Approximations of permutation-symmetric vertex couplings in quantum graphs

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Abstract. We consider boundary conditions at the vertex of a star graph which make Schrödinger operators on the graph self-adjoint, in particular, the two-parameter family of such conditions invariant with respect to permutations of graph edges. It is proved that the corresponding operators can be approximated in the norm-resolvent sense by elements of another Schrödinger operator family on the same graph in which the δ coupling is imposed at the vertex and an additional point interaction is placed at each edge provided the coupling parameters are properly chosen.

1. Introduction

There is no necessity to describe here *in extenso* what quantum graphs are and why they are important; if such a need nevertheless arises we can refer to papers from the dawns of the history [**RS53**], from the times of new beginning in the eighties [**GP88, EŠ89**], to more recent work containing a rich bibliography [**KS99, Ku04**], and last not least, to the other contributions making this volume.

As in the most of the mentioned work, the object of our interest here are Schrödinger operators on metric graphs; we neglect external fields and consider a free spinless particle on the graph, with the Hamiltonian which acts as $H\psi_j = -\psi_j''$, where ψ_j denotes the wave function at the jth edge. It is known for longtime [**EŠ89**] that in order to make H self-adjoint, a vertex joining n graph edges may be characterized by boundary conditions involving n^2 real parameters; they have the form of a linear relation between $\Psi(0)$, the column vector of the boundary values at the vertex (identified conventionally with the origin of the coordinates), and $\Psi'(0)$, the vector of the derivatives, taken all in the outgoing direction.

A general and elegant form of these boundary conditions was found in [**KS99**]: any vertex in which n edges meet can be described by a pair of $n \times n$ matrices A, B such that $\operatorname{rank}(A, B) = n$ and the product AB^* is self-adjoint. The self-adjointness

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is guaranteed if the corresponding boundary values satisfy the condition

$$A\Psi(0) + B\Psi'(0) = 0\,. \tag{1.1}$$

Moreover, soon after several authors [**FT00, Ha00, KS00**] pointed out that the matrix pair in (1.1) can be made unique by choosing

$$A = U - I\,, \quad B = i(U + I) \tag{1.2}$$

with a unitary U; a nontrivial coupling between the edges corresponds naturally to the situation when the matrix U is non-diagonal. A simple proof of this fact for $n = 2$ was given in [**FT00**] and extended to any n in [**CE04**].

While the conditions (1.1) ensure self-adjointness of quantum graph Hamiltonians, or in physical terms conservation of probability current in the vertex, they say nothing about a *physical meaning of such a vertex coupling*. A natural way to address the last question is to investigate approximations of a quantum graph by more realistic systems with no free parameters. An example is a quantum particle living in a configuration space in the form of a thin tube-like domain; one can consider a family of such domains shrinking to the given graph. A solution to this problem at the level of eigenvalue convergence was found [**KZ01, RS01, Sa01, EP05**] in the situation that the tube-like domain supports Laplacian with Neumann boundary conditions (or similar operators), and an extension to the resolvent convergence has been announced [**Po05**]. These results, however, gave a partial answer to the problem stated above because the limit leads to the *free boundary conditions*,

$$\psi_j(0) = \psi_k(0)\,,\; j,k = 1,\dots,n\,, \quad \sum_{j=1}^{n} \psi_j'(0) = 0\,, \tag{1.3}$$

only. It is hoped that other approximating families, say, using Dirichlet Laplacians, geometrically induced and/or external potentials, could yield different vertex couplings, but this problem is difficult and no such results are known at present.

A less ambitious program aims at approximating vertex couplings by means of Schrödinger operators on the graph itself, using suitable families of scaled potential, regular or singular. This is relatively easy as long as we attempt to approximate couplings with wavefunctions continuous at the vertex, i.e. the one-parameter family of the so-called δ *couplings* [**Ex96a**] described by the conditions

$$\psi_j(0) = \psi_k(0) =: \psi(0)\,,\; j,k = 1,\dots,n\,, \quad \sum_{j=1}^{n} \psi_j'(0) = \alpha\psi(0) \tag{1.4}$$

with $\alpha \in \mathbb{R}$, which is obtained from (1.1) and (1.2) by choosing $U = \frac{2}{n+i\alpha}J - I$, where J and I are the $n \times n$ matrix whose all entries are equal to one and the unit matrix, respectively. The procedure is analogous to the approximation of δ interaction of the line [**AGHH**]: one starts with the conditions (1.3) and adds at each edge a naturally scaled potential, $V_{j,\epsilon}(x) = \frac{1}{\epsilon}V_j\left(\frac{x}{\epsilon}\right)$ for some $V_j \in L^1$. A norm-resolvent limit then yields the δ coupling with $\alpha := \sum_j \int V_j(x)\,\mathrm{d}x$ [**Ex96b**].

The situation is more complicated if the wavefunctions are discontinuous at the vertex. The simplest example of such a situation is the so-called δ_s' *coupling*,

$$\psi_j'(0) = \psi_k'(0) =: \psi'(0)\,,\; j,k = 1,\dots,n\,, \quad \sum_{j=1}^{n} \psi_j(0) = \beta\psi'(0) \tag{1.5}$$

with $\beta \in \mathbb{R}$. An inspiration can be found in the way in which Cheon and Shigehara [**CS98a, CS98b**] approximated formally the δ' interaction on the line using a nonlinearly scaled family of δ interactions – their argument was later shown to yield a norm-resolvent convergence and to lead to approximations in terms of regular potentials [**AN00, ENZ01**]. It was shown in [**CE04**][1] that the CS-type method can be used to approximate the δ'_s coupling for any n, the approximating operator domains having functions continuous at the vertex. The aim of this paper is to show that *this result can be extended to couplings* with discontinuous wavefunctions *which are invariant with respect to permutations of the graphs edges*: an approximation using a δ coupling at the vertex and an n-tuple of δ interactions at the edges approaching the vertex will be derived for all such couplings[2]. We will see that in the generic case the idea of [**CS98a**] has a direct, albeit rather tedious extension to the graph case, while for two one-parameter subfamilies the choice of coupling parameters requires a modification. Extensions to more general boundary conditions inspired by [**SMC99**] and approximations by regular potentials are left to a subsequent publication.

2. Permutation symmetric vertex couplings

As in the previous work cited above we consider a *star graph* Γ consisting of n halflines meeting at a single vertex. The corresponding Hilbert space is thus $\mathcal{H} = \bigoplus_{j=1}^n L^2(\mathbb{R}_+)$. A general Hamiltonian describing a free particle living on the graph is a self-adjoint extension of the operator H_0 acting as $H_0\psi_j = -\psi_j''$ on functions $\Psi = \{\psi_j\} \in \bigoplus_{j=1}^n W^{2,2}(\mathbb{R}_+)$ satisfying the conditions $\psi_j(0) = \psi_j'(0) = 0$; each such extension is specified by a boundary condition (1.1) at the vertex.

Since the action of these operators at each component of the wavefunction is the same, symmetry properties of the extensions are given by those of the boundary conditions. We will be interested in the permutation-invariant extensions, first introduced in [**EŠ89**], which form a two-parameter family.

Proposition 2.1. *The boundary conditions (1.1) are permutation invariant if and only if the matrix U in (1.2) equals*

$$U = aI + bJ \tag{2.1}$$

with complex coefficients a, b satisfying the relations

$$|a| = 1 \quad \text{and} \quad |a + nb| = 1. \tag{2.2}$$

Proof. The condition (1.1) is permutation invariant *iff* it is satisfied at the same time by the vectors $P\Psi(0)$ and $P\Psi'(0)$ for any $P \in \mathcal{S}_n$. Multiplying it by P^{-1} from the left we get

$$(P^{-1}UP - I)\Psi(0) + i(P^{-1}UP + I)\Psi'(0) = 0\,,$$

the matrix $P^{-1}UP$ being obviously unitary. In view of the uniqueness of the parametrization (1.2) the property is equivalent to $P^{-1}UP = U$. Next we notice that a simultaneous permutation of the rows and columns leaves the diagonal elements on the diagonal, and the off-diagonal ones off the diagonal; since $P^{-1}UP = U$

[1] A small correction is due with respect to this paper: in the statement and proof of Theorem 1 β/n should be replaced everywhere by β.

[2] We will be concerned with nontrivial couplings only corresponding to non-diagonal matrices U; in the opposite case the task is reduced to a much simpler halfline problem – cf. [**FCT02**].

has to be satisfied for any $P \in \mathcal{S}_n$ it follows that $U = aI + bJ$ for some $a, b \in \mathbb{C}$. The conditions restricting the values of a, b follow from the unitarity of U,

$$(UU^*)_{ij} = |a|^2\delta_{ij} + 2\Re(a\bar{b}) + n|b|^2 = \delta_{ij}\,,$$

which yields the relations

$$|a|^2 + 2\Re(a\bar{b}) + n|b|^2 = 1\,, \quad 2\Re(a\bar{b}) + n|b|^2 = 0\,,$$

for $i = j$ and $i \neq j$, respectively. Substituting from the second to the first one we get $|a|^2 = 1$. Finally, using $|a + nb|^2 = |a|^2 + 2n\Re(a\bar{b}) + n^2|b|^2$ we see that the left-hand side of the second relation is a multiple of $|a + nb|^2 - |a|^2$. □

For definiteness we will denote in the following the self-adjoint extension corresponding to fixed a, b as $H^{a,b}$. Notice that the boundary conditions described by Proposition 2.1 can be also written more explicitly as the following system,

$$(a-1)\psi_j(0) + b\sum_{k=1}^{n}\psi_k(0) + i(a+1)\psi_j'(0) + ib\sum_{k=1}^{n}\psi_k'(0) = 0\,, \quad j = 1, \dots, n\,,$$

which shows, in particular, that $(a-1)\psi_j(0) + i(a+1)\psi_j'(0)$ is independent of j. To get a useful equivalent formulation we subtract the kth one of these condition from the jth one obtaining

$$(a-1)\left(\psi_j(0) - \psi_k(0)\right) + i(a+1)\left(\psi_j'(0) - \psi_k'(0)\right) = 0\,, \quad j, k = 1, \dots, n\,, \tag{2.3}$$

while summing all of them gives

$$(a - 1 + nb)\sum_{k=1}^{n}\psi_k(0) + i(a + 1 + nb)\sum_{k=1}^{n}\psi_k'(0) = 0\,. \tag{2.4}$$

Examples 2.2. We have already mentioned that $a = -1$ and $b = \frac{2}{n+i\alpha}$ describes the δ coupling (1.4), similarly $a = 1$ and $b = \frac{2}{i\beta - n}$ corresponds to the δ_s' coupling (1.5). Another example is the δ' coupling [**Ex96a**],

$$\sum_{j=1}^{n}\psi_j'(0) = 0\,, \quad \psi_j(0) - \psi_k(0) = \frac{\beta}{n}(\psi_j'(0) - \psi_k'(0))\,,\ j, k = 1, \dots, n\,, \tag{2.5}$$

referring to $a = \frac{i\beta+n}{i\beta-n}$ and $b = \frac{2}{n-i\beta}$, and its dual counterpart δ_p with the roles of functions and derivatives interchanged for which $a = \frac{n-i\alpha}{n+i\alpha}$ and $b = -\frac{2}{n+i\alpha}$.

3. Approximation: a heuristic argument

Let us describe the family we will employ to approximate permutation-symmetric Hamiltonians $H^{a,b}$. Let us recall that we will consider all operators of this class with the exception of those with a δ coupling, i.e. with the wavefunctions continuous at the vertex, because for the latter we have the natural approximation described in the introduction. We denote by $H_{u,v}(d)$ the operator which is obtained from $H_{u,0} := H^{-1,2/(n+iu)}$ by adding a δ interaction of strength v to each edge at the distance d from the vertex; it is the same scheme which was used in the particular case of δ_s' treated in [**CE04**]. The aim of the present section is to derive formally how the values of the parameters u, v as functions of d should be chosen.

At the vertex the boundary condition defining $H_{u,v}(d)$ are of the form (1.4) with α replaced by u, while the added δ interactions are characterized by

$$\psi_j(d+) = \psi_j(d-) =: \psi_j(d)\,, \ \ \psi_j'(d+) - \psi_j'(d-) = v\psi_j(d)\,, \quad j = 1, \dots, n\,. \tag{3.1}$$

To find relations between the boundary values, we employ Taylor expansion

$$\psi_j(d) = \psi_j(0) + d\psi_j'(0) + \mathcal{O}(d^2)\,, \quad \psi_j'(d-) = \psi_j'(0+) + \mathcal{O}(d)\,, \quad j = 1,\dots,n\,; \tag{3.2}$$

we want to choose u, v to get the relations (2.3) and (2.4) in the limit $d \to 0+$. The first one of the relations (3.2) together with the continuity at the vertex imply

$$\psi_j(d) - \psi_k(d) = d\left(\psi_j'(0) - \psi_k'(0)\right) + \mathcal{O}(d^2)\,. \tag{3.3}$$

Furthermore, the second one of the relations (3.2) in combination with (3.1) tell us that the difference $\psi_j'(0+) - \psi_k'(0+)$ is equal to

$$\psi_j'(d-) - \psi_k'(d-) + \mathcal{O}(d) = \psi_j'(d+) - \psi_k'(d+) - v(\psi_j(d) - \psi_k(d)) + \mathcal{O}(d)$$

giving thus $d\left(\psi_j'(d+) - \psi_k'(d+)) - v(\psi_j(d) - \psi_k(d))\right) + \mathcal{O}(d^2)$ as the value of the left-hand side in (3.3), which can be rewritten as

$$(1 + dv)\,(\psi_j(d) - \psi_k(d)) - d\left(\psi_j'(d+) - \psi_k'(d+)\right) = \mathcal{O}(d^2)\,.$$

This should give $(a-1)(\psi_j(0) - \psi_k(0)) + i(a+1)(\psi_j'(0+) - \psi_k'(0+)) = 0$ in the limit $d \to 0+$. As we have mentioned above, the case of a δ interaction in which we have $a = -1$ is excluded, hence we are allowed to require $\frac{1+dv}{d} = \frac{a-1}{i(a+1)}$. This in turn yields the following relation for the parameter v,

$$v = -\frac{1}{d} - i\frac{a-1}{a+1}\,; \tag{3.4}$$

notice that it is real-valued in view of the condition $|a| = 1$, because

$$i\frac{a-1}{a+1} = i\frac{|a|^2 + 2i\Im a - 1}{|a+1|^2} = -2\frac{\Im a}{|a+1|^2} \in \mathbb{R}\,.$$

It remains to find u. We employ again the first of the relations (3.2) together with both the vertex conditions (1.4) for $\alpha = u$ rewriting in this way $\sum_{j=1}^n \psi_j(d)$ as

$$n\psi(0) + d\sum_{j=1}^n \psi_j'(0+) + \mathcal{O}(d^2) = \frac{n}{u}\sum_{j=1}^n \psi_j'(0+) + d\sum_{j=1}^n \psi_j'(0+) + \mathcal{O}(d^2)$$

As before we use (3.1) and (3.2) to eliminate $\psi_j'(0+)$,

$$\sum_{j=1}^n \psi_j'(0+) = \sum_{j=1}^n \psi_j'(d-) + \mathcal{O}(d) = \sum_{j=1}^n \psi_j'(d+) - v\sum_{j=1}^n \psi_j(d) + \mathcal{O}(d)$$

Substituting into the expression for $\sum_{j=1}^n \psi_j(d)$ we get after a simple manipulation

$$\left(1 + v\left(\frac{n}{u} + d\right)\right)\sum_{j=1}^n \psi_j(d) = \left(\frac{n}{u} + d\right)\left(\sum_{j=1}^n \psi_j'(d+) + \mathcal{O}(d)\right) + \mathcal{O}(d^2)\,.$$

using the value of v given by (3.4) we find that the quantity

$$\left(\left(\frac{1}{d} + i\frac{a-1}{a+1}\right)\frac{n}{u} + i\frac{a-1}{a+1}d\right)\sum_{j=1}^n \psi_j(d) + \left(\frac{n}{u} + d\right)\left(\sum_{j=1}^n \psi_j'(d+) + \mathcal{O}(d)\right) \tag{3.5}$$

behaves as $\mathcal{O}(d^2)$ in the limit $d \to 0+$. We will look for u having a stronger singularity than v assuming $\frac{1}{u} = \mathcal{O}(d^2)$; then the last claim simplifies as follows,

$$\left(\frac{1}{d^2}\frac{n}{u} + i\frac{a-1}{a+1}\right)\sum_{j=1}^n \psi_j(d) + \sum_{j=1}^n \psi_j'(d+) = \mathcal{O}(d)\,.$$

This is required to give the condition (2.4) in the limit $d \to 0+$ which happens if

$$\frac{1}{d^2}\frac{n}{u} + i\frac{a-1}{a+1} = \frac{a-1+nb}{i(a+1+nb)}\,,$$

provided the two denominators containing the coupling parameters do not vanish. The first one is zero for the δ coupling which we have excluded from the outset, the second one vanishes *iff* a, b correspond to the δ_p coupling described in Examples 2.2. It is also clear that in view of the conditions $|a| = 1$ a $|a + nb| = 1$ the fractions $\frac{a-1+nb}{a+1+nb}$ and $\frac{a-1}{a+1}$ are purely imaginary. This motivates us to choose

$$u = i\frac{n}{d^2}\left(\frac{a-1+nb}{a+1+nb} + \frac{a-1}{a+1}\right)^{-1} \tag{3.6}$$

assuming that the expression in the parentheses is nonzero which is true as long as

$$a(a+nb) \neq 1\,. \tag{3.7}$$

The parameter u defined by (3.6) is, of course, real and $u = \mathcal{O}(d^{-2})$ as $d \to 0+$; this concludes our search for the approximating operator family in the generic case.

It remains to carry on the heuristic argument for the two excluded one-parameter subfamilies, the δ_p coupling and the one violating the condition (3.7). We will show that the coupling of the δ interactions at the graph arms can be preserved, it is only necessary to change the function u describing the vertex. Let us first suppose that the latter has a stronger singularity at $d = 0$, for instance,

$$u = \frac{\zeta}{d^3} \tag{3.8}$$

for a fixed nonzero $\zeta \in \mathbb{R}$ (in fact, one can replace d^3 by d^ν for any $\nu > 2$). Substituting this into (3.5) we get a condition which in the limit $d \to 0+$ yields

$$i\frac{a-1}{a+1}\sum_{j=1}^{n}\psi_j(0) + \sum_{j=1}^{n}\psi_j'(0) = 0\,.$$

The left-hand side makes sense since $a \neq -1$ and it is easy to check that if (3.7) is not valid, i.e. $a + nb = a^{-1}$, the last relation is equivalent to (2.4). On the other hand, to deal with the δ_p coupling we take u with a pole singularity,

$$u = -\frac{n}{d}\,. \tag{3.9}$$

The second term in (3.5) then vanishes and we find that $\sum_{j=1}^{n}\psi_j(0) = \mathcal{O}(d^2)$ which gives in the limit $d \to 0+$ the condition (2.4) for the particular case of δ_p.

4. The main result

Now we are ready to formulate and prove our main result.

Theorem 4.1. *Given complex numbers $a \neq -1$ and $b \neq 0$ satisfying the conditions (2.2), define $u = u(d)$ and $v = v(d)$ for $d > 0$ as in the previous section, i.e. by the relations* (3.6), (3.8), (3.9), *and* (3.4), *respectively; then the operators $H_{u,v}(d)$ converge to $H^{a,b}$ in the norm resolvent topology as $d \to 0+$.*

Proof. To begin with we observe that the permutation symmetry of the boundary conditions (2.3) and (2.4) allows us to simplify the task by reducing it to independent halfline problems. To this aim let us find the spectrum of the matrix U. Since the latter equals $U = aI + bJ$ it is sufficient to look at the matrix J

which has rank one, and thus zero is its eigenvalue of multiplicity $n-1$ corresponding to vectors with vanishing component sum; the remaining simple eigenvalue is n. The corresponding eigenvalues of U are a and $a+nb$, respectively, with the same multiplicities. Consider first the generic case where these eigenvalues are not inverse to each other by (3.7) and none of them equals to -1.

The symmetry allows us to decompose the operator in question, $H^{a,b}$ on the Hilbert space $\mathcal{H}=\bigoplus_{j=1}^n L^2(\mathbb{R}^+)=L^2(\mathbb{R}^+)\otimes\mathbb{C}^n$, into orthogonal sum of two components. The first one denoted as $H^{(1)a,b}$ acts at the "scalar" subspace isomorphic to $L^2(\mathbb{R}^+)\otimes\mathbb{C}$ consisting of functions $\Psi\in\mathcal{H}$ which are symmetric with respect to permutations, $\psi_j(x)=\psi_k(x)$ for all $j,k=1,\dots,n$; it is characterized by the boundary conditions $(a+nb-1)\Psi(0)+i(a+nb+1)\Psi'(0)=0$. The other one for which we use the symbol $H^{(n-1)a,b}$ acts on the orthogonal complement which is isomorphic to $L^2(\mathbb{R}^+)\otimes\mathbb{C}^{n-1}$ consisting of $\Psi\in\mathcal{H}$ with vanishing component sum. The action of $H^{(n-1)a,b}$ on all linear combinations $\sum_{j=1}^n c_j\psi_j(x)$ is identical and the boundary conditions are $(a-1)\Psi(0)+i(a+1)\Psi'(0)=0$.

In the same way one can decompose the approximating operators $H_{u,v}(d)$. The part $H^{(1)}_{u,v}(d)$ acts on the "scalar" subspace of functions invariant with respect to permutations, the boundary conditions being $\Psi'(0)=\frac{u}{n}\Psi(0)$. The remaining component acts on "(n-1)-dimensional vector functions" being isomorphic to $n-1$ copies of the "scalar" problem with Dirichlet boundary conditions.

We will use the fact that the resolvents of all the involved operators can be constructed explicitly using a standard ODE result in combination with Krein's formula. Let us consider first the part independent of the coupling at the vertex. For a fixed k from the upper complex halfplane the Green function of the Laplacian on the halfline with Dirichlet condition at the origin is

$$\mathcal{G}_k(x,y)=\frac{1}{k}\sin(kx_<)^{ikx_>}=\frac{1}{\kappa}\sinh(\kappa x_<)^{-\kappa x_>}\,,$$

where we denote conventionally $x_<=\min\{x,y\}$, $x_>=\max\{x,y\}$, and $\kappa=-ik$. The δ interaction at the point $x=d$ represents a rank-one perturbation of the above free resolvent, and corresponding Green's function is found easily with the help of (3.1) as in [**AGHH**, Sec. I.3] or [**CE04**] to be equal to

$$\mathcal{G}_k^v(x,y)=\mathcal{G}_k(x,y)+\frac{\mathcal{G}_k(x,d)\mathcal{G}_k(d,y)}{-v^{-1}-\mathcal{G}_k(d,d)}\,. \tag{4.1}$$

Next we have to find the Green function of the approximated operator. Following [**We**, Sec. 8.4] we need a solution of the equation $-\psi''=k^2\psi$ satisfying the condition $(a-1)\psi(0)+i(a+1)\psi'(0)=0$ and its Wronskian with $\phi(x)=\mathrm{e}^{-\kappa x}$; this yields

$$\mathcal{G}_{i\kappa}^a(x,y)=\frac{(i(a-1)\sinh\kappa x_<+\kappa(a+1)\cosh\kappa x_<)\,\mathrm{e}^{-\kappa x_>}}{\kappa(i(a-1)+\kappa(a+1))}\,. \tag{4.2}$$

Now we are going to show that $\mathcal{G}_{i\kappa}^v$ converges to $\mathcal{G}_{i\kappa}^a$ pointwise as $d\to 0+$. First we suppose that both arguments are not smaller than d; without loss of generality we may put $d\le x\le y$ rewriting the difference $\mathcal{G}_{i\kappa}^v(x,y)-\mathcal{G}_{i\kappa}^a(x,y)$ as

$$\frac{\mathrm{e}^{-\kappa y}}{\kappa}\left(\sinh\kappa x+\frac{\mathrm{e}^{-\kappa x}\sinh^2\kappa d}{\frac{\kappa d(a+1)}{a+1+id(a-1)}-\mathrm{e}^{-\kappa d}\sinh\kappa d}-\frac{i(a-1)\sinh\kappa x+\kappa(a+1)\cosh\kappa x}{i(a-1)+\kappa(a+1)}\right)$$

The sum of the first and the third term in the bracket equals $\frac{-\kappa(a+1)\mathrm{e}^{-\kappa x}}{i(a-1)+\kappa(a+1)}$ being independent of d. In the second term we use the expansion $\sinh(x) = x + \mathcal{O}(x^3)$ and obtain after simple manipulations

$$\frac{\mathrm{e}^{-\kappa x}\sinh^2\kappa d}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \mathrm{e}^{-\kappa d}\sinh\kappa d} = \mathrm{e}^{-\kappa x}\left(\frac{\kappa(a+1)}{i(a-1)+\kappa(a+1)} + \mathcal{O}(d^2)\right),$$

hence the terms non-vanishing in the limit cancel and there is a $K>0$ such that

$$|\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)| < K\,\mathrm{e}^{-\kappa x}\mathrm{e}^{-\kappa y}\,d^2$$

holds for all $d<x<y$, and by the same argument also for $d<y<x$.

Next we suppose that $x \le d \le y$ when the Green's function difference is

$$\frac{\mathrm{e}^{-\kappa y}}{\kappa}\left(\sinh\kappa x + \frac{\sinh\kappa x\,\mathrm{e}^{-\kappa d}\sinh^2\kappa d}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \mathrm{e}^{-\kappa d}\sinh\kappa d} - \frac{i(a-1)\sinh\kappa x + \kappa(a+1)\cosh\kappa x}{i(a-1)+\kappa(a+1)}\right)$$

We want to show that the expression in the bracket is uniformly bounded in x,d provided d is small enough; we may suppose that $d<1$. The first and the third term are bounded in view of the continuity at $x=0$, the middle one is easily found to be $\mathcal{O}(1)$ as $d\to 0+$. Hence there is an $L>0$ independent of x,y and d such that $|\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)| < L\,\mathrm{e}^{-\Re(\kappa)y}$, and in a similar way one can estimate the Green function difference for $y \le d < x$. It remains to deal with the case when both argument are less than d, say $x<y<d$; we may again suppose that $d<1$. The middle term in the above expression is then replaced by

$$\frac{\sinh\kappa x\,\sinh\kappa y\,\mathrm{e}^{-2\kappa d}}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \mathrm{e}^{-\kappa d}\sinh\kappa d}$$

and one checks easily that there is an $M>0$ independent of x,y and d such that the pointwise estimate $|\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)| < M$ holds.

These bounds allow us to estimate Hilbert-Schmidt norm of the difference,

$$\left\|R_{H^{(n-1)}_{u,v}(d)}(k^2) - R_{H^{(n-1)a,b}}(k^2)\right\|_2^2 = \int_0^\infty\int_0^\infty |\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)|^2\,\mathrm{d}x\mathrm{d}y$$
$$= \int_0^d\int_0^d |\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)|^2\,\mathrm{d}x\mathrm{d}y + \int_0^d\int_d^\infty |\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)|^2\,\mathrm{d}x\mathrm{d}y$$
$$+ \int_d^\infty\int_0^d |\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)|^2\,\mathrm{d}x\mathrm{d}y + \int_d^\infty\int_d^\infty |\mathcal{G}^v_{i\kappa}(x,y) - \mathcal{G}^a_{i\kappa}(x,y)|^2\,\mathrm{d}x\mathrm{d}y$$

It is straightforward to check that last integral does not exceed the value

$$\int_d^\infty\int_d^\infty \left(\mathrm{e}^{-\Re(\kappa)x}\mathrm{e}^{-\Re(\kappa)y}Kd^2\right)^2\mathrm{d}x\mathrm{d}y \le \left(\frac{K}{2\Re(\kappa)}\right)^2 d^4,$$

and similarly the first one and the middle two are estimated by $4M^2d^2$ and $\frac{L^2}{2\Re(\kappa)}\,d$, respectively, which means that

$$\lim_{d\to 0+}\left\|R_{H^{(n-1)}_{u,v}(d)}(k^2) - R_{H^{(n-1)a,b}}(k^2)\right\|_2^2 = 0,$$

and the same is *a fortiori* true for the operator norm. This concludes the argument for the first component of the operator.

The proof for the "scalar" component is similar, just a bit more complicated, so we can skip some details. First we construct the Green function for the δ coupling with the parameter $u \in \mathbb{R}$ projected on the subspace of functions with coinciding components; in a similar way as above we find that the resolvent kernel equals

$$\mathcal{G}^{u}_{i\kappa}(x,y) = \frac{\mathrm{e}^{-\kappa x_>}}{\kappa\left(\frac{u}{n}+\kappa\right)}\left(\frac{u}{n}\sinh\kappa x_< + \kappa\cosh\kappa x_<\right)\,.$$

An analogous construction for the (negative) Laplacian with the boundary condition $(a+nb-1)\psi(0)+i(a+nb+1)\psi'(0)=0$ at the origin gives the Green function

$$\mathcal{G}^{a,b}_{i\kappa}(x,y) = \frac{\mathrm{e}^{-\kappa y}\;\left(i(a+nb-1)\sinh\kappa x + \kappa(a+nb+1)\cosh\kappa x\right)}{\kappa(i(a+nb-1)+\kappa(a+nb+1))}\,.$$

Finally the resolvent kernel of the approximating function, to be compared with the last expression, is obtained again from $\mathcal{G}^{u}_{i\kappa}(x,y)$ by means of Krein's formula

$$\mathcal{G}^{u,v}_{i\kappa}(x,y) = \mathcal{G}^{u}_{i\kappa}(x,y) + \frac{\mathcal{G}^{u}_{i\kappa}(x,d)\mathcal{G}^{u}_{i\kappa}(d,y)}{-v^{-1}-\mathcal{G}^{u}_{i\kappa}(d,d)}\,.$$

To estimate the Green function difference we assume again first that $d \le x \le y$, so

$$\begin{aligned}
&\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y) \\
&\quad= \frac{\mathrm{e}^{-\kappa y}}{\kappa}\left(\frac{\frac{u}{n}\sinh\kappa x+\kappa\cosh\kappa x}{\frac{u}{n}+\kappa} + \frac{\frac{\mathrm{e}^{-\kappa x}}{\left(\frac{u}{n}+\kappa\right)^2}\left(\frac{u}{n}\sinh\kappa d+\kappa\cosh\kappa d\right)^2}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \frac{\mathrm{e}^{-\kappa d}}{\frac{u}{n}+\kappa}\left(\frac{u}{n}\sinh\kappa d+\kappa\cosh\kappa d\right)}\right. \\
&\qquad\left. - \frac{i(a+nb-1)\sinh\kappa x+\kappa(a+nb+1)\cosh\kappa x}{i(a+nb-1)+\kappa(a+nb+1)}\right)
\end{aligned}$$

where we have used (3.4) and u should be substituted from (3.6). Our aim is to find the behavior of this expression for small d using the expansion

$$\cosh(x) = 1+\mathcal{O}(x^2)\,, \qquad \frac{1}{1+x} = 1-x+\mathcal{O}(x^2) \quad \text{as } x\to 0.$$

The first term gives

$$\frac{\frac{u}{n}\sinh\kappa x+\kappa\cosh\kappa x}{\frac{u}{n}+\kappa} = \sinh\kappa x + \mathrm{e}^{-\kappa x}\,\mathcal{O}(d^2)\,,$$

for the second one we get after a straightforward but tedious computation

$$\mathrm{e}^{-\kappa x}\left(\frac{\kappa(a+1+nb)}{\kappa(a+1+nb)+i(a-1+nb)} + \mathcal{O}(d)\right)\,,$$

and the third is independent of d; putting everything together we find

$$\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y) = \mathrm{e}^{-\kappa x}\mathrm{e}^{-\kappa y}\,\mathcal{O}(d)\,,$$

because the non-vanishing terms cancel again. In other words, there is a $K'>0$ independent of x,y and d such that the following inequality

$$\left|\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y)\right| < K'\mathrm{e}^{-\kappa x}\mathrm{e}^{-\kappa y}\,d \tag{4.3}$$

holds. In the "mixed" case, $x \le d \le y$, we have

$$\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y) = \frac{\mathrm{e}^{-\kappa y}}{\kappa}\left(\frac{\frac{u}{n}\sinh\kappa x + \kappa\cosh\kappa x}{\frac{u}{n}+\kappa} + \right.$$
$$+\frac{\frac{\mathrm{e}^{-\kappa d}}{\left(\frac{u}{n}+\kappa\right)^2}\left(\frac{u}{n}\sinh\kappa x + \kappa\cosh\kappa x\right)\left(\frac{u}{n}\sinh\kappa d + \kappa\cosh\kappa d\right)}{\kappa\frac{d(a+1)}{a+1+id(a-1)} - \frac{\mathrm{e}^{-\kappa d}}{\frac{u}{n}+\kappa}\left(\frac{u}{n}\sinh\kappa d + \kappa\cosh\kappa d\right)}$$
$$\left. - \frac{i(a+nb-1)\sinh\kappa x + \kappa(a+nb+1)\cosh\kappa x}{i(a+nb-1)+\kappa(a+nb+1)}\right).$$

The first and the third term at the right-hand side are obviously bounded independently of x, y and d, and in the same way as above one can check that the second one is $\mathcal{O}(1)$ as $d \to 0+$, hence there is an L' independent of x, y and $d < 1$ such that

$$\left|\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y)\right| < \mathrm{e}^{-\Re(\kappa)y}\, L'\,.$$

The same is naturally true if the roles of x and y are interchanged. It remains to analyze the situation when both x, y do not exceed d, say $x \le y \le d$, when

$$\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y) = \frac{\mathrm{e}^{-\kappa y}}{\kappa}\,\frac{\frac{u}{n}\sinh\kappa x + \kappa\cosh\kappa x}{\frac{u}{n}+\kappa}$$
$$+\frac{\frac{\mathrm{e}^{-2\kappa d}}{\kappa\left(\frac{u}{n}+\kappa\right)^2}\left(\frac{u}{n}\sinh\kappa x + \kappa\cosh\kappa x\right)\left(\frac{u}{n}\sinh\kappa y + \kappa\cosh\kappa y\right)}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \frac{\mathrm{e}^{-\kappa d}}{\frac{u}{n}+\kappa}\left(\frac{u}{n}\sinh\kappa d + \kappa\cosh\kappa d\right)}$$
$$-\frac{\mathrm{e}^{-\kappa y}}{\kappa}\,\frac{i(a+nb-1)\sinh\kappa x + \kappa(a+nb+1)\cosh\kappa x}{i(a+nb-1)+\kappa(a+nb+1)}\,.$$

In the same way as above one establishes existence of an $M' > 0$ independent of x, y and $d < 1$ such that

$$\left|\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y)\right| < M'\,.$$

Using these bounds and repeating the above Hilbert-Schmidt estimate we get

$$\lim_{d\to 0+}\left\|R_{H^{(1)}_{u,v}(d)}(k^2) - R_{H^{(1)a,b}}(k^2)\right\|_2^2 = 0\,,$$

which implies the analogous limiting relation for the operator norm of the resolvent difference which we set out to prove.

In the remaining two cases it is sufficient to consider the "scalar" component because the orthogonal complement does not contain the parameter u. Take first the case when the condition (3.7) is violated. If the variables satisfy $d \le x \le y$ we can rewrite the Green function difference using (3.8) as

$$\mathcal{G}^{u,v}_{i\kappa}(x,y) - \mathcal{G}^{a,b}_{i\kappa}(x,y) = \frac{\mathrm{e}^{-\kappa y}}{\kappa}\left(\frac{\frac{\zeta}{n}\sinh\kappa x + \kappa d^3\cosh\kappa x}{\frac{\zeta}{n}+\kappa d^3}\right.$$
$$+\frac{\frac{\mathrm{e}^{-\kappa x}}{\left(\frac{\zeta}{n}+\kappa d^3\right)^2}\left(\frac{\zeta}{n}\sinh\kappa d + \kappa d^3\cosh\kappa d\right)^2}{\frac{\kappa d(a+1)}{a+1+id(a-1)} - \frac{\mathrm{e}^{-\kappa d}}{\frac{\zeta}{n}+\kappa d^3}\left(\frac{\zeta}{n}\sinh\kappa d + \kappa d^3\cosh\kappa d\right)}$$
$$\left.-\frac{i(a+nb-1)\sinh\kappa x + \kappa(a+nb+1)\cosh\kappa x}{i(a+nb-1)+\kappa(a+nb+1)}\right);$$

expanding the first two terms at the right-hand side we establish existence of a $K' > 0$ independent of x, y and d such that the inequality (4.3) holds. In a similar way one proceeds when one or both arguments are smaller than d. The same can be done in the δ_p case where the resolvent difference for $d \le x \le y$ is

$$\mathcal{G}_{i\kappa}^{u,v}(x,y) - \mathcal{G}_{i\kappa}^{a,b}(x,y) = \frac{\mathrm{e}^{-\kappa y}}{\kappa}\Bigg(\frac{-\sinh\kappa x + \kappa d\cosh\kappa x}{-1+\kappa d} + \frac{\frac{\mathrm{e}^{-\kappa x}}{(-1+\kappa d)^2}\left(-\sinh\kappa d + \kappa d\cosh\kappa d\right)^2}{\kappa\frac{d(a+1)}{a+1+id(a-1)} - \frac{\mathrm{e}^{-\kappa d}}{-1+\kappa d}(-\sinh\kappa d + \kappa d\cosh\kappa d)} - \sinh\kappa x\Bigg)$$

and the other variable combinations are dealt with analogously. The Hilbert-Schmidt estimate is the same as in the generic case; this concludes the proof. □

5. Concluding remarks

We have mentioned in the introduction that approximation including singular couplings can be used an intermediate step in a search for approximations based on regular potentials. In this sense δ coupling and δ interactions are preferable because in this case we already know how to make the second step; hence our result paves way to a complete potential approximation of permutation symmetric couplings.

In particular, comparing with [**CE04**] we do not need δ_p coupling to approximate δ', and the δ_p itself can be approximated by δ interactions. We have seen, however, that in this case the central singularity is of a pole type with respect to d similarly as the couplings of the δ's at graph edges. This illustrates the exceptional character of δ_p which is in a sense akin to δ, with the roles of the "scalar" and $(n-1)$-components interchanged. The remaining one-parameter family of couplings violating the condition (3.7) needs, on the contrary, a stronger singularity with respect to d at the vertex. The reason of this behavior is not clear; this underlines one more time the fact that our present understanding to the zoology of vertex couplings in quantum graphs is still far from satisfactory.

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