talking about, so that ambiguity does not arise.

For $I_i^{(1)}$ such that $|D_i| = 1,\ D_i = \{z\}$ we define

$$g^+_{D_i} = \sum_{\omega\in G_z^+(I_i^{(1)})} \prod_{b\in\mathbb{B}(I_i^{(1)})} p_b^{\omega_b} q_b^{1-\omega_b}$$

$$d^+_{D_i} = \sum_{\omega\in B_z^+(I_i^{(1)})} \prod_{b\in\mathbb{B}(I_i^{(1)})} p_b^{\omega_b} q_b^{1-\omega_b},$$

and for $I_i^{(1)}$ such that $|D_i| > 1$ we define

$$g^+_{D_i} \equiv 0$$

$$d^+_{D_i} = \sum_{\omega\in\bigcap_{u\in D(I_i^{(1)})} B(I_u^{(0)}),} \ \prod_{b\in\mathbb{B}(I_i^{(1)})} p_b^{\omega_b} q_b^{1-\omega_b}.$$

For $I^{(1)} = \{I_i^{(1)}\}_i$ denoting by $\mu^w_{L,\,I^{(1)}} \equiv \mu^w_{L,\{I_i^{(1)}\}}$ the marginal of μ^w_L, where all variables corresponding to the edges $b \in \mathbb{B}(I_i^{(1)})$ for all i are integrated out, we get:

$$\prod_i \Big(p^{|I_i^{(1)}|}\mathbb{I}_{G(I_i^{(1)})} + \zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}\Big)\mu^w_{L,\,I^{(0)}}$$

$$= \prod_i \Big[\Big(p^{|I_i^{(1)}|}\mathbb{I}_{G(I_i^{(1)})} + \zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}\Big)\Big(g^+_{D_i} + d^+_{D_i}\Big)\Big]\,\mu^w_{L,\{I_i^{(1)}\}}$$

$$= \prod_i \Big(\gamma(I_i^{(1)}) + \Delta(I_i^{(1)})\Big)\mu^w_{L,\{I_i^{(1)}\}},$$

where

$$\gamma(I_i^{(1)}) \;=\; p^{|I_i^{(1)}|}\mathbb{I}_{G(I_i^{(1)})}\Big(g^+_{D_i}+d^+_{D_i}\Big)+\zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}g^+_{D_i},$$

$$\Delta(I_i^{(1)}) \;=\; \zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}d^+_{D_i},$$

and thus we can conclude 1-step of decomposition:

$$\mu^w_L=\sum_{\{D_s\}_s}c(\{D_s\})\prod_i\Big(\gamma(I_i^{(1)})+\Delta(I_i^{(1)})\Big)\mu^w_{L,\{I_i^{(1)}\}}, \tag{3.12}$$

where the sum $\sum_{\{D_s\}_s}$ is taken over all possible choices of defected sets such that $|D_s|\geq 1$, for all values of s, and the constants $c(\{D_s\})$ are such that $\sum_{\{D_s\}_s}c(\{D_s\})=1$. At this point we again emphasize that at higher scales D_i's will represent the indices of defected blocks, and not the blocks themselves.

Comments.

1)

$$\gamma(I_i^{(1)})(\omega)>0\;\Rightarrow\;\Delta(I_i^{(1)})(\omega)=0.$$

$$\Delta(I_i^{(1)})(\omega)>0\;\Rightarrow\;\gamma(I_i^{(1)})(\omega)=0.$$

2) $\gamma(I_i^{(1)})(\omega)>0$ implies the existence of a pedestal connecting the left boundary of $I_i^{(1)}$ to its right boundary.

3) $\Delta(I_i^{(1)})(\omega)>0$ does imply the absence of a pedestal-like oriented open path connecting the left boundary of $I_i^{(1)}$ to its right boundary, though generally it does not imply absence of *some* open oriented path connecting the left boundary of $I_i^{(1)}$ to its right boundary.

Estimates of $\Delta(I_i^{(1)})$. We have

$$\Delta(I_i^{(1)})(\omega)\leq\zeta_{D_i}(I_i^{(1)})d^+_{D_i}(\omega)=\begin{cases}\zeta_{D_i(\omega)}(I_i^{(1)})P(B^+_z(I_i^{(1)})) & \text{if } D=\{z\},\\[2ex] \zeta_{D_i(\omega)}(I_i^{(1)}) & \text{if } |D|>1.\end{cases}$$

From (2.9) we have

$$P(B^+_z(I_i^{(1)}))\leq l_1^{-\beta(\alpha'-1)(1-\eta)},$$

Remark. In order to control estimates on $\mathscr{C}_{I_i^{(k)}}(\omega)$ through the inductive steps, we will use the following bound

$$\mathscr{C}_{I_i^{(k)}}(\omega)\leq\widetilde{\mathscr{C}}_{I_i^{(k)}}(\omega),$$

where $\widetilde{\mathscr{C}}_{I_i^{(k)}}(\omega)$ is the number of disjoint connected open components within $I_i^{(k)}$ created only by nearest neighbor edges in the configuration ω.

In the case $|D|=1$ we get

$$\sum_\omega \kappa^{\mathscr{C}_{I_i^{(1)}}(\omega)}\Delta(I_i^{(1)})\mathbb{I}_{|D|=1}(\omega) \le \kappa^{\mathscr{C}_{I_i^{(1)}}(\omega)}(\kappa^{-2}l_1^{-\delta})^{|D|}l_1^{-\beta(\alpha'-1)(1-\eta)} \le l_1^{-\delta-\beta(\alpha'-1)(1-\eta)},$$

and in the case $|D|>1$ we get

$$\sum_\omega \kappa^{\mathscr{C}_{I_i^{(1)}}(\omega)}\Delta(I_i^{(1)})\mathbb{I}_{|D|>1}(\omega) \le l_1^{-|D|\delta} \le l_1^{-2\delta}.$$

On the other hand we have trivial, but still useful bound

$$\sum_\omega \left[\zeta_{D_i}(I_i^{(1)})\mathbb{I}_{D_i(I_i^{(1)})}g^+_{D_i}\right](\omega) \le \kappa^{-1}.$$

This completes the first step.

To proceed forward we will need the following definition

$$k^* = \inf\{k:\ p^{l_k-(2\lfloor l_k^{\alpha'/\alpha}\rfloor+6l_{k-1})} \le \kappa^{-1}\}. \tag{3.13}$$

k-th step. Assume that we have decomposition of μ_L up to level $k-1$, i.e.

$$\mu_L^w = \sum_{I_*^{(k-1)}\in\mathscr{I}^{k-1}} c(\ I^{(k-1)}) \prod_{I_i^{(k-1)}\in\ I^{k-1}} \Big(\gamma(I_i^{(k-1)})+\Delta(I_i^{(k-1)})\Big)\mu_{L,\ I_*^{(k-1)}}, \tag{3.14}$$

where:

• the sum $\sum_{I_*^{(k-1)}\in\mathscr{I}^{k-1}}$ is taken over all possible triples $I_*^{(k-1)} = \big(I^{(k-1)}, J^{(k-1)}, D^{(k-1)}\big)$ formed by the partition $I^{(k-1)}$, which can be obtained as the $k-1$-st step of the itinerary, the itinerary $J^{(k-1)}$, along which $I^{(k-1)}$ was obtained, and finally, the collection of indices $D^{(k-1)} = \{D_i^{(k-1)}\}_i$, which assigns where the $(k-2)$-level defects are located in each block $I_j^{(k-1)}$ of $I^{(k-1)}$;

• $\sum_{I^{(k-1)}\in\mathscr{I}^{k-1}} c(\ I^{(k-1)}) = 1$;

• Functions $\gamma(I_i^{(k-1)}) \equiv \gamma(I_i^{(k-1)}, D_i^{(k-1)})$ and $\Delta(I_i^{(k-1)}) \equiv \Delta(I_i^{(k-1)}, D_i^{(k-1)})$ satisfy the following properties:

1)

$$\gamma(I_i^{(k-1)})(\omega) > 0 \ \Rightarrow\ \Delta(I_i^{(k-1)})(\omega) = 0,$$

$$\Delta(I_i^{(k-1)})(\omega) > 0 \ \Rightarrow\ \gamma(I_i^{(k-1)})(\omega) = 0,$$

2) $\gamma(I_i^{(k-1)})(\omega) > 0$ implies existence of the $(k-1)$-level pedestal, connecting left boundary of $I_i^{(k-1)}$ to its right boundary,

3) $\Delta(I_i^{(k-1)})(\omega) > 0$ does imply absence of pedestal-like oriented open path connecting left boundary of $I_i^{(k-1)}$ to its right boundary. If $\Delta(I_i^{(k-1)})(\omega) > 0$, we say that $I_i^{(k-1)}$ is defected for ω;

4)

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_i^{(k-1)}}(\omega)} \Delta(I_i^{(k-1)})(\omega) \leq l_{k-1}^{-\delta}; \tag{3.15}$$

5) Estimates on $\gamma(I_i^{(k-1)})$. The pedestal constructing algorithm described in Sect. 2 implies that each term $\gamma(I_i^{(k-1)})$ will have the following structure:

$$\gamma(I_i^{(k-1)}) = \sum_{\varphi=0}^{k-1} \gamma_\varphi(I_i^{(k-1)}),$$

where φ is the highest level of the defects present in $\gamma_\varphi(I_i^{(k-1)})$. Speaking informally, this is so, because in our approach to construct pedestals connecting the left boundary of $I_i^{(k-1)}$ to its right boundary, we do it in two possible ways: or we take pedestals in each sub-block $I_j^{(k-2)}$, $I_j^{(k-2)} \subset I_i^{(k-1)}$ and take union of them over all such j, or we take pedestals in all sub-blocks $I_j^{(k-2)} \subset I_i^{(k-1)}$, but one, which is defected, and "fix" this defect by finding an open edge of an appropriate length, which connects the left and the right pedestals of $I_i^{(k-1)}$ (see the definition of pedestals in Sect. 3). The first way to obtain pedestals contributes to $\gamma(I_i^{(k-1)})$ with a term which is product of $\gamma(I_j^{(k-2)})$, and the second way contributes to $\gamma(I_i^{(k-1)})$ with a term which is product of $\gamma(I_j^{(k-2)})$ over blocks which are not defected times the indicator of the event that there exists an open edge with certain properties, connecting left and right pedestals.

Each term $\gamma_\varphi(I_i^{(k-1)})$ itself can be represented as

$$\gamma_\varphi(I_i^{(k-1)}) = \sum_{\phi} \gamma_{\varphi,\phi}(I_i^{(k-1)}),$$

where, for the fixed φ, the sum $\sum_\phi$ is taken over all non-empty sets ϕ of indices ν of $(\varphi+1)$-level blocks $I_\nu^{(\varphi+1)}$, where φ-level defects are located, and which are compatible with the fixed itinerary. Thus, slightly abusing notations, we can write:

$$\gamma(I_i^{(k-1)}) = \sum_{\varphi=0}^{k-1} \sum_{\phi} \gamma_{\varphi,\phi}(I_i^{(k-1)}). \tag{3.16}$$

Estimates. If $k \leq k^*$, then for each $\varphi \geq 1$ and any ϕ we have:

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_i^{(k-1)}}(\omega)} \gamma_{\varphi,\phi}(I_i^{(k-1)})(\omega) \leq 1. \tag{3.17}$$

5b) If $k-1 \geq k^*$ the estimate (3.17) holds for all $\varphi \geq 0$ and any ϕ;

- The measure $\mu_{L,\ I_*^{(k-1)}} \equiv \mu_{L, I^{(k-1)}, J^{(k-1)}}$ is the marginal of $\mu_{L,\ I_*^{(k-2)}}$, where all variables corresponding to the edges $b \in \mathbb{B}(I_i^{(k-1)})$ for all i, are integrated out.

Remark. It is very important to keep in mind that the functions $\gamma(I_i^{(k-1)}) \equiv \gamma(I_i^{(k-1)}, D_i^{(k-1)})$ and $\Delta(I_i^{(k-1)}) \equiv \Delta(I_i^{(k-1)}, D_i^{(k-1)})$ dependent on $D_i^{(k-1)}$. To make formulas lighter, we omit them from the notation.

The inductive step. We proceed as follows: For each fixed triple $I_*^{(k-1)} = \big(I^{(k-1)}, J^{(k-1)}, D^{(k-1)}\big)$ we take the corresponding product

$$\prod_{I_i^{(k-1)} \in\ I^{(k-1)}} \Big(\gamma(I_i^{(k-1)}) + \Delta(I_i^{(k-1)})\Big)\mu_{L,\ I_*^{(k-1)}} \tag{3.18}$$

in (3.14). Our goal now is to re-write (3.18) as a convex sum of product-type expressions, where the sum will be taken over all possible choices of $I^{(k)}$ admissible from $I^{(k-1)}$ as the k-th step of the itinerary $J^{(k-1)}$, and products will be taken over all k-blocks from $I^{(k)}$. Moreover, in each term of this sum the measure $\mu_{L,\ I_*^{(k-1)}}$ will be re-written as $\mu_{L,\ I_*^{(k)}} \equiv \mu_{L, I^{(k)}, J^{(k)}}$, which is the marginal of $\mu_{L,\ I_*^{(k-1)}}$, where all variables corresponding to the edges $b \in \cup_i \mathbb{B}(I_i^{(k)})$, are integrated out and incorporated into the product in an appropriate way.

We begin by fixing $I^{(k-1)} \in \mathscr{I}^{k-1}$, and writing:[2]

$$\begin{aligned}
\prod_{I_i^{(k-1)} \in I^{(k-1)}} \Big(\gamma(I_i^{(k-1)}) + \Delta(I_i^{(k-1)})\Big) &= \prod_{x_i^{(k)}} \prod_{s=j_i^k+1}^{j_{i+1}^k} \Big(\gamma(I_s^{(k-1)}) + \Delta(I_s^{(k-1)})\Big) \\
&= \prod_{x_i^{(k)}} \Big(\prod_{s=j_i^k+1}^{j_{i+1}^k} \gamma(I_s^{(k-1)}) + \sum_{\substack{D_i \subseteq [j_i^k+1, j_{i+1}^k] \\ |D_i| \geq 1}} \Big[\prod_{s' \in [j_i^k+1, j_{i+1}^k] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big] \Big) \\
&= \prod_{x_i^{(k)}} \sum_{\substack{D_i \subseteq [j_i^k+1, j_{i+1}^k] \\ |D_i| \geq 1}} c_{j_{i+1}^k - j_i^k - 1}^{|D_i|} \Big(\prod_{s=j_i^k+1}^{j_{i+1}^k} \gamma(I_s^{(k-1)}) + c_{j_{i+1}^k - j_i^k - 1}^{-|D_i|} \prod_{s' \in [j_i^k+1, j_{i+1}^k] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big) \\
&= \sum_{\{D_i\}_i} c\left(\{D_i\}_i\right) \prod_{x_i^{(k)}} \Big(\prod_{s=j_i^k+1}^{j_{i+1}^k} \gamma(I_s^{(k-1)}) + c_{j_{i+1}^k - j_i^k - 1}^{-|D_i|} \prod_{s' \in [j_i^k+1, j_{i+1}^k] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big)
\end{aligned} \tag{3.19}$$

where the sum in the right hand side of the last equality is taken over all possible collections $\{D_i\}_i$ of non-empty sets of indices $D_i \subseteq [j_i^k+1, j_{i+1}^k]$, $|D_i| \geq 1$, which are the indices of the defected $(k-1)$-level blocks in the segment $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)}$, $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)} \cap I(l_{M_L}) \neq \emptyset$, and $c\left(\{D_i\}_i\right) = \prod_i c_{j_{i+1}^k - j_i^k - 1}^{|D_i|}$ is such that

$$\sum_{\{D_i\}_i} c\left(\{D_i\}_i\right) = 1.$$

[2]For simplicity, we write j_i^k instead of $\hat{j}_i^k$ (see construction of k-blocks on Sect. 3, for definition).

Re-arrangement of factors at step k.

i) Boundary shift case. Once $\{D_i\}_i$ is fixed, we select those indices i for which $|D_i| = 1,\ D_i = \{u\}$ and $u \leq j^k_{x^k_i + \lfloor l_k^{\alpha'/\alpha} \rfloor}$ or $u \geq j^k_{x^k_{i+1} - \lfloor l_k^{\alpha'/\alpha} \rfloor}$. If $u \geq j^k_{x^k_{i+1} - \lfloor l_k^{\alpha'/\alpha} \rfloor}$ (resp. $u \leq j^k_{x^k_i + \lfloor l_k^{\alpha'/\alpha} \rfloor}$) and $D_{i+1} \cap [j^k_{x^k_{i+1}}, j^k_{x^k_{i+1}+2\lfloor l_k^{\alpha'/\alpha} \rfloor}] = \emptyset$ (resp. $D_{i-1} \cap [j^k_{x^k_i - 2\lfloor l_k^{\alpha'/\alpha} \rfloor}, j^k_{x^k_i}] = \emptyset$), then

$$\begin{aligned}
&\Big(\prod_{s=j^k_i+1}^{j^k_{i+1}} \gamma(I_s^{(k-1)}) + c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s' \in [j^k_i+1, j^k_{i+1}] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big) \\
\times\ &\Big(\prod_{s=j^k_{i+1}+1}^{j^k_{i+2}} \gamma(I_s^{(k-1)}) + c^{-|D_{i+1}|}_{j^k_{i+2}-j^k_{i+1}-1} \prod_{s' \in [j^k_{i+1}+1, j^k_{i+2}] \setminus D_{i+1}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_{i+1}} \Delta(I_{s''}^{(k-1)}) \Big) \\
=\ &\Big(\prod_{s=j^k_i+1}^{j^k_{x^k_{i+1}+\lfloor l_k^{\alpha'/\alpha} \rfloor}} \gamma(I_s^{(k-1)}) + c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s' \in [j^k_i+1, j^k_{x^k_{i+1}+\lfloor l_k^{\alpha'/\alpha} \rfloor}] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big) \\
\times\ &\Big(\prod_{s=j^k_{x^k_{i+1}+\lfloor l_k^{\alpha'/\alpha} \rfloor}+1}^{j^k_{i+2}} \gamma(I_s^{(k-1)}) + c^{-|D_{i+1}|}_{j^k_{i+2}-j^k_{i+1}-1} \prod_{s' \in [j^k_{x^k_{i+1}+\lfloor l_k^{\alpha'/\alpha} \rfloor}+1, j^k_{i+2}] \setminus D_{i+1}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_{i+1}} \Delta(I_{s''}^{(k-1)}) \Big)
\end{aligned} \tag{3.20}$$

and respectively

$$\begin{aligned}
&\Big(\prod_{s=j^k_{i-1}+1}^{j^k_i} \gamma(I_s^{(k-1)}) + c^{-|D_{i-1}|}_{j^k_i-j^k_{i-1}-1} \prod_{s' \in [j^k_{i-1}+1, j^k_i] \setminus D_{i-1}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_{i-1}} \Delta(I_{s''}^{(k-1)}) \Big) \\
\times\ &\Big(\prod_{s=j^k_i+1}^{j^k_{i+1}} \gamma(I_s^{(k-1)}) + c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s' \in [j^k_i+1, j^k_{i+1}] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big) = \\
=\ &\Big(\prod_{s=j^k_{i-1}+1}^{j^k_{x^k_i-\lfloor l_k^{\alpha'/\alpha} \rfloor}} \gamma(I_s^{(k-1)}) + c^{-|D_{i-1}|}_{j^k_i-j^k_{i-1}-1} \prod_{s' \in [j^k_{i-1}+1, j^k_{x^k_i-\lfloor l_k^{\alpha'/\alpha} \rfloor}] \setminus D_{i-1}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_{i-1}} \Delta(I_{s''}^{(k-1)}) \Big) \\
\times\ &\Big(\prod_{s=j^k_{x^k_i-\lfloor l_k^{\alpha'/\alpha} \rfloor}+1}^{j^k_{i+1}} \gamma(I_s^{(k-1)}) + c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s' \in [j^k_{x^k_i-\lfloor l_k^{\alpha'/\alpha} \rfloor}+1, j^k_{i+1}] \setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_i} \Delta(I_{s''}^{(k-1)}) \Big)
\end{aligned} \tag{3.21}$$

ii) Two blocks merge case. For those indices i such that $|D_i| = 1,\ D_i = \{u\},\ u \geq j^k_{x^k_{i+1}-\lfloor l_k^{\alpha'/\alpha}\rfloor}$ and $D_{i+1} \cap [j^k_{x^k_{i+1}}, j^k_{x^k_{i+1}+2\lfloor l_k^{\alpha'/\alpha}\rfloor}] \neq \emptyset$ we write

$$
\begin{aligned}
&\Big(\prod_{s=j_i^k+1}^{j_{i+1}^k} \gamma(I_s^{(k-1)}) + c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s'\in[j_i^k+1,j_{i+1}^k]\setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_i} \Delta(I_{s''}^{(k-1)})\Big) \\
&\times \Big(\prod_{s=j_{i+1}^k+1}^{j_{i+2}^k} \gamma(I_s^{(k-1)}) + c^{-|D_{i+1}|}_{j^k_{i+2}-j^k_{i+1}-1} \prod_{s'\in[j_{i+1}^k+1,j_{i+2}^k]\setminus D_{i+1}} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_{i+1}} \Delta(I_{s''}^{(k-1)})\Big) \\
&= c_2\Big(\prod_{s=j_i^k+1}^{j_{i+2}^k} \gamma(I_s^{(k-1)}) + c_2^{-1} c^{-|D_{i+1}|}_{j^k_{i+2}-j^k_{i+1}-1} \prod_{s'\in[j_i^k+1,j_{i+1}^k]\setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_i} \Delta(I_{s''}^{(k-1)}) \prod_{s=j_{i+1}^k+1}^{j_{i+2}^k} \gamma(I_s^{(k-1)})\Big) \\
&+ c_2\Big(\prod_{s=j_i^k+1}^{j_{i+2}^k} \gamma(I_s^{(k-1)}) + c_2^{-1} c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} \prod_{s=j_i^k+1}^{j_{i+1}^k} \gamma(I_s^{(k-1)}) \prod_{s'\in[j_{i+1}^k+1,j_{i+2}^k]\setminus D_{i+1}} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_{i+1}} \Delta(I_{s''}^{(k-1)})\Big) \\
&+ c_2^2\Big(\prod_{s=j_i^k+1}^{j_{i+2}^k} \gamma(I_s^{(k-1)}) + c_2^{-2} c^{-|D_i|}_{j^k_{i+1}-j^k_i-1} c^{-|D_{i+1}|}_{j^k_{i+2}-j^k_{i+1}-1} \times \\
&\times \prod_{s'\in[j_i^k+1,j_{i+1}^k]\setminus D_i} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_i} \Delta(I_{s''}^{(k-1)}) \prod_{s'\in[j_{i+1}^k+1,j_{i+2}^k]\setminus D_{i+1}} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_{i+1}} \Delta(I_{s''}^{(k-1)})\Big) \qquad (3.22)
\end{aligned}
$$

iii) No modification case. If during the adjustment procedure the segment $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)}$, for some i, remains unchanged, then the corresponding term in (3.19) is not modified.

As long as the adjustment procedure calls on the rearrangement of blocks we continue to make modifications of the products following the rules above, and taking into account all modifications which were performed up to this current step.

When all modifications are completed, and taking into account (3.20), (3.21) and (3.22), we can write the r.h.s. of the last equality in (3.19):

$$
\sum_{(I^k,\{D_r\}_r)} \prod_{I_r^{(k)}} c(I_r^{(k)}) \prod_{I_r^{(k)}} \Big(\prod_{s:I_s^{(k-1)}\subset I_r^{(k)}} \gamma(I_s^{(k-1)}) + \widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \prod_{\substack{s:I_s^{(k-1)}\subset I_r^{(k)} \\ s\notin D_r}} \gamma(I_{s'}^{(k-1)}) \prod_{s''\in D_r} \Delta(I_{s''}^{(k-1)})\Big) \tag{3.23}
$$

where:

- the sum $\sum_{(I^k,\{D_r\}_r)}$ is taken over all possible choices of pairs $(I^k, \{D_r\}_r)$ such that

1) $I^{(k)} \equiv I^{(k)}(\{D_i\}_i) = \{I_r^{(k)}\}_r$ is admissible from $I^{(k-1)}$, which is fixed (see before (3.19)), and assuming that the set of indices of defected $(k-1)$-level blocks in $I^{(k-1)}$ equals to $\cup_i D_i$;

2) $\{D_r\}_r$ is the collection of sets of indices, where D_r is the set of indices of defected $(k-1)$-level blocks within $I_r^{(k)}$;

- the constants $\prod_{I_r^{(k)}} c(I_r^{(k)})$ depend on the all itinerary $J_r^{(k)}$, and

$$\sum_{(I^k,\{D_r\}_r)} \prod_{I_r^{(k)}} c(I_r^{(k)}) = 1;$$

- $\widehat{\zeta}_{D_r}^{-1}(I_r^{(k)})$ satisfies the following bound:

$$\widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \leq c_2^{-4} c_{Cl_{k-1}^{\alpha-1}}^{-|D_r|} \leq \frac{16}{\ln^4 2}\left(\frac{C\lfloor l_{k-1}^{\alpha-1}\rfloor}{\ln 2}\right)^{|D_r|}. \tag{3.24}$$

To show (3.24) observe, that making all modifications of products, the largest term appears when three adjacent intervals merge (recall, that no more than three can merge), and in the corresponding modification we take the "worst" term, which corresponds to the last term in (3.22). Assume that merge $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)}$, $\bigcup_{s=j_{i+1}^k+1}^{j_{i+2}^k} I_s^{(k-1)}$, and $\bigcup_{s=j_{i+2}^k+1}^{j_{i+3}^k} I_s^{(k-1)}$. Applying repeatedly (3.22) first to the terms corresponding to $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)}$ and $\bigcup_{s=j_{i+1}^k+1}^{j_{i+2}^k} I_s^{(k-1)}$, and then to $\bigcup_{s=j_i^k+1}^{j_{i+1}^k} I_s^{(k-1)} \cup \bigcup_{s=j_{i+1}^k+1}^{j_{i+2}^k} I_s^{(k-1)}$ and $\bigcup_{s=j_{i+2}^k+1}^{j_{i+3}^k} I_s^{(k-1)}$, we get that that the largest corresponding constant is

$$c_2^{-4} c_{j_{i+1}^k - j_i^k - 1}^{-|D_i|} c_{j_{i+2}^k - j_{i+1}^k - 1}^{-|D_{i+1}|} c_{j_{i+3}^k - j_{i+2}^k - 1}^{-|D_{i+3}|}. \tag{3.25}$$

On the other hand the estimate (2.6) gives the uniform bound:

$$j_{i+1}^k - j_i^k - 1 \leq \frac{3l_k + 6l_{k-1}}{l_{k-1} - (2\lfloor l_{k-1}^{\alpha'/\alpha}\rfloor + 6l_{k-2})}, \tag{3.26}$$

which, together with (3.25) implies (3.24).

To continue, for $I_i^{(k)}$ such that $|D(I_i^{(k)})| = 1,\ D(I_i^{(1)}) = \{z\}$ we define

$$\begin{aligned} g_{D_i}^+ &= \sum_{\omega\in G_z^+(I_i^{(k)})} \prod_{b\in\mathbb{B}(I_i^{(k)})} p_b^{\omega_b} q_b^{1-\omega_b}, \\ d_{D_i}^+ &= \sum_{\omega\in B_z^+(I_i^{(k)})} \prod_{b\in\mathbb{B}(I_i^{(k)})} p_b^{\omega_b} q_b^{1-\omega_b}, \end{aligned}$$

and for $I_i^{(k)}$ such that $|D(I_i^{(k)})| > 1$ we define

$$\begin{aligned} g_{D_i}^+ &\equiv 0, \\ d_{D_i}^+ &= \sum_{\omega\in\cap_{u\in D(I_i^{(k)})} B(I_u^{(k-1)})} \prod_{b\in\mathbb{B}(I_i^{(k)})} p_b^{\omega_b} q_b^{1-\omega_b}. \end{aligned}$$

For $I^{(k-1)}$ fixed and $I^{(k)} = \{I_r^{(k)}\}_k$, obtained from $I^{(k-1)}$, with the collection of defect indices $D \equiv D^{(k-1)} = \{D_r^{(k-1)}\}_r = \{D_r\}_r$ we write

$$\prod_{I_r^{(k)}} \Big(\prod_{s: I_s^{(k-1)} \subset I_r^{(k)}} \gamma(I_s^{(k-1)}) + \widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \prod_{\substack{s: I_s^{(k-1)} \subset I_r^{(k)} \\ s \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_r(I_r^{(k)})} \Delta(I_{s''}^{(k-1)}) \Big) \mu^w_{L, I_*^{k-1}}$$

$$= \prod_{I_r^{(k)}} \Big(\prod_{s: I_s^{(k-1)} \subset I_r^{(k)}} \gamma(I_s^{(k-1)}) + \widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \prod_{\substack{s: I_s^{(k-1)} \subset I_r^{(k)} \\ s \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_r(I_r^{(k)})} \Delta(I_{s''}^{(k-1)}) \Big) \times$$

$$\times (g^+_{D_r} + d^+_{D_r}) \mu^w_{L, I_*^{(k)}}$$

$$= \prod_{I_r^{(k)}} \Big(\gamma(I_r^{(k)}) + \Delta(I_r^{(k)}) \Big) \mu^w_{L, I_*^{(k)}}, \tag{3.27}$$

where

$$\gamma(I_r^{(k)}) = \prod_{s: I_s^{(k-1)} \subset I_r^{(k)}} \gamma(I_s^{(k-1)}) \Big(g^+_{D_r} + d^+_{D_r} \Big) +$$

$$+ \widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \prod_{\substack{s: I_s^{(k-1)} \subset I_r^{(k)} \\ s \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_r(I_r^{(k)})} \Delta(I_{s''}^{(k-1)}) g^+_{D_r} \tag{3.28}$$

$$\Delta(I_r^{(k)}) = \widehat{\zeta}_{D_r}^{-1}(I_r^{(k)}) \prod \gamma(I_{s'}^{(k-1)}) \prod_{s'' \in D_r(I_r^{(k)})} \Delta(I_{s''}^{(k-1)}) d^+_{D_r}. \tag{3.29}$$

and $\mu_{L, I_*^{(k)}}$, is the marginal of $\mu_{L, I_*^{(k-1)}}$, with all variables corresponding to the edges $b \in \mathbb{B}(I_i^{(k)})$ for all i, integrated out.

Thus, taking (3.27) into account, (3.14) can be continued:

$$\mu^w_L = \sum_{I^{k-1} \in \mathscr{I}^{k-1}} c(\ I^{k-1}) \sum_{\{D_r\}_r} \prod_{I_r^{(k)}} c(I_r^{(k)}) \prod_{I_r^{(k)}} \Big(\gamma(I_r^{(k)}) + \Delta(I_r^{(k)}) \Big) \mu^w_{L, I_*^{(k)}}$$

$$= \sum_{I^{k-1} \in \mathscr{I}^k} c(\ I^k) \prod_{I_r^{(k)} \in \mathscr{I}^k} \Big(\gamma(I_r^{(k)}) + \Delta(I_r^{(k)}) \Big) \mu^w_{L, I_*^{(k)}} \tag{3.30}$$

Estimates of $\Delta(I_i^{(k)})$. First of all we observe, that using (3.24) and (3.29) we have

$$\Delta(I_i^{(k)})(\omega) \le \frac{16}{\ln^4 2} \left(\frac{C \lfloor l_{k-1}^{\alpha-1} \rfloor}{\ln 2} \right)^{|D_r|} \prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)})(\omega) \prod_{s'' \in D_r(I_r^{(k)})} \Delta(I_{s''}^{(k-1)})(\omega) d^+_{D_r}(\omega). \tag{3.31}$$

Case $k > k^*$. Recall, that for the fixed $L > 0$ we need to perform only finitely many steps of the induction, namely up to $k \le \widehat{M}_L$. Using (3.16), the term $\prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)})$ in (3.31) can be written

$$\prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \gamma(I_{s'}^{(k-1)}) = \prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \sum_{\varphi=0}^{k-1} \sum_{\phi} \gamma_{\varphi,\phi}(I_{s'}^{(k-1)}) = \sum\nolimits^{\{(\varphi_{s'},\phi_{s'})\}_{s'}} \prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)}),$$

where the sum $\sum^{\{(\varphi_{s'},\phi_{s'})\}_{s'}}$ is taken over all possible choices of the collections $\{(\varphi_{s'},\phi_{s'})\}_{s': I_{s'}^{(k-1)} \subset I_r^{(k)},\, s' \notin D_r(I_r^{(k)})}$, compatible with the given itinerary.

Next we will estimate

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_r^{(k)}}(\omega)} \Delta(I_i^{(k)})(\omega). \tag{3.32}$$

Using (2.9), (3.15), (3.17) and (3.31), we get that in the case $|D| = 1$ (assume that $D_r(I_r^{(k)}) = \{s''\}$), for any fixed collection $\{(\varphi_{s'},\phi_{s'})\}_{s': I_{s'}^{(k-1)} \subset I_r^{(k)},\, s' \notin D_r(I_r^{(k)})}$ the corresponding term in right hand side of (3.32) is bounded from above by:

$$\begin{aligned}
&\sum_{\omega} \Big[\kappa^{\mathscr{C}_{I_r^{(k)}}(\omega)} \frac{16}{\ln^4 2} \left(\frac{C\lfloor l_{k-1}^{\alpha-1} \rfloor}{\ln 2} \right) \Delta(I_{s''}^{(k-1)})(\omega) d_{D_r}^{+}(\omega) \prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \neq s''}} \gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega) \Big] \\
\le\ & \frac{16C\lfloor l_{k-1}^{\alpha-1} \rfloor}{\ln^5 2} \Big(\sum_{\omega} \kappa^{\mathscr{C}_{I_{s''}^{(k-1)}}(\omega)} \Delta(I_{s''}^{(k-1)})(\omega) d_{D_r}^{+}(\omega) \Big) \prod_{\substack{s': I_{s'}^{(k-1)} \subset I_r^{(k)} \\ s' \notin D_r(I_r^{(k)})}} \Big(\sum_{\omega} \kappa^{\mathscr{C}_{I_{s'}^{(k-1)}}(\omega)} \gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega) \Big) \\
\le\ & l_{k-1}^{-\delta+\alpha-1} l_{k-1}^{-\beta(1-\eta)(\alpha'-1)} \le l_k^{-\delta},
\end{aligned} \tag{3.33}$$

where for the first inequality we used the inequality (3.35) below, and the fact that events $\gamma > 0$ or $\gamma = 0$ and $\Delta > 0$ or $\Delta = 0$ are conditionally independent for distinct blocks, given that their boundary is fixed, for the second inequality we use (2.9), (3.15) and (3.17) and the fact that on the set $\{\omega : \gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega) > 0\}$ the quantity $\mathscr{C}_{I_{s'}^{(k-1)}}(\omega)$ is constant, and for the last inequality we used (2.19).

In the case $|D| > 1$, for any fixed collection $\{(\varphi_{s'},\phi_{s'})\}_{s': I_{s'}^{(k-1)} \subset I_r^{(k)},\, s' \notin D_r(I_r^{(k)})}$ the corresponding term on the right hand side of (3.32) is bounded from above by:

$$\sum_{\omega}\Big[\kappa^{\mathscr{C}_{I_r^{(k)}}(\omega)}\frac{16}{\ln^4 2}\left(\frac{C\lfloor l_{k-1}^{\alpha-1}\rfloor}{\ln 2}\right)^{|D_r|}\prod_{s''\in D_r(I_r^{(k)})}\Delta(I_{s''}^{(k-1)})(\omega)d_{D_r}^{+}(\omega)\prod_{\substack{s':I_{s'}^{(k-1)}\subset I_r^{(k)}\\ s'\notin D_r(I_r^{(k)})}}\gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega)\Big]$$

$$\begin{aligned}
&\le \frac{16}{\ln^4 2}\left(\frac{C\lfloor l_{k-1}^{\alpha-1}\rfloor}{\ln 2}\right)^{|D_r|}\prod_{\substack{s':I_{s'}^{(k-1)}\subset I_r^{(k)}\\ s'\notin D_r(I_r^{(k)})}}\Big(\sum_{\omega}\kappa^{\mathscr{C}_{I_{s'}^{(k-1)}}(\omega)}\gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega)\Big)\\
&\quad\times\prod_{s''\in D_r(I_r^{(k)})}\Big(\sum_{\omega}\kappa^{\mathscr{C}_{I_{s''}^{(k-1)}}(\omega)}\Delta(I_{s''}^{(k-1)})(\omega)\ d_{D_r}^{+}(\omega)\Big)\\
&\le l_{k-1}^{(\alpha-1-\delta)|D_r|}\le l_k^{-\delta}, \qquad (3.34)
\end{aligned}$$

where for the last inequality we used (2.18) and the fact that

$$\prod_{\substack{s':I_{s'}^{(k-1)}\subset I_r^{(k)}\\ s'\notin D_r(I_r^{(k)})}}\kappa^{-\mathscr{C}_{I_{s'}^{(k-1)}}}\prod_{s''\in D_r(I_r^{(k)})}\kappa^{-\mathscr{C}_{I_{s''}^{(k-1)}}}\le\kappa^{-\mathscr{C}_{I_r^{(k)}}} \qquad (3.35)$$

Since the events $\{\omega:\gamma_{\varphi_{s'},\phi_{s'}}(I_{s'}^{(k-1)})(\omega)>0\}$ form a partition of the event $\{\gamma(I_{s'}^{(k-1)})>0\}$ form a partition of the event $\{\gamma(I_{s'}^{(k-1)})>0\}$, from (3.33) and (3.34) we get the first desired estimate:

$$\sum_{\omega}\kappa^{\mathscr{C}_{I_r^{(k)}}(\omega)}\Delta(I_i^{(k)})(\omega)\le l_k^{-\delta}. \qquad (3.36)$$

To get the estimate for $\sum_\omega \kappa^{-\mathscr{C}_{I_r^{(k)}}}\gamma_{\varphi,\phi}(I_r^{(k)})(\omega)$, observe that for the first term in its representation (3.28) the following holds:

$$\sum_{\omega}\Big[\kappa^{-\mathscr{C}_{I_r^{(k)}}}\prod_{s:I_s^{(k-1)}\subset I_r^{(k)}}\gamma_{\varphi_s,\phi_s}(I_s^{(k-1)})\Big(g^{+}_{D_r(I_r^{(k)})}+d^{+}_{D_r(I_r^{(k)})}\Big)\Big]\le 1. \qquad (3.37)$$

In order to estimate the second term

$$\sum_{\omega}\Big[\kappa^{-\mathscr{C}_{I_r^{(k)}}}\widehat{\zeta}_{D_r}^{-1}(I_r^{(k)})\Delta(I_{s''}^{(k-1)})g^{+}_{D_r(I_r^{(k)})}\prod_{\substack{s:I_s^{(k-1)}\subset I_r^{(k)}\\ s\notin D_r(I_r^{(k)})}}\gamma_{\varphi_s,\phi_s}(I_s^{(k-1)})\Big] \qquad (3.38)$$

observe that in this case $D(I_r^{(k)})=\{s''\}$, $I_{s''}^{(k-1)}$ is defected and the event $H_{s''}(I_r^{(k)})$ occurs. This means that for the chosen itinerary during the last adjustment the block $I_{s''}^{(k-1)}$ is not obtained via merging of two (or three) previous level blocks, since this, by the definition, would imply creation of the block on which the event B occurs. This immediately implies $\widehat{\zeta}_D^{-1}(I_r^{(k)})=1$, and therefore, again using (3.15) and (3.17), (3.38) can be upper bounded by 1, and we get the second desired estimate:

$$\sum_{\omega} \kappa^{-\mathscr{C}_{I_i^{(k)}}(\omega)} \gamma_{\varphi,\phi}(I_r^{(k)})(\omega) \leq 1. \tag{3.39}$$

Case $k \leq k^*$. We define

$$\Phi(I_i^{(k)}) = \{s : \gamma_{0,\phi}(I_s^{(k-1)}) > 0\},$$

i.e. $\Phi(I_i^{(k)})$ is the set of indices s of sub-blocks $I_s^{(k-1)}$ such that for given ω all nearest neighbor edges are open in $I_s^{(k-1)}$.

If $\Phi^c = \emptyset$, it means that we are taking the configuration ω, such that all nearest neighbor bonds in $I_i^{(k)}$ are open. Thus the best estimate which we can get in the case $k < k^*$ for $\varphi = 0$ is:

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_i^{(k)}}} \gamma_{0,\phi}(I_i^{(k)})(\omega) < \kappa, \tag{3.40}$$

On the other hand if $k = k^*$, then by definition of k^* (see (3.13)) we have that $p^{I_i^{(k^*)}} < \kappa^{-1}$, and it follows:

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_i^{(k)}}} \gamma_{0,\phi}(I_i^{(k)})(\omega) < 1$$

In this case to get (3.39) for $\varphi > 0$ we follow the same procedure as in the previous case.

If $\Phi^c \neq \emptyset$, we can not use (3.40) to obtain estimates (3.33), (3.34), (3.37) and (3.39). But in this case it is enough to observe, that for blocks $I_s^{(k-1)}$, $s \in \Phi$ trivially holds

$$\sum_{\omega} \gamma_{0,\phi}(I_s^{(k-1)})(\omega) \leq 1, \tag{3.41}$$

and, since these blocks are "entirely connected" by nearest neighbor edges, they do not contribute to the total amount of distinct connected clusters $\mathscr{C}_{I_i^{(k)}}$ in $I_i^{(k)}$ (notice that this is not true in general, but here we are using the fact that blocks which are neighbors to $I_s^{(k-1)}$, $s \in \Phi$ are also having pedestals). So we have:

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_i^{(k)}}} \prod_{\substack{s: I_s^{(k-1)} \subset I_i^{(k)} \\ s \in \Phi}} \gamma_{0,\phi_s}(I_s^{(k-1)})(\omega) \prod_{\substack{s'': I_{s''}^{(k-1)} \subset I_i^{(k)} \\ s'' \in \Phi^c}} \gamma_{\varphi_{s''}\phi_{s''}}(I_{s''}^{(k-1)})(\omega) \leq 1. \tag{3.42}$$

The above inequality follows from (3.41) applied to the first product and the inductive assumption applied to the second product. Using (3.42) the estimates (3.33), (3.34), (3.37) and (3.39) now can be deduced in the same manner as in the case $k > k^*$.

The final step. (3.36) with $k = \widehat{M}_L$ (see (2.2)), gives

$$\sum_{\omega} \kappa^{\mathscr{C}_{I_r^{(\widehat{M}_L)}}(\omega)} \Delta(I_r^{(\widehat{M}_L)})(\omega) \leq l_{\widehat{M}_L}^{-\delta}, \tag{3.43}$$

and (3.30) with $k = \widehat{M}_L$, and (3.43) imply that

$$\mu^w_{\kappa,L}\big((-\infty,-L] \rightsquigarrow [L,+\infty)\big) \geq 1 - l^{-\delta}_{\widehat{M}_L}. \tag{3.44}$$

Since through Sect. 2 and 3 we assumed that

$$\lfloor l^{\alpha'/\alpha}_{\widehat{M}_L} \rfloor + 2 < L, \tag{3.45}$$

we get that there exists an open oriented path π, staring at some vertex $x \in [-l_{\widehat{M}_L}, -L]$ and ending at some vertex $y \in [L, l_{\widehat{M}_L}]$, and which uses open edges of length at most $\lfloor l^{\alpha'/\alpha}_{\widehat{M}_L} \rfloor$.

4 Cluster of the origin

The open oriented path π, described in the last paragraph of the previous section, starting at some vertex $x \in [-l_{\widehat{M}_L}, -L]$ and ending at some vertex $y \in [L, l_{\widehat{M}_L}]$, and which uses open edges of length at most $\lfloor l^{\alpha'/\alpha}_{\widehat{M}_L} \rfloor$ doesn't need to go through the origin 0. We still assume that $\lfloor l^{\alpha'/\alpha}_{\widehat{M}_L} \rfloor + 2 < L$. Next we give the uniform lower bound for the measure of the event that the origin is connected to $[L, +\infty)$.

Cluster of the origin. Recalling that $j^k_z = \min\{j : z \in I^{(k-1)}_j\}$, for any $1 \leq k \leq \widehat{M}_L - 1$ we define the following events:

$$\psi^{(k)} = \bigcap_{i=j^k_0 - \lfloor l_1^{\alpha'/\alpha} \rfloor}^{j^k_0 + \lfloor l_1^{\alpha'/\alpha} \rfloor} G(I^{(k-1)}_i) \tag{4.1}$$

and consider

$$\Psi_L = \bigcap_{j=1}^{\widehat{M}_L - 1} \psi^{(j)}. \tag{4.2}$$

The occurrence of $\bigcap_{s=1}^n \psi^{(s)}$ implies that the origin $\mathbf{0}$ belongs to the pedestal $\Upsilon(I^{(n-1)}_{j_0})$ for any $1 \leq n \leq \widehat{M}_L - 1$, and therefore on the event Ψ_L the origin $\mathbf{0}$ belongs to the open cluster connecting $(-\infty, -L]$ to $[L, +\infty)$.

Finally, using (3.36) and FKG, and under our choice of parameters α, α', δ we get

$$\mu^w_{\kappa,L}(\Psi_L) \geq \prod_{k=1}^{\widehat{M}_L} (1 - l^{-\delta}_{k-1})^{\lfloor l_k^{\alpha'/\alpha} \rfloor} > c(\alpha, \alpha', \delta, l_1) > 0.$$

To complete the proof of the Theorem 1.1 we still need to consider the case $L \leq \lfloor l^{\alpha'/\alpha}_{\widehat{M}_L} \rfloor + 2$. In this case $M_L = \widehat{M}_L - 1$, and first we need to obtain the estimate (3.44) for $L' = l_{M_L}$. Observe that L' satisfies $\lfloor l_{\widehat{M}_{L'}}^{\alpha'/\alpha} \rfloor + 2 < L'$, and we can apply all arguments of previous sections. Once such estimate is obtained, we can repeat the construction of the events as in (4.1) and (4.2), obtaining uniform lower bound in L for the probability of the event that there exists an open oriented path connecting $(-\infty, -L]$ to $[L, +\infty)$. Moreover, due to the construction of the events, such path will be going through the origin. This completes the proof of the Theorem 1.1. $\square$

The proof of the Theorem 1.2 goes through the same lines, and uses arguments very similar to the arguments of the present Section, and therefore we omit it.

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