SUM RULES AND SPECTRAL MEASURES OF SCHRÖDINGER OPERATORS WITH L^2 POTENTIALS

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Abstract. Necessary and sufficient conditions are presented for a positive measure to be the spectral measure of a half-line Schrödinger operator with square integrable potential.

1. Introduction

In this paper, we will discuss which measures occur as the spectral measures for half-line Schrödinger operators with certain decaying potentials. Let us begin with the appropriate definitions.

A potential $V \in L^2_{\text{loc}}(\mathbb{R}^+)$ (where $\mathbb{R}^+ = [0,\infty)$) is said to be limit point at infinity if

$$H = -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

together with a Dirichlet boundary condition at the origin, $u(0) = 0$, defines a self-adjoint operator on $L^2(\mathbb{R}^+)$, without the need for a boundary condition at infinity. This is what we will mean by a Schrödinger operator. (In some sections, we will also treat $V \in L^1_{\text{loc}}$ where we feel that this generality may be of use to others.)

The spectral theory of such operators was first described by Weyl and subsequently refined by many others. We will now sketch the parts of this theory that are required to state our results; fuller treatments can be found elsewhere (e.g., [6, 25, 43]).

The name 'limit point' was coined by Weyl for the following property, which is equivalent to that given above: for all $z \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique function $\psi \in L^2(\mathbb{R}^+)$ so that $-\psi'' + V\psi = z\psi$ and $\psi(0) = 1$. The value of $\psi'(0)$ is denoted $m(z)$ and is termed the (Weyl) m-function. It is an analytic function of z. Of course, by homogeneity, one has that

$$m(z) = \frac{\psi'(0)}{\psi(0)} \tag{1.2}$$

where ψ is *any* L^2 solution of $-\psi'' + V\psi = z\psi$. This will prove the more convenient definition.

Simple Wronskian calculations show that $m(z)$ has a positive imaginary part whenever $\text{Im}(z) > 0$. Therefore, by the Herglotz Representation Theorem, there is

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a unique positive measure $d\rho$ so that

$$\int \frac{d\rho(E)}{1+E^2} < \infty \tag{1.3}$$

and

$$m(z) = \int \left[\frac{1}{E-z} - \frac{E}{1+E^2}\right] d\rho(E) + \operatorname{Re} m(i). \tag{1.4}$$

Uniqueness follows from the fact that

$$d\rho(E) = \operatorname*{w-lim}_{\varepsilon\downarrow 0} \tfrac{1}{\pi} \operatorname{Im} m(E+i\varepsilon)\, dE. \tag{1.5}$$

Moreover, the boundary values $m(E+i0)$ exist almost everywhere and are equal to the Radon-Nikodym derivative, $\frac{d\mu}{dE}$.

At first sight, (1.4) does not permit us to recover $\operatorname{Im} m$ from $d\rho$ without first knowing $\operatorname{Re} m(i)$. Actually, it can be recovered from the asymptotic [1, 15]

$$m(z) = \sqrt{-z} + o(1) \tag{1.6}$$

that holds as $|z| \to \infty$ along rays at a small angle to the negative real axis. (If the support of $d\rho$ is bounded from below, it holds as $z \to -\infty$.)

Just as V determines $d\rho$, so ρ (or $m(z)$) determines V. This is a famous result of Gel'fand–Levitan [14, 22, 23]; see also Remling [28] and Simon [33].

As we have described, each potential gives rise to a spectral measure, which also determines V. Our main goal in this paper is to give necessary and sufficient conditions in terms of ρ for $V \in L^2(\mathbb{R}^+)$. Our model here is a result we proved recently [19] for Jacobi matrices, the discrete analog of Schrödinger operators. (Other precursors will be discussed later). To properly frame our result, we recall briefly the Jacobi matrix result. A Jacobi matrix is a semi-infinite tridiagonal matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \\ 0 & a_2 & b_3 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} \tag{1.7}$$

viewed as an operator on $\ell^2(\mathbb{Z}_+) = \ell^2(\{1,2,3,\dots\})$. The spectral measure here is defined by

$$m(E) = \langle \delta_1, (J-E)^{-1}\delta_1\rangle = \int \frac{d\mu(x)}{x-E}. \tag{1.8}$$

We write J_0 for the Jacobi matrix with $b_n \equiv 0$, $a_n \equiv 1$.

Our earlier result is:

Theorem 1.1 ([19]). *$J - J_0$ is Hilbert-Schmidt, that is,*

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty, \tag{1.9}$$

if and only if the spectral measure $d\mu$ obeys

(i) (Blumenthal–Weyl) $\operatorname{supp}(d\mu) = [-2,2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$ *with* $E_1^+ > E_2^+ > \cdots > 2$ *and* $E_1^- < E_2^- < \cdots < -2$ *with* $\lim_{j\to\infty} E_j^\pm = \pm 2$ *if* $N_\pm = \infty$.

(ii) (Normalization) *μ is a probability measure.*

(iii) (Lieb–Thirring Bound)

$$\sum_{\pm,j}(|E_j^{\pm}|-2)^{3/2}<\infty \tag{1.10}$$

(iv) (Quasi-Szegő Condition) *Let* $d\mu_{\rm ac}(E)=f(E)\,dE$. *Then*

$$\int_{-2}^{2}\log(f(E))\sqrt{4-E^2}\,dE>-\infty \tag{1.11}$$

We have changed the ordering of the conditions relative to [19] in order to facilitate comparison with Theorem 1.2 below.

To state the result for Schrödinger operators, we need some further preliminaries. Let $d\rho_0$ be the free spectral measure (i.e., for $V=0$), it is

$$d\rho_0(E)=\pi^{-1}\chi_{[0,\infty)}(E)\sqrt{E}\,dE. \tag{1.12}$$

For any positive measure ρ, we define a signed measure $d\nu$ on $(1,\infty)$ by

$$\frac{2}{\pi}\int f(k^2)k\,d\nu(k)=\int f(E)[d\rho(E)-d\rho_0(E)]. \tag{1.13}$$

Notice that $d\nu$ is parameterized by momentum, k, rather than energy, $E=k^2$. This is actually the natural independent variable for what follows (in [19] we used z defined by $E=z+z^{-1}$). We will write w for the m-function in terms of k:

$$w(k)=m(k^2). \tag{1.14}$$

With this notation,

$$\frac{d\nu}{dk}=\mathrm{Im}[w(k+i0)]-k \tag{1.15}$$

at a.e. point $k\in(1,\infty)$. Here $\frac{d\nu}{dk}$ is the Radon-Nikodym derivative of the a.c. part of ν, which hay have a singular part as well.

We will eventually prove (see Section 9) that if $V\in L^2$, then

$$\|\nu\|^2_{\ell^2(M)}:=\sum\big[|\nu|([n,n+1])\big]^2<\infty \tag{1.16}$$

and as a partial converse, if $d\rho$ is supported in $[-a,\infty)$ and (1.16) holds, then $d\rho$ is the spectral measure for a potential $V\in L^2_{\rm loc}$.

We also need to introduce the long- and short-range parts of the Hardy–Littlewood maximal function [16, 30],

$$(M\nu)(x)=\sup_{L>0}\frac{1}{2L}\,|\nu|([x-L,x+L]) \tag{1.17}$$

$$(M_s\nu)(x)=\sup_{0<L\le 1}\frac{1}{2L}\,|\nu|([x-L,x+L]) \tag{1.18}$$

$$(M_l\nu)(x)=\sup_{1\le L}\frac{1}{2L}\,|\nu|([x-L,x+L]). \tag{1.19}$$

The main theorem of this paper is

Theorem 1.2. *A positive measure* $d\rho$ *on* $\mathbb{R}$ *is the spectral measure associated to a* $V\in L^2(\mathbb{R}^+)$ *if and only if*

(i) (Weyl) $\mathrm{supp}(d\rho)=[0,\infty)\cup\{E_j\}_{j=1}^N$ *with* $E_1<E_2<\cdots<0$ *and* $E_j\to 0$ *if* $N=\infty$.

(ii) (Normalization)

$$\int \log\biggl[1+\biggl(\frac{M_s\nu(k)}{k}\biggr)^2\biggr] k^2\,dk<\infty \tag{1.20}$$

(iii) (Lieb–Thirring)

$$\sum_j |E_j|^{3/2}<\infty \tag{1.21}$$

(iv) (Quasi-Szegő)

$$\int_0^\infty \log\biggl[\frac{1}{4}\,\frac{d\rho}{d\rho_0}+\frac{1}{2}+\frac{1}{4}\,\frac{d\rho_0}{d\rho}\biggr]\sqrt{E}\,dE<\infty \tag{1.22}$$

Remarks. 1. It may be surprising that we have replaced the innocuous normalization condition in the Jacobi case by (1.20). The reason is the following: $\mu(\mathbb{R})=1$ in the Jacobi case is the condition that μ is the spectral measure of some Jacobi matrix. In this theorem, we do not presume a priori that $d\rho$ is spectral measure. We will eventually see that (1.20) implies that ρ is the spectral measure of an L^2_{loc} potential. Indeed, (1.20) has additional information that we will need to control high energy pieces.

2. The name of condition (i) was chosen because the fact that it is implied by $V\in L^2$ is an immediate consequence of Weyl's Theorem on the invariance of the essential spectrum under (relatively) compact perturbations.

3. Bounds on sums of powers of eigenvalues in terms of the L^p norm of the potential are usually referred to as Lieb–Thirring Inequalities in deference to their exhaustive work on this question, [24]. However, the particular case that appears in Theorem 1.2 was first observed by Gardner, Greene, Kruskal, and Miura; see [13, p. 115].

4. The argument of log in (1.22) has the form

$$\tfrac14\,\lambda+\tfrac12+\tfrac14\,\lambda^{-1}=[\tfrac12\,(\lambda+\lambda^{-1})]^2\geq 1 \tag{1.23}$$

so the integrand is nonnegative. This is significantly different from (1.11) where the integrand can have both signs, and the finiteness of the measure implies one sign is automatically finite so we do not have to worry about oscillations. In our case, an oscillating integrand would present severe difficulties because spectral measures are not finite.

5. Theorem 1.2 implies that if $V\in L^2$, then $\sigma_{\text{ac}}(H)=[0,\infty)$. This is a result of Deift–Killip [8].

We will prove Theorem 1.2 in two parts. First, we prove an equivalence of $V\in L^2$ and a set of conditions that has an unsatisfactory element. Then we will show the conditions of Theorem 1.2 are equivalent to those of this Theorem 1.3. Both portions are lengthy.

The intermediate theorem requires one further object. Let

$$F(q)=\pi^{-1/2}\int_{p\geq 1} p^{-1}e^{-(q-p)^2}\,d\nu(p). \tag{1.24}$$

By (1.3), this integral is absolutely convergent.

Theorem 1.3. *A positive measure $d\rho$ on $\mathbb{R}$ is the spectral measure associated to a $V\in L^2(\mathbb{R}^+)$ if and only if*

(i) (Weyl) $\operatorname{supp}(d\rho) = [0,\infty) \cup \{E_j\}_{j=1}^N$ *with* $E_1 < E_2 < \cdots < 0$ *and* $E_j \to 0$ *if* $N = \infty$.

(ii) (Local Solubility)

$$\int_0^\infty |F(q)|^2\, dq < \infty \tag{1.25}$$

(iii) (Lieb–Thirring)

$$\sum_j |E_j|^{3/2} < \infty \tag{1.26}$$

(iv) (Strong Quasi-Szegő)

$$\int \log\biggl[\frac{|w(k+i0)+ik|^2}{4k\operatorname{Im} w(k+i0)}\biggr] k^2\, dk < \infty \tag{1.27}$$

Remarks. 1. Notice that

$$\begin{aligned}
|w(k+i0)+ik|^2 &\geq |\operatorname{Im} w(k+i0)+k|^2 \\
&\geq |\operatorname{Im} w(k+i0)+k|^2 - |\operatorname{Im} w(k+i0)-k|^2 \\
&= 4k\operatorname{Im} w(k+i0)
\end{aligned}$$

so the argument in the log in (1.27) is at least 1 and the integrand is strictly positive, so the integral either converges or is $+\infty$.

2. There is a significant difference between (1.27) and (1.11). Since (1.27) involves M and not just $\operatorname{Im} w$, the singular part of $d\rho$ enters in both (ii) and (iv). Still, as we shall see, the restriction on $d\rho_{\mathrm{s}}$ is mild.

3. The occurrence of w in (1.27) means that if one starts with $d\rho$, it is difficult to check this condition—one first has to calculate the Hilbert transform (conjugate function) of $d\rho$. Considering the example

$$d\rho = d\rho_0 + \sum_{j=1}^\infty c_j \delta(E-j^2) dE$$

shows that (1.25) is not strong enough to allow the replacement of (1.27) by the weaker Quasi-Szegő condition (1.22). Specifically, (1.25) only requires $c_j \in \ell^2$ while (1.27) implies $c_j \in \ell^1$. The relation of the two Quasi-Szegő conditions and their connection to $\operatorname{Re} w(k)$ is discussed in Section 8.

The advantage of the maximal function is that it involves no cancellation; we see plainly that (1.20) is a statement about the size of $d\rho - d\rho_0$.

4. The name 'Local Solubility' comes from the fact that this condition (plus the fact that support of $d\rho$ is bounded from below) guarantees that $d\rho$ is the spectral measure for some L^2_{loc} potential. See Section 6.

5. We will prove Theorem 1.3 in Section 7, and then use it to prove Theorem 1.2 in Sections 8–11.

There are significant differences from Theorem 1.1, both in the form of Theorem 1.3 and its proof. Understanding the difficulties that led to these differences is illuminating. To understand the issues, we recall that Theorem 1.1 was proven by showing a general sum rule, dubbed the P_2 sum rule:

$$Q(d\mu) + \sum_{\pm,j} F(E_j^\pm) = \tfrac14 \sum_j b_j^2 + \tfrac12 \sum_j G(a_j) \tag{1.28}$$

where

$$Q(d\mu) = \frac{1}{4\pi}\int_{-2}^{2} \log\biggl(\frac{\sqrt{4-E^2}}{2\operatorname{Im} m(E+i0)}\biggr)\sqrt{4-E^2}\,dE \tag{1.29}$$

$$G(a) = a^2 - 1 - \log|a|^2 \tag{1.30}$$

$$F(\beta+\beta^{-1}) = \tfrac{1}{4}\,[\beta^2 - \beta^{-2} - \log\beta^4], \quad |\beta|>1. \tag{1.31}$$

To prove Theorem 1.1, one proves (1.28) is always true (both sides may be infinite) and then notes $Q(d\mu)<\infty$ if and only if (1.11) holds, $F(E_j^\pm) = (|E_j^\pm|-2)^{3/2} + (O(|E_j^\pm|-2)^2)$, and $G(a) = 2(a-1)^2 + O((a-1)^3)$, so $\sum_{\pm,j} F(E_j^\pm)<\infty$ if and only if (1.10) holds and the right side of (1.9) is finite if and only if (1.7) holds.

For nice J's, (e.g., $b_n=0$ and $a_n=1$ for n large), (1.28) is a combination of two sum rules of Case [3, 4]. For general J's, it is proven in Killip–Simon [19] with later simplifications of parts of the proof in [34, 37].

The difficulties in extending this strategy in the continuous case were several:

(i) The translation of the normalization condition $\mu(\mathbb{R})=1$ is not clear. We needed a condition that guaranteed $d\rho$ is the spectral measure associated to a reasonable V, preferably belonging to L^2_{loc}. We sought to express this in terms of the divergence of $\rho(-\infty,R)$ as $R\to\infty$. As it turned out, the A-function approach to the inverse spectral problem, [15, 33], leads quickly and conveniently to the condition (1.25), which is perfect for us.

(ii) The natural half-line sum rules in the Schrödinger case invariably lead to terms involving $V(0)$ or worse still, $V'(0)$. This is clearly unacceptable for one seeking $V\in L^2$ conditions.

(iii) The half-line sum rules also lead to terms that, like (1.11), have an integrand that has a variable sign. In (1.11), the fact that $\int f(E)\,dE\le 1$ implies uniform control on $\int \log_+(f(E))\sqrt{4-E^2}\,dE$ and so the terms of the 'wrong' sign (where $f(E)>1$) present no problem. But in the whole-line case where $\rho(\mathbb{R})=\infty$, terms of opposite signs could involve difficult to control cancellations.

The resolution of difficulties (ii) and (iii) was to fall back to the whole-line sum rule used in [8]. The penalty is that the Strong Quasi-Szegő condition, (1.27), little resembles the Quasi-Szegő condition of our earlier theorem, (1.11). It is this disappointment that led us to push on and find Theorem 1.2.

Whole-line sum rules date to the original inverse-scattering solution of the KdV equation, [13]. Consider the operator

$$L_0 = -\frac{d^2}{dx^2} + \chi_{(0,\infty)}(x)V(x) \tag{1.32}$$

acting on $L^2(\mathbb{R})$ with eigenvalues $E_j^{(0)}$. The well-known sum rule is, [8, 13, 46],

$$\tfrac{1}{8}\int_0^\infty V(x)^2\,dx = \tfrac{2}{3}\sum_j (E_j^{(0)})^{3/2} + Q \tag{1.33}$$

where

$$Q = \frac{1}{\pi}\int_0^\infty \log\biggl[\frac{|w(k+i0)+ik|^2}{4k\operatorname{Im} w(k+i0)}\biggr]k^2\,dk. \tag{1.34}$$

As in [19], we will need to prove it in much greater generality than was known previously. Essentially, assuming that w is the m-function of an L^2_{loc} potential V, we will prove (1.33) always holds although both sides may be infinite.

If one notes that the half-line and whole-line eigenvalues interlace,

$$E_j^{(0)} \leq E_j \leq E_{j+1}^{(0)}, \tag{1.35}$$

it is clear that (1.33) proves Theorem 1.2.

As was the case [19, 34, 37], the key to the proof of (1.33) is a 'step-by-step' sum rule, that is, a result that, in essence, is the difference of (1.33) for L_0 and for

$$L_t = -\frac{d^2}{dx^2} + \chi_{(t,\infty)}(x)V(x) \tag{1.36}$$

and which always holds. A second important ingredient is the semicontinuity of Q.

In Section 2, we will discuss a relative Wronskian which is the analogue of the product of m-functions used implicitly in [19, 37] and explicitly in [34] to prove a multi-step sum rule. In Section 3, as an aside, we will re-express this relative Wronskian as a perturbation determinant. In Section 4, we prove the step-by-step sum rule. In Section 5, we prove lower semicontinuity of the quasi-Szegő term. In Section 6, we discuss (1.25) and, in particular, show it implies μ is the spectral measure of a locally L^2 potential. Section 7 completes the proof of (1.33) and of Theorem 1.3. Sections 8–11 prove Theorem 1.2 given Theorem 1.3.

The earliest theorem of the type presented here is Verblunsky's form [45] of Szegő's theorem [35, 41]. Let us elaborate. The orthogonal polynomials associated to a measure on the unit circle obey a recurrence and the coefficients that appear in this recurrence are known as the Verblunsky coefficients. The result just mentioned says that the Verblunsky coefficients are square summable if and only if the logarithm of the density of the a.c. part of the measure is integrable. In fact, there is a sum rule relating these quantities.

One of the more interesting spectral consequences of Szego's theorem is the construction by Totik [44] (see also Simon [35]) that given any measure supported on the circle, there is an equivalent measure whose recursion coefficients lie in all ℓ^p $(p > 2)$. We expect that the results and techniques of the current paper will provide tools allowing one to carry this result over to Schrödinger operators (although it seems likely that ℓ^p will be replaced by $\ell^p(L^2)$ rather than by L^p.

Kreĭn systems give a continuum analogue for orthogonal polynomials on the unit circle. The corresponding version of Szegő's Theorem can be found in [21]; though for proofs, see [38]. Using a continuum analogue of the Geronimus relations, Kreĭn's Theorem gives results for potentials of the form $V(x) = a(x)^2 \pm a'(x)$ with $a \in L^2$. Note that the operators associated to such potentials are automatically positive—there are no bound states. For a further discussion of the application of Kreĭn systems to Schrödginer operators, see [10, 11, 12].

More recently, Sylvester and Winebrenner [40] studied the scattering for the Helmholtz equation on a half-line and obtained necessary and sufficient conditions (in terms the reflection coefficient) for square integrability of the derivative of the wave speed. Applying appropriate Liouville transformations connects this work to the study of Schrödinger operators with potentials $V(x) = a(x)^2 \pm a'(x)$, just as for Kreĭn systems. Our methods parallel their work in places, particularly with regard to the semicontinuity properties of Q discussed in Section 5. However, dealing with bound states adds to the complexity of our case.

As mentioned earlier, [8] proved that $\sigma_{\mathrm{ac}}(H) = [0,\infty)$ for $H = -\frac{d^2}{dx^2} + V$ with $V \in L^2$. Earlier work by Christ, Kiselev, and Remling, [20, 5, 27], settled the case $V(x) \leq C(1+|x|^2)^{-\alpha}$ for $\alpha > \frac{1}{2}$ by entirely different means. The most recent development in this direction is the use of sum rules by Rybkin, [29], to prove $\sigma_{\mathrm{ac}}(H) = [0,\infty)$ for potentials of the form $V = f + g'$ with $f, g \in L^2$.

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2. The Relative Wronskian

In this section, we will consider $V \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ for which the operator

$$H = -\frac{d^2}{dx^2} + V \tag{2.1}$$

with boundary condition $u(0) = 0$ is essentially selfadjoint and has $\sigma_{\mathrm{ess}}(H) \subset [0,\infty)$. As noted in the Introduction, for any $k \in \mathbb{C}_+$ with $k^2 \notin \sigma(H)$, there is unique solution $\psi_+(x,k)$ of

$$-\psi'' + V\psi = k^2\psi \tag{2.2}$$

which is L^2 at $+\infty$ and $\psi_+(0) = 1$. By the above assumption on σ_{ess}, this extends to a meromorphic function of k in $\mathbb{C}_+$ with poles exactly at the negative eigenvalues of H. Moreover, the poles are simple.

Let us define

$$W(x,k) = e^{-ikx}\psi_+'(x,k) + ike^{-ikx}\psi_+(x,k). \tag{2.3}$$

W is the Wronskian of $\psi_+(x,k)$ and $\psi_-^{(0)}(x,k) \equiv e^{-ikx}$, which is the solution of

$$-\psi'' = k^2\psi \tag{2.4}$$

that is L^2 at $-\infty$ (recall $\operatorname{Im} k > 0$). Note that $W(x,k)$ is a meromorphic function of k, an absolutely continuous function of x, and is easily seen to obey

$$\frac{\partial}{\partial x} W(x,k) = e^{-ikx}\psi_+(x,k)V(x). \tag{2.5}$$

The zeros of $k \mapsto W(x_0,k)$ are precisely those points where one can find a $c \in \mathbb{C}$ for which

$$u(x) = \begin{cases} \psi_+(x,k), & x \ge x_0 \\ ce^{-ikx}, & x \le x_0 \end{cases} \tag{2.6}$$

is a C^1 function, that is, $W(x_0,k) = 0$ if and only if k^2 is an eigenvalue of the operator L_t of (1.33) with $t = x_0$. In particular, all zeros lie on the imaginary axis: $k = i\kappa$ with $\kappa > 0$.

We will use $\kappa_1(x) > \kappa_2(x) > \cdots$ to indicate the zeros of $W(x,k)$ so that $-\kappa_j(x)^2$ are the negative eigenvalues of L_x.

We define the *relative Wronskian* by

$$a_x(k) = \frac{W(x,k)}{W(0,k)}. \tag{2.7}$$

For each x, it is a meromorphic function of k. Like the m-function—and unlike $W(x,\cdot)$—it is independent of the normalization $\psi_+(0,k) = 1$. By the above, we have

Proposition 2.1. *The poles of $a_x(k)$ are simple and lie at those points $k = -i\kappa_j(0)$ for which $-\kappa_j(0)^2$ is an eigenvalue of L_0. The zeros are also simple and lie on the set $k = -i\kappa_j(x)$ where $-\kappa_j(x)^2$ are the eigenvalues of L_x. (In the event that a point lies in both sets, there is neither a pole nor a zero—they cancel one another.)*

Next, we note that

Proposition 2.2. *For* $\operatorname{Re} k \neq 0$, $a_x(k)$ *is absolutely continuous in* x. *Moreover,* $\log[a_x(k)]$, *defined so that it is continuous in* x *with* $\log(a_{x=0}(k)) = 0$, *obeys*

$$\frac{d}{dx} \log[a_x(k)] = V(x)(ik + w(k;x))^{-1} \tag{2.8}$$

where

$$w(k,x) = \frac{\psi_+'(x,k)}{\psi_+(x,k)} \tag{2.9}$$

is the m*-function associated to the operator* L_x *restricted to* $[x,\infty)$.

Proof. As a ratio of nonvanishing absolutely continuous functions, $a_x(k)$ is absolutely continuous, and then so is its log. By (2.5),

$$\frac{d}{dx} \log[a_x(k)] = \frac{e^{-ikx}\psi_+(x,k)V(x)}{W(x,k)}.$$

As

$$\frac{W(x,k)}{e^{-ixk}\psi_+(x,k)} = \frac{\psi_+'(x,k)}{\psi_+(x,k)} + ik = w(k;x) + ik,$$

(2.8) is immediate. □

Proposition 2.3. (a) *For* $k \in \mathbb{C}_+$ *with* $\operatorname{Re} k \neq 0$,

$$\bigl|\log[a_x(k)]\bigr| \leq |\operatorname{Re} k|^{-1} \int_0^x |V(y)|\, dy. \tag{2.10}$$

(b) *Fix* $K > 0$. *Then there exist* $R > 0$ *and* C *so, for all* k *in* $\mathbb{C}_+$ *with* $|k| > R$ *and all* x *in* $[0,K]$,

$$\bigl|\log[a_x(k)]\bigr| \leq C|k|^{-1}. \tag{2.11}$$

Proof. (a) If $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$, then $\operatorname{Im} w > 0$ and so $|ik + w(k;x)|^{-1} \leq |\operatorname{Re} k|^{-1}$. Thus (2.10) follows from (2.8).

(b) By [15], uniformly for $x \in [0,K]$, $w(k;x) - ik \to 0$ as $|k| \to \infty$ with $\arg(-ik) \leq \frac{\pi}{4}$. This plus (2.8) implies that (2.11) holds uniformly in $x \in (0,K)$ and $|\arg(-ik)| \leq \frac{\pi}{4}$. By (2.10), it holds for $\arg k \in (0,\frac{\pi}{4}) \cup (\frac{3\pi}{4},\pi)$. □

Proposition 2.4. *Suppose that for some* $k_0 \in (0,\infty)$, $\lim_{\varepsilon\downarrow 0} w(k_0+i\varepsilon;0) \equiv w(k_0+i0;0)$ *exists and* $\operatorname{Im} w(k_0+i0;0) \in (0,\infty)$. *Then for all* x, $\lim_{\varepsilon\downarrow 0} w(k_0+i\varepsilon;x) \equiv w(k_0+i0;x)$ *exists and* $\lim_{\varepsilon\downarrow 0} a_x(k_0+i\varepsilon) \equiv a_x(k_0+i0)$ *exists. Moreover,*

$$|a_x(k_0+i0)|^2 = \frac{T(k_0,0)}{T(k_0,x)} \tag{2.12}$$

where

$$T(k_0,x) = \frac{4k_0 \operatorname{Im} w(k_0+i0;x)}{|w(k_0+i0;x)+ik_0|^2}. \tag{2.13}$$

Proof. As $\operatorname{Im} w(k_0+i0;0) \in (0,\infty)$, we may also take the vertical limit $\psi_+(x,k_0+i0)$; indeed, this is just the solution to (2.2) with $\psi(0)=1$, $\psi'(0) = \operatorname{Im} w(k_0+i0;0)$, and $k = k_0$.

As $\operatorname{Im} w(k+i0) > 0$, $\psi_+(x,k_0+i0)$ is not a complex multiple of a real-valued solution and so cannot have any zeros. Thus $\lim_{\varepsilon\downarrow 0} w(k_0+i\varepsilon;x)$ exists. Similarly, by (2.3), $W(x,k_0+i\varepsilon)$ has a limit and

$$|W(x,k_0+i0)| = |\psi_+(x,k_0+i0)|\,|w(k_0+i0;x)+ik|. \tag{2.14}$$

As $\psi_+(\,\cdot\,, k_0+i0)$ and $\overline{\psi_+(\,\cdot\,, k_0+i0)}$ obey the same equation, their Wronskian is a constant (in x), that is

$$|\psi_+(x,k_0+i0)|^2 \operatorname{Im} w(x;k_0+i0) = C_{k_0}. \tag{2.15}$$

The definition (2.13) and (2.14), (2.15) imply

$$|W(x,k_0+i0)|^2 = \frac{4C_{k_0}k_0}{T(k_0,x)}. \tag{2.16}$$

Thus (2.7) implies (2.12). □

We write the letter T in (2.13) because, as we will see, it represents the transmission probability of stationary scattering theory.

Our final result in this section is

Proposition 2.5. *Let $V \in L^2_{\text{loc}}([0,\infty))$ and suppose $\sigma_{\text{ess}}(H) \subset [0,\infty)$. Then as $\kappa \to \infty$ (real κ),*

$$\log[a_x(i\kappa)] = -\frac{1}{2\kappa}\int_0^x V(y)\,dy + \frac{1}{8\kappa^3}\int_0^x V(y)^2\,dy + o(\kappa^{-3}) \tag{2.17}$$

with an error uniform in x for $x \in [0,K]$ for any K.

Proof. By [15, 33],

$$w(i\kappa,x) = -\kappa - \int_0^1 V(x+y)e^{-2\kappa y}\,dy + o(\kappa^{-1}) \tag{2.18}$$

with the $o(\kappa^{-1})$ uniform in x for $x \in [0,K]$. Notice that the integral in (2.18) is $o(\kappa^{-1/2})$ since $V \in L^2_{\text{loc}}$. Thus

$$[w(i\kappa,x)-\kappa]^{-1} = (-2\kappa)^{-1} + (2\kappa)^{-2}\int_0^1 V(x+y)e^{-2\kappa y}\,dy + o(\kappa^{-3}). \tag{2.19}$$

To get this, note that one error term is $O(\kappa^{-3})o(\kappa^{-1})$ and by the fact that the integral in (2.18) is a priori $o(\kappa^{-1/2})$, the other is $O(\kappa^{-3})O(\kappa^{-1/2})^2$.

Thus, by integrating (2.8), the proposition will follow once we show

$$\lim_{\kappa\to\infty} \kappa \int_0^x V(y)\int_0^1 V(y+s)e^{-2\kappa s}\,ds = \tfrac{1}{2}\int_0^x V(y)^2\,dy \tag{2.20}$$

for all $V \in L^2(0,x+1)$. To prove this, note first that it is trivial if V is continuous since then, $\kappa\int_0^1 V(y+s)e^{-2\kappa s}\,ds = \frac{1}{2}[V(y)+o(1)]$ for each y uniformly in y in $[0,x]$. Moreover,

$$\begin{aligned}\biggl|\int_0^x f(y)\biggl(\int_0^1 g(y+s)\kappa e^{-2\kappa s}\,ds\biggr)dy\biggr| \\ \le \tfrac{1}{2}\biggl(\int_0^x |f(y)|^2\,dy\biggr)^{1/2}\biggl(\int_0^{x+1}|g(y)|^2\,dy\biggr)^{1/2}\end{aligned} \tag{2.21}$$

so an approximation theorem goes from V continuous to general V in $L^2(0,x+1)$.

To prove (2.21), use the Schwartz inequality and

$$\biggl\|\int_0^1 g(\,\cdot\,+s)\kappa e^{-2\kappa s}\,ds\biggr\|_2 \le \int_0^1 \kappa e^{-2\kappa s}\|g(\,\cdot\,+s)\|_2\,ds \le \tfrac{1}{2}\biggl(\int_0^{x+1}|g(y)|^2\,dy\biggr)^{1/2}$$

where $\|\cdot\|_2$ is $L^2(0,x)$ norm. □

3. Perturbation Determinants: An Aside

In this section, we provide an alternate definition of $a_x(k)$ which we could have used (and, indeed, initially did use) to define and prove the basic properties of this function. The definition as a perturbation determinant makes the similarity to the Jacobi matrix theory stronger. Expressions of suitable Wronskians as Fredholm determinants go back to Jost and Pais [17]. We will not use this alternate definition again in this paper, but felt it is suggestive and should be useful for other purposes.

We will write $\mathfrak{I}_1$ for the space of trace-class operators with the usual norm: $\|A\| = \operatorname{Tr}(|A|)$.

We need one preliminary:

Proposition 3.1. *Let V be in $L^1_{\text{loc}}((0,\infty))$ and consider $L = -\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R})$ (with a boundary condition at infinity if V is limit circle there). Fix $0 < K < \infty$ and view $L^2([0,K])$ as functions (and multiplication operators) on all of $\mathbb{R}$ that happen to vanish outside this interval.*

Given $z \in \mathbb{C}_+$, the mapping $f \mapsto f(L-z)^{-1}$ is continuous and differentiable from $L^2(0,K)$ into the trace class operators.

Proof. Let L_D be the operator with a Dirichlet boundary condition added at $x = K$, that is, $L_D = L_D^- \oplus L_D^+$ with L_D^- on $L^2(-\infty,K)$ with $u(K) = 0$ boundary conditions and L_D^+ on $L^2(K,\infty)$ with $u(K) = 0$ boundary conditions and the same boundary condition at infinity as L.

Let $u_\pm$ solve $-u'' + Vu = zu$ with u_- square-integrable at $-\infty$ and u_+, L^2 at $+\infty$ (or, obeying H's boundary condition at infinity if V is limit circle). Let φ be given by

$$\varphi(x) = \begin{cases} u_+(K)u_-(x), & x \le K \\ u_-(K)u_+(x), & x \ge K \end{cases} \tag{3.1}$$

and normalize u_- so that $W(u_+,u_-) = 1$. Then, standard formulae for Green's functions [6] show that with $G(x,y)$, the integral kernel of $(L-z)^{-1}$ and $G_D(x,y)$ that of $(L_D - z)^{-1}$,

$$G(x,y) - G_D(x,y) = (u_+(K)u_-(K))^{-1}\varphi(x)\varphi(y). \tag{3.2}$$

Since φ is bounded on $[0,K]$, $f\varphi \in L^2$ and so

$$f[(L-z)^{-1} - (L_D - z)^{-1}]$$

is a bounded rank one operator, and so trace class. Thus it suffices to prove $f(L_D - z)^{-1} = f(L_D^- - z)^{-1} \oplus 0$ is trace class, and so that $f(L_D^- - z)^{-1}$ is trace class on $L^2(-\infty,K)$.

Similarly, adding a boundary condition at $x = 0$ is rank one, so with H_D the operator on $L^2(0,K)$ with $u(0) = u(K) = 0$ boundary conditions, it suffices to prove that $f(H_D - z)^{-1}$ is trace class.

As $V \restriction [0,K]$ is in L^1, H_D is bounded from below, and so by adding a constant to V, we can suppose $H_D \ge 0$. Thus it suffices to show that $f(H_D + 1)^{-1}$ is trace class. Write

$$\begin{aligned} f(H_D+1)^{-1} &= \int_0^\infty e^{-t} f e^{-tH_D}\,dt \\ &= \int_0^\infty e^{-t}(fe^{-tH_D/2})(e^{-tH_D/2})\,dt. \end{aligned}$$

By general principles (see [32]), the integral kernel of $e^{-tH/2}$, call it $P_t(x,y)$, obeys

$$\begin{aligned} P_t(x,y) &\leq Ct^{-1/2}\exp\biggl(\frac{-(x-y)^2}{Dt}\biggr), && t\leq 1 \\ &\leq C, && t\geq 1. \end{aligned}$$

¿From this it follows that for any $g\in L^2(0,K)$,

$$\|ge^{-sH_D}\|_2 \leq C\|g\|_{L^2}(1+|s|^{-1/4})$$

and $\|\cdot\|_2$ a Hilbert-Schmidt norm. Thus

$$\begin{aligned} \|f(H_D+1)^{-1}\|_1 &\leq \int_0^\infty e^{-t}\,\|fe^{-tH/2}\|_2\,\|e^{-tH/2}\|_2\,dt \\ &< \infty. \end{aligned}$$

The proof of continuity and differentiability in f follows from these estimates. □

Remark. The use of Dirichlet decoupling and semigroup estimates to get trace class results goes back to Deift–Simon [9].

Corollary 3.2. *If L_t is given by* (1.36) *and $z\in\mathbb{C}_+$, then*

$$X_t = (L_t - z)(L_0 - z)^{-1} - 1 \in \mathfrak{I}_1. \tag{3.3}$$

Moreover, if $V(x)$ is continuous in a one-sided neighborhood of $x=0$, $\operatorname{Tr}(X_t)$ is differentiable at $t=0$ and

$$\frac{d}{dt}\operatorname{Tr}(X_t)\bigg|_{t=0} = V(0)G(0,0) \tag{3.4}$$

where G is the integral kernel of the operator $(L_0-z)^{-1}$.

Proof. We have that

$$L_t - L_0 = V\chi_{[0,t]} \tag{3.5}$$

so (3.3) follows from Proposition 3.1. By the continuity assumption, X_t has a piecewise continuous integral kernel $X_t(x,y) = V(x)\chi_{[0,t]}(x)G(x,y)$, so (see, e.g., Theorem 3.9 in Simon [31])

$$\operatorname{Tr}(X_t) = \int_0^t V(x)G(x,x)\,dx$$

from which (3.4) follows. □

The main result in this section is

Theorem 3.3. *Let $V\in L^2_{\mathrm{loc}}(0,\infty)$ with $\sigma_{\mathrm{ess}}(H)=[0,\infty)$. Then for*

$$k\in\mathbb{C}_+\backslash\{k=-i\kappa \mid -\kappa^2\in\sigma(L_0)\}$$

we have

$$a_x(k) = \det[(L_x-k^2)/(L_0-k^2)]. \tag{3.6}$$

Proof. By continuity, we can suppose $\operatorname{Re}k\neq 0$. Similarly, by Proposition 3.1, we can suppose V is continuous on $[0,x]$.

Let $\tilde a_s(k)$ be the right-hand side of (3.6). If we prove that for $0<t<x$,

$$\frac{d}{dt}\log[\tilde a_t(k)] = V(t)[ik + w(k;t)]^{-1} \tag{3.7}$$

then, by (2.8) and $\tilde{a}_0(k) \equiv 1$, we could conclude (3.6).

As

$$\frac{d}{dA} \log\bigl[\det(1+A)\bigr]\bigg|_{A=0} = \operatorname{Tr}(A)$$

and for t near t_0,

$$\log \tilde{a}_t(k) = \log \tilde{a}_{t_0}(k) + \log\det[(L_t - k^2)/(L_{t_0} - k^2)], \tag{3.8}$$

(3.4) and (3.8) imply that

$$\frac{d}{dt} \log[\tilde{a}_t(k)]\bigg|_{t_0} = V(t_0)G(t_0,t_0)$$

where G is the integral kernel for $(L_{t_0} - k^2)^{-1}$. This leads to (3.7) after writing the Green's function in terms of ψ_+ and $\psi_-^{(0)}$. □

4. The Step-by-Step Sum Rule

In this section, we will prove a general step-by-step sum rule for all $V \in L^2_{\text{loc}}([0,\infty))$ that involves $\int_0^x V(y)^2\,dy$. We begin with a preliminary: Recall (see Proposition 2.1) that $\kappa_j(t) \geq 0$ is defined so that $\kappa_1(t) \geq \kappa_2(t) \geq \cdots$ and $\{-\kappa_j(t)^2\}_{j=1}^{N(t)}$ are the negative eigenvalues of L_t and $\kappa_j(t) = 0$ if $j > N(t)$, which may be infinite.

Proposition 4.1. *For any $V \in L^2_{\text{loc}}([0,\infty))$ and $t \in (0,\infty)$,*

$$\sum_j |\kappa_j(t)^2 - \kappa_j(0)^2| < \infty. \tag{4.1}$$

Proof. Let $A(s) = -\frac{d^2}{dx^2} + \chi_{[t,\infty)}V + s\chi_{[0,t)}V$ so $A(0) = L_t$ and $A(1) = L_0$. Let $E_j(s)$ denote the negative eigenvalues $B(s)$ with $E_1 \leq E_2 \leq \cdots$ and $E_j(s) = 0$ if $j > N(s)$, the number of negative eigenvalues of $A(s)$. Let $\psi_j(s)$ be the corresponding normalized eigenvectors. Pick $a > 0$ so for all $s \in [0,1]$, $A(s) \geq 1 - a$.

By first-order eigenvalue perturbation [26, 18] (a.k.a. the Feynman–Hellman theorem) if $j \leq N(s)$:

$$\begin{aligned}\frac{d}{ds} E_j(s) &= \langle \psi_j(s), \chi_{[0,t)} V \psi_j(s)\rangle \\ &= (E_j(s)+a)\langle (A(s)+a)^{-1/2}\psi_j(s), \chi_{[0,t]}V(A(s)+a)^{-1/2}\psi_j(s)\rangle\end{aligned}$$

so

$$\left|\frac{dE_j(s)}{ds}\right| \leq 2a|\langle \psi_j(s), (A(s)+a)^{-1/2}\chi_{[0,t]}V(A(s)+a)^{-1/2}\psi_j(s)\rangle| \tag{4.2}$$

and thus

$$\begin{aligned}\sum_{j=1}^{N(s)} \left|\frac{dE_j(s)}{ds}\right| &\leq 2a\|\chi_{[0,t]}|V|^{1/2}(A(s)+a)^{-1/2}\|_2^2 \\ &\leq C \end{aligned}\tag{4.3}$$

where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. (4.3), which indicates a C independent of $s \in [0,1]$, follows from an estimate that

$$\|\chi_{[0,t]}|V|^{1/2}(A(s)+a)^{-1/2}\|_2^2 = \int_0^t |V(x)|^2(A(s)+a)^{-1}(x,x), dx$$

$$\leq C.$$

(4.3) implies

$$\sum_{j=1}^{\infty} |E_j(s) - E_j(u)| \leq C|s-u|$$

which for $s=1$, $u=0$ is (4.1). □

Remark. One may also prove this proposition using the $\mathfrak{I}_1 \to L^1$ bound for the Kreĭn spectral shift function. Indeed, the proof of this general result follows along the general lines given above.

We can use this to define the Blaschke product that we will need to deal with the zeros and poles of $a_t(k)$:

Proposition 4.2. *Let*

$$B_t(k) = \prod_j \biggl\{ \biggl[\frac{k + i\kappa_j(0)}{k - i\kappa_j(0)} \, \frac{k - i\kappa_j(t)}{k + i\kappa_j(t)} \biggr] \exp\biggl[-\frac{2i}{k} \bigl(\kappa_j(0) - \kappa_j(t)\bigr) \biggr] \biggr\}. \tag{4.4}$$

Then:

(i) *The infinite product converges on* $\mathbb{C}_+ \backslash \{\kappa_j^{(0)}\}_{j=0}^{N(0)}$.

(ii) $B_t(k)$ *has a continuation to* $\bar{\mathbb{C}}_+ \backslash [\{\kappa_j(0)\}_{j=0}^{N} \cup \{0\}]$ *and*

$$k \in \mathbb{R}\backslash\{0\} \Rightarrow |B_t(k)| = 1. \tag{4.5}$$

(iii) *For* $k \notin i\mathbb{R}$,

$$|\log|B_t(k)|| \leq C|\mathrm{Re}\, k|^{-2}. \tag{4.6}$$

(iv) *Uniformly for* $\arg(y) \leq \frac{\pi}{4}$,

$$\log[B_t(iy)] = \frac{2}{3y^3} \sum_j [\kappa_j(0)^3 - \kappa_j(t)^3] + O(|y|^{-5}) \tag{4.7}$$

as $|y| \to \infty$.

Proof. Let $\kappa, \lambda > 0$. Define

$$F(k;\kappa,\lambda) = \log\biggl[\frac{k + i\kappa}{k - i\kappa} \, \frac{k - i\lambda}{k + i\lambda} \biggr] - \frac{2i}{k} (\kappa - \lambda). \tag{4.8}$$

Then

$$F(k;\lambda,\lambda) = 0$$

and, by a straightforward computation,

$$\frac{\partial}{\partial\kappa} F(k;\kappa,\lambda) = -\frac{2i\kappa^2}{k(k^2+\kappa^2)}. \tag{4.9}$$

It follows for $k \in \mathbb{C}$ with $\pm ik \notin [\min(\kappa,\lambda) \max(\kappa,\lambda)]$,

$$|F(k;\kappa,\lambda)| \leq 2 \int_{\min(\kappa,\lambda)}^{\max(\kappa,\lambda)} \frac{\mu^2}{|k|\,|k^2+\mu^2|} \, d\mu \tag{4.10}$$

The right side is invariant under $k \to \bar{k}$, so suppose $\mathrm{Im}\, k \geq 0$. Then $\frac{\mu}{|k+i\mu|} \leq 1$, so

$$|F(k;\kappa,\lambda)| \leq \frac{\max(\kappa,\lambda)^2 - \min(\kappa,\lambda)^2}{|k| \, \inf\{|k - i\mu| \mid \mu \in \pm(\min(\kappa,\lambda), \max(\kappa,\lambda))\}}. \tag{4.11}$$

We can thus prove:

(i) By (4.11), if $k \notin \{0\} \cup \{i\kappa_j(0)\} \cup \{-i\kappa_j(t)\} \equiv Q$, we have for all n sufficiently large that

$$|F(k;\kappa_n(0),\kappa_n(t))| \leq C_k |\kappa_n(0)^2 - \kappa_n(t)^2|$$

so, by (4.1), the product (4.4) converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash Q$.

(ii) The above argument shows B has analytic continuation across $\mathbb{R}\backslash\{0\}$. Since the continuation is given by a convergent product, and the finite products have magnitude 1 on $\mathbb{R}$, that is true of B on $\mathbb{R}\backslash\{0\}$.

(iii) From (4.11) and $\inf\{|k - i\mu| \mid k \in \dots\} \geq \mathrm{Re}|k|$, we have

$$|F(k;\kappa,\lambda)| \leq \frac{|\kappa^2 - \lambda^2|}{|\mathrm{Re}\, k|^2}$$

which, given (4.1), implies (4.6).

(iv) By (4.9) for y real and large,

$$\begin{aligned} \frac{\partial}{\partial\kappa} F(iy,\kappa,\lambda) &= \frac{2\kappa^2}{y(y^2-\kappa^2)} \\ &= \frac{2\kappa^2}{y^3} + O\biggl(\frac{\kappa^4}{y^5}\biggr) \end{aligned}$$

so (4.7) holds by integrating and using

$$\int_{\kappa_j(t)}^{\kappa_j(0)} 2\mu^2\, d\mu = \tfrac{2}{3}\, [\kappa_j(0)^3 - \kappa_j(t)^3]. \qquad \square$$

Let $a_t(k)$ be given by (2.7) and $B_t(k)$ by (4.4). The two functions are analytic in $\mathbb{C}_+$ and have the same zeros and poles, so

$$g_t(k) = \log\biggl[\frac{a_t(k)}{B_t(k)}\biggr] \tag{4.12}$$

is analytic in $\mathbb{C}_+$. We define g_t by taking the branch of log which is real for $k = i\kappa$ with κ large.

Proposition 4.3. (i) *$a_t(k)$ is analytic in $\mathbb{C}_+$.*

(ii) *For a.e. $k \in \mathbb{R}_+$, $\lim_{\varepsilon\downarrow 0} g_t(k+i\varepsilon) \equiv g_t(k)$ exists and if $\mathrm{Im}\, m(k^2+i0) > 0$, then*

$$\mathrm{Re}\, g_t(k) = \tfrac{1}{2}\, \log\biggl[\frac{T(k,0)}{T(k,t)}\biggr] \tag{4.13}$$

with T given by (2.13).

(iii) *For each $\varepsilon > 0$,*

$$\mathrm{Im}\, k > \varepsilon \Rightarrow |g_t(k)| \leq C_\varepsilon |k|^{-1}. \tag{4.14}$$

(iv) *For all $k \in \mathbb{C}_+$, $\mathrm{Re}\, k \neq 0$,*

$$|g_t(k)| \leq C[|\mathrm{Re}\, k|^{-1} + |\mathrm{Re}\, k|^{-2}]. \tag{4.15}$$

(v) *As $y \to \infty$ along the real axis,*

$$g_t(iy) = ay^{-1} + by^{-3} + o(y^{-3}) \tag{4.16}$$

with coefficients

$$a = -\tfrac{1}{2} \int_0^t V(x)\, dx \tag{4.17}$$

$$b = \tfrac{1}{8}\int_0^t V(x)^2\,dx - \tfrac{2}{3}\sum_j [\kappa_j(0)^3 - \kappa_j(t)^3]. \tag{4.18}$$

Proof. (i) is discussed in the definition.

(ii) This combines Proposition 2.4 and (4.5).

(iii) This follows from (2.10), (2.11), (4.6), (4.7), and the continuity (and so, boundedness) of g_t on compact subsets of $\mathbb{C}_+$.

(iv) This combines (2.10) and (4.6).

(v) This combines (2.17) and (4.7). □

We are now ready for the nonlocal step-by-step sum rule.

Theorem 4.4. *Suppose* $V \in L^2_{\text{loc}}(\mathbb{R}^+)$ *and* $\operatorname{Im} m(E+i0) > 0$ *for almost every* $E > 0$. *Then for any* $y_0, y_1 \in (0,\infty)$,

$$\operatorname{Re}\biggl[g_t(iy_0) - \frac{y_1 g_t(iy_1)}{y_0}\biggr] = \int_0^\infty \frac{(y_0^2 - y_1^2)\xi^2}{y_0(\xi^2+y_0^2)(\xi^2+y_1^2)}\,\log\biggl[\frac{T(\xi,0)}{T(\xi,t)}\biggr]\,\frac{d\xi}{\pi} \tag{4.19}$$

where g_t *is given by* (4.12) *and* T, *by* (2.13).

Proof. If h is a bounded harmonic function on $\mathbb{C}_+$ with a continuous extension to $\bar{\mathbb{C}}_+$, then for $y > 0$,

$$h(x+iy) = \frac{y}{\pi}\int \frac{h(\xi)}{(\xi-x)^2+y^2}\,d\xi. \tag{4.20}$$

This Poisson representation is standard [30, 39] and follows by noting that the difference of the two sides is a harmonic function on $\mathbb{C}_+$ vanishing on $\mathbb{R}$ so, by the reflection principle, a restriction of a bounded harmonic function on $\mathbb{C}$ vanishing on $\mathbb{R}$ and so 0 by Liouville's Theorem.

As $\operatorname{Re} g_t(k)$ is a bounded harmonic function on $\{k \mid \operatorname{Im} k \geq \varepsilon\}$, we have for all $y > 0$ and $\varepsilon > 0$,

$$\operatorname{Re} g_t(x+iy+i\varepsilon) = \frac{y}{\pi}\int \frac{\operatorname{Re} g_t(\xi+i\varepsilon)}{(\xi-x)^2+y^2}\,dw \tag{4.21}$$

and therefore,

$$\operatorname{Re} g_t(iy_0+i\varepsilon) - \frac{y_1}{y_0}\operatorname{Re} g_t(iy_1+i\varepsilon) = \int Q(\xi,y_0,y_1)\operatorname{Re} g_t(\xi+i\varepsilon)\,d\xi \tag{4.22}$$

where

$$\begin{aligned} Q(\xi,y_1,y_0) &= \frac{1}{\pi}\biggl[\frac{y_0}{\xi^2+y_0^2} - \frac{y_1}{y_0}\,\frac{y_1}{\xi^2+y_1^2}\biggr] \\ &= \frac{1}{\pi}\,\frac{y_0^2-y_1^2}{y_0}\,\frac{\xi^2}{(\xi^2+y_0^2)(\xi^2+y_1^2)}. \end{aligned}$$

By (4.15), uniformly in ε,

$$|\operatorname{Re} g_t(\xi+i\varepsilon)| \leq C[|\xi|^{-2} + |\xi|^{-1}]$$

and clearly,

$$|Q(\xi)| \leq C_{y_0,y_1}\,\frac{\xi^2}{1+\xi^4}$$

so, by the dominated convergence theorem, we can take $\varepsilon \downarrow 0$ in (4.21). The left side converges to the left side of (4.19) and, by (4.13) and $\operatorname{Re} g_t(-\bar{k}) = \operatorname{Re} g_t(k)$, the right side converges to the right side of (4.19). □

Here is the step-by-step version of the Faddeev–Zhabat sum rule (1.33):

Theorem 4.5 (Step-by-Step Faddeev–Zhabat Sum Rule)**.** *Suppose* $V \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ *and* $\operatorname{Im} m(E+i0) > 0$ *for almost every* $E > 0$*. For any* $t > 0$*,*

$$\tfrac{1}{8}\int_0^t V(x)^2\,dx = \tfrac{2}{3}\sum_j [\kappa_j(0)^3 - \kappa_j(t)^3] + \lim_{y\to\infty}\int_0^\infty P(\xi,y)\log\left[\frac{T(\xi,t)}{T(\xi,0)}\right] d\xi \quad (4.23)$$

where

$$P(\xi,y) = \frac{1}{\pi}\left[\frac{4\xi^2 y^4}{(\xi^2+4y^2)(\xi^2+y^2)}\right]. \quad (4.24)$$

Proof. By (4.16), with b given by (4.18),

$$y^3[g_t(2iy) - \tfrac{1}{2}\,g_t(iy)] = b[\tfrac{1}{8} - \tfrac{1}{2}] + o(1)$$

so, by (4.19),

$$b = \lim_{y\to\infty} -\frac{8}{3}\left[\frac{y^3[(2y)^2-(y)^2]}{2\pi y}\right]\int_0^\infty \frac{\xi^2}{(\xi^2+4y^2)(\xi^2+y^2)}\,\log\left[\frac{T(\xi,0)}{T(\xi,t)}\right] d\xi \quad (4.25)$$

which is (4.23). □

Remarks. 1. As $\lim_{y\to\infty} P(\xi,y) = \frac{1}{\pi}\xi^2$, formally, (4.23) is just a difference of (1.34) for L_0 and L_t.

2. In the preceding theorems, the assumption that $\operatorname{Im} m(E+i0) > 0$ for almost every $E > 0$ was only used to allow us to apply Proposition 2.4 to obtain a simpler expression for the boundary values of $a_t(k)$. The assumption may be removed if one is willing to replace the ratio $T(\xi,t)/T(\xi,0)$ by the limiting value of the relative Wronskian.

5. Lower Semicontinuity of the Quasi-Szegő Terms

For any $V \in L^1_{\mathrm{loc}}(0,\infty)$, we can define (in the limit circle case after picking a boundary condition at infinity) $T(k,0)$ by (2.13) for a.e. $k \in (0,\infty)$ and then

$$Q(V) = -\frac{1}{\pi}\int_0^\infty \log[T(k,0)]k^2\,dk. \quad (5.1)$$

Since $T \le 1$, $-\log[T] \ge 0$ and the integral can only diverge to ∞, so $Q(M)$ is always defined although it may be infinite. The main result in this section is:

Theorem 5.1. *Let* V_n, V *be a sequence in* $L^2_{\mathrm{loc}}((0,\infty))$*. Let* V *be limit point at infinity. Suppose*

$$\int_0^a |V_n(x) - V(x)|^2\,dx \to 0 \quad (5.2)$$

for each $a > 0$*. Then*

$$Q(V) \le \liminf Q(V_n). \quad (5.3)$$

Remarks. 1. As noted in the Introduction, this is related to results in Sylvester–Winebrenner [40]. However, they have no bound states and $|r(k)| \le 1$ in the upper half-plane. This fails in our case and our argument will need to be more involved.

2. It is interesting that the analogue in the Jacobi case [19] used semicontinuity of the entropy and this result comes from weak semicontinuity of the L^p-norm.

3. It is not hard to see that this result holds if L^2_{loc} is replaced by L^1_{loc} and the $|\dots|^2$ in (5.2) is replaced by $|\dots|^1$. Basically, one still has strong resolvent

convergence in that case. But the argument is simpler in the $L^2_{\rm loc}$ case we need, so that is what we state.

We will prove this theorem in several steps. We will write $w_n(k)$ and $w(k)$ for the m-functions (parameterized by momentum) associated to V_n and V respectively.

Proposition 5.2. *Let V_n, V obey the hypothesis of Theorem 5.1. Then for all k with $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$, one has $w_n(k) \to w(k)$ as $n \to \infty$.*

Proof. Let H (resp., H_n) be the operator $u \mapsto -u''+Vu$ on $L^2(0,\infty)$ with boundary condition $u(0)=0$ at $x=0$ and, if need be, a boundary condition at ∞ for some n if the corresponding H_n is limit circle at ∞.

By the standard construction of these operators, H being limit point at infinity has $D \equiv \{u \in C_0^\infty([0,\infty)) \mid u(0)=0\}$ as an operator core. ([25, Theorem X.7] has the result essentially if V is continuous, but the proof works if V is $L^2_{\rm loc}$. Essentially, any $\varphi \in [(H+i)[D]]^\perp$ solves $-\varphi''+V\varphi = i\varphi$ with $\varphi(0)=0$ and that cannot be L^2; it follows that $\overline{H \pm i[D]} = L^2$ which is essential selfadjointness.)

Let $f = (H-k^2)\varphi$ with $\varphi \in D$. Then

$$\begin{aligned} \|[(H_n-k^2)^{-1} - (H-k^2)^{-1}]f\| &= \|(H_n-k^2)^{-1}(V_n-V)\varphi\| \\ &\le |\operatorname{Im} k^2|^{-1}\|(V_n-V)\varphi\| \to 0 \end{aligned}$$

by (5.2), so we have strong resolvent convergence.

If $\varphi \in L^2(0,a)$ and $\psi = (H_n-k^2)^{-1}\varphi$, then for $x>a$,

$$w_n(k,x) = \frac{\psi_n'(x)}{\psi_n(x)}$$

and so, for $x>0$, we have $w_n(k,x) \to w(k,x)$.

Differentiating (2.9) with respect to x and using (2.2) leads to the Riccati equation

$$\frac{dw}{dx} = k^2 - V(x) - w^2. \tag{5.4}$$

By combining this with (5.2), one can deduce $w_n(k) \to w(k)$. $\square$

We now define the reflection coefficient (for now, a definition; we will discuss its connection with reflection at the end of the section) by

$$r_n(k) = \frac{ik - w_n(k)}{ik + w_n(k)}. \tag{5.5}$$

The following bound is clearly relevant.

Proposition 5.3. *Let $k = |k|e^{i\eta}$ with $\eta \in [0, \frac{\pi}{2})$, $|k| \neq 0$. Then*

$$\sup_{z \in \mathbb{C}_+} \left| \frac{ik-z}{ik+z} \right| = \left(\frac{1+\sin(\eta)}{1-\sin(\eta)} \right)^{1/2}. \tag{5.6}$$

Proof. $z \mapsto \frac{ik-z}{ik+z}$ is a fractional linear transformation which takes $z=-ik \in \mathbb{C}_-$ to infinity since $\operatorname{Re} k > 0$ if $\eta \in [0, \frac{\pi}{2})$. Thus $\mathbb{C}_+$ is mapped into the interior of the circle $\{\frac{ik-x}{ik+x} \mid x \in \mathbb{R}\} \cup \{-1\}$. By replacing k by $k/|k|$, we can suppose $|k|=1$. Let

$$f(x) = \left| \frac{x - ie^{i\eta}}{x + ie^{i\eta}} \right|^2.$$

Straightforward calculus shows that $f'(x) = 0$ exactly at $x = \pm 1$. Since $|f| \to 1$ as $x \to \pm\infty$, we see the maximum of $f(x) = (1 + x^2 + 2x \sin \eta)/(1 + x^2 - 2x \sin \eta)$ occurs at $x = 1$ and is $(1 + \sin(\eta))/(1 - \sin(\eta))$. $\square$

Lemma 5.4. *Let f_n and f_∞ be a sequence of functions on $\mathbb{D}$, the open disk, with*

$$\sup_{z \in \mathbb{D}, n} |f_n(z)| < \infty. \tag{5.7}$$

Let $f_n(z) \to f_\infty(z)$ for all $z \in \mathbb{D}$. Let $f_n(e^{i\theta})$ be the a.e. radial limit of $f_n(re^{i\theta})$ and similarly for $f_\infty(e^{i\theta})$. Then $f_n(e^{i\theta}) \to f_\infty(e^{i\theta})$ weak-$*$, that is, for all $g \in L^1(\partial\mathbb{D})$,

$$\int_0^{2\pi} g(e^{i\theta}) f_n(e^{i\theta}) \frac{d\theta}{2\pi} \to \int_0^{2\pi} g(e^{i\theta}) f_\infty(e^{i\theta}) \frac{d\theta}{2\pi}. \tag{5.8}$$

Proof. By (5.7), it suffices to prove (5.8) for $g(e^{i\theta}) = e^{ik\theta}$ for all k. But for H^∞ functions (see [30]), $\int e^{ik\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0$ if $k > 0$ and $\int e^{-ik\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = f^{(k)}(0)/k!$. Pointwise convergence in $\mathbb{D}$ and boundedness implies convergence of all derivatives inside $\mathbb{D}$. $\square$

Theorem 5.5. *Let $r_n(k)$ be given by (5.5) for $\operatorname{Im} k > 0$. Then for a.e. $k \in (0, \infty)$, $r_n(k) = \lim_{\varepsilon \downarrow 0} r_n(k + i\varepsilon)$ exists and obeys*

$$|r_n(k)| \leq 1, \qquad (k > 0). \tag{5.9}$$

Moreover, for any g in $L^1(a, b)$ with $0 < a < b < \infty$, we have that

$$\int_a^b g(k) r_n(k)\, dk \to \int_a^b g(k) r(k)\, dk \tag{5.10}$$

and that for $1 \leq p < \infty$,

$$\liminf_{n \to \infty} \int_a^b |r_n(k)|^p k^2\, dk \geq \int_a^b |r(k)|^p k^2\, dk. \tag{5.11}$$

Proof. Pick $0 < c < a < b < d < \infty$. Let Q be the semidisk in $\mathbb{C}_+$ with flat edge (c, d). Let $\varphi : \mathbb{D} \to Q$ be a conformal map. Since

$$\sup_{k \in Q} \arg(k) < \frac{\pi}{2},$$

we have

$$\sup_{n, k \in Q} |r_n(k)| < \infty \tag{5.12}$$

by Proposition 5.3. We can thus apply Lemma 5.4 to $r_n \circ \varphi$ and so conclude (5.10). (5.9) follows from Proposition 5.3 for $\eta = 0$.

Note that (5.10) implies $r_n \to r$ in the weak topology on $L^p((a, b), k^2\, dk)$. Thus (5.11) is just an expression of the fact that the norm on a Banach space is weakly lower semicontinuous. $\square$

Proof of Theorem 5.1. Notice that

$$T(k_0, 0) + |r(k_0, 0)|^2 = 1. \tag{5.13}$$

Thus

$$\begin{aligned} -\log[T] &= -\log(1 - |r|^2) \\ &= \sum_{m=1}^\infty \frac{|r|^{2m}}{m}. \end{aligned} \tag{5.14}$$

(5.11) implies that for each m and $0 < a < b < \infty$,

$$\int_a^b \left[\frac{|r(k)|^{2m}}{m}\right] k^2\,dk \leq \liminf \int_a^b \left[\frac{|r_n(k)|^{2m}}{m}\right] k^2\,dk,$$

which becomes

$$\int_a^b \left[\sum_{m=1}^{M} \frac{|r|^{2m}}{m}\right] k^2\,dk \leq \liminf \int_a^b \left[\sum_{m=1}^{M} \frac{|r_n|^{2m}}{m}\right] k^2\,dk$$

so, by (5.14),

$$-\frac{1}{\pi}\int_a^b \log[T(k_0,0)]k^2, dk \leq \liminf \biggl\{-\frac{1}{\pi}\int_0^\infty \log[T_n(k_0,0)]k^2\,dk\biggr\}.$$

Now take $a \downarrow 0$ and $b \to \infty$. $\square$

We end this section with a sketch of an alternate approach to Theorem 5.1. We present this approach because it is rooted in the physics of scattering. Since we have a direct proof, we do not produce all the technical details—indeed, one is missing. The argument is in a sequence of steps:

Step 1. Let L be the whole-line problem obtained by setting $V = 0$ on $(-\infty, 0)$. Let j be a C^∞ function with $0 \leq j \leq 1$ and $j(x) = 0$ if $x > 0$ and $j(x) = 1$ if $x < -1$. Let J be multiplication by j. Then, by [7],

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itL} J e^{-itL} P_{\text{ac}}(L) = P_\ell^\pm(L) \tag{5.15}$$

exist and are invariant projections for L. $L \restriction \operatorname{ran}(P_\ell^\pm)$ is absolutely continuous and has spectrum $[0,\infty)$ with multiplicity 1.

Step 2.

$$P_\ell^-(L)P_\ell^+(L)P_\ell^-(L) \equiv R_\ell^-(L) \tag{5.16}$$

is a positive operator on $\operatorname{ran}(P_\ell^-)$ which commutes with $L \restriction \operatorname{ran}(P_\ell^-(L))$ and so, by the simplicity of the spectrum of this operator, it is multiplication by a function $R_L(E)$. Since $0 \leq R_\ell^-(L) \leq 1$, as a function, $0 \leq R(E) \leq 1$. R is discussed in [7].

Step 3. By computations related to those in [40],

$$R_L(k) = |r(k)|^2 \tag{5.17}$$

with r given by (5.5).

Step 4. We believe that for $V_n \to V$ in the sense of Theorem 5.1, one has for a dense set of vectors uniformity in n of the limit in (5.15), but we have not nailed down the details. If true, one has

$$\operatorname*{w-lim}_{n\to\infty} R_\ell^-(L_n) = R_\ell^-(L). \tag{5.18}$$

Step 5. By (5.17), $|r_n(k)|^2 \to |r(k)|^2$ weakly as L^∞-functions (i.e., when smeared with $g \in L^1(a,b)$) on $[a,b]$ for any $0 < a < b < \infty$. By the weak semicontinuity of the norm, (5.11) holds for $p \geq 2$.

Step 6. Get semicontinuity of $Q(V)$ from (5.11) for $p \geq 2$, as we do in the above proof.

6. Local Solubility

In this section, we will study (1.25) and describe its relation to $d\rho$ being the spectral measure of some $V \in L^2_{\rm loc}$. We will prove:

Theorem 6.1. *Let $d\rho$ be a measure obeying* (1.3) *and condition* (i) *of Theorem 1.3. Define F by* (1.24) *and suppose* (1.25) *holds. Then $d\rho$ is the spectral measure of some $V \in L^2_{\rm loc}$.*

Theorem 6.2. *Let $d\rho$ be the spectral measure on a potential in L^2. Then* (1.25) *holds, that is, $F \in L^2(\mathbb{R}^+)$.*

The key will be to prove that essentially, $\widehat{F}(\alpha)$, the Fourier transform of F, is $e^{-\frac{1}{4}\alpha^2} A(\alpha)$, where $A(\alpha)$ is the A-function introduced by Simon [33] and studied further by Gesztesy–Simon [15].

We will, first and foremost, use formula (1.21) from [15]:

$$A(\alpha) = -2 \int \lambda^{-1/2} \sin\bigl(2\alpha\sqrt{\lambda}\bigr)\, d(\rho - \rho_0)(\lambda) \tag{6.1}$$

where $\lambda^{-1/2}\sin(2\alpha\sqrt{\lambda})$ is interpreted as $|\lambda|^{-1/2}\sinh(2\alpha\sqrt{|\lambda|})$ if $\lambda < 0$ and (6.1) holds in distributional sense. We will also need the following (eqn. (1.16) of [15]):

$$|A(\alpha) - V(\alpha)| \leq \biggl| \int_0^\alpha |V(y)|\, dy \biggr|^2 \exp\biggl(\alpha \int_0^\alpha |V(y)|\, dy \biggr) \tag{6.2}$$

proven in [33] for regular V's and in (1.16) of [15] for $V \in L^1_{\rm loc}$. Finally, we need the following result of Remling [28], which can also be proven by the Gel'fand–Levitan method:

Proposition 6.3. *If a measure $d\rho$ obeys* (1.3) *and the distribution* (6.1) *lies in $L^1_{\rm loc}[0,\infty)$, then $d\rho$ is the spectral measure of a potential $V \in L^1_{\rm loc}[0,\infty)$.*

We can relate A and F as distributions. Let

$$A(\alpha) = A_S(\alpha) + A_L(\alpha) \tag{6.3}$$

where A_S is the integral over $\lambda < 1$ and A_L over $\lambda \geq 1$. Since

$$\int_{-\infty}^{\infty} \pi^{-1/2} e^{-q^2} e^{-iq\alpha} = e^{-\alpha^2/4}, \tag{6.4}$$

(1.13), (1.24), and (6.1) immediately imply

$$e^{-\alpha^2/4} A_L(\alpha) = i[\widehat{F}(2\alpha) - \widehat{F}(-2\alpha)]. \tag{6.5}$$

For $p \geq 1$ and $q \leq 0$, we have $e^{-(p-q)^2} \leq e^{-p^2} e^{-q^2}$. Combining this with

$$\int_{p \geq 1} e^{-p^2}\, d|\nu|(p) < \infty,$$

which follows from (1.3), we obtain that for $q \leq 0$,

$$F(q) \leq C e^{-q^2}. \tag{6.6}$$

Proof of Theorem 6.1. By (1.25) and (6.6), $F \in L^2(\mathbb{R})$ and hence $\widehat{F} \in L^2(\mathbb{R})$. By (6.5), $A_L(\alpha) \in L^2_{\rm loc}$. By (6.1), $A_S(\alpha)$ is bounded on bounded intervals, so $A(\alpha) \in L^2_{\rm loc}$. By Remling's Theorem (Proposition 6.3), $d\rho$ is the spectral measure of some $V \in L^1_{\rm loc}$. By (6.2), $|A(\alpha) - V(\alpha)|$ is bounded on bounded intervals, so $A \in L^2_{\rm loc} \Rightarrow V \in L^2_{\rm loc}$. ∎

To prove Theorem 6.2, we need the following elementary fact:

Proposition 6.4. *If T is a tempered distribution on $(1,\infty)$ which is real and* $\operatorname{Im}\widehat{T}(\alpha)\in L^2$*, then* $T\in L^2$.

Proof. We begin by noting that if $h\in L^2(0,\infty)$, then

$$\int_{-\infty}^{\infty} |\operatorname{Re}\widehat{h}(\alpha)|^2\,d\alpha = \int_{-\infty}^{\infty} |\operatorname{Im}\widehat{h}(\alpha)|^2\,d\alpha \tag{6.7}$$

since $\overline{\widehat{h}(\alpha)}=\widehat{\tilde h}(\alpha)$, where $\tilde h(x)=h(-x)$, and thus, by the Plancherel theorem,

$$\int_{-\infty}^{\infty} h(\alpha)^2\,d\alpha = \int_{-\infty}^{\infty} \tilde h(x)h(x)\,dx = 0 \tag{6.8}$$

implying (6.7). In particular,

$$\int_{-\infty}^{\infty} |\widehat{h}(\alpha)|^2\,d\alpha = 2\int |\operatorname{Im}\widehat{h}(\alpha)|^2\,d\alpha. \tag{6.9}$$

Given T, pick C^∞ g with $g(x)=g(-x)$, $|g(x)|\le 1$, $g(x)=1$ for $|x|$ small, $\operatorname{supp}(y)\subset[-1,1]$, and $\int g(x)\,dx=1$. Define

$$g_L(x)=g\biggl(\frac{x}{L}\biggr) \qquad r_\delta(x)=\delta^{-1}g\biggl(\frac{x}{\delta}\biggr) \tag{6.10}$$

and note that $r_\delta * (g_L T)\in L^2$, supported in $(0,\infty)$ for $\delta<1$, and since

$$[r_\delta * (g_L T)]^\wedge(\alpha) = \widehat{h}_1(\alpha\delta)(\widehat{g}_L * T)(\alpha) \tag{6.11}$$

and $\widehat{h}_1,\widehat{g}_L$ are real, we have

$$\int |\operatorname{Im}[r_\delta * (g_L T)]^\sim(\alpha)|^2\,d\alpha \le \int |\operatorname{Im}\widehat{T}|^2\,d\alpha.$$

Thus, by (6.9) and the Plancherel theorem,

$$\int |r_\delta * g_L T(x)|^2\,dx \le 2\int |\operatorname{Im}\widehat{T}(\alpha)|^2\,d\alpha$$

so $T\in L^2$ by taking $\delta\downarrow 0$ and $L\to\infty$. □

Proof of Theorem 6.2. If $V\in L^2$, then

$$\int_0^\alpha |V(y)|\,dy \le \biggl(\int_0^\infty |V(y)|^2\,dy\biggr)^{1/2}\alpha^{1/2} \tag{6.12}$$

so (6.2) says that

$$|A(\alpha)-V(\alpha)| \le C\alpha^2\exp(C\alpha^{3/2}) \tag{6.13}$$

and thus, $e^{-\alpha^2/2}A(\alpha)\in L^2$.

From (6.1),

$$|A_S(\alpha)| \le e^{C\alpha} \tag{6.14}$$

so $e^{-\alpha^2/2}A_S(\alpha)\in L^2$, and thus, $e^{-\alpha^2/2}A(\alpha)\in L^2$. By (6.5) and the fact that F is real-valued, it follows that $\operatorname{Im}\widehat{F}\in L^2$.

F is not supported on $(1,\infty)$, but by (6.6) and boundedness on $(0,1)$, $F=F_1+F_2$, where F_2 is supported on $(1,\infty)$ and $F_1\in L^2$. Thus, $\operatorname{Im}\widehat{F_1}\in L^2$, so $\operatorname{Im}\widehat{F}_2\in L^2$. By Proposition 6.4, $F_2\in L^2$, that is, (1.25) holds. □

7. Proof of Theorem 1.3

Here we will use the results of the last three sections to prove Theorem 1.3. We use the strategy of [19] as refined in [37] and [34]. We treat each direction of the theorem in a separate subsection.

$\mathbf{V \in L^2 \Rightarrow}$ (i)–(iv). As $V \in L^2$, $V(H_0+1)^{-1}$ is compact, and thus (i) holds by Weyl's Theorem. (ii) is just Theorem 6.2.

Fix $R < \infty$ and let

$$V^{(R)}(x) = \begin{cases} V(x), & 0 \le x \le R \\ 0, & x > R \end{cases} \tag{7.1}$$

so $L_t = H_0$ if $t > R$. Thus applying Theorem 4.5 to $V^{(R)}$ with $t > R$ gives

$$\tfrac{1}{2} \int_0^R V(x)^2 = \tfrac{2}{3} \sum_j [\kappa_j^{(R)}]^3 + \lim_{y\to\infty} \int_0^\infty P(\xi, y) \log\biggl(\frac{1}{T^{(R)}(\xi,0)}\biggr)\, d\xi. \tag{7.2}$$

By rewriting P as

$$P(\xi, y) = \frac{\xi^2}{\pi}\biggl[\frac{1}{(1+\frac{\xi^2}{4y^2})(1+\frac{\xi^2}{y^2})}\biggr],$$

we see that it is monotone increasing in y. As the integrand $\log(\frac{1}{T}) \ge 0$, the monotone convergence theorem implies

$$\tfrac{1}{8} \int_0^R V(x)^2\, dx = \tfrac{2}{3} \sum_{j=1}^{N(R)} \bigl[\kappa_j^{(R)}\bigr]^3 + Q(V^{(R)}). \tag{7.3}$$

Now take $R \to \infty$. Theorem 5.1 controls $Q(V^{(R)})$ and since $\kappa_j^{(R)}$ converge individually to the κ_j associated to V, we have

$$\sum_{j=1}^\infty \kappa_j^3 \le \liminf_{R\to\infty} \sum_{j=1}^\infty \bigl[\kappa_j^{(R)}\bigr]^3 \tag{7.4}$$

(a trivial instance of Fatou's Lemma). Thus (7.3) becomes

$$\tfrac{1}{8} \int_0^\infty V(x)^2\, dx \ge \tfrac{2}{3} \sum_{j=1}^\infty \kappa_j^3 + Q(V). \tag{7.5}$$

In particular, $V \in L^2$ implies $Q < \infty$, that is, (1.27) holds. As $\kappa_j^3 = [E_j^{(0)}]^{3/2}$, so $\sum [E_j^{(0)}]^{3/2} < \infty$. By (1.35), this implies (1.26). $\square$

(i)–(iv) $\mathbf{\Rightarrow V \in L^2}$. By Theorem 6.2, $d\rho$ is the spectral measure of a $V \in L^2_{\text{loc}}$ so, in particular, (4.23) holds. Since $-\kappa_j(t)^3 \le 0$ and $\log[T(\xi,t)] \le 0$, this implies that

$$\tfrac{1}{8} \int_0^t V(x)^2\, dx \le \tfrac{2}{3} \sum_j \kappa_j(0)^3 + \lim_{y\to\infty} \int_0^\infty P(\xi, y) \log\biggl(\frac{1}{T(\xi,0)}\biggr)\, d\xi. \tag{7.6}$$

By the same monotone convergence argument used in the first part of the proof,

$$\tfrac{1}{8} \int_0^t V(x)^2\, dx \le \tfrac{2}{3} \sum_j [E_j^{(0)}]^{3/2} + Q. \tag{7.7}$$

Taking $t \to \infty$, we see $V \in L^2$ and that (7.7) holds with $t = \infty$. $\square$

Our proof shows that the Faddeev–Zhabat sum rule, (1.33), holds for any $V \in L^2(0,\infty)$. Rewriting Q in terms of the reflection coefficient (see (5.14)) and fixed on $(-R,\infty)$ with $R<\infty$, one can obtain (1.33) for $V \in L^2(-\infty,\infty)$ by using the ideas in [37].

8. Isolating $\mathrm{Re}(w)$

The next four sections are devoted to deducing Theorem 1.3 from Theorem 1.2. This amounts to showing that the two lists of conditions are equivalent. (They are not equivalent item by item, only collectively.)

The role of this section is to prove that

$$\text{Strong Quasi-Szegő} \Leftrightarrow \text{Quasi-Szegő} + (R<\infty) \tag{8.1}$$

where

$$R = \int_0^\infty \log\biggl\{1+\biggl(\frac{\mathrm{Re}\, w}{k}\biggr)^2\biggr\} k^2\, dk. \tag{8.2}$$

Note that the strong quasi-Szegő condition involves both the real and imaginary parts of w, whereas the quasi-Szegő condition depends only on $\mathrm{Im}\, w$ and R only on $\mathrm{Re}\, w$. Hence the title of this section.

We now present an outline of the proof of Theorem 1.3: under the assumption of the Weyl and Lieb-Thirring conditions, we prove

(i) Strong Quasi-Szegő $\Rightarrow$ Quasi-Szegő (this section).
(ii) Strong Quasi-Szegő + Local Solubility $\Rightarrow$ Normalization (see Section 11).
(iii) Quasi-Szegő + Normalization $\Rightarrow$ $R<\infty$ (see Section 11), so, by (8.1), Quasi-Szegő + Normalization $\Rightarrow$ Strong Quasi-Szegő.
(iv) Normalization $\Rightarrow$ Local Solubility (see Section 9).

The first two statements show that the conditions in Theorem 1.3 imply those in Theorem 1.2, the second pair proves the converse.

Lemma 8.1. *For any $f \in \mathbb{C}$ and any $0<\epsilon\le 1$,*

$$\log(1+|f|^2) \le \begin{cases} \epsilon^{-1}\log(1+\epsilon|f|^2) \\ \log(1+\epsilon|f|^2)+\log(1+\epsilon^{-1}). \end{cases}$$

Moreover, if $\epsilon=(1+\delta)^{-2}$ and $\delta\ge 6$, then

$$\log(1+\epsilon^{-1}) \le 6\log(\tfrac14\delta+\tfrac12+\tfrac14\delta^{-1}).$$

Proof. The first inequality follows from the concavity of $F: x\mapsto \log(1+x|f|^2)$:

$$\epsilon\log(1+|f|^2) = (1-\epsilon)F(0)+\epsilon F(1) \le F(\epsilon) = \log(1+\epsilon|f|^2).$$

The second inequality follows from

$$1+|f|^2 \le 1+\epsilon|f|^2+\epsilon^{-1}+|f|^2 = (1+\epsilon|f|^2)(1+\epsilon^{-1})$$

by taking logarithms. For the last inequality, notice that

$$1+(1+\delta)^2 = \delta^2+2\delta+2 \le \delta^2+4\delta+6+4\delta^{-1}+\delta^{-2} = 16(\tfrac14\delta+\tfrac12+\tfrac14\delta^{-1})^2$$

and since $\delta\ge 6$, we have $2\le \frac14\delta+\frac12+\frac14\delta^{-1}$. Therefore,

$$1+(1+\delta)^2 \le (\tfrac14\delta+\tfrac12+\tfrac14\delta^{-1})^6,$$

which gives the result. $\square$

Theorem 8.2. *Using the notations*

$$SQS = \int \log\left[\frac{|w(k+i0)+ik|^2}{4k\,\mathrm{Im}\,w(k+i0)}\right] k^2\,dk, \tag{8.3}$$

$$QS = \int_0^\infty \log\left[\frac{1}{4}\frac{d\rho}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\frac{d\rho_0}{d\rho}\right]\sqrt{E}\,dE, \tag{8.4}$$

and R as in (8.2)*, we have $QS \le SQS \le QS + R$ and $R \le 55\,SQS$. In particular, $QS + R < \infty \Leftrightarrow SQS < \infty$.*

Proof. The bulk of the proof rests on the following calculation:

$$\frac{|w(k+i0)+ik|^2}{4k\,\mathrm{Im}\,w(k+i0)} = \frac{\mathrm{Im}\,w}{4k} + \frac{1}{2} + \frac{k}{4\,\mathrm{Im}\,w} + \frac{k}{4\,\mathrm{Im}\,w}\left(\frac{\mathrm{Re}\,w}{k}\right)^2 \tag{8.5}$$

$$= \left(\frac{\delta}{4} + \frac{1}{2} + \frac{1}{4\delta}\right)\left[1 + \frac{1}{(1+\delta)^2}\left(\frac{\mathrm{Re}\,w}{k}\right)^2\right] \tag{8.6}$$

where $\delta = \frac{\mathrm{Im}\,w}{k} = \frac{d\rho}{d\rho_0}$. Taking logarithms and integrating immediately shows that $QS \le SQS \le QS + R$.

To prove $R \le 55\,SQS$, we make use of the following notation:

$$\delta = \frac{d\rho}{d\rho_0}, \quad \epsilon = (1+\delta)^{-2}, \quad f(k) = \frac{\mathrm{Re}\,w}{k}, \quad \text{and} \quad A = \{E : \delta > 6\}.$$

Notice that from the calculation above,

$$SQS = \int \log(\tfrac{1}{4}\delta + \tfrac{1}{2} + \tfrac{1}{4}\delta^{-1})k^2\,dk + \int \log(1+\epsilon|f|^2)k^2\,dk.$$

Combining this with Lemma 8.1 gives

$$\begin{aligned}
R &= \int_0^\infty \log(1+|f|^2)k^2\,dk \\
&\le \int_A \log(1+\epsilon|f|^2)k^2\,dk + \int_A \log(1+\epsilon^{-1})k^2\,dk + \\
&\qquad + 49\int_{A^c} \log(1+\epsilon|f|^2)k^2\,dk \\
&\le 49\,SQS + 6\int_A \log(\tfrac{1}{4}\delta + \tfrac{1}{2} + \tfrac{1}{4}\delta^{-1})k^2\,dk \\
&\le 55\,SQS.
\end{aligned}$$

The number 49 appears because on A^c, $\delta \le 6$ which implies $\epsilon^{-1} \le 49$. □

9. The Normalization Conditions

In this section, we will prove that

$$\text{Normalization} \;\Rightarrow\; (1.16) \;\Rightarrow\; \text{Local Solubility}$$

(cf. step (iv) in the strategy of Section 8). This then implies that $d\rho$ is the spectral measure of a potential $V \in L^2_{\mathrm{loc}}$ by Theorem 6.1.

Proposition 9.1. *Let $d\nu$ be any real signed measure on $[0,\infty)$ and define $M_l\nu$ by* (1.19)*. Then the following are equivalent:*

$$M_l\nu \in L^2(dk) \tag{9.1}$$

$$|\nu|([n,n+1]) \in \ell^2 \tag{9.2}$$

$$\int \log\biggl[1 + \biggl(\frac{M_l\nu}{k}\biggr)^2\biggr] k^2\, dk < \infty. \tag{9.3}$$

Proof. It is not difficult to see that (9.1) $\Rightarrow$ (9.2):

$$\sum \bigl[|\nu|([n,n+1])\bigr]^2 \leq \int_0^\infty \bigl[|\nu|([k-1,k+1])\bigr]^2\, dk \leq 4\int_0^\infty \bigl[M_l\nu(k)\bigr]^2\, dk. \tag{9.4}$$

To prove the converse, let us write $\nu_n = |\nu|([n,n+1])$. Then, for any $k \in [n,n+1]$,

$$M_l\nu(k) = \sup_{L\geq 1} \frac{|\nu|([k-L,k+L])}{2L} \leq \sup_{m\geq 0} \frac{3}{2m+1} \sum_{j=n-m}^{n+m} \nu_j. \tag{9.5}$$

Indeed, one may take m to be the integer in $[L, L+1)$. As the discrete maximal operator in (9.5) is ℓ^2 bounded, we may deduce

$$\int_0^\infty \bigl[M_l\nu(k)\bigr]^2\, dk \leq \sum_n \Bigl[\sup_{k\in[n,n+1]} M_l\nu(k)\Bigr]^2 \leq C\sum_n \nu_n^2. \tag{9.6}$$

This proves (9.2) $\Rightarrow$ (9.1).

As $\log(1+x^2) \leq x^2$,

$$\int \log[1 + k^{-2}M_l\nu(k)^2]\, k^2\, dk \leq \int [M_l\nu(k)]^2\, dk,$$

which proves (9.1) $\Rightarrow$ (9.3).

We will finish the proof by showing that (9.3) implies (9.2). For each $k \in [n,n+1]$, it follows directly from the definition that $\frac{1}{2}\nu_n \leq M_l\nu(k)$. Thus

$$\sum_n n^2 \log\biggl[1 + \frac{\nu_n^2}{4(n+1)^2}\biggr] \leq \int \log\biggl[1 + \biggl(\frac{M_l\nu}{k}\biggr)^2\biggr] k^2\, dk, \tag{9.7}$$

which shows that $\nu_n \geq (n+1)$ only finitely many times. For the remaining values of n, one need only apply the estimate $\log(1+x) \geq \frac{1}{2}x$ for $x \in [0,1]$, which follows by comparing derivatives, to see that $\nu_n \in \ell^2$. $\square$

Theorem 9.2. *If*

$$\int \log\biggl[1 + \biggl(\frac{M_s\nu}{k}\biggr)^2\biggr] k^2\, dk < \infty,$$

then the equivalent conditions of Proposition 9.1 hold.

Proof. The result follows by the reasoning used to prove (9.3) $\Rightarrow$ (9.2): For all $k \in [n,n+1]$,

$$|\nu|([n,n+1]) \leq |\nu|([k-1,k+1]) \leq 2M_s\nu(k).$$

Thus (9.7) holds with $M_s\nu$ in place of $M_l\nu$ and the argument given above may be continued from there. $\square$

Theorem 9.3. *If* (9.2) *holds, then so does Local Solubility, that is,* (1.25). *In particular, by Theorem 9.2,*

Normalization $\Rightarrow$ Local Solubility.

Remark. This is step (iv) of the strategy in Section 8.

Proof. By the definition (1.24),

$$F(q) = 2\pi^{-1/2} \int_{p\geq 1} e^{-(p-q)^2}\, d\nu(p). \tag{9.8}$$

As $(n+x-q)^2 \geq (n-q)^2 - 2|x||n-q|$ for $|x|<1$,

$$|F(q)| \leq 2\pi^{-1/2} \sum_{n=1}^{\infty} e^{-(n-q)^2} e^{2|n-q|} |\nu|([n,n+1]). \tag{9.9}$$

Thus by Young's inequality for sums, (9.2) $\Rightarrow F \in L^2$. □

We conclude this section with a result we will need in Section 11. In the proof, we will use the following simple inequality: for $\delta \in [0,1]$,

$$\log[\tfrac{1}{4}\,\delta + \tfrac{1}{2} + \tfrac{1}{4}\,\delta^{-1}] \geq \tfrac{1}{4}(\delta-1)^2. \tag{9.10}$$

As equality holds when $\delta = 1$, the result follows by differentiating:

$$\frac{\delta - 1}{\delta(\delta+1)} \leq \frac{1}{2}(\delta - 1).$$

Theorem 9.4. *If* (*Strong*) *Quasi-Szegő and Local Solubility hold, then* ν *obeys* (9.2). *In particular, by Theorem 1.3, this follows for* $V \in L^2$.

Proof. Let us recall that the Quasi-Szegő condition says

$$\int_0^\infty \log\left[\frac{1}{4}\,\frac{d\rho}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\,\frac{d\rho_0}{d\rho}\right] k^2\, dk < \infty. \tag{9.11}$$

(By Theorem 8.2, this is also implied by the strong quasi-Szegő condition.)

Let us decompose $d\nu = d\nu_+ - d\nu_-$ where $d\nu_\pm$ are both positive measures. The definition of $d\nu$, (1.13), shows that for $k>1$,

$$\frac{d\rho}{d\rho_0} = 1 + \frac{1}{k}\,\frac{d\nu}{dk}.$$

Moreover, $d\nu_-$ is absolutely continuous; in fact, (1.15) shows $\frac{d\nu_-}{dk} \leq k$.

Let us restrict the integral (9.11) to the essential support of $d\nu_-$, that is, where $\frac{d\rho}{d\rho_0} \leq 1$. Using (9.10), we deduce that

$$\int_0^\infty \left|\frac{d\nu_-}{dk}\right|^2 dk < \infty \tag{9.12}$$

and hence that $|\nu_-|([n,n+1]) \in \ell^2$. To complete the proof, we need to deduce the same result for ν_+.

The local solubility condition says $F \in L^2$ where F is defined as in (9.8). The first sentence of Theorem 9.3 says

$$F_-(q) = \int_{p\geq 1} e^{-(p-q)^2}\, d\nu_-(p) \ \in L^2(dq)$$

and so $F \in L^2$ implies

$$F_+(q) = \int_{p\geq 1} e^{-(p-q)^2}\, d\nu_+(p) \ \in L^2(dq).$$

For $q \in [n,n+1]$, we have $F_+(q) \geq e^{-1}\nu_+([n,n+1])$ and thus may conclude $\nu_+([n,n+1]) \in \ell^2$. □

10. Harmonic Analysis Preliminaries

For harmonic functions in the half-plane, it is well known that the conjugate function belongs to L^p $(0 < p < \infty)$ if and only if the same is true for the non-tangential maximal function. The first direction appears already in the paper of Hardy and Littlewood that introduced the maximal function [16, Theorem 27]. The other direction, which is much harder, is due to Burkholder, Gundy, and Silverstein [2]. The purpose of this section is to present an analogous theorem with a peculiar replacement for L^p. Theorem 2 of [2] covers this situation perfectly if one is willing to consider the maximal Hilbert transform; we are not. However, this does resolve one direction; for the other, we will use subharmonic functions in the manner of [16].

We will use the following notation: $f \lesssim g$ means $f \le Cg$ for some absolute constant C, whereas $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

Proposition 10.1. *Let $d\sigma$ be a compactly supported positive measure on $\mathbb{R}$,*

$$\int \log\bigl[1 + |H\sigma|^2\bigr]\,dx \lesssim \int \log\bigl[1 + |M\sigma|^2\bigr]\,dx. \tag{10.1}$$

Proof. This is a special case of [2, Theorem 2]. It is also amenable to the good-λ approach discussed in textbooks: [39, §V.4] or [42, §XIII]. ☐

As noted earlier, Burkholder, Gundy, and Silverstein do not provide the converse inequality; indeed as they note, in the generality they treat, the result is false without switching to the maximal Hilbert transform. Nevertheless, the function $x \mapsto \log[1 + x^2]$ grows sufficiently quickly that the result is true. We divide the proof into two propositions.

Proposition 10.2. *There is a λ_0 so that for any finite positive measure $d\sigma$ on $\mathbb{R}$,*

$$\int_{\{M\sigma > \lambda_0\}} \log\bigl[1 + |M\sigma|^2\bigr]\,dx \lesssim \int \log\bigl[1 + (H\sigma)^2 + \bigl(\tfrac{d\sigma}{dx}\bigr)^2\bigr]\,dx. \tag{10.2}$$

In particular, $|\{M\sigma > 2\lambda_0\}| \lesssim \mathrm{RHS}(10.2)$.

Proof. Let $u(z) + iv(z) = \int d\sigma(x)/(x - z)$ denote the Cauchy integral of $d\sigma$, then

$$F(z) = \log[1 + u(z) + iv(z)]$$

is analytic—$u \ge 0$ because it is the Poisson integral of a positive measure. In particular, $|F|^{1/2}$ is subharmonic. Now as $|F(z)| \ge \log|1 + u(z)|$,

$$\log\bigl[1 + [M\sigma](x)\bigr] \lesssim \sup_{y>0} \log\bigl[1 + u(x + iy)\bigr] \tag{10.3}$$

$$\le \sup_{y>0} |F(x + iy)| \tag{10.4}$$

$$\lesssim \bigl\{[M|F|^{1/2}](x)\bigr\}^2. \tag{10.5}$$

Elementary calculations show $|\mathrm{Re}\,F| \le \log(1 + u + |v|)$ and $|\mathrm{Im}\,F| \le \frac{\pi}{2}$; therefore,

$$\log\bigl[1 + M\sigma\bigr] \lesssim \Bigl\{1 + M\sqrt{\log[1 + u + |v|]}\Bigr\}^2 \lesssim 1 + \Bigl\{M\sqrt{\log[1 + u + |v|]}\Bigr\}^2.$$

¿From this, one may deduce that for λ_1 sufficiently large,

$$\sqrt{\log\bigl[1 + M\sigma\bigr]} \lesssim M\sqrt{\log[1 + u + |v|]} \tag{10.6}$$

on the set where $\log[1 + M\sigma] \ge \lambda_1$.

Interpolating between the L^∞ and L^2 bounds on M shows that

$$\int_{Mf>\lambda} |Mf|^2\,dx \lesssim \int_{|f|>\lambda/2} |f|^2\,dx.$$

Combining this with (10.6), we see that for $\lambda_0 \geq e^{\lambda_1}$ and ϵ sufficiently small,

$$\int_{\{M\sigma>\lambda_0\}} \log\bigl[1+M\sigma\bigr]\,dx \lesssim \int_{\{|u+iv|>\epsilon\}} \log[1+u+|v|]\,dx.$$

To obtain (10.2), we need merely note that $\log(1+x) \approx \log(1+x^2)$ on any interval $[a,\infty)$ with $a>0$. $\square$

Proposition 10.3. *For any finite positive measure $d\sigma$ on $\mathbb{R}$,*

$$\int \log\bigl[1+(M\sigma)^2\bigr]\,dx \lesssim \int \log\bigl[1+(H\sigma)^2+\bigl(\tfrac{d\sigma}{dx}\bigr)^2\bigr]\,dx. \tag{10.7}$$

Proof. By Proposition 10.2, it suffices to prove

$$\int_{\{M\sigma\leq\lambda_0\}} |M\sigma|^2\,dx \lesssim \int \log\bigl[1+(H\sigma)^2+(\tfrac{d\sigma}{dx})^2\bigr]\,dx. \tag{10.8}$$

Let $\Omega=\{M\sigma>4\lambda_0\}$, $d\sigma_1=\chi_\Omega d\sigma$, and $d\sigma_2=\chi_{\Omega^c}d\sigma$. We will prove (10.8) by writing $M\sigma \leq M\sigma_1+M\sigma_2$.

It is a well-known property of the maximal function that

$$\sigma(\{M\sigma>4\lambda_0\}) \lesssim \lambda_0|\{M\sigma>2\lambda_0\}|.$$

Combining this with Proposition 10.2, shows that $\|\sigma_1\|=\sigma(\Omega)\lesssim \mathrm{RHS}(10.8)$. Consequently, by the weak-type L^1 bound on the maximal operator,

$$\int_{\{M\sigma\leq\lambda_0\}} |M\sigma_1|^2\,dx \lesssim \int_0^{\lambda_0} \frac{\|\sigma_1\|}{\lambda} 2\lambda\,d\lambda \lesssim \mathrm{RHS}(10.8).$$

Now we turn to bounding $M\sigma_2$. On Ω^c, we know that $d\sigma$ must be absolutely continuous and its Radon-Nikodym derivative is bounded by $4\lambda_0$. Therefore, L^2 boundedness of the maximal operator implies

$$\int |M\sigma_2|^2\,dx \lesssim \int_{\{M\sigma\leq4\lambda_0\}} |\tfrac{d\sigma}{dx}|^2\,dx \lesssim \int \log\bigl[1+\bigl(\tfrac{d\sigma}{dx}\bigr)^2\bigr]\,dx,$$

which completes the proof. $\square$

Putting the previous propositions together, we obtain the following

Theorem 10.4. *If σ is a positive measure of compact support, then*

$$\int \log\Bigl[1+|H\sigma|^2+(\tfrac{d\sigma}{dx})^2\Bigr]\,dx \approx \int \log\Bigl[1+|M\sigma|^2\Bigr]\,dx. \tag{10.9}$$

11. Taming $\mathrm{Re}\,m$

The purpose of this section is to prove Corollary 11.3 below and so complete the proof of Theorem 1.2 as laid out in Section 8.

Let H_s denote the short-range Hilbert transform: $H_s\sigma = K*\sigma$ where

$$K(x)=\begin{cases}0, & |x|>1\\ \frac{1}{\pi}[x^{-1}-x], & |x|<1\end{cases}$$

and let $H_l = H - H_s$ denote the long-range Hilbert transform: $H_l\sigma = K * \sigma$ with

$$K(x) = \begin{cases} \frac{1}{\pi}x^{-1}, & |x| > 1 \\ \frac{1}{\pi}x, & |x| < 1. \end{cases}$$

Note that both H_s and H_l are Calderón–Zygmund operators and so bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. As in the Introduction, we define short- and long-range maximal operators:

$$[M_s\sigma](x) = \sup_{L\leq 1} \frac{|\sigma|\big([x-L, x+L]\big)}{2L},$$

and for M_l, the supremum is taken over $L \geq 1$. Naturally, both truncated maximal operators are L^p-bounded for $1 < p \leq \infty$.

We will use the notation

$$\|\mu\|^2_{\ell^2(M)} = \sum_n \Big[|\mu|\big([n, n+1]\big)\Big]^2$$

as introduced in (1.16). Obviously, $\|\mu\|^2_{\ell^2(M)} \leq \|\mu\|^2$.

Lemma 11.1. *Let $F(k) = (1+k^2)^{-1}$. For each complex measure $\mu \in \ell^2(M)$,*

$$\int \big[\Phi * |d\mu|\big]^2\, dk \lesssim \|\mu\|^2_{\ell^2(M)}, \quad \int |M_l\mu|^2\, dk \lesssim \|\mu\|^2_{\ell^2(M)}, \quad \int |H_l\mu|^2\, dk \lesssim \|\mu\|^2_{\ell^2(M)}.$$

Proof. All three inequalities follow by replacing $|d\mu|$ by its average on each of the intervals $[n, n+1]$. This operation changes $\Phi * |d\mu|$ and $M_l\mu$ by no more than a factor of two. For H_l, it introduces an error which can be bounded by $\Phi * |d\mu|$. We then use the L^2 boundedness of the appropriate operator. $\square$

Theorem 11.2. *If μ is a positive measure on $\mathbb{R}$ with $\|\mu\|^2_{\ell^2(M)} < \infty$, then*

$$\int \log\Big[1 + |H\mu|^2 k^{-2}\Big]\,(1+k^2)\,dk < \infty \tag{11.1}$$

if and only if

$$\int \log\Big[1 + |M_s\mu|^2 k^{-2}\Big]\,(1+k^2)\,dk < \infty. \tag{11.2}$$

Proof. As neither integral can diverge on any compact set, we can restrict our attention to $k > 1$.

We begin by proving that (11.2) implies (11.1). Given a compactly supported positive measure $d\sigma$, Theorem 10.4 and Lemma 11.1 show that

$$\begin{aligned}
\int_n^{n+1} \log\Big[1 + |H_s\sigma|^2\Big]\,dk &\lesssim \int_n^{n+1} \log\Big[1 + |H\sigma|^2\Big] + |H_l\sigma|^2\,dk \\
&\lesssim \|\sigma\|^2_{\ell^2(M)} + \int_n^{n+1} \log\Big[1 + |H\sigma|^2\Big]\,dk \\
&\lesssim \|\sigma\|^2_{\ell^2(M)} + \int \log\Big[1 + |M\sigma|^2\Big]\,dk \\
&\lesssim \|\sigma\|^2_{\ell^2(M)} + \int \log\Big[1 + |M_s\sigma|^2\Big]\,dk + \int |M_l\sigma|^2\,dk \\
&\lesssim 2\|\sigma\|^2_{\ell^2(M)} + \int \log\Big[1 + |M_s\sigma|^2\Big]\,dk.
\end{aligned}$$

Let us choose $\sigma = (1+n^2)^{-1/2} d\mu_n$ where $d\mu_n$ is the restriction of $d\mu$ to the interval $[n-1, n+2]$. Combining the above with Lemma 11.1 gives

$$\begin{aligned}
\sum (1+n^2) \int_n^{n+1} \log\Big[1 + \tfrac{1}{n^2+1} |H\mu|^2\Big]\, dk& \\
&\lesssim \|\mu\|^2_{\ell^2(M)} + \sum (1+n^2) \int_n^{n+1} \log\Big[1 + \tfrac{1}{n^2+1} |H_s\mu|^2\Big]\, dk \\
&\lesssim \|\mu\|^2_{\ell^2(M)} + \sum (1+n^2) \int_{n-3}^{n+4} \log\Big[1 + \tfrac{1}{n^2+1} |M_s\mu|^2\Big]\, dk \\
&\lesssim \|\mu\|^2_{\ell^2(M)} + \int \log\Big[1 + k^{-2} |M_s\mu|^2\Big] (1+k^2)\, dk.
\end{aligned}$$

The proof that (11.1) implies (11.2) is a little more involved because the Hilbert transform is positivity preserving.

Let ϕ be a smooth bump which is supported on $[-2, 3]$ and is equal to 1 on $[-1, 2]$. We will write $\phi_n(x)$ for $\phi(x-n)$. Elementary calculations show that

$$\Big| [H_s(\phi d\sigma)](x) - \phi(x)[H_s\sigma](x) \Big| \lesssim \|\sigma\| \Phi(x) \tag{11.3}$$

where $\Phi(x) = (1+x^2)^{-1}$. Using Theorem 10.4, Lemma 11.1, and then (11.3),

$$\begin{aligned}
\int_n^{n+1} \log\Big[1 + |M_s\sigma|^2\Big]\, dk &\le \int_n^{n+1} \log\Big[1 + |M(\phi_n d\sigma)|^2\Big]\, dk \\
&\lesssim \int \log\Big[1 + |H(\phi_n d\sigma)|^2 + |\phi_n \tfrac{d\sigma}{dk}|^2\Big]\, dk \\
&\lesssim \int \log\Big[1 + |H_s(\phi_n d\sigma)|^2\Big]\, dk + \|\sigma\|^2 \\
&\lesssim \int \log\Big[1 + \phi_n^2 |H_s\sigma|^2\Big]\, dk + \|\sigma\|^2
\end{aligned}$$

By choosing $\sigma = (1+n^2)^{-1/2} d\mu_n$ where $d\mu_n$ is the restriction of $d\mu$ to the interval $[n-4, n+5]$, the proof may be completed in much the same manner as was used to prove the opposite implication. ☐

It is now easy to complete the outline from Section 8.

Corollary 11.3. *In the nomenclature of Theorems 1.2, 1.3, and 8.2,*

$$\textit{Normalization} \Rightarrow R < \infty \tag{11.4}$$

$$\textit{Strong Quasi-Szegő} \ + \ \textit{Local Solubility} \Rightarrow \textit{Normalization}. \tag{11.5}$$

Proof. We begin with (11.4). As the m-function associated to the free operator is purely imaginary on the spectrum, we have that for all $k > 0$,

$$\operatorname{Re} w(k) = \int_{(-\infty,1]} \frac{d\rho(E) - d\rho_0(E)}{E - k^2} + \frac{2}{\pi} \int \frac{\xi d\nu(\xi)}{\xi^2 - k^2} \tag{11.6}$$

$$= f(k) + \frac{1}{\pi} \int \frac{d\nu(\xi)}{\xi + k} + \frac{1}{\pi} \int \frac{d\nu(\xi)}{\xi - k} \tag{11.7}$$

$$= f(k) + [H\mu](k) \tag{11.8}$$

where $f(k)$ is defined to be the first term on the RHS of (11.6) and $d\mu$ is defined by

$$\int g(k)\,d\mu(k) = \int g(k)\,d\nu(k) + \int g(-k)\,d\nu(k).$$

By Theorem 9.2, Normalization implies $\nu \in \ell^2(M)$ and hence $\mu \in \ell^2(M)$; thus we may apply Theorem 11.2 to see that (11.1) holds. As

$$\log[1+(x+y)^2] \lesssim x^2 + \log[1+y^2]$$

and $|f(k)| \lesssim (k-1)^{-1}$ for $k > 1$, we see that this is sufficient to deduce $R < \infty$.

We now turn to (11.5). By Theorem 8.2, we know that $R < \infty$ and so by the calculation above, (11.1) holds. From the proof of Theorem 9.4 we are guaranteed that $d\mu$, defined as above, belongs to $\ell^2(M)$. Thus we may apply Theorem 11.2 to deduce that the normalization condition holds. □

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