Recurrence for persistent random walks in two dimensions

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Abstract

We discuss the question of recurrence for persistent, or Newtonian, random walks in $\mathbb{Z}^2$, i.e., random walks whose transition probabilities depend both on the walker's position and incoming direction. We use results by Tóth and Schmidt–Conze to prove recurrence for a large class of such processes, including all "invertible" walks in elliptic random environments. Furthermore, rewriting our Newtonian walks as ordinary random walks in a suitable graph, we gain a better idea of the geometric features of the problem, and obtain further examples of recurrence.

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1 Introduction

A persistent random walk (PRW) in a lattice is a second order Markov chain on that lattice. In other words, if $\{X_n\}_{n\in\mathbb{N}}$ is a realization of the chain (a *walk*), the transition probabilities at time n depend not only on the position X_n, but also on the incoming direction $X_n - X_{n-1}$ (in a sense, the walker's "velocity"). For this reason, one can term this type of random walk 'Newtonian'.

In this note we treat exclusively PRWs in $\mathbb{Z}^2$ with nearest-neighbor transitions, or *jumps*, but we consider both space-homogeneous and inhomogeneous transition probababilities (in jargon, *environments*). In the latter framework, we pay particular attention to random environments (REs).

Our goal is to give general ideas and specific techniques to prove recurrence for these random walks. In particular, using results by Schmidt and Conze [S, C] and by Tóth [T], we establish recurrence for a large class or PRWRE. Also, we present an *ad hoc* construction that maps our nearest-neighbor Newtonian random walks in $\mathbb{Z}^2$ into

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first order random walks in a certain graph Γ. For some examples, this isomorphism easily entails recurrence, or lack thereof. We hope that this construction can be fertile ground for further study.

We lay down the notation: A persistent random walk in $\mathbb{Z}^2$ is a Markov chain P^ω on $\mathbb{Z}^2 \times \Delta$, where $\Delta := \{\pm e_1, \pm e_2\}$ (e_1, e_2 are the generators of $\mathbb{Z}^2$) and ω, the environment, is an element of

$$\mathcal{E} := M_\Delta^{\mathbb{Z}^2} = \{\omega = \{\omega_x\}_{x\in\mathbb{Z}^2} \mid \omega_x \in M_\Delta\} \tag{1.1}$$

Here M_Δ denotes the set of all 4×4 stochastic matrices indexed by the elements of Δ. If $\xi := (x_0, d_0) \in \mathbb{Z}^2 \times \Delta$ is an initial condition, the Markov chain is uniquely determined by the laws

$$P_\xi^\omega\{(X_0, D_0) = (x_0, d_0)\} = 1; \tag{1.2}$$

$$P_\xi^\omega\{(X_{n+1}, D_{n+1}) = (x', d') \,|\, (X_n, D_n) = (x, d)\} = \begin{cases} \omega_x(d, d'), & \text{if } x' = x + d'; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

The environment and the walk are called *homogeneous* if $\omega_x =$ constant.

If a probability Π is defined on $\mathcal{E}$—more precisely on Σ, the infinite-product σ-algebra naturally induced by definition (1.1)—and ω is a random element of $(\mathcal{E}, \Sigma, \Pi)$, then we have a measured family (an *ensemble*) $\{P^\omega\}_{\omega\in\mathcal{E}}$ of PRWs. In this case we speak of 'persistent random walk in a random environment'. As it is physically reasonable, we will always assume Π to be invariant for the natural action of $\mathbb{Z}^2$ on $\mathcal{E}$. In fact, Π is generally taken ergodic for that action; e.g., Π can be the product of $\mathbb{Z}^2$ copies of a probability measure on M_Δ (case of i.i.d. transition probabilities).

Persistent random walks were introduced by Szász and Tóth in the 1980s [SzT, T] as stochastic models for systems of a particle moving chaotically in a ν-dimensional space under the influence of a certain force, for example the gradient of random potential. Specializing to $\nu = 2$ and observing Fig. 1, call $L_x^{(d)}$ the side of the unit square centered in $x \in \mathbb{Z}^2$ that has d as its inner normal. Indicating by $q, p \in \mathbb{R}^2$, respectively, the position and the momentum of the particle, consider the cross section

$$\mathcal{N} = \bigcup_{\substack{x\in\mathbb{Z}^2 \\ d\in\Delta}} \mathcal{N}_x^{(d)} := \bigcup_{\substack{x\in\mathbb{Z}^2 \\ d\in\Delta}} \left\{(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2 \;\middle|\; q \in L_x^{(d)},\ p \cdot d > 0\right\}. \tag{1.4}$$

If $T : \mathcal{N} \longrightarrow \mathcal{N}$ is the first-return map generated by the dynamics of the particle, then, in general,

$$(q, p) \in \mathcal{N}_x^{(d)} \quad \Longrightarrow \quad T(q, p) \in \mathcal{N}_{x+f}^{(f)}, \text{ for some } f \in \Delta \tag{1.5}$$

(excluding of course those orbits that intersect $L_{x+f}^{(f)}$ tangentially or in a corner). If one assumes that, relative to some distribution of initial conditions, the dynamics

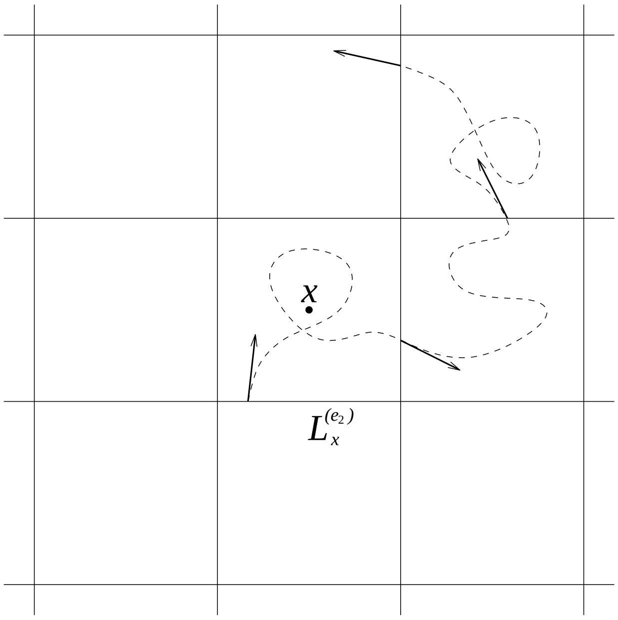

FIGURE 1: Construction of the cross-section $\mathcal{N}$. $L_x^{(e_2)}$ is the bottom side of the unit square centered in $x \in \mathbb{Z}^2$.

inside each unit square is so chaotic that the particle's outgoing side appears to depend only on the incoming side and not on the exact incoming coordinates, then one is approximately justified in replacing a true orbit $\{T^n(q,p)\}$ of the map with a realization $\{(X_n, D_n)\}$ of the random walk.

We call a PRW *isotropic* if, $\forall x \in \mathbb{Z}^2$, $d' \in \Delta$,

$$\sum_{d \in \Delta} \omega_x(d, d') = 1; \tag{1.6}$$

that is, the ω_x are doubly stochastic matrices. In this case, the process can be inverted by means of another PRW. Furthermore, the family $\{P_\xi^\omega\}_{\xi \in \mathbb{Z}^2 \times \Delta}$ is time-invariant in the following sense: For all positive integers k,

$$\begin{aligned} &P_\xi^\omega\{(X_1, D_1) = (x_1, d_1), \dots, (X_n, D_n) = (x_n, d_n)\} = \\ &= \sum_\eta P_\eta^\omega\{(X_k, D_k) = \xi, (X_{k+1}, D_{k+1}) = (x_1, d_1), \dots, (X_{k+n}, D_{k+n}) = (x_n, d_n)\}. \end{aligned} \tag{1.7}$$

This type of situation simulates, for instance, a Hamiltonian system defined by a random potential $V_\omega(q)$ that is zero on every $L_x^{(d)}$ (cf. the random Lorentz gases of [Le]). In this system, if one takes into account the conservation of energy (redefining $\mathcal{N}_x^{(d)}$ in (1.4) with the additional condition $|p|^2 =$ constant), one sees that the Liouville measure induces on $\mathcal{N}$ an invariant measure μ that assigns the same weight to every $\mathcal{N}_x^{(d)}$, whence the term 'isotropic'. Clearly $\mu(\mathcal{N}) = \infty$.

A PRW is said to be *elliptic* if there is an $\varepsilon > 0$ such that, $\forall x, d, d'$,

$$\omega_x(d, d') \geq \varepsilon. \tag{1.8}$$

A PRWRE is elliptic if (1.8) holds with an ε independent of $\omega \in \mathcal{E}$.

The question of recurrence for ordinary (i.e., first order) RWRE is settled in dimension one but open and intensively pursued for $\nu \geq 2$ [Z]. The seminal work of Schmidt [S] and Conze [C] has shown that, in dimension two, recurrence is implied by the Central Limit Theorem (plus ergodicity). This result has been established only in particular cases, the most famous of which are perhaps Lawler's 'balanced' walks [La] (where the transition probabilities in the directions v and $-v$ are the same). We also mention the improvement by Komorowski and Olla [KO] for certain 'two-fold stochastic' walks. It should be noted that all these results require the ellipticity condition, the analog of (1.8) for ordinary RWRE.

This is the plan of the paper: In Section 2 we lay out some basic definitions together with a simple recurrence result for homogeneous environments. In Section 3 we prove that all isotropic elliptic PRWRE (and more) are recurrent. In Section 4 we make a step further by introducing Γ, the dual graph for nearest-neighbor PRWs in $\mathbb{Z}^2$. PRWs in $\mathbb{Z}^2$ become first order RWs in Γ. We see that Γ is not outrageously complicated and, in some cases, recurrence is easy to establish. Finally, in the Appendix, we present a recurrence result for the Manhattan lattice which will be used (and generalized) throughout Section 4.

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2 Preliminary definitions and homogeneous walks

In our Newtonian RW it is necessary to distinguish between *absolute* directions in $\mathbb{Z}^2$ and *relative* directions to the motion of the random walker. We call the former East, North, West, and South, and use the symbols

$$\mathbf{E} := e_1, \qquad \mathbf{N} := e_2, \qquad \mathbf{W} := -e_1, \qquad \mathbf{S} := -e_2. \tag{2.1}$$

For the latter, we use the terms Right, Forward, Left, and Backward. These are functions of a given direction $d \in \Delta$; precisely, if $d^\perp \in \Delta$ is such that $(d, d^\perp)$ has the same orientation as (e_1, e_2), they are defined as

$$\mathbf{R}(d) := -d^\perp, \qquad \mathbf{F}(d) := d, \qquad \mathbf{L}(d) := d^\perp, \qquad \mathbf{B}(d) := -d. \tag{2.2}$$

In ascending order of strength, the following are the properties that we are interested in, for our random walks:

Definition 2.1 *The PRW in the environment $\omega \in \mathcal{E}$ is said to have **asymptotic velocity** $v \in \mathbb{R}^2$ if, $\forall \xi \in \mathbb{Z}^2 \times \Delta$,*

$$\lim_{n\to+\infty} \frac{X_n}{n} = v \qquad P_\xi^\omega\text{-almost surely.}$$

*In case $v \neq 0$, one says that the walk is **ballistic**.*

Definition 2.2 *The PRW in ω is **diffusive** if there exist two constants $C_2 > C_1 > 0$ such that, $\forall \xi \in \mathbb{Z}^2 \times \Delta$,*

$$C_1\, n \leq E_\xi^\omega \left(|X_n|^2 \right) \leq C_2\, n,$$

*where E_ξ^ω is the expectation w.r.t. P_ξ^ω. If, in addition, $X_n/\sqrt{n}$ tends in distribution to a non-degenerate centered Gaussian, then we say that the walk satisfies the **Central Limit Theorem (CLT)** with zero asymptotic velocity.*

Definition 2.3 *The PRW in ω is **recurrent** if, $\forall \xi \in \mathbb{Z}^2 \times \Delta$,*

$$P_\xi^\omega\{(X_n, D_n) = \xi = (X_0, D_0), \text{ for some } n > 0\} = 1.$$

*By the Markov property, this clearly implies that, almost surely, $(X_n, D_n) = (X_0, D_0)$ for infinitely many n. On the other hand, the walk is **transient** if, for every finite $B \subset \mathbb{Z}^2 \times \Delta$, there exists almost surely an $m > 0$ such that $(X_n, D_n) \notin B$ $\forall n \geq m$.*

Remark 2.4 In the case of random environment $(\mathcal{E}, \Sigma, \Pi)$, all these definitions are meant for Π-a.e. $\omega \in \mathcal{E}$.

In the homogeneous case, that is, when ω_x = constant, these properties are rather easily investigated with the methods of finite Markov chains. Nevertheless, for the sake of completeness, we present here a rather general result for that case.

Theorem 2.5 *Assume that a homogeneous PRW, (i.e., $\omega_x = \omega_0$ $\forall x \in \mathbb{Z}^2$) is defined by an irreducible aperiodic stochastic matrix $\omega_0 \in M_\Delta$. A necessary and sufficient condition for the walk to satisfy the CLT with zero asymptotic velocity, and be recurrent, is that*

$$\begin{aligned} \frac{1 - \omega_0(\mathbf{E},\mathbf{E}) - \omega_0(\mathbf{W},\mathbf{E})}{\omega_0(\mathbf{N},\mathbf{E}) + \omega_0(\mathbf{S},\mathbf{E})} &= \frac{1 - \omega_0(\mathbf{E},\mathbf{W}) - \omega_0(\mathbf{W},\mathbf{W})}{\omega_0(\mathbf{N},\mathbf{W}) + \omega_0(\mathbf{S},\mathbf{W})} = \\ = \frac{\omega_0(\mathbf{E},\mathbf{N}) + \omega_0(\mathbf{W},\mathbf{N})}{1 - \omega_0(\mathbf{N},\mathbf{N}) - \omega_0(\mathbf{S},\mathbf{N})} &= \frac{\omega_0(\mathbf{E},\mathbf{S}) + \omega_0(\mathbf{W},\mathbf{S})}{1 - \omega_0(\mathbf{N},\mathbf{S}) - \omega_0(\mathbf{S},\mathbf{S})}. \end{aligned} \tag{2.3}$$

If the walk is isotropic, as per definition (1.6), the above condition is clearly verified.

Proof. As the environment is homogeneous, fix without loss of generality $\xi = (x_0, d_0) = (0, e_1)$. The statistical properties of

$$X_n = \sum_{k=0}^{n-1} D_k \tag{2.4}$$

are simply inferred from the statistical properties of the finite Markov chain $\{D_n\}$, governed by the stochastic matrix ω_0. It is well known that a finite, irreducible, aperiodic Markov chain satisfies the CLT for the number of visits in each state. More precisely, if $\pi = (\pi^{\mathbf{E}}, \pi^{\mathbf{N}}, \pi^{\mathbf{W}}, \pi^{\mathbf{S}})$ is the (unique) stationary vector of ω_0, determined by

$$\sum_{d\in\Delta} \pi^d \, \omega_0(d, d') = \pi^{d'} \qquad \forall d' \in \Delta, \tag{2.5}$$

and $C_n = (C_n^{\mathbf{E}}, C_n^{\mathbf{N}}, C_n^{\mathbf{W}}, C_n^{\mathbf{S}})$ is the so-called *counting vector*, given by

$$C_n^d := \# \{ 0 \le k \le n-1 \mid D_k = d \} \, ; \tag{2.6}$$

then

$$\frac{C_n - n\pi}{\sqrt{n}} \tag{2.7}$$

converges in distribution, as $n \to +\infty$, to a multivariate centered Gaussian [RS]. By linearity, then, $(X_n - nV)/\sqrt{n}$, with

$$V := \pi^{\mathbf{E}} \, \mathbf{E} + \pi^{\mathbf{N}} \, \mathbf{N} + \pi^{\mathbf{W}} \, \mathbf{W} + \pi^{\mathbf{S}} \, \mathbf{S}, \tag{2.8}$$

also converges to a centered Gaussian. Therefore, the RW will satisfy the CLT as per Definition 2.2 if and only if $V = 0$, that is, if and only if $\pi^{\mathbf{E}} = \pi^{\mathbf{W}}$ and $\pi^{\mathbf{N}} = \pi^{\mathbf{S}}$.

Let us prove the sufficiency of condition (2.3). Denote by $\lambda \in [0, +\infty]$ the value of the four equal expressions in (2.3). For each such λ, there is a unique pair $p_1, p_2 \ge 0$ such that

$$p_1 + p_2 = \frac{1}{2}, \qquad \frac{p_2}{p_1} = \lambda. \tag{2.9}$$

Is it easy to check that (2.3) is equivalent to $p := (p_1, p_2, p_1, p_2)$ verifying (2.5), and thus being equal to π. This last sentence also shows the necessity of (2.3) if $V = 0$, that is, if π is of the form (p_1, p_2, p_1, p_2).

As for the recurrence, let us reinterpret our Markov chain as the dynamical system $(\Delta^{\mathbb{N}}, \sigma, P)$, where σ is the left shift on $\Delta^{\mathbb{N}}$ and P is the Markov probability induced on $\Delta^{\mathbb{N}}$ by the stochastic matrix ω_0 and the stochastic vector π. It is well known [RS] that if ω_0 is irreducible and aperiodic (a.k.a. ergodic) then this system is ergodic. Furthermore, by the above, the *cocycle* (2.4) satisfies the CLT with zero mean. Hence, by Conze's theorem on the recurrence of dynamical cocycles [C], $X_n = 0$, P-a.s., for infinitely many n. For at least one (and thus infinitely many) such n, it must also be that $D_n = e_1$ (by the Markov property and the fact that, if $(X_n, D_n) = (0, d)$, there is a positive probability that $(X_k, D_k) = (0, e_1)$ for some $k > n$). Q.E.D.

3 Tóth environments

A large class of PRWRE can be proven recurrent by combining the Schmidt–Conze theorem with a previous result by Tóth [T]. The latter extends Kipnis and Varadhan's proof of the invariant principle [KV] (namely, convergence to Brownian motion in the proper scaling) to isotropic elliptic PRWRE, and others.

To properly state the theorem, we need a couple of definitions: For $y \in \mathbb{Z}^2$ and $\omega = \{\omega_x\}_{x\in\mathbb{Z}^2}$,

$$(\tau_y\,\omega)_x := \omega_{x+y} \tag{3.1}$$

represents the natural action of $\mathbb{Z}^2$ on $\mathcal{E}$. Also, let W denote the transition matrix of the standard random walk, namely,

$$W(d, d') = \frac{1}{4} \qquad \forall d, d' \in \Delta. \tag{3.2}$$

Theorem 3.1 *Assume that* $\{\tau_y\}$ *is an ergodic group of automorphisms on* $(\mathcal{E}, \Sigma, \Pi)$. *Suppose also that,* Π*-a.s.,* ω *is isotropic and,*

$$\forall x \in \mathbb{Z}^2, \qquad \|\omega_x - W\| \le 1 - \varepsilon, \tag{3.3}$$

for some $\varepsilon > 0$ *(*$\|\cdot\|$ *is the standard operator norm for* 4×4 *matrices). Then the associated PRWRE verifies the CLT and is recurrent.*

Proof. First of all, due to the translation-invariance of Π, we can assume that $X_0 = 0$. Secondly, we incorporate the randomness of the environment into a unique Markov chain that describes all walks in all environments. This Markov chain is defined on the state space $\Delta \times \mathcal{E}$ and its realizations are denoted by $\{(D_n, \Omega_n)\}$. It is uniquely determined by the laws

$$\mathbb{P}\{(D_0, \Omega_0) \in A \times B\} = \frac{|A|}{4}\,\Pi(B); \tag{3.4}$$

$$\mathbb{P}\{(D_{n+1}, \Omega_{n+1}) = (d', \omega') \,|\, (D_n, \Omega_n) = (d, \omega)\} = \begin{cases} \omega_0(d, d'), & \text{if } \omega' = \tau_{d'}\,\omega; \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

In (3.4), $|A|$ is the cardinality of $A \subseteq \Delta$, and $B \in \Sigma$. In the theory of RWRE, this process is sometimes called 'the point of view of the particle'.

As in the proof of Theorem 2.5, we exploit the associated dynamical system

$$\left([\Delta \times \mathcal{E}]^{\mathbb{N}}, \bar{\sigma}, \mathbb{P}\right), \tag{3.6}$$

where $\bar{\sigma}$ is the left shift on $[\Delta \times \mathcal{E}]^{\mathbb{N}}$. Using (1.6) and the translation-invariance of Π in (3.5), we see that $(|\cdot| \otimes \Pi)/4$ (where $|\cdot|$ is the counting measure on Δ) is a stationary measure on $\Delta \times \mathcal{E}$. Since this is the same as the initial distribution chosen in (3.4), we conclude that $\mathbb{P}$ is a stationary measure for the entire process, which is equivalent to saying that (3.6) is a measure-preserving (non-invertible) dynamical system.

Remark 3.2 The non-invertibility is merely our choice. Since (1.6) implies that the process can be defined for negative times too, using the transposed environment, one could have introduced an invertible dynamics on $[\Delta \times \mathcal{E}]^{\mathbb{Z}}$, also preserving $\mathbb{P}$. In this case, Schmidt's result on recurrent cocycles [S] would apply as well as Conze's [C].

In [T] it is proved that this dynamical system is ergodic and, more importantly, that the associated RW satisfies the invariance principle. This means that, defining $R_n(0) := 0$ and, for $j = 1, 2, \ldots, n$,

$$R_n\left(\frac{j}{n}\right) := \frac{1}{\sqrt{n}} \sum_{k=0}^{j-1} D_k, \tag{3.7}$$

and extending the definition to all $t \in [0,1]$ by means of linear interpolation, the distribution of the random path $R_n : [0,1] \longrightarrow \mathbb{R}^2$ converges weakly to that of a Brownian motion. That is, if f is a continuous function of the space $\mathcal{C}([0,1];\mathbb{R}^2)$ of all such paths (endowed with the usual weak* topology), then

$$\int_{[\Delta\times\mathcal{E}]^{\mathbb{N}}} f(R_n)\, d\mathbb{P} - \int_{\mathcal{C}([0,1];\mathbb{R}^2)} f\, d\mathcal{W}_S \;\longrightarrow\; 0, \tag{3.8}$$

as $n \to +\infty$. Here $\mathcal{W}_S$ is the 2D Wiener measure with zero drift and variance S (by isotropy S must be a positive multiple of the identity).

It is then a trivial corollary that X_n, which is a cocycle of (3.6) by means of (2.4), satisfies the CLT. This and the ergodicity of (3.6) imply, via [C], that $X_n = 0$ for infinitely many n, $\mathbb{P}$-a.s.

We need to prove that also $(X_n, D_n) = (0, D_0)$, for infinitely many n, $\mathbb{P}$-a.s. For a.e. $\{(D_n, \Omega_n)\} \in [\Delta \times \mathcal{E}]^{\mathbb{N}}$ it makes sense to denote by k the smallest $n > 0$ such that $X_n = 0$, and set

$$\hat{\sigma}\{(D_n, \Omega_n)\} := \bar{\sigma}^k\{(D_n, \Omega_n)\}. \tag{3.9}$$

The map $\hat{\sigma}$ preserves $\mathbb{P}$ because $\bar{\sigma}$ does. Therefore, by the Poincaré Theorem applied to $([\Delta \times \mathcal{E}]^{\mathbb{N}}, \hat{\sigma}, \mathbb{P})$, a point that starts in $\{D_0 = d_0\}$ will return there at infinitely many times k with $X_k = 0$, yielding the sought assertion.

Which, as it is evident, is equivalent to the saying that, for every $\xi = (0, d_0)$ and Π-a.e. $\omega \in \mathcal{E}$, $\{(X_n, D_n)\}$ is recurrent w.r.t. P_ξ^ω. Q.E.D.

The next lemma shows that the strongest requirement in condition (3.3) is not the strictness of the bound, but its uniformity.

Lemma 3.3 *If $Q, Q^T \in M_\Delta$ then $\|Q - W\| \le 1$. The inequality is strict if and only if $Q^T Q$ is irreducible and aperiodic.*

Remark 3.4 The irreducibility and aperiodicity of $Q^T Q$ are implied by a number of conditions given directly on Q:

1. *All the entries of Q are positive* (which, for want of a better term, we may call 'local ellipticity'). As a matter of fact, it is enough that *no column of Q contains more than 1 zero*. This implies that the entries of $Q^T Q$ are positive.

2. *Q is irreducible, aperiodic and normal.* In this case, the entries of $(Q^T Q)^k = (Q^T)^k Q^k$ are positive, for k large enough.

3. *Q is irreducible, aperiodic, and its diagonal entries are positive.* In fact, in this case, Q can be rewritten as $Q = \rho\mathbf{1} + \bar{Q}$, where $\rho > 0$, $\mathbf{1}$ is the identity, and all the entries of $\bar{Q}$ are non-negative. Therefore, the entries of

$$(Q^T Q)^k = (\rho\mathbf{1} + \bar{Q}^T)Q \cdots (\rho\mathbf{1} + \bar{Q}^T)Q = \rho^k Q^k + \cdots \tag{3.10}$$

are positive, for large k (the remainder terms all have non-negative entries).

It would be a mistake, however, to conjecture that for all doubly stochastic, irreducible, aperiodic Q, $\|Q - W\| < 1$. An exception, for example, is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}. \tag{3.11}$$

PROOF OF LEMMA 3.3. We start by noting that W is an orthogonal projection in $\mathbb{C}^4$. The hypothesis that Q is doubly stochastic is equivalent to

$$QW = WQ = W. \tag{3.12}$$

This implies that the eigenspaces of W, namely, $E := \mathrm{span}\{(1,1,1,1)\}$ and $E^\perp$, are invariant w.r.t. Q and Q^T. The action of $Q - W$, relative to the decomposition $\mathbb{C}^4 = E \oplus E^\perp$, is easy to calculate, using that Q is stochastic:

$$Q - W = (Q - 1)|_E \oplus Q|_{E^\perp} = Q|_{E^\perp}. \tag{3.13}$$

Now

$$\|Q|_{E^\perp}\|^2 = \|(Q^T Q)|_{E^\perp}\| \le 1 \tag{3.14}$$

by the Perron-Frobenius Theorem applied to the self-adjoint stochastic matrix $Q^T Q$. By the same theorem, $Q^T Q$ is irreducible and aperiodic if and only if its eigenvalues for the subspace $E^\perp$ have all modulus < 1, i.e., if and only if $\|(Q^T Q)|_{E^\perp}\| < 1$. Q.E.D.

As an illustration of the strength and limitations of Theorem 3.1, let us consider a few examples. Clearly, the theorem applies to all isotropic and elliptic random environments.

Moving on, for all $x \in \mathbb{Z}^2$, produce i.i.d. random vectors $(\zeta_x, \zeta'_x, \zeta''_x)$ with $\zeta_x + \zeta'_x + \zeta''_x = 1$. Assume nothing about the probability law for the stochastic vector, except that its components must be larger than a certain $\varepsilon > 0$. Then set

$$(\forall d \in \Delta) \qquad \omega_x(d, \mathbf{F}) = \zeta_x, \quad \omega_x(d, \mathbf{L}) = \zeta'_x, \quad \omega_x(d, \mathbf{R}) = \zeta''_x, \quad \omega_x(d, \mathbf{B}) = 0. \tag{3.15}$$

where we have abbreviated notation (2.2) as $\mathbf{F} = \mathbf{F}(d)$, $\mathbf{L} = \mathbf{L}(d)$, etc. This construction defines a RE $(\mathcal{E}, \Sigma, \Pi)$ that is evidently isotropic and verifies the hypotheses of Theorem 3.1—condition (3.3) holds via Remark 3.4, pt. 1 (or 3).

The same holds if the roles of $\mathbf{F}$ and $\mathbf{B}$ are swapped in (3.15).

On the other hand, if, Π-a.s., $\omega_x(d, \pm d) = 0 \; \forall x, d$, then Theorem 3.1 does not apply because $\omega_x^T \omega_x$ is reducible and $\|\omega_s - W\| = 1$. We may call any such environment 'Left-Right environment' (more about this in Section 4).

Even conceding on this last example and similar ones, the real shortcomings of Theorem 3.1 are two—or maybe three. First, the uniformity of assumption (3.3), which seems too much of an absolute requirement for a probabilistic problem (on the other hand, in the realm of ordinary RWs in 2D, this is mirrored by the ellipticity condition, which is an hypothesis of all general recurrence results known to this day [Z]). Second, and more important, Theorem 3.1 does not cover non-isotropic REs. Third, and least relevant, it implies nothing about a given, specific, inhomogeneous environment.

In the next section we improve the situation somewhat.

4 The dual graph

Suppose that we want to visualize a persistent RW in $\mathbb{Z}^2$ as a first order RW in $\mathbb{Z}^2 \times \Delta$. We quickly realize that the most immediate mental image of $\mathbb{Z}^2 \times \Delta$, that is, four stacked copies of $\mathbb{Z}^2$, is not really the best to work with because, due to the costraints (1.3), from a given $(x, d) \in \mathbb{Z}^2 \times \Delta$ the walker can only transit to the four other sites $(x + d', d')$. Once we draw, for every x, d, d', an oriented edge from (x, d) to $(x + d', d')$, we are left with a rather labyrinthic picture.

A better choice, it seems, is to consider Γ, the dual of the graph G whose vertices are the points of $\mathbb{Z}^2$ and whose oriented edges are the pairs of nearest-neighbor relations. See Fig. 2 for the construction of Γ.

If we consider the picture of Γ given in Fig. 3, we can make the following observations:

1. The diagonal edges of Γ (those that are followed when the particle goes Left or Right) form two connected subgraphs M_1 and M_2, each isomorphic to the Manhattan lattice $\mathbb{M}$ (see Appendix).

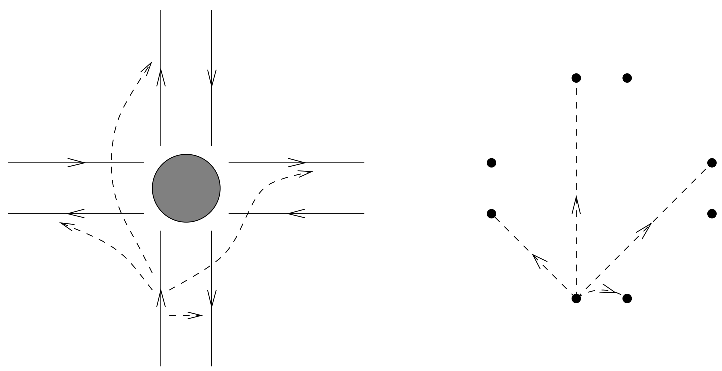

FIGURE 2: Construction of Γ: The nearest-neighbor edges of $\mathbb{Z}^2$ are turned into the vertices of Γ, and the relations of possible consecutiveness between those edges are turned into the (oriented) edges of Γ. Only one elementary building block of Γ is shown here; see Fig. 3 for a wider view.

2. Calling B the set of all short horizontal or vertical edges (corresponding to the particle going Backward) and F the set of all long horizontal or vertical edges (particle going Forward), we see that each edge in $B \cup F$ connects M_1 to M_2 or viceversa. That is, $B \cup F = L_+ \cup L_-$, where L_+ is the set of all edges leading from M_1 to M_2 and L_- is the set of all edges leading from M_2 to M_1.

3. Although Γ is invariant for the action of $\mathbb{Z}^2$, the partition $M_1 \cup M_2 \cup L_+ \cup L_-$ of its edges is only invariant for the action of

$$\mathbb{Z}^2_{\text{even}} := \left\{ (x^1, x^2) \in \mathbb{Z}^2 \;\middle|\; x^1 + x^2 \in 2\mathbb{Z} \right\}. \tag{4.1}$$

   Actually, for $x \in \mathbb{Z}^2_{\text{odd}} := \mathbb{Z}^2 \setminus \mathbb{Z}^2_{\text{even}}$,

$$\upsilon_x(M_1) = M_2, \quad \upsilon_x(M_2) = M_1, \quad \upsilon_x(L_+) = L_-, \quad \upsilon_x(L_-) = L_+, \tag{4.2}$$

   where $\{\upsilon_x\}$ is the natural action of $\mathbb{Z}^2$ on Γ. This shows that $\mathbb{Z}^2_{\text{even}}$ is the most complete symmetry for the problem.

The above considerations suggest that Γ is better viewed as a three-dimensional object in which the two Manhattan lattices lie one on top of the other, with opposite orientations, as in Fig. 4.

One notices that, removing B and F from Γ or, equivalently, setting $\omega_x(d, \pm d) = 0 \;\forall x, d$, splits $\mathbb{Z}^2 \times \Delta$ into two parts that are never connected by an orbit of the RW. These are the Left-Right environments mentioned at the end of Section 3.

If furthermore $\omega_x(d, \pm d^\perp) = 1/2 \;\forall x, d$ (making the environment homogeneous), the walk on either component is isomorphic to a standard walk in the Manhattan lattice (see Appendix). Thus the whole system, which we call 'symmetric Left-Right

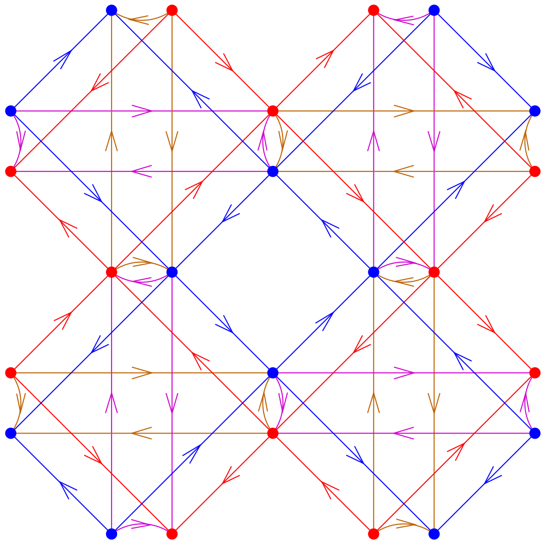

FIGURE 3: A larger portion of Γ. The two embedded Manhattan lattices M_1, M_2 are colored blue and red, respectively. Edges that lead from M_1 to M_2 (that is, from blue to red), denoted L_+ in the text, are colored in magenta; edges that lead from M_2 to M_1 (red to blue), denoted L_-, are colored in brown.

RW', is recurrent. This is a well-known fact and can be proved in several different ways (cf. also [Le]).

One can construct *ad hoc* inhomogeneous PRWs whose recurrence properties are quickly obtained from these elementary results.

4.1 Inhomogeneous Forward probability

Let $\{\zeta_x\}_{x\in\mathbb{Z}^2}$ be a collection of numbers from $[0,1]$. For $x \in \mathbb{Z}^2_{\mathrm{even}}$ define,

$$\begin{array}{llll} (\text{if } d = \mathbf{N}) & \omega_x(d,\mathbf{F}) = \zeta_x, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, & \omega_x(d,\mathbf{B}) = 0; \\ (\text{if } d \neq \mathbf{N}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, & \omega_x(d,\mathbf{B}) = 0. \end{array} \tag{4.3}$$

(Once again, the dependence of $\mathbf{R}, \mathbf{F}, \mathbf{L}, \mathbf{B}$ on d, as per (2.2), has been omitted.) Similarly, for $x \in \mathbb{Z}^2_{\mathrm{odd}}$,

$$\begin{array}{llll} (\text{if } d = \mathbf{S}) & \omega_x(d,\mathbf{F}) = \zeta_x, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, & \omega_x(d,\mathbf{B}) = 0; \\ (\text{if } d \neq \mathbf{S}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, & \omega_x(d,\mathbf{B}) = 0. \end{array} \tag{4.4}$$

In other words, this is a modification of the symmetric Left-Right walk in the following sense: When the particle reaches a point of $\mathbb{Z}^2_{\mathrm{even}}$ from the South, it has some probability to proceed Forward, that is, go North; when it reaches a point of $\mathbb{Z}^2_{\mathrm{odd}}$ from the North, it has some probability to go South.

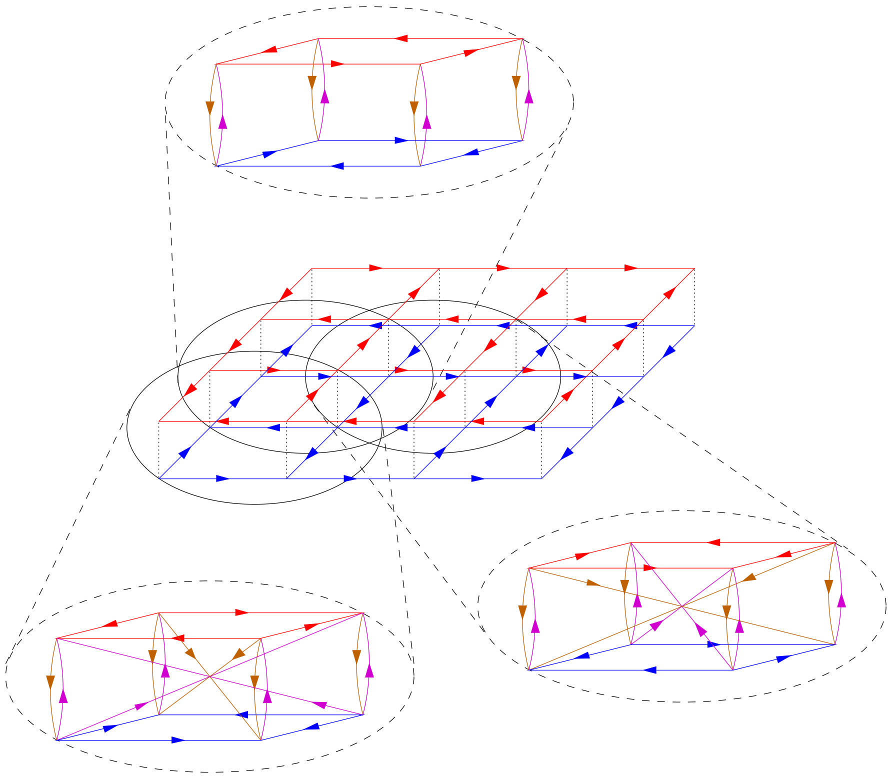

FIGURE 4: Three-dimensional rendering of Γ. Here the graph is embedded in the (z^1, z^2, z^3)-space and its vertices are the points of $\mathbb{Z}^2 \times \{0, 1\}$. M_1 and M_2 lie in the planes $\{z^3 = 0\}$ and $\{z^3 = 1\}$, respectively. Note that all "cubes" do not have the same structure in terms of L_+ and L_- (see blowups). Indeed this structure is $(2\mathbb{Z})^2$-periodic in the displayed rendering (corresponding to $\mathbb{Z}^2_{\text{even}}$-periodicity for the problem).

Notice that these extra Forward displacements need not be *statistically balanced*. This means that, defining the *local drift* as

$$\delta_\omega(x, d) := \sum_{d' \in \Delta} \omega_x(d, d')\, d', \tag{4.5}$$

its $\mathbb{Z}^2$-average

$$\lim_{\Lambda \nearrow Z^2} \frac{1}{4|\Lambda|} \sum_{\substack{x \in \Lambda \\ d \in \Delta}} \delta_\omega(x, d) \tag{4.6}$$

may be different from zero. (Here Λ is, e.g., a square whose side length is going to $+\infty$, and $|\Lambda|$ is the cardinality of Λ). Nonetheless, one can prove the following

Proposition 4.1 *If, for at least one $x \in \mathbb{Z}^2_{\text{even}}$, $\zeta_x > 0$ and, for at least one $y \in \mathbb{Z}^2_{\text{odd}}$, $\zeta_y > 0$, the PRW in the environment defined above is recurrent for all initial conditions.*

Proof. In this proof and for the remainder of the section we use the 3D rendering of Γ shown in Fig. 4. Namely, we think of Γ as embedded in $\mathbb{R}^3$ and having its vertices in the points of $\mathbb{Z}^2 \times \{0, 1\}$. Thus, for example, M_i ($i = 1, 2$) is embedded in $\mathbb{R}^2 \times \{i - 1\}$, L_+ is the set of all edges that "point upwards", B is the set of all vertical edges, and so on. Notice also that the action of v_x, as introduced in (4.2), corresponds to a translation by the vector $(x^1 + x^2, -x^1 + x^2, 0)$, where $(x^1, x^2) = x$.

Studying assumptions (4.3)-(4.4) with an eye on Fig. 4, we see that the only non-horizontal edges that count (i.e., that can be traveled on with positive probability) are those parallel to the vector $(1, 1, 1)$. We call them *effective* edges of $L_+ \cup L_-$. All others are as good as removed.

Denoting by (z^1, z^2, z^3) the coordinate system in $\mathbb{R}^3$, define the function $\psi : \mathbb{Z}^2 \times \{0, 1\} \longrightarrow \mathbb{Z}^2$ as

$$\psi(z^1, z^2, z^3) := (z^1 - z^3, z^2 - z^3). \tag{4.7}$$

If $\{Z_n\}_{n\in\mathbb{N}}$ is a realization of our RW in Γ, with $Z_0 = (0, 0, 0)$, set $Y_n := \psi(Z_n)$. Geometrically this operation amounts to shifting M_2 by the vector $(-1, -1, 0)$ (thus aligning the orientations of M_2 and M_1) and projecting the walk vertically onto M_1. See Fig. 5.

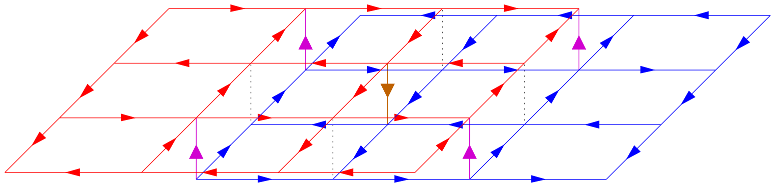

Figure 5: Illustration of the proof of Proposition 4.1.

It is clear that $\{Y_n\}$ can only have same-site or nearest-neighbor jumps, the former case happening when, and only when, Z_n shifts level. Define $n_0 := 0$ and, recursively,

$$n_m := \min \{k > n_{m-1} \mid Y_k \neq Y_{k+1}\}. \tag{4.8}$$

Then set $Y'_m := Y_{n_m}$. This reparametrization has the effect of deleting those times when the Y-walker does not move. Manifestly, $\{Y'_m\}$ is the standard RW in the Manhattan lattice (at each site, the two outgoing edges are equivalent, in terms of probability).

By the result in the Appendix, $\{Y'_m\}$ is recurrent. Therefore, for infinitely many $n > 0$, $Y_n = (0, 0)$, that is, either $Z_n = (0, 0, 0)$ or $Z_n = (1, 1, 1)$. If the second possibility happened all the time, this would be equivalent to infinitely many failed attempts at traveling from $(1, 1, 1)$ to $(0, 0, 0)$. Since, by the Markov property and the hypothesis, these attempts are i.i.d. with positive probability, this situation can only occur with probability zero. Q.E.D.

It is clear from the previous proof that one can conjure up an apparently similar environment with substantially different recurrence properties. If $\{\zeta_x\} \subset [0,1]$ as above, define, for $x \in \mathbb{Z}^2_{\text{even}}$,

$$\begin{array}{llll} (\text{if } d = \mathbf{N}, \mathbf{S}) & \omega_x(d,\mathbf{F}) = \zeta_x, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, & \omega_x(d,\mathbf{B}) = 0; \\ (\text{if } d = \mathbf{W}, \mathbf{E}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, & \omega_x(d,\mathbf{B}) = 0. \end{array} \tag{4.9}$$

Also, for $x \in \mathbb{Z}^2_{\text{odd}}$,

$$(\forall d \in \Delta) \qquad \omega_x(d,\mathbf{F}) = 0, \quad \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, \quad \omega_x(d,\mathbf{B}) = 0. \tag{4.10}$$

Notice that in this case the space average (4.6) of the local drift is zero. Yet the following is true:

Proposition 4.2 *Suppose that $\zeta_x > 0$ for at least one $x \in \mathbb{Z}^2_{\text{even}}$. Then the recurrence of the above-defined random walk is determined by its initial condition $\xi = (x_0, d_0)$. Namely, if*

$$x_0 \in \mathbb{Z}^2_{\text{even}},\ d_0 = \mathbf{E}, \mathbf{W} \qquad \textit{or} \qquad x_0 \in \mathbb{Z}^2_{\text{odd}},\ d_0 = \mathbf{N}, \mathbf{S}$$

the walk is recurrent; otherwise it is non-recurrent. In either case the walk has zero asymptotic velocity.

Proof. Assumptions (4.9)-(4.10) imply that all the effective non-horizontal edges belong to L_+ (cf. Fig. 4). Therefore, if the walk lands in M_2, it cannot go back to M_1. Furthermore, by hypothesis, all walks eventually land in M_2 (because—see the Appendix—the standard RW in M_1 goes everywhere a.s., and thus will keep visiting sites x with $\zeta_x > 0$ until it finally rises to M_2).

The proof is now immediate, as the assertion of Proposition 4.2 reads: If the initial condition is in M_2, the RW is recurrent; if it is in M_1, the walk is not recurrent. The null asymptotic velocity is a consequence of the properties of the stardard RW in M_2. Q.E.D.

4.2 Inhomogeneous Backward probability

The examples presented above are not exceptional, and what can be done with an inhomogeneous Forward probability can be done with an inhomogeneous Backward probability as well.

For instance, take once again a collection $\{\zeta_x\}$ of numbers from $[0,1]$. For $x \in \mathbb{Z}^2_{\text{even}}$, set

$$\begin{array}{llll} (\text{if } d = \mathbf{N}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, & \omega_x(d,\mathbf{B}) = \zeta_x; \\ (\text{if } d \neq \mathbf{N}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, & \omega_x(d,\mathbf{B}) = 0. \end{array} \tag{4.11}$$

For $x \in \mathbb{Z}^2_{\mathrm{odd}}$, set

$$\begin{array}{llll} (\text{if } d = \mathbf{S}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = (1-\zeta_x)/2, & \omega_x(d,\mathbf{B}) = \zeta_x; \\ (\text{if } d \neq \mathbf{S}) & \omega_x(d,\mathbf{F}) = 0, & \omega_x(d,\mathbf{L}) = \omega_x(d,\mathbf{R}) = 1/2, & \omega_x(d,\mathbf{B}) = 0. \end{array} \tag{4.12}$$

As for the model of Proposition 4.1, the local drift need not be statistically balanced.

Proposition 4.3 *Suppose that, for at least one $x \in \mathbb{Z}^2_{\mathrm{even}}$, $\zeta_x > 0$ and, for at least one $y \in \mathbb{Z}^2_{\mathrm{odd}}$, $\zeta_y > 0$. Suppose also that there is no $x \in \mathbb{Z}^2_{\mathrm{even}}$ such that $\zeta_x = \zeta_{x+\mathbf{S}} = 1$. Then the PRW in the environment defined above is recurrent for all initial conditions.*

PROOF. As anticipated, in this proof too we regard Γ as embedded in $\mathbb{R}^3$, with coordinates (z^1, z^2, z^3).

By assumption, in this case, the effective edges of $L_+ \cup L_-$ are those connecting $(z^1, z^2, 0)$ to $(z^1, z^2, 1)$ or viceversa, with $(z^1, z^2) \in (2\mathbb{Z})^2$. Define $\bar\psi : \mathbb{Z}^2 \times \{0,1\} \longrightarrow \mathbb{Z}^2$ as

$$\bar\psi(z^1, z^2, z^3) := (z^1 + z^3, z^2 + z^3). \tag{4.13}$$

In analogy with (4.7), $\bar\psi$ induces on Γ a translation of M_2 by the vector $(1,1,0)$ (again, this aligns the orientations of M_2 and M_1), followed by a vertical projection onto M_1. See Fig. 6.

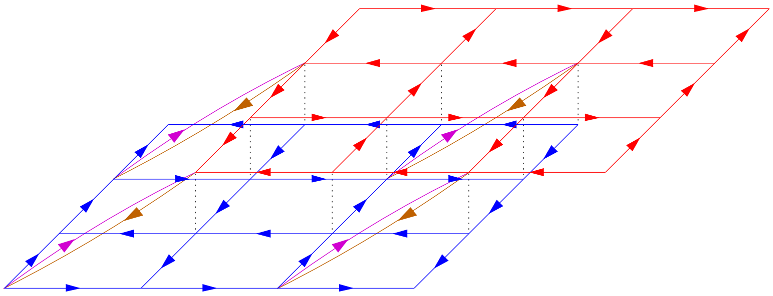

FIGURE 6: Illustration of the proof of Proposition 4.3.

The non-horizontal effective edges are projected into diagonals of those unit squares of $\mathbb{Z}^2$ whose lower left corner belongs to $(2\mathbb{Z})^2$. (These squares correspond, in the original problem, to the sites of $\mathbb{Z}^2_{\mathrm{even}}$; compare Figs. 3 and 4). These are the squares that, in the proof given in the Appendix, are each collapsed to a single point (see Fig. 7).

This suggests the final step of our proof. For $\{Z_n\}$ a realization of our RW and φ as in (A.2), set $Y_n := (\varphi \circ \bar\psi)(Z_n)$. It is clear that $\{Y_n\}$ has many "dead times", i.e., times n at which $Y_n = Y_{n+1}$. The second hypothesis of Proposition 4.3, excluding the possibility that the Z-walker remains trapped in a pair of twin edges of B, implies that all sequences of consecutive dead times are finite with probability 1. Hence we can eliminate them, as in the proof of Proposition 4.1, by introducing $Y'_m := Y_{n_m}$, where the reparametrization is the same as (4.8).

It is very easy to check that $\{Y'_m\}$ is the standard RW in 2D. Its recurrence yields the recurrence of $\{Z_n\}$ by the same argument as in the the proof of Proposition 4.1. Q.E.D.

It is trivial, at this point, to construct an environment with inhomogeneous, statistically balanced, Backward probabilities, for which the conclusions of Proposition 4.2 hold. We will not bore the reader with that.

4.3 Random Left-Right environment

In all the previous examples we have assumed that, at each site, the probability for the walker to go Left or Right was the same. We needed this symmetry to somehow project the given RW into a standard RW in the Mahnattan lattice, or similar. With the next and last example, we go beyond this assumption and show that the same techniques can be pushed further, with the help of some results from the theory of ordinary RWRE in $\mathbb{Z}^2$.

For $0 < \varepsilon < 1/2$ and x ranging in $\mathbb{Z}^2_{\text{even}}$, let (ζ_x, ζ'_x) be i.i.d. random vectors from $[\varepsilon, 1-\varepsilon]^2$, w.r.t. to a probability distribution whose details are of no relevance. To each realization of $\{\zeta_x, \zeta'_x\}$ associate an environment $\omega \in \mathcal{E}$ as follows. For $x \in \mathbb{Z}^2_{\text{even}}$,

$$(\forall d \in \Delta) \qquad \omega_x(d, \mathbf{L}) = \omega_x(d, \mathbf{R}) = 1/2 \tag{4.14}$$

and

$$\begin{array}{ll} \omega_{x+\mathbf{E}}(\mathbf{E}, \mathbf{N}) = \omega_{x+\mathbf{W}}(\mathbf{W}, \mathbf{S}) = \zeta_x, & \omega_{x+\mathbf{E}}(\mathbf{E}, \mathbf{S}) = \omega_{x+\mathbf{W}}(\mathbf{W}, \mathbf{N}) = 1 - \zeta_x; \\ \omega_{x+\mathbf{N}}(\mathbf{N}, \mathbf{E}) = \omega_{x+\mathbf{S}}(\mathbf{S}, \mathbf{W}) = \zeta'_x, & \omega_{x+\mathbf{N}}(\mathbf{N}, \mathbf{W}) = \omega_{x+\mathbf{S}}(\mathbf{S}, \mathbf{E}) = 1 - \zeta'_x. \end{array} \tag{4.15}$$

Finally, for all $x \in \mathbb{Z}^2$ and $d \in \Delta$,

$$\omega_x(d, d) = \omega_x(d, -d) = 0. \tag{4.16}$$

The resulting RE is thus of the Left-Right type and is denoted, as usual, by $(\mathcal{E}, \Sigma, \Pi)$.

Proposition 4.4 *The PRWRE defined above verifies the CLT and is recurrent.*

PROOF. As discussed earlier, in a Left-Right environment the edges of $L_+ \cup L_-$ are not effective and the walk ranges entirely within M_1 or M_2, depending on the initial condition. Upon closer inspection of (4.15), we realize that the transition probabilities for the edges of M_1 are determined solely by $\{\zeta_x\}$, whereas the probabilities for the edges of M_2 are governed by $\{\zeta'_x\}$.

Without loss of generality, assume that the initial condition is $z_0 = (0, 0, 0)$, which lies in M_1. This corresponds, in the original picture, to $(x_0, d_0) = (0, \mathbf{N}) = \xi$.

We use the same technique as in the Appendix to map our PRWRE $\{Z_n\}$ into a known RWRE in $\mathbb{Z}^2$. For

$$\bar{\varphi}(z^1, z^2, z^3) := \left(\left[\frac{z^1}{2} \right], \left[\frac{z^2}{2} \right] \right), \tag{4.17}$$

set $Y_n := \bar{\varphi}(Z_{2n})$. Let us derive the transition probabilities that the laws of $\{Z_n\}$ induce on $\{Y_n\}$. Call y the element of $\mathbb{Z}^2$ associated to $(x, d) \in \mathbb{Z}^2_{\mathrm{even}} \times \{\mathbf{N}, \mathbf{S}\}$ via the correspondence $(x, d) \mapsto z \mapsto y$. With the visual aid of Figs. 7 and 3 we verify that

$$\begin{aligned}
P^\omega_\xi\{Y_{n+1} = y + \mathbf{E} \,|\, Y_n = y\} &= \omega_x(d, \mathbf{E})\, \omega_{x+\mathbf{E}}(\mathbf{E}, \mathbf{N}) \;=\; \zeta_x/2; \\
P^\omega_\xi\{Y_{n+1} = y + \mathbf{N} \,|\, Y_n = y\} &= \omega_x(d, \mathbf{W})\, \omega_{x+\mathbf{W}}(\mathbf{W}, \mathbf{N}) \;=\; (1 - \zeta_x)/2; \\
P^\omega_\xi\{Y_{n+1} = y + \mathbf{W} \,|\, Y_n = y\} &= \omega_x(d, \mathbf{W})\, \omega_{x+\mathbf{W}}(\mathbf{W}, \mathbf{S}) \;=\; \zeta_x/2; \\
P^\omega_\xi\{Y_{n+1} = y + \mathbf{S} \,|\, Y_n = y\} &= \omega_x(d, \mathbf{E})\, \omega_{x+\mathbf{E}}(\mathbf{E}, \mathbf{S}) \;=\; (1 - \zeta_x)/2.
\end{aligned} \tag{4.18}$$

(We restrict our attention to $(x, d) \in \mathbb{Z}^2_{\mathrm{even}} \times \{\mathbf{N}, \mathbf{S}\}$ because, with the chosen initial condition, $(X_{2n}, D_{2n}) \in \mathbb{Z}^2_{\mathrm{even}} \times \{\mathbf{N}, \mathbf{S}\}$ $\forall n > 0$.)

Equations (4.18) show that the environment for the Y-walk is balanced, in the sense of Lawler, and elliptic. It is known that, Π-a.s., this ordinary RW in $\mathbb{Z}^2$ satisfies the CLT and is recurrent [La, Z]. As for for the original walk, the CLT holds because the time-reparametrization between $\{Y_n\}$ and $\{Z_n\}$, or $\{(X_n, D_n)\}$, is simply linear; and the recurrence is given by the same argument used in the last few proofs. Q.E.D.

A Appendix: The Manhattan lattice

The (two-dimensional) *Manhattan lattice* is defined as the oriented graph $\mathbb{M}$ whose vertex set is $\mathbb{Z}^2$ and whose edge set is

$$\begin{aligned}
E_{\mathbb{M}} \;:=\; & \left\{ \left[(x^1, x^2) \,,\, (x^1 + \epsilon(x^2), x^2) \right] \,\middle|\, (x^1, x^2) \in \mathbb{Z}^2 \right\} \cup \\
& \left\{ \left[(x^1, x^2) \,,\, (x^1, x^2 + \epsilon(x^1)) \right] \,\middle|\, (x^1, x^2) \in \mathbb{Z}^2 \right\},
\end{aligned} \tag{A.1}$$

where $\epsilon(j) = 1$ or -1, depending on $j \in \mathbb{Z}$ being even or odd, respectively. See Fig. 7.

The standard random walk in $\mathbb{M}$ is the one for which, at each node $x \in \mathbb{Z}^2$, the random walker has probability $1/2$ of following either outgoing edge based in x. We present a proof by Guillotin [G] that this RW has the same statistical features as the standard RW in 2D—in particular it satisfies the CLT and is recurrent. For $x = (x^1, x^2) \in \mathbb{Z}^2$, define

$$\varphi(x) := \left(\left[\frac{x^1}{2} \right], \left[\frac{x^2}{2} \right] \right), \tag{A.2}$$

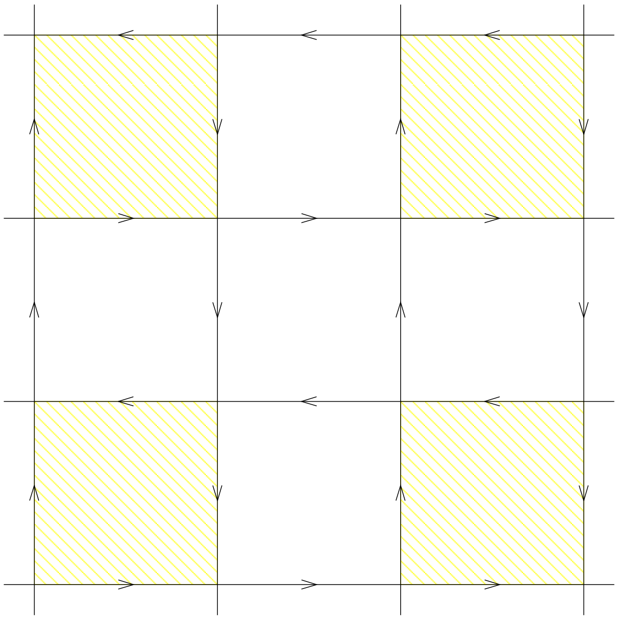

FIGURE 7: The Manhattan lattice $\mathbb{M}$. If the four vertices of each yellow square are identified and time is accelerated by a factor 2, the standard random walk in $\mathbb{M}$ becomes the standard random walk in $\mathbb{Z}^2$.

where $[j]$ is the largest integer less than or equal to $j \in \mathbb{Z}$. If $\{X_n\}_{n\in\mathbb{N}}$ is a realization of the RW in $\mathbb{M}$, define $Y_n := \varphi(X_{2n})$. Denote by Φ the transformation $\{X_n\} \mapsto \{Y_n\}$, whose geometric meaning is explained in the caption of Fig. 7.

Fixing $X_0 = (0,0)$ makes Φ invertible, and a little thought convinces one that the endomorphism of probability spaces that Φ induces between the two RWs in $\mathbb{Z}^2$ is actually an automorphism. At any rate, it is evident that $\{Y_n\}$ is the stardard nearest-neighbor RW in $\mathbb{Z}^2$, which immediately implies the CLT for $\{X_n\}$.

As for the recurrence, one has that, a.s., $Y_n = (0,0)$ for infinitely many n. For those n, either $X_{2n} = (0,0)$ or $X_{2n} = (1,1)$ (*tertium non datur*, as $2n$ is even). By the Markov property, the latter cannot happen all the time, because there is a positive probability of going from $(1,1)$ to $(0,0)$.

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