Construction of a specification from its singleton part

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Abstract: We state a construction theorem for specifications starting from single-site conditional probabilities (singleton part). We consider general single-site spaces and kernels that are absolutely continuous with respect to a chosen product measure (free measure). Under a natural order-consistency assumption and weak non-nullness requirements we show existence and uniqueness of the specification extending the given singleton part. We determine conditions granting the continuity of the specification. In addition, we show that, within a class of measures with suitable support properties, consistency with singletons implies consistency with the full specification.

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1 Introduction

A *specification* on a product space of the form $E^{\mathbb{Z}^d}$ is a family of probability kernels labelled by the finite subsets $\Lambda \subset \mathbb{Z}^d$ satisfying the requirements of a consistent system of conditional probabilities. They are the central objects of mathematical statistical mechanics, see, for instance, Georgii (1988). In this paper we determine conditions that guarantee the (re)construction of a specification from single-site conditional probabilities (*singletons*). Such a scenario yields an interesting simplification of the theory of specifications, and sets it in a framework analogous to that of discrete-time stochastic processes, traditionally defined and characterized by properties of single-site transition probabilities.

The issue of singleton characterization of specifications stems already from Dobrushin's (1968) seminal work. His remarks were taken up by Flood and Sullivan (1980) and lead to Theorem (1.33) in Georgii (1988). These references studied the reconstruction problem, namely how to recover a pre-existing specification starting from the singletons or from a subspecification. More recently, Dachian and Nahapetian (2001, 2004), and us (Fernández and Maillard, 2004) have addressed the more general construction problem under different hypotheses. Our present results extend and complement those of these references. Moreover, our proof, while inspired from existing proofs, offers an alternative formulation that, we believe, clarifies the algebraic and measure-theoretical properties involved.

We work with general single-spin spaces and consider singletons that are absolutely continuous with respect to a pre-established product measure (*free measure*). This is the natural framework from the physical point of view. We demand two key conditions, besides the obvious finiteness and normalization requirements: **(H1)** some degree of non-nullness, and **(H2)** a compatibility condition. The former is the extension, to our framework, of Dachian and Nahapetian's (2004) *very weak positivity.* An alternative form of non-nullness was demanded in our previous work (see the appendix in Fernández and Maillard, 2004). It is a strong form of non-nullness but allowing the existence of local exclusion rules ("grammars"). Condition (H2) is the adaptation, in the absence of strict positivity, of the compatibility identity (A.9) in Fernández and Maillard (2004). This is a partially integrated condition that, for finite spins, coincide with the pointwise condition imposed by Dachian and Nahapetian (2004) (defining what they call *1-point specifications*).

Under these conditions we show (Theorem 4.1) that there exists a unique specification that is absolutely continuous with respect to the free measure and whose single-site probabilities coincide with the given singletons. The proof provides a recursive construction of this specification [formulas (4.5)–

(4.6)]. Our scheme makes no use of the possible continuity of each singleton with respect to exterior configurations. As such, it is equally applicable to Gibbsian (Kozlov, 1974 and Sullivan, 1973) and non-Gibbsian (van Enter, Fernández and Sokal, 1993) theories. Nevertheless we determine a natural condition ensuring that the continuity of singletons lead to a continuous specification. Furthermore, in the third part of our theorem, we establish conditions under which a measure is consistent with the so-constructed specification if and only it is consistent with the original singletons.

We illustrate our results with two types of examples. First, we present a simple model showing the actual meaning of the different hypotheses. Second, we show instances where singleton consistency is strictly weaker than consistency with the full specification. This justifies the need of some minimal hypotheses, like the ones contained in part **(III)** of our theorem, to guarantee the equivalence between both levels of consistency. We also present a rather detailed comparison of our theorem with preceding results. The key issue is the degree of "nullness" allowed for the specification, that is, the presence of "excluded configurations" leading to zero probability weights. As we discuss, the theorem below, as well as the results by Dachian and Nahapetian (2001, 2004), are suited to situations where the exclusions come from asymptotic reasons. In contrast, the theorem in Fernández and Maillard (2004) is more appropriate for local exclusion rules.

2 Preliminaries

We consider a general measurable space $(E, \mathcal{E})$ and the product space $\Omega = E^{\mathbb{Z}^d}$ for $d \geq 1$ (*configuration space*), endowed with the product σ-algebra $\mathcal{F} = \mathcal{E}^{\mathbb{Z}^d}$. Our notation will be fairly standard. We shall denote $\mathcal{P}(\Omega, \mathcal{F})$ the set of probability measures on $(\Omega, \mathcal{F})$. Support sites will be indicated with subscripts, if different from the whole of $\mathbb{Z}^d$. For example, if $U \subset \mathbb{Z}^d$ we denote $\Omega_U = E^U$ and $\mathcal{F}_U$ the sub-σ-algebra of $\mathcal{F}$ generated by the cylinders with base in Ω_U. Likewise $\sigma \in \Omega$, $\sigma_\Lambda \in \Omega_\Lambda$. "Concatenated" configurations will be denoted as customary: If $\Lambda, \Delta \subset \mathbb{Z}^d$ are disjoint, $x_\Lambda \sigma_\Delta \in \Omega_{\Lambda \cup \Delta}$ is the configuration coinciding with x_Λ on Λ and with σ_Δ on Δ, while $x_\Lambda \sigma_\Delta \omega$ is the configuration in Ω which in addition is equal to ω on $(\Lambda \cup \Delta)^{\mathrm{c}}$. One-site sets will be labelled just by the site, for instance we shall write ω_i instead of $\omega_{\{i\}}$. For each $U \subset \mathbb{Z}^d$, its cardinal will be denoted $|U|$, its indicator function $\mathbb{1}_U$ and the set of its finite subsets $\mathcal{S}(U)$. We shall abbreviate $\mathcal{S} \triangleq \mathcal{S}(\mathbb{Z}^d)$. For $\Lambda \subset \mathbb{Z}^d$ and $i \in \Lambda$ we denote $\Lambda_i^* \triangleq \Lambda \setminus \{i\}$ ($|\Lambda| \geq 1$). Throughout this paper we adopt the convention "$1/\infty = 0$".

We recall that a *measure kernel* on $\mathcal{F} \times \Omega$ is a map $\gamma(\,\cdot \mid \cdot\,) : \mathcal{F} \times \Omega \to \mathbb{R}$

such that $\gamma(\,\cdot \mid \omega)$ is a measure for each $\omega \in \Omega$ while $\gamma(A \mid \cdot\,)$ is $\mathcal{F}$–measurable for each event $A \in \mathcal{F}$. If each $\gamma(\,\cdot \mid \omega)$ is a probability measure the kernel is called a *probability kernel*. To obtain cleaner formulas we shall adopt operator-like notations to handle kernels. Thus, for kernels γ and $\widetilde{\gamma}$ and non-negative measurable functions f and ρ, we shall denote:

- $\gamma(f)$ for the measurable function $\int f(\eta)\, \gamma(d\eta \mid \cdot\,)$.
- $\gamma\, \widetilde{\gamma}$ for the composed kernel defined by $(\gamma\, \widetilde{\gamma})(f) = \gamma(\widetilde{\gamma}(f))$.
- $\rho\, \gamma$ for the product kernel defined by $(\rho\, \gamma)(f) = \gamma(\rho f)$.

The following are the only two definitions needed for this paper.

Definition 2.1
A **specification** *on* $(\Omega, \mathcal{F})$ *is a family of probability kernels* $\{\gamma_\Lambda\}_{\Lambda \in \mathcal{S}}$ *such that for all* Λ *in* $\mathcal{S}$,

(a) $\gamma_\Lambda(A \mid \cdot\,) \in \mathcal{F}_{\Lambda^c}$ *for each* $A \in \mathcal{F}$.

(b) $\gamma_\Lambda(B \mid \omega) = 1\!\!1_B(\omega)$ *for each* $B \in \mathcal{F}_{\Lambda^c}$ and $\omega \in \Omega$.

(c) *For each* $\Delta \in \mathcal{S}$ *with* $\Delta \supset \Lambda$,

$$\gamma_\Delta\, \gamma_\Lambda \;=\; \gamma_\Delta\,. \tag{2.1}$$

The last property is called *consistency*. It is stronger than the *almost* consistency of the finite-volume conditional probabilities of a measure on $(\Omega, \mathcal{F})$. Without further requirements, this strengthening is usually illusory: If $(E, \mathcal{E})$ is a standard Borel space, each measure on $(\Omega, \mathcal{F})$ is consistent with some specification (Sokal, 1981). Matters become more delicate if in addition kernels are requested to be continuous with respect to their second variable, that is if the Feller property is imposed. Consistency with a continuous specification is the hallmark of Gibbsianness. See, for instance, van Enter, Maes and Shlosman (2000) for a survey of the different notions and issues arising when this continuity is absent.

Definition 2.2
A probability measure μ *on* $(\Omega, \mathcal{F})$ *is said to be* **consistent** *with a specification* $\{\gamma_\Lambda\}_{\Lambda \in \mathcal{S}}$ *if*

$$\mu\, \gamma_\Lambda \;=\; \mu \quad \text{for every } \Lambda \in \mathcal{S}. \tag{2.3}$$

The family of these measures will be denoted $\mathcal{G}\big(\{\gamma_\Lambda\}_{\Lambda \in \mathcal{S}}\big)$.

3 Main hypotheses

Throughout this paper we fix a family $(\lambda^i)_{i\in\mathbb{Z}^d}$ of *a priori* (non-negative) measures on $(E,\mathcal{E})$. Its choice is in general canonically dictated by the structure of the single-spin space E. For instance, if E admits some group structure all λ_i are chosen equal to the corresponding Haar measure (the "more symmetric" measure). For each $\Lambda \subset \mathbb{Z}^d$, let $\lambda^\Lambda \triangleq \bigotimes_{i\in\Lambda} \lambda^i$ (*free measure on* Λ) and λ_Λ denote the kernel (*free kernel on* Λ) defined by

$$\lambda_\Lambda(h \mid \omega) \;=\; \left(\lambda^\Lambda \otimes \delta_{\omega_{\Lambda^c}}\right)(h) \;=\; \int h(\sigma_\Lambda\omega)\, \lambda^\Lambda(d\sigma_\Lambda) \tag{3.1}$$

for every measurable function h and configuration ω.

Except in part **(III)** of our Theorem 4.1, the measures λ^i are not required to be normalized or even finite. The lack of normalization is the only aspect that could prevent the family $(\lambda_\Lambda)_{\Lambda\in\mathcal{S}}$ from being a specification. Indeed, this family satisfies (a) and (b) of Definition 2.1 and, furthermore, the following factorization property:

$$\lambda_{\Lambda\cup\Delta} \;=\; \lambda_\Lambda \lambda_\Delta\;, \tag{3.2}$$

for each pair of disjoint sets $\Lambda, \Delta \subset \mathbb{Z}^d$. If the kernels are normalized, this is a strengthening of the consistency condition (c) above.

We shall construct specifications by multiplying each kernel λ_Λ by a suitable measurable function ρ_Λ. The resulting kernels can be interpreted as *dependent* or *interacting* kernels. A family $(\rho_\Lambda)_{\Lambda\in\mathcal{S}}$ yielding an interacting kernel is called a λ-*modification* in Georgii's (1988) treatise (see, specially, Section 1.3). If E is countable and each λ^i is (a multiple of) the counting measure, every specification is obtained in this form.

Our specifications will be built starting from a family of single-site kernels of the form $\rho_i\lambda_i$, $i \in \mathbb{Z}$. The following definitions state the crucial hypotheses granting the feasibility of our construction.

Definition 3.3
A family $\{\rho_i\}_{i\in\mathbb{Z}^d}$, of $\mathcal{F}$-measurable functions $\rho_i : \Omega \to [0,\infty[$ satisfies **hypothesis (H1)** *if for each $\omega \in \Omega$, $j \in \mathbb{Z}^d$ and $V \in \mathcal{S}(\{j\}^c)$, there exists $x_j \in \Omega_j$ such that*

$$\rho_j\left(x_j\sigma_V\omega\right) > 0, \quad \forall\, \sigma_V \in \Omega_V\;, \tag{3.4}$$

and, for every $i \in \mathbb{Z}^d : i \neq j$,

$$\inf\left\{\lambda_i\Big(\rho_i\,\rho_j^{-1}\Big)\left(x_j\sigma_V\omega\right) : \sigma_V \in \Omega_V\right\} > 0 \tag{3.5}$$

and

$$\sup\left\{\lambda_i\Big(\rho_i\,\rho_j^{-1}\Big)\left(x_j\sigma_V\omega\right) : \sigma_V \in \Omega_V\right\} < \infty\;. \tag{3.6}$$

We denote

$$b(j,V,\omega) \triangleq \Big\{x_j \in \Omega_j \text{ satisfying (3.4)–(3.6)}\Big\} \tag{3.7}$$

and

$$B(j,V) \ \triangleq\ \Big\{\omega \in \Omega : \omega_j \in b(j,V,\omega)\Big\}\,. \tag{3.8}$$

Furthermore for every $W \in \mathcal{S}(V^{\mathrm{c}})$,

$$b(V,W,\omega) \ \triangleq\ \Big\{x_V \in \Omega_V : x_k \in b(k, V_k^* \cup W, \omega) \text{ for every } k \in V\Big\}\,. \tag{3.9}$$

If E is finite, hypothesis (H1) is exactly the condition of very weak positivity introduced by Dachian and Nahapetian (2004). Our sets $b(j,V,\omega)$ correspond to *good* ("*bonnes*") configurations at the site j given ω outside $V \cup \{j\}$, while the $B(j,V)$ correspond to configurations that are good in a more global sense. In both cases, this "goodness" must be uniform with respect to the configurations in V. The product structure of the sets $b(\Lambda,W,\omega)$, embodied in definition (3.9), is essential for our procedure and prevents its immediate extension to other than product spaces.

Definition 3.10
A family $\{\rho_i\}_{i\in\mathbb{Z}^d}$, of $\mathcal{F}$-measurable functions $\rho_i : \Omega \to [0,\infty[$ satisfies **hypothesis (H2)** *if for each i,j in $\mathbb{Z}^d$ and $\omega \in \Omega$, the following is true:*

(a) *For each $x_i \in b(i,\{j\},\omega)$ and $x_j \in b(j,\{i\},\omega)$,*

$$\frac{\rho_i(\omega)\,\rho_j(x_i\omega)}{\rho_i(x_i\omega)\,\lambda_j\left(\rho_j\,\rho_i^{-1}\right)(x_i\omega)} \ =\ \frac{\rho_j(\omega)\,\rho_i(x_j\omega)}{\rho_j(x_j\omega)\,\lambda_i\left(\rho_i\,\rho_j^{-1}\right)(x_j\omega)}\,. \tag{3.11}$$

(b) *The map $R_i^j : \Omega \longrightarrow]0,+\infty]$ defined by*

$$R_i^j(\omega) \ =\ \left(\frac{\rho_i}{\rho_j} \times \lambda_j\left(\rho_j\,\rho_i^{-1}\right)\right)(x_i\omega) \tag{3.12}$$

is independent of the choice of $x_i \in b(i,\{j\},\omega)$ and hence defines a $\mathcal{F}_{\{i\}^{\mathrm{c}}}$-measurable map.

Let us pause to discuss the meaning and motivation of these hypotheses. The conditions (3.4)–(3.6) in (H1) imply that the denominators in (3.11) and the numerator in (3.12) are neither zero nor infinity. The denominator can be zero in the latter, in which case $R_i^j(\omega) = \infty$.

As the reader will see, R_i^j is what is needed to fulfill the identity

$$\rho_{\{i,j\}}(\omega) \ =\ \frac{\rho_j(\omega)}{R_i^j(\omega)}\,. \tag{3.13}$$

Due to the $i \leftrightarrow j$ symmetry of the LHS, this identity must be accompanied by the consistency requirement

$$\frac{\rho_i(\omega)}{R_i^j(\omega)} \;=\; \frac{\rho_j(\omega)}{R_j^i(\omega)} \;. \tag{3.14}$$

Under strict positivity hypotheses, identity (3.13) holds with

$$R_i^j(\omega) \;=\; \lambda_i\left(\rho_i\,\rho_j^{-1}\right)(\omega)\;, \tag{3.15}$$

as exploited in Georgii (1988), Theorem (1.33), or in Fernández and Maillard (2004), Appendix. The consistency condition (3.14) is imposed as a further hypothesis in the latter reference, while it is automatic in the former because the singletons are known to come from a specification in the first place. A look to our arguments in the aforementioned appendix convinced us that to extend them to weakly positive cases we should at least start from the following desideratum:

(i) Identities (3.13) and (3.14) must be true.

(ii) Definition (3.15) must be verified whenever the RHS is meaningful.

(iii) R_i^j must be $\mathcal{F}_{\{i\}^c}$-measurable [as in (3.15)].

The quantity $\lambda_j\left(\rho_j\,\rho_i^{-1}\right)(\omega)$ is well defined whenever $\omega_i = x_i \in b(i,\{j\},\omega)$. In this case the validity of (3.14) and (iii) of the desideratum implies

$$\frac{\rho_i(x_i\omega)}{R_i^j(\omega)} \;=\; \frac{\rho_j(x_i\omega)}{\lambda_j\left(\rho_j\,\rho_i^{-1}\right)(x_i\omega)}\;. \tag{3.16}$$

This explains (3.12). With this definition of R_i^j, identity (3.11) is exactly (3.14). Our theorem below shows that, in fact, the above desideratum is basically all that is needed to make a successful construction.

4 Main Result

Theorem 4.1
Let $\{\rho_i\}_{i\in\mathbb{Z}^d}$ be a family of $\mathcal{F}$-measurable functions $\rho_i : \Omega \to [0,\infty[$ satisfying:

(a) *For every i in $\mathbb{Z}^d$,*

$$\lambda_i(\rho_i \mid \omega) \;=\; 1\;, \tag{4.2}$$

for all $\omega \in \Omega$.

(b) *Hypotheses* **(H1)** and **(H2)**.

Then there exists a family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ *of measurable functions* $\rho_\Lambda : \Omega \to [0,\infty[$, *with* $\rho_{\{i\}} = \rho_i$, *such that the family of kernels* $\{\rho_\Lambda \lambda_\Lambda\}_{\Lambda\in\mathcal{S}}$ *is a specification. Furthermore:*

(I) *If*

$$\lambda^j\big(b(j,V,\omega)\big) > 0 \tag{4.3}$$

for each $\omega\in\Omega$, $V\in\mathcal{S}$ *and* $j\in V^{\rm c}$, *then there exists exactly one family* $\{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ *with the above property.*

(II) *Suppose that* E *is a topological space and* $\mathcal{E}$ *its borelian* σ*-algebra, and consider the product topology for* Ω. *If the functions* ρ_i *are sequentially continuous and for each* $V\in\mathcal{S}$ *and* $j\in V^{\rm c}$ *there exists* $x_j \in \bigcap_\omega b(j,V,\omega)$ *such that*

$$\int \sup_\omega \Big[(\rho_i\, \rho_j^{-1})(\sigma_i x_j \omega)\Big]\, \lambda^i(d\sigma_i) \;<\; \infty \tag{4.4}$$

for all $i\in V$, *then the functions* ρ_Λ, $\Lambda\in\mathcal{S}$, *are sequentially continuous.*

Explicitly, the functions ρ_Λ *are recursively defined throughout the identity*

$$\rho_{\Theta\cup\Gamma}(\omega) \;=\; \frac{\rho_\Theta(\omega)}{R^\Gamma_\Theta(\omega)}\;, \tag{4.5}$$

valid for every $\Theta\in\mathcal{S}$, $\Gamma\in\mathcal{S}(\Theta^{\rm c})$, $\omega\in\Omega$, *where*

$$R^\Gamma_\Theta(\omega) \;=\; \bigg(\frac{\rho_\Theta}{\rho_\Gamma} \times \lambda_\Gamma\left(\rho_\Gamma\, \rho_\Theta^{-1}\right)\bigg)\,(x_\Theta\omega)\;, \tag{4.6}$$

is independent of the choice of $x_\Theta \in b(\Theta,\Gamma,\omega)$.

(III) *In this part we suppose that*

$$\lambda^i(\Omega_i) \;=\; 1 \quad \textit{for every } i\in\mathbb{Z}^d\;. \tag{4.7}$$

[As remarked below, this is not a big loss of generality.] Let $\mathcal{N}$ *be the set of probability measures* μ *on* $(\Omega,\mathcal{F})$ *such that*

$$\mu\,\lambda_j\big[B(j,V)^{\rm c}\big] = 0 \quad \textit{for every } V\in\mathcal{S} \textit{ and } j\in V^{\rm c}\;. \tag{4.8}$$

Then, within this class, consistency is equivalent to consistency with singletons:

$$\mathcal{N}\cap\mathcal{G}\big(\{\rho_\Lambda\,\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}\big) \;=\; \mathcal{N}\cap\Big\{\mu\in\mathcal{P}(\Omega,\mathcal{F}) : \mu\,(\rho_i\,\lambda_i) = \mu,\ i\in\mathbb{Z}^d\Big\}. \tag{4.9}$$

Remarks

4.10 In particular, if $x_\Theta \in b(\Theta, \Gamma, \omega)$, formulas (4.5)–(4.6) yield

$$\rho_{\Theta\cup\Gamma}(x_\Theta\omega) \;=\; \frac{\rho_\Gamma(x_\Theta\omega)}{\lambda_\Gamma\left(\rho_\Gamma\,\rho_\Theta^{-1}\right)(x_\Theta\omega)}\;, \tag{4.11}$$

a formula already present in Theorem (1.33) of Georgii (1988).

4.12 A simple recursive argument shows that the translation invariance of the measures λ^i and the functions ρ_i imply that of the functions ρ_Λ.

4.13 When E is finite and each λ^i is the counting measure, results **(I)** and **(II)** were obtained by Dachian and Nahapetian (2004). In the strictly positive case (everybody is good) we recover the results of the appendix of Fernández and Maillard (2004). See Section 5 for more details.

4.14 As remarked by Georgii (1988, Remark (1.28) (3)), the normalization condition (4.7) is equivalent to the existence of functions $r_i(\omega_i) > 0$ with $0 < \lambda^i(r_i) < \infty$. Indeed, the definition $\widetilde{\rho}_i = \rho_i/r_i$ leads to the identity $\rho_i\,\lambda_i = \widetilde{\rho}_i\,\widetilde{\lambda}_i$ with $\widetilde{\lambda}^i(\Omega_i) = 1$. Such functions r_i exist, for instance, if the measures λ^i are σ-finite.

4.15 If E is compact, then usually both the measures λ^i and the functions ρ_i are bounded. In such a situation the continuity of ρ_Λ and h implies the continuity of $(\rho_\Lambda\lambda_\Lambda)\,(h \mid \cdot)$ and the specification $\{\rho_\Lambda\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}$ is a *Feller specification*. If E is finite such specifications are also *quasilocal*. See van Enter, Fernández and Sokal (1993) for a survey of these notions and their relation to Gibbsianness.

4.16 Preston (2004), in an unpublished preprint, proves a rather strong result related to our part **(III)**. The author takes a reconstructive point of view — the functions ρ_i come from a pre-existing specification — and determines conditions under which consistency coincides with singleton consistency. His framework is more general than ours in that a product structure is not demanded. On the other hand, the hypothesis imposed by Preston to the consistent measure involves all kernels, and not only the singletons as in (4.8).

As an illustration of our results, we present a family of singletons satisfying the hypotheses of Theorem 4.1 but not fitting any of the existing (re)construction schemes. The main value of this example is to provide a concrete manifestation of the different hypotheses of the theorem.

Example 4.17
Let $E = [0,1]$ and $\mathcal{E}$ be its Borel σ-algebra. For each $i \in \mathbb{Z}$ we take λ^i equal to the Lebesgue measure and define

$$\rho_i(\omega) \;=\; \begin{cases} 2\mathbb{1}_{[0,1/2]}(\omega_i) & \text{if } |\{j : \omega_j > 1/2\}| = \infty \;, \\ 2\mathbb{1}_{]1/2,1]}(\omega_i) & \text{otherwise} \;. \end{cases} \tag{4.18}$$

Let us see that these functions satisfy the hypotheses of Theorem 4.1. The measurability of each ρ_i and the normalization (4.2) are readily verified. We check (H1) and (H2) for ω such that $|\{j : \omega_j > 1/2\}| = \infty$, the complementary case is analogous. For such ω we see that for all $j \in \mathbb{Z}^d$ and $V \in \mathcal{S}(\{j\}^{\mathrm{c}})$:

(i) $\rho_j(\sigma_V\omega) > 0$ for all $\sigma_V \in \Omega_V$ if and only if $\omega_j \in [0,1/2]$.

(ii) If $\omega_j \in [0,1/2]$ then $\lambda_i\left(\rho_i\rho_j^{-1}\right)(\sigma_V\omega) = 1/2$ for all $\sigma_V \in \Omega_V$.

It follows that (H1) is verified with $b(j,V,\omega) = [0,1/2]$. Furthermore,

$$R_i^j(\omega) \;=\; \begin{cases} 1/2 & \text{if } \omega_j \in [0,1/2] \\ \infty & \text{otherwise} \;, \end{cases} \tag{4.19}$$

satisfying **(H2)(b)**, and, if $x_i, x_j \in [0,1/2]$,

$$\begin{aligned} \frac{\rho_i(\omega)\,\rho_j(x_i\omega)}{\rho_i(x_i\omega)\,\lambda_j\left(\rho_j\,\rho_i^{-1}\right)(x_i\omega)} \;&=\; \frac{\rho_j(\omega)\,\rho_i(x_j\omega)}{\rho_j(x_j\omega)\,\lambda_i\left(\rho_i\,\rho_j^{-1}\right)(x_j\omega)} \\ &=\; \begin{cases} 4 & \text{if } \omega_i,\omega_j \in [0,1/2] \\ 0 & \text{otherwise} \;, \end{cases} \end{aligned} \tag{4.20}$$

in agreement with hypotheses **(H2)(a)**. We observe that our construction indeed leads to

$$\rho_\Lambda(\omega) \;=\; \begin{cases} 2^{|\Lambda|}\mathbb{1}_{[0,1/2]^\Lambda}(\omega_\Lambda) & \text{if } |\{j : \omega_j > 1/2\}| = \infty \;, \\ 2^{|\Lambda|}\mathbb{1}_{]1/2,1]^\Lambda}(\omega_\Lambda) & \text{otherwise} \;. \end{cases} \tag{4.21}$$

The following example shows that without condition of the type (4.8), singleton consistency may not be equivalent to consistency.

Example 4.22
We consider the spin-1/2 setting, namely $E = \{-1,1\}$, $\mathcal{E}$ discrete, λ^i Bernoulli-1/2. Let us denote A^+ the set of configurations for which the $+1$ spins percolate. We define the following families of singleton functions

$$\rho_i(\omega) = \begin{cases} \mathbb{1}_{\{\omega_i=+1\}}(\omega) & \text{if } \omega \in A^+ \;, \\ \mathbb{1}_{\{\omega_i=-1\}}(\omega) & \text{otherwise} \end{cases} \tag{4.23}$$

and

$$\widetilde{\rho}_i(\omega) = \begin{cases} 1\!\!1_{\{\omega_i=-1\}}(\omega) & \text{if } \omega \in A^+ \ , \\ 1\!\!1_{\{\omega_i=+1\}}(\omega) & \text{otherwise} \ . \end{cases} \tag{4.24}$$

It is not hard to check that our theorem applies for both families of functions. The specification resulting for the first family has

$$\mathcal{N} \cap \mathcal{G}\big(\{\rho_\Lambda \, \lambda_\Lambda\}_{\Lambda\in\mathcal{S}}\big) \;=\; \mathcal{G}\big(\{\rho_\Lambda \, \lambda_\Lambda\}_{\Lambda\in\mathcal{S}}\big) \;=\; \big\{\delta_{\underline{-1}}, \delta_{\underline{+1}}\big\} \ , \tag{4.25}$$

where $\delta_{\underline{\pm 1}}$ is the measure concentrated on the configuration $\omega_i = \pm 1$. In contrast, for the second family

$$\mathcal{N} \cap \mathcal{G}\big(\{\widetilde{\rho}_\Lambda \, \lambda_\Lambda\}_{\Lambda\in\mathcal{S}}\big) \;=\; \mathcal{G}\big(\{\widetilde{\rho}_\Lambda \, \lambda_\Lambda\}_{\Lambda\in\mathcal{S}}\big) \;=\; \emptyset \ . \tag{4.26}$$

Nevertheless, if $d \geq 2$, we can construct a measure μ that satisfies

$$\mu(\rho_i \, \lambda_i) \;=\; \mu \;=\; \mu(\widetilde{\rho}_i \, \lambda_i) \tag{4.27}$$

without being consistent with either specification. [Obviously, μ violates (4.8).] Indeed, let μ^+ and μ^- be the "+" and "−" Ising states at low enough temperature so that

$$\mu^+(A^+) \;=\; 1 \quad \text{and} \quad \mu^-(A^+) \;=\; 0 \ . \tag{4.28}$$

Then the measure

$$\mu \;=\; \frac{1}{2}\,\mu^+ + \frac{1}{2}\,\mu^- \ . \tag{4.29}$$

satisfies

$$\mu\big((\rho_i \, \lambda_i)(\{\omega_i = 1\})\big) \;=\; \mu(A^+) \;=\; \frac{1}{2} \tag{4.30}$$

and

$$\mu\big((\widetilde{\rho}_i \, \lambda_i)(\{\omega_i = 1\})\big) \;=\; \mu\big((A^+)^{\mathrm{c}}\big) \;=\; \frac{1}{2} \ . \tag{4.31}$$

On the other hand, as μ is invariant under spin flips,

$$\mu\big(\{\omega_i = 1\}\big) \;=\; \frac{1}{2} \ , \tag{4.32}$$

so (4.27) is satisfied. The multisite kernels $\rho_\Lambda \, \lambda_\Lambda$ and $\widetilde{\rho}_\Lambda \, \lambda_\Lambda$ are concentrated on configurations that are identically $+1$ or -1 within Λ. Hence the measure μ can not be consistent with them because it gives non-zero measure to every cylinder.

5 Comparison with previous results

Our results are the generalization of those obtained by Dachian and Nahapetian (2004) in the case of finite single-site space E and each λ^i equals to the counting measure. In this framework our hypotheses **(H1)** reduces to the positivity requirement (3.4). A family of single-site weights $\{\rho_i\}_{i\in\mathbb{Z}^d}$ satisfying such a property is termed *very weakly positive* by the authors. In our notation, their result (obtained by combining their Proposition 18, Theorem 19 and Theorem 21) is the following

Proposition 5.1 (Dachian and Nahapetian (2004))
Let $\{\rho_i\}_{i\in\mathbb{Z}^d}$ be a family of very weakly positive probability weights which are normalized in the sense that

$$\lambda^i(\rho_i) \equiv 1 . \tag{5.2}$$

Then:

(I) *There exists a unique family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ of measurable functions $\rho_\Lambda : \Omega \to [0,\infty[$, with $\rho_{\{i\}} = \rho_i$, such that the family of kernels $\{\rho_\Lambda\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}$ is a specification if and only if*

$$\begin{aligned}\rho_i(x_j\, u_i\, \omega)\, \rho_j(u_j\, u_i\, \omega)\, \rho_i(x_i\, u_j\, \omega)\, \rho_j(x_j\, x_i\, \omega) \\ = \;\rho_j(x_i\, u_j\, \omega)\, \rho_i(u_i\, u_j\, \omega)\, \rho_j(x_j\, u_i\, \omega)\, \rho_i(x_i\, x_j\, \omega)\;,\end{aligned} \tag{5.3}$$

for every $i, j \in \mathbb{Z}^d$, $u_i \in \Omega_i$, $u_j \in \Omega_j$, $x_i \in b(i, \{j\}, \omega)$ and $x_j \in b(j, \{i\}, \omega)$.

(II) *The functions $\{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ are continuous if and only if the functions $\{\rho_i\}_{i\in\mathbb{Z}^d}$ are.*

For finite state-space E, we claim that this proposition coincides with parts **(I)** and **(II)** of Theorem 4.1. To prove this, it suffices to show that (5.3) is equivalent to our hypotheses **(H2)**. In fact, the "only if" part of Proposition 5.1 implies that (5.3) is satisfied whenever **(H2)** is. Thus, we only need to show that (5.3) implies our condition **(H2)**. This is easily seen. Indeed, if we divide both sides of (5.3) by $\rho_i(x_i\, u_j\, \omega)\, \rho_j(x_j\, u_i\, \omega)$ and sum them over $u_i \in \Omega_i$ and $u_j \in \Omega_j$, we obtain thanks to the normalization (5.2),

$$\rho_i(x_i\, x_j\, \omega) \sum_{u_j\in\Omega_j} \frac{\rho_j(x_i\, u_j\, \omega)}{\rho_i(x_i\, u_j\, \omega)} \;=\; \rho_j(x_j\, x_i\, \omega) \sum_{u_i\in\Omega_i} \frac{\rho_i(x_j\, u_i\, \omega)}{\rho_i(x_i\, u_j\, \omega)}\;, \tag{5.4}$$

that is

$$\rho_i(x_i\, x_j\, \omega)\, \lambda_j(\rho_j\, \rho_i^{-1})(x_i\, \omega) \;=\; \rho_j(x_j\, x_i\, \omega)\, \lambda_i(\rho_i\, \rho_j^{-1})(x_j\, \omega)\;. \tag{5.5}$$

Dividing term-by-term (5.3) by (5.5), we arrive to (3.11).

A related but complementary result is contained in Appendix A of Fernández Maillard (2004). In this appendix, the configuration space Ω is an arbitrary subset of $E^{\mathbb{Z}^d}$, for a general measurable space E. Thus, Ω may describe local exclusion rules or grammars. Kernels are supposed to be positives on the whole of Ω and therefore our hypotheses **(H2)** becomes

$$\frac{\rho_i}{\lambda_i\left(\rho_i\,\rho_j^{-1}\right)}(\omega) \;=\; \frac{\rho_j}{\lambda_j\left(\rho_j\,\rho_i^{-1}\right)}(\omega)\;, \tag{5.6}$$

for every i, j in $\mathbb{Z}^d$ and every $\omega \in \Omega$. The result is

Proposition 5.7 (Fernández and Maillard (2004))
Let $\{\rho_i\}_{i\in\mathbb{Z}^d}$ be a family of measurable functions which are normalized — $\lambda^i(\rho_i) \equiv 1$ — and satisfy (5.6) and the following bounded-positivity properties. For every $i, j \in \mathbb{Z}^d$,

$$\inf_{\omega\in\Omega} \lambda_j\left(\rho_j\,\rho_i^{-1}\right)(\omega) \;>\; 0\;, \tag{5.8}$$

and

$$\sup_{\omega\in\Omega} \lambda_j\left(\rho_j\,\rho_i^{-1}\right)(\omega) \;<\; +\infty\;. \tag{5.9}$$

Then:

(I) *There exists a unique family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ of measurable functions $\rho_\Lambda : \Omega \to [0,\infty[$, with $\rho_{\{i\}} = \rho_i$, such that the family of kernels $\{\rho_\Lambda\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}$ is a specification.*

(II) *If the functions ρ_i are continuous and $\int \sup_\omega(\rho_i\,\rho_j^{-1})(\sigma_i\omega_{\{i\}^c})\,\lambda^i(d\sigma_i) < \infty$ for all $i, j \in \mathbb{Z}^d$, then the functions ρ_Λ, and thus the specification γ, are continuous.*

(III) $\mathcal{G}(\{\rho_\Lambda\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}) = \left\{\mu \in \mathcal{P}(\Omega,\mathcal{F}) : \mu(\rho_i\,\lambda_i) = \mu \text{ for all } i \in \mathbb{Z}^d\right\}$.

(IV) *For each $\Lambda \in \mathcal{S}$ there exist constants $C_\Lambda, D_\Lambda > 0$ such that $C_\Lambda\,\rho_k(\omega) \le \rho_\Lambda(\omega) \le D_\Lambda\,\rho_k(\omega)$ for all $k \in \Lambda$ and all $\omega \in \Omega$.*

If Ω is the full product space $E^{\mathbb{Z}^d}$ and every configuration is allowed — $b(j, V, \omega) = \Omega_j\ \forall\ V \in \mathcal{S},\ j \in V^{\mathrm{c}},\ \omega \in \Omega$ — Proposition 5.7 and Theorem 4.1 coincide. But, otherwise, these two results have different ranges of application. Indeed, models with local exclusion rules (for instance, no two nearest neighbor simultaneously occupied) are covered by Proposition 5.7, but do not satisfy the hypotheses of Theorem 4.1. The reason is that each $b(j, V, \omega)$ is empty if $V \neq \emptyset$ or if ω violates the exclusion rules. On the other hand, models with "asymptotic" exclusion rules, like in Example 4.17, fall outside the scope of Proposition 5.7.

6 Proofs

We need tree lemmas to build the proof of our theorem. We start with the crucial one showing that the algorithm (4.5)–(4.6) recursively leads to multi-site generalizations of hypotheses (H1) and (H2).

Lemma 6.1
Let $\{\rho_i\}_{i\in\mathbb{Z}^d}$ be a family of $\mathcal{F}$-measurable functions $\rho_i : \Omega \to [0,\infty[$ satisfying hypotheses (H1) and (H2). Then

(1) *The equations*

$$\rho_{\Lambda\cup\{i\}}(\omega) \;=\; \frac{\rho_\Lambda(\omega)}{R^i_\Lambda(\omega)}\;, \tag{6.2}$$

$$R^i_\Lambda(\omega) \;=\; \left(\frac{\rho_\Lambda}{\rho_i}\times\lambda_i\left(\rho_i\,\rho_\Lambda^{-1}\right)\right)(x_\Lambda\omega)\;\;, \tag{6.3}$$

$i\notin\Lambda$, recursively define for each $\Lambda\in\mathcal{S}$ measurable functions $\rho_\Lambda : \Omega \to [0,+\infty[$ and $R^i_\Lambda : \Omega \longrightarrow]0,+\infty]$, the latter being independent of the choice of $x_\Lambda \in b(\Lambda,\{i\},\omega)$.

(2) *The functions defined above satisfy that for each $\omega\in\Omega$, $V\in\mathcal{S}(\Lambda^{\mathrm{c}})$, $x_\Lambda\in b(\Lambda,V,\omega)$,*

$$\rho_\Lambda\left(x_\Lambda\sigma_V\omega\right) > 0 \quad \forall\,\sigma_V\in\Omega_V\;, \tag{6.4}$$

and, for each $j\in\Lambda$ and $i\in\mathbb{Z}^d$, $i\neq j$,

$$\inf\left\{\lambda_i\left(\rho_i\,\rho_j^{-1}\,\lambda_j\left(\rho_j\,\rho^{-1}_{\Lambda^*_j}\right)\right)(x_\Lambda\sigma_V\omega) : \sigma_V\in\Omega_V\right\} > 0 \tag{6.5}$$

and

$$\sup\left\{\lambda_i\left(\rho_i\,\rho_j^{-1}\,\lambda_j\left(\rho_j\,\rho^{-1}_{\Lambda^*_j}\right)\right)(x_\Lambda\sigma_V\omega) : \sigma_V\in\Omega_V\right\} < \infty\;, \tag{6.6}$$

with the convention that $\rho_\emptyset\equiv 1$.

(3) *More generally,*

$$\rho_{\Theta\cup\Gamma}(\omega) \;=\; \frac{\rho_\Theta(\omega)}{R^\Gamma_\Theta(\omega)}\;, \tag{6.7}$$

with

$$R^\Gamma_\Theta(\omega) \;=\; \left(\frac{\rho_\Theta}{\rho_\Gamma}\times\lambda_\Gamma\left(\rho_\Gamma\,\rho_\Theta^{-1}\right)\right)(x_\Theta\omega)\;, \tag{6.8}$$

for each $\Theta\in\mathcal{S}$, $\Gamma\in\mathcal{S}(\Theta^{\mathrm{c}})$ and $\omega\in\Omega$. The RHS of (6.8) is independent of $x_\Theta\in b(\Theta,\Gamma,\omega)$.

Proof We will prove the Lemma by induction over $|\Lambda| \geq 1$. In (3) we assume $\Theta \cup \Gamma = \Lambda \cup \{i\}$ for some $i \notin \Lambda$. Note that, in particular, (3) implies that the value of the functions ρ_Λ do not depend on the order in which the sites of Λ are swept during the recursive construction.

The initial inductive step is immediate: If $\Lambda = \{j\}$, item (1) amounts to the identity (3.13) (with $i \leftrightarrow j$) and (6.3) is just the definition of R^i_j. Item (2) coincides with hypothesis (H1) while item (3) is the identity (3.14) which remains valid even if some numerator is zero or some denominator is infinity.

Suppose now (1)–(3) valid for all finite subsets of $\mathbb{Z}^d$ involving up to n sites. Consider $\Lambda \in \mathcal{S}$ of cardinality $n+1$, $i \notin \Lambda$, $V \in \mathcal{S}$ with $V \subset \Lambda^{\mathrm{c}}$ and some $x_\Lambda \in b(\Lambda, V, \omega)$. We observe that, by the very definition of $b(\Lambda, V, \omega)$ [see (3.9)],

$$x_j \in b(j, \Lambda^*_j \cup V, \omega) \quad \text{and} \quad x_{\Lambda^*_j} \in b(\Lambda^*_j, V \cup \{j\}, \omega) \tag{6.9}$$

for each site $j \in \Lambda$. The leftmost statement implies, by hypothesis (H1), that

$$\rho_j(x_j \sigma_{\Lambda^*_j \cup V}\omega) > 0, \quad \forall\, \sigma_{\Lambda^*_j \cup V} \in \Omega_{\Lambda^*_j \cup V}\ , \tag{6.10}$$

and, if $i \neq j$,

$$\left.\begin{matrix}\inf\\ \sup\end{matrix}\right\} \Big\{ \lambda_i \left(\rho_i\, \rho_j^{-1}\right)(x_j \sigma_{\Lambda^*_j \cup V}\omega) : \sigma_{\Lambda^*_j \cup V} \in \Omega_{\Lambda^*_j \cup V} \Big\} \left\{\begin{matrix} > 0 \\ < \infty\ . \end{matrix}\right. \tag{6.11}$$

On the other hand, the rightmost statement in (6.9) and the inductive hypothesis (2) imply that

$$\left.\begin{matrix}\inf\\ \sup\end{matrix}\right\} \Big\{ \lambda_j \left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)(x_{\Lambda^*_j} \sigma_{V \cup \{i\}}\omega) : \sigma_{V \cup \{i\}} \in \Omega_{V \cup \{i\}} \Big\} \left\{\begin{matrix} > 0 \\ < \infty\ . \end{matrix}\right. \tag{6.12}$$

Proof of (2) Combining (6.10) and (6.12) we see that the quotient

$$\rho_\Lambda\,(x_\Lambda \sigma_V \omega) \ \triangleq\ \frac{\rho_j\,(x_\Lambda \sigma_V \omega)}{\lambda_j \left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)(x_\Lambda \sigma_V \omega)} \tag{6.13}$$

satisfies

$$0 \ <\ \rho_\Lambda\,(x_\Lambda \sigma_V \omega) \ <\ \infty\ , \tag{6.14}$$

while (6.10) and (6.12) imply that

$$\left.\begin{matrix}\inf\\ \sup\end{matrix}\right\} \Big\{ \lambda_i \left(\rho_i\, \rho_j^{-1}\, \lambda_j(\rho_j\, \rho_{\Lambda^*_j}^{-1})\right)(x_\Lambda \sigma_V \omega) : \sigma_V \in \Omega_V \Big\} \left\{\begin{matrix} > 0 \\ < \infty\ . \end{matrix}\right. \tag{6.15}$$

Together (6.13) and (6.15) yield

$$\left.\begin{matrix}\inf\\ \sup\end{matrix}\right\} \Big\{ \lambda_i \left(\rho_i\, \rho_\Lambda^{-1}\right)(x_\Lambda \sigma_V \omega) : \sigma_V \in \Omega_V \Big\} \left\{\begin{matrix} > 0 \\ < \infty\ . \end{matrix}\right. \tag{6.16}$$

Proof of (1) We consider now $V = \{i\}$ with $i \notin \Lambda$. Inequalities (6.16) and the symmetry relation (6.14) imply that $\big(\rho_\Lambda \lambda_i(\rho\, \rho_\Lambda^{-1})\big)\,(x_\Lambda \sigma_i \omega) > 0$ for all $\sigma_i \in b(\Lambda, \{i\}, \omega)$ and thus it makes sense to define

$$R^i_\Lambda(\omega) \;=\; \left(\frac{\rho_\Lambda}{\rho_i} \times \lambda_i\left(\rho_i\, \rho_\Lambda^{-1}\right)\right)(x_\Lambda \omega) \tag{6.17}$$

which may be infinite but, due to (6.14) and (6.16), is never zero. We conclude that the function $\rho_{\Lambda\cup\{i\}}$ defined by (6.2) takes values on $[0,\infty[$.

We must prove that definition (6.17) is indeed independent of the choice of $x_\Lambda \in b(\Lambda, \{i\}, \omega)$. We analyze first the case $R^i_\Lambda(\omega) < \infty$. For each $j \in \Lambda$ and each $\sigma_i \in \Omega_i$ we have, by the inductive hypothesis (1),

$$\rho_\Lambda(x_\Lambda \sigma_i \omega) \;=\; \frac{\rho_j}{\lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)}(x_\Lambda \sigma_i \omega)\;. \tag{6.18}$$

Furthermore, combining (6.17) and (6.18) we obtain

$$R^i_\Lambda(\omega) \;=\; \left(\frac{\rho_j}{\rho_i \times \lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)} \times \lambda_i\left(\frac{\rho_i \times \lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)}{\rho_j}\right)\right)(x_\Lambda \omega)\;. \tag{6.19}$$

We now use (3.14), namely $\rho_i/R^j_i \;=\; \rho_j/R^i_j$, and make use of the $\mathcal{F}_{\{i\}^c}$-measurability of R^j_i to pass it through the λ_i-integration. We get

$$\begin{aligned} R^i_\Lambda(\omega) \;&=\; \left(\frac{R^i_j}{\lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)} \times \lambda_i\left(\frac{\lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)}{R^i_j}\right)\right)(x_\Lambda \omega) \\ &=\; \left(\frac{R^i_j}{\lambda_j\left(\rho_j\, \rho_{\Lambda^*_j}^{-1}\right)} \times \lambda_{\{i,j\}}\left(\frac{\rho_j\, \rho_{\Lambda^*_j}^{-1}}{R^i_j}\right)\right)(x_\Lambda \omega)\;. \end{aligned} \tag{6.20}$$

In the last equality we used the factorization property (3.2) of the free kernel and the $\mathcal{F}_{j^c}$-measurability of R^i_j. The final expression is manifestly independent of the actual value of x_j. Since j is an arbitrary site of Λ, we conclude that R^i_Λ is $\mathcal{F}_{\Lambda^c}$-measurable.

Let us turn now to the case $R^i_\Lambda(\omega) = \infty$. This happens if, and only if, $\rho_i(x_\Lambda \omega) = 0$. We must prove that, in this case, $\rho_i(\widetilde{x}_j x_{\Lambda^*_j} \omega) = 0$ for any $j \in \Lambda$ and any $\widetilde{x}_j \in \Omega_j$ such that $\widetilde{x}_j x_{\Lambda^*_j} \in b(\Lambda, \{i\}, \omega)$. But, by the definition of $b(\Lambda, \{i\}, \omega)$, for every $j \in \Lambda$

$$R^j_i(x_\Lambda \omega) < \infty \quad \text{and} \quad \rho_j(x_\Lambda \omega) > 0 \tag{6.21}$$

and

$$R_i^j(\widetilde{x}_j x_{\Lambda_j^*}\omega) < \infty \quad \text{and} \quad \rho_j(\widetilde{x}_j x_{\Lambda_j^*}\omega) > 0 \ . \tag{6.22}$$

We can now establish the following chain of implications:

$$\begin{aligned} \rho_i(x_\Lambda\omega) = 0 \quad &\Longrightarrow \quad R_j^i(x_\Lambda\omega) = \infty \\ &\Longrightarrow \quad R_j^i(\widetilde{x}_j x_{\Lambda_j^*}\omega) = \infty \quad \Longrightarrow \quad \rho_i(\widetilde{x}_j x_{\Lambda_j^*}\omega) = 0 \ . \end{aligned} \tag{6.23}$$

The first implication results from (6.21) and the symmetry relation (3.14), the second one is a consequence of the $\mathcal{F}_{\{j\}^c}$-measurability of R_j^i and the last one follows from (6.22) and (3.14).

Proof of (3) We consider Θ and Γ disjoint, non-empty, with $|\Theta \cup \Gamma| = n+1$, for $n \geq 2$ (the case $n = 1$ was analyzed at the begining). We have to prove that if $\Theta \cup \Gamma = \widetilde{\Theta} \cup \widetilde{\Gamma}$ with $\widetilde{\Theta}$ and $\widetilde{\Gamma}$ disjoint, then

$$\frac{\rho_\Theta}{R_\Theta^\Gamma} \ = \ \frac{\rho_{\widetilde{\Theta}}}{R_{\widetilde{\Theta}}^{\widetilde{\Gamma}}} \ . \tag{6.24}$$

As the argument is symmetric in Θ and Γ we can assume that $|\Theta| \geq 2$, in which case, modulo iteration, it is enough to prove that for $k \in \Theta$

$$\frac{\rho_\Theta}{R_\Theta^\Gamma} \ = \ \frac{\rho_{\Theta_k^*}}{R_{\Theta_k^*}^{\Gamma\cup\{k\}}} \ . \tag{6.25}$$

Let us fix some $\omega \in \Omega$ and $x_\Theta \in b(\Theta, \Gamma, \omega)$. The inductive definition (6.7)–(6.8) immediately yields the identity

$$\frac{\rho_\Theta(\omega)}{\rho_\Theta(x_\Theta\omega)} \ = \ \frac{\rho_{\Theta_k^*}(\omega)\rho_k(x_{\Theta_k^*}\omega)}{\rho_{\Theta_k^*}(x_{\Theta_k^*}\omega)\rho_k(x_\Theta\omega)} \ . \tag{6.26}$$

In addition we need the following identity

$$\lambda_\Gamma\left(\rho_\Gamma\rho_\Theta^{-1}\right)(x_\Theta\omega) \ = \ \lambda_\Gamma\left(\rho_\Gamma\rho_k^{-1}\right)(x_\Theta\omega)\,\lambda_{\Gamma\cup\{k\}}\left(\rho_{\Gamma\cup\{k\}}\rho_{\Theta_k^*}^{-1}\right)(x_{\Theta_k^*}\omega) \ . \tag{6.27}$$

This is proved as follows. We start from the relation

$$\left(\rho_\Gamma\,\rho_\Theta^{-1}\right)(x_\Theta\omega) \ = \ \left(\rho_\Gamma\,\rho_k^{-1}\,\lambda_k\left(\rho_k\,\rho_{\Theta_k^*}^{-1}\right)\right)(x_\Theta\,\omega) \tag{6.28}$$

which is an immediate consequence of the inductive hypotheses (6.7)–(6.8) [see (4.11)]. As $x_k \in b(k, \Gamma\cup\Theta_k^*, \omega)$, the LHS is well defined for every ω_Γ. We can, therefore, integrate both sides and conclude that

$$\lambda_\Gamma\left(\rho_\Gamma\,\rho_\Theta^{-1}\right)(x_\Theta\omega) \ = \ \lambda_\Gamma\left(\rho_\Gamma\,\rho_k^{-1}\,\lambda_k\left(\rho_k\,\rho_{\Theta_k^*}^{-1}\right)\right)(x_\Theta\,\omega) \ . \tag{6.29}$$

Next we observe that

$$\frac{\rho_\Gamma(x_\Theta \sigma_\Gamma \omega)}{\rho_k(x_\Theta \sigma_\Gamma \omega)} \;=\; \frac{\lambda_\Gamma \left(\rho_\Gamma\, \rho_k^{-1}\right)(x_\Theta \omega)}{R_k^\Gamma(x_\Theta \sigma_\Gamma \omega)} \tag{6.30}$$

for all $\sigma_\Gamma \in \Omega_\Gamma$. Again, this is a consequence of the inductive validity of (6.7)–(6.8) which, in particular, also implies that if $x_\Theta \in b(\Theta, \Gamma, \omega)$,

$$R_\Gamma^k(x_\Theta \omega) \;=\; \lambda_\Gamma \left(\rho_\Gamma\, \rho_k^{-1}\right)(x_\Theta \omega)\;. \tag{6.31}$$

To obtain (6.27) we must insert (6.30) into (6.29) and use that by (3.2) $\lambda_\Gamma \lambda_k = \lambda_{\Gamma\cup\{k\}}$.

The combination of (6.26) and (6.27) yields, thanks to the inductive definition of $\rho_{\Gamma\cup\{k\}}(x_{\Theta_k^*}\omega)$,

$$\frac{\rho_\Theta}{R_\Theta^\Gamma}(\omega) \;=\; \frac{\rho_{\Theta_k^*}(\omega)\, \rho_{\Gamma\cup\{k\}}(x_{\Theta_k^*}\omega)}{\rho_{\Theta_k^*}(x_{\Theta_k^*}\omega)\, \lambda_{\Gamma\cup\{k\}} \left(\rho_{\Gamma\cup\{k\}} \rho_{\Theta_k^*}^{-1}\right)(x_{\Theta_k^*}\omega)}\;. \tag{6.32}$$

Due to the inductive definition (6.8) of $R_{\Theta_k^*}^{\Gamma\cup\{k\}}$, the RHS of (6.32) is precisely the RHS of (6.25). This concludes the proof of (3), at least when $R_\Theta^\Gamma(\omega) < \infty$. But in fact the argument leading to identity (6.32) remains valid also when $R_\Theta^\Gamma(\omega)$ is infinite. In this case we have the following chain of implications:

$$R_\Theta^\Gamma(\omega) = \infty \quad\Longrightarrow\quad \rho_{\Gamma\cup\{k\}}(x_{\Theta_k^*}\omega) \;=\; 0 \quad\Longrightarrow\quad R_{\Theta_k^*}^{\Gamma\cup\{k\}}(\omega) = \infty\;. \tag{6.33}$$

The first implication is due to (6.32) while the second one follows from the inductive definition of $R_{\Theta_k^*}^{\Gamma\cup\{k\}}$. Display (6.33) proves (6.25) when $R_\Theta^\Gamma(\omega) = \infty$.

The proof that $R_\Theta^\Gamma(x_\Theta\omega)$ is independent of $x_\Theta \in b(\Theta, \Gamma, \omega)$ is completely analogous to the preceding proof of (1). We leave to the reader the pleasure of obtaining a formula similar to (6.20) and a chain of implications similar to (6.23) but changing $\Lambda \to \Theta$ and $i \to \Gamma$. $\square$

The following is a rather elementary property of conditional expectations.

Lemma 6.34
Let $\{\gamma_\Lambda\}_{\Lambda\in\mathcal{S}}$ be a specification, then for each $\Lambda \in \mathcal{S}$, $\Gamma \in \mathcal{S}_{\Lambda^c}$ and bounded measurable functions f, g,

$$\gamma_{\Lambda\cup\Gamma}\Big[f\, \gamma_\Lambda\big(\gamma_\Gamma(g)\big)\Big] \;=\; \gamma_{\Lambda\cup\Gamma}\Big[g\, \gamma_\Gamma\big(\gamma_\Lambda(f)\big)\Big]\;. \tag{6.35}$$

Proof By the consistency of the specification and the $\mathcal{F}_{\Lambda^c}$-measurability of $\gamma_\Lambda\big(\gamma_\Gamma(g)\big)$, we have

$$\begin{aligned}\gamma_{\Lambda\cup\Gamma}\Big[f\,\gamma_\Lambda\big(\gamma_\Gamma(g)\big)\Big] &= \gamma_{\Lambda\cup\Gamma}\Big[\gamma_\Lambda\left(f\,\gamma_\Lambda\big(\gamma_\Gamma(g)\big)\right)\Big]\\ &= \gamma_{\Lambda\cup\Gamma}\Big[\gamma_\Lambda(f)\,\gamma_\Lambda\Big(\gamma_\Gamma(g)\Big)\Big]\end{aligned}$$

Similarly, the $\mathcal{F}_{\Lambda^c}$-measurability of $\gamma_\Lambda(f)$ and the consistency of the specification give

$$\begin{aligned}\gamma_{\Lambda\cup\Gamma}\Big[\gamma_\Lambda(f)\,\gamma_\Lambda\Big(\gamma_\Gamma(g)\Big)\Big] &= \gamma_{\Lambda\cup\Gamma}\Big[\gamma_\Lambda\Big(\gamma_\Lambda(f)\,\gamma_\Gamma(g)\Big)\Big]\\ &= \gamma_{\Lambda\cup\Gamma}\Big[\gamma_\Lambda(f)\,\gamma_\Gamma(g)\Big]\end{aligned}$$

Identity (6.35) follows from the $f \leftrightarrow g$ symmetry of the last expression. □

Our last lemma is the basis of the proof of part III of the theorem. For every $V \in \mathcal{S}$ such that $|V| \geq 2$, let us define

$$B_V \triangleq \bigcap_{i\in V} B(i, V_i^*)\ . \tag{6.38}$$

Lemma 6.39
Let $V \in \mathcal{S}$.

(1) *For every $W \in \mathcal{S}(V^c)$,*

$$\rho_{V\cup W} = \frac{\rho_V}{\lambda_V(\rho_V\,\rho_W^{-1})} = \frac{\rho_W}{\lambda_W(\rho_W\,\rho_V^{-1})} \quad \textit{on } B_{V\cup W}\ . \tag{6.40}$$

(2) *For every $j \in V^c$*

$$B(j, V) \in \mathcal{F}_{V^c}\ . \tag{6.41}$$

(3) *If $\mu \in \mathcal{N}$, then*

- **(a)** $\mu\,\lambda_V\big(B(k, V_k^*)^c\big) = 0$ *for every $k \in V$,*
- **(b)** $\mu\,\lambda_V(B_V^c) = 0$.

Furthermore, if μ satisfies the singleton consistency

$$\mu\Big((\rho_i\,\lambda_i)(h)\Big) = \mu(h) \quad \textit{for every } i \in \mathbb{Z}^d\ , \tag{6.42}$$

then

(c) $\mu\big(B(j,V)^{\rm c}\big) = 0$ *for every* $j \in V^{\rm c}$,

(d) $\mu(B_V^{\rm c}) = 0$.

Proof

(1) Let $\omega \in B_{V\cup W}$. Then $\omega_V \in b(V,W,\omega)$ and $\omega_W \in b(W,V,\omega)$. Hence by Lemma 6.1 (3), we have the claim.

(2) It suffices to combine (3.4)–(3.8)

(3)(a) We apply part (2):

$$\begin{aligned}\mu\,\lambda_V\,(B(k,V_k^*)^{\rm c}) &= \mu\,\lambda_k\,\lambda_{V_k^*}\big(B(k,V_k^*)^{\rm c}\big)\\ &= \mu\,\lambda_k\,\big(B(k,V_k^*)^{\rm c}\,\lambda_{V_k^*}(\Omega)\big)\\ &= 0\,, \end{aligned}\tag{6.43}$$

where we use the fact that the measure $\lambda^{V_k^*}$ is finite.

(3)(c) Since $\mu \in \mathcal{N}$ satisfies SC we have

$$\mu\big(B(j,V)^{\rm c}\big) \;=\; \mu\,\lambda_j\big(1\!\!1_{B(j,V)^{\rm c}}\,\rho_j\big) \;=\; 0\,. \tag{6.44}$$

(3)(b)–(d) In view of part (3)(a)–(c), the proof is a consequence of the following observation. For a measure ν

$$\nu\big(B(k,V_k^*)\big) \;=\; 0 \;\;\forall\, k\in V \quad\Longrightarrow\quad \nu\big(B_V^{\rm c}\big) \;=\; 0\,. \tag{6.45}$$

This follows from the inequality

$$\nu\big(B_V^{\rm c}\big) \;=\; \nu\Big(\bigcup_{k\in V} B(k,V_k^*)^{\rm c}\Big) \;\le\; \sum_{k\in V}\nu\big(B(k,V_k^*)^{\rm c}\big).\quad\Box \tag{6.46}$$

Proof of Theorem 4.1

We consider the functions ρ_Λ constructed in the previous Lemma 6.1 and a bounded measurable function h. We will prove, by induction over $|\Lambda|$, where $\Lambda\in\mathcal{S}$, that

(P1) ρ_Λ is normalized;

(P2) For each $\Gamma\subset\Lambda$

$$\Big(\rho_\Lambda\,\lambda_\Lambda\Big)\Big((\rho_\Gamma\,\lambda_\Gamma)\,(h)\Big) \;=\; (\rho_\Lambda\,\lambda_\Lambda)(h)\,. \tag{6.47}$$

(P3) If (4.3) holds, every specification in Λ of the form $\{\widetilde{\rho}_\Gamma\lambda_\Gamma : \Gamma\subset\Lambda\}$ such that

$$\Big(\widetilde{\rho}_\Lambda\,\lambda_\Lambda\Big)\Big((\rho_i\,\lambda_i)\,(h)\Big) \;=\; (\widetilde{\rho}_\Lambda\,\lambda_\Lambda)(h)\,,\;\forall i\in\Lambda \tag{6.48}$$

satisfies that, for each $\omega\in\Omega_\Lambda$,

$$\widetilde{\rho}_\Lambda(\xi_\Lambda\omega) = \rho_\Lambda(\xi_\Lambda\omega) \quad \text{for } \lambda^\Lambda\text{-a.a. } \xi_\Lambda\in\Omega_\Lambda\,. \tag{6.49}$$

(P4) If all the functions ρ_i, $i \in \mathbb{Z}$, are continuous and (4.4) holds, then each function ρ_Λ is continuous and for all $i \in \Lambda^{\rm c}$ there exists $x_\Lambda \in \bigcap_\omega b(\Lambda, i, \omega)$ such that

$$\int \sup_\omega \left(\rho_i\, \rho_\Lambda^{-1}\right) (\sigma_i x_\Lambda \omega)\, \lambda^i(d\sigma_i) \;<\; \infty\,. \tag{6.50}$$

(P5) If $\mu \in \mathcal{N}$ (recall (4.8)) and satisfies singleton consistency (6.42), then

$$\mu\Big((\rho_\Lambda\, \lambda_\Lambda)(h)\Big) \;=\; \mu(h). \tag{6.51}$$

The case $|\Lambda| = 1$ is straightforward: (P1) is just the singleton normalization (4.2), (P2), (P3) and (P5) are trivially true while (P4) is (4.4). We take now $\Lambda \in \mathcal{S}$ with $|\Lambda| \geq 2$ and assume that (P1)–(P5) are verified by all its non-trivial subsets.

Proof of (P1) Let $\omega \in \Omega$ and $k \in \Lambda$. By the factorization property (3.2) of λ_Λ and the definition of ρ_Λ we have that

$$\lambda_\Lambda\left(\rho_\Lambda\right)(\omega) \;=\; \lambda_k \left(\lambda_{\Lambda_k^*}\Big(\frac{\rho_{\Lambda_k^*}}{R^k_{\Lambda_k^*}}\Big)\right)(\omega)\,. \tag{6.52}$$

Therefore, by the $\mathcal{F}_{(\Lambda_k^*)^{\rm c}}$-measurability of $R^k_{\Lambda_k^*}$ and the inductive normalization (P1),

$$\lambda_\Lambda\left(\rho_\Lambda\right)(\omega) \;=\; \lambda_k\left(\frac{\lambda_{\Lambda_k^*}\left(\rho_{\Lambda_k^*}\right)}{R^k_{\Lambda_k^*}}\right)(\omega) \;=\; \lambda_k\left(\frac{1}{R^k_{\Lambda_k^*}}\right)(\omega)\,.$$

Replacing

$$R^k_{\Lambda_k^*}(\omega) \;=\; \left(\frac{\rho_{\Lambda_k^*}}{\rho_k}\, \lambda_k\left(\rho_k\, \rho_{\Lambda_k^*}^{-1}\right)\right)(x_{\Lambda_k^*}\omega)\,,$$

for any $x_{\Lambda_k^*} \in b(\Lambda_k^*, \{k\}, \omega)$, we readily obtain $\lambda_\Lambda\left(\rho_\Lambda\right)(\omega) = 1$.

Proof of (P2) It suffices to show that for some $i \in \Lambda$

$$\Big(\rho_\Lambda\, \lambda_\Lambda\Big)\Big(\left(\rho_{\Lambda_i^*}\, \lambda_{\Lambda_i^*}\right)(h)\Big) \;=\; (\rho_\Lambda\, \lambda_\Lambda)(h)\,. \tag{6.53}$$

Indeed, such an identity combined with the inductive hypothesis (P2) yields that for Γ strictly contained in Λ,

$$\Big(\rho_\Lambda\, \lambda_\Lambda\Big)\Big(\left(\rho_\Gamma\, \lambda_\Gamma\right)(h)\Big) \;=\; \Big(\rho_\Lambda\, \lambda_\Lambda\Big)\Big(\left(\rho_{\Lambda_i^*}\, \lambda_{\Lambda_i^*}\right)\Big(\left(\rho_\Gamma\, \lambda_\Gamma\right)(h)\Big)\Big) \;=\; (\rho_\Lambda\, \lambda_\Lambda)(h)\,, \tag{6.54}$$

as needed. To prove(6.53) we use the definitions of λ_Λ and ρ_Λ to write

$$\Big(\rho_\Lambda\,\lambda_\Lambda\Big)\Big(\big(\rho_{\Lambda_i^*}\,\lambda_{\Lambda_i^*}\big)\,(h)\Big) \;=\; \lambda_i\left(\lambda_{\Lambda_i^*}\left(\frac{\rho_{\Lambda_i^*}}{R^i_{\Lambda_i^*}}\lambda_{\Lambda_i^*}(\rho_{\Lambda_i^*}\,h)\right)\right)\;. \tag{6.55}$$

Since $R^i_{\Lambda_i^*}$ is $\mathcal{F}_{\Lambda_i^{*c}}$-measurable and $\lambda_{\Lambda_i^*}(\rho_{\Lambda_i^*}) = 1$ [inductive (P1)], it follows that

$$\Big(\rho_\Lambda\,\lambda_\Lambda\Big)\Big(\big(\rho_{\Lambda_i^*}\,\lambda_{\Lambda_i^*}\big)\,(h)\Big) \;=\; \lambda_i\left(\lambda_{\Lambda_i^*}\left(\frac{\rho_{\Lambda_i^*}\,h}{R^i_{\Lambda_i^*}}\right)\right) \;=\; \big(\rho_\Lambda\,\lambda_\Lambda\big)\,(h)\;. \tag{6.56}$$

Proof of (P3) We pick $k\in\Lambda$ and apply Lemma 6.34 to the specification $\{\widetilde{\rho}_\Gamma\lambda_\Gamma : \Gamma\subset\Lambda\}$ for $f\equiv 1\!\!1_{A_\Lambda}$ and $g\equiv 1\!\!1_{B_\Lambda}$ with $A_\Lambda, B_\Lambda\in\mathcal{F}_\Lambda$. We obtain

$$\begin{aligned}&\int \widetilde{\rho}_{\Lambda_k^*}(\xi_\Lambda\omega)\,\widetilde{\rho}_k(\xi_k x_{\Lambda_k^*}\omega)\,\widetilde{\rho}_\Lambda(x_\Lambda\omega)\,1\!\!1_{A_\Lambda}(\xi_\Lambda)\,1\!\!1_{B_\Lambda}(x_\Lambda)\,\lambda^\Lambda(d\xi_\Lambda)\,\lambda^\Lambda(dx_\Lambda)\\ &=\;\int \widetilde{\rho}_k(x_\Lambda\omega)\,\widetilde{\rho}_{\Lambda_k^*}(x_{\Lambda_k^*}\xi_k\omega)\,\widetilde{\rho}_\Lambda(\xi_\Lambda\omega)\,1\!\!1_{A_\Lambda}(\xi_\Lambda)\,1\!\!1_{B_\Lambda}(x_\Lambda)\,\lambda^\Lambda(dx_\Lambda)\,\lambda^\Lambda(d\xi_\Lambda)\;,\end{aligned} \tag{6.57}$$

for every $\omega\in\Omega_{\Lambda^c}$. Each member of the preceding equality defines a probability measure over the product σ-algebra $\mathcal{F}_\Lambda\otimes\mathcal{F}_\Lambda$. This σ-algebra is generated by the π-system, $\{A_\Lambda\times B_\Lambda : A_\Lambda,\ B_\Lambda\in\mathcal{F}_\Lambda\}$. As both sides coincide on these system, they must be equal as probability measures and, with the aid of the inductive hypothesis (P3) we conclude that

$$\rho_{\Lambda_k^*}(\xi_\Lambda\omega)\,\rho_k(\xi_k x_{\Lambda_k^*}\omega)\,\widetilde{\rho}_\Lambda(x_\Lambda\omega) \;=\; \rho_k(x_\Lambda\omega)\,\rho_{\Lambda_k^*}(x_{\Lambda_k^*}\xi_k\omega)\,\widetilde{\rho}_\Lambda(\xi_\Lambda\omega)\;, \tag{6.58}$$

for $\lambda^\Lambda\times\lambda^\Lambda$-a.a. $(\xi_\Lambda, x_\Lambda)\in\Omega_\Lambda\times\Omega_\Lambda$. Since by assumption each λ^j charges $b(j,\Lambda_j^*,\omega)$, identity (6.58) must be verified for some choice of $x_j\in b(j,\Lambda_j^*,\omega)$. In this case the factors of $\widetilde{\rho}_\Lambda$ in the RHS of (6.58) are non-zero and we can solve

$$\widetilde{\rho}_\Lambda(\xi_\Lambda\omega) \;=\; \frac{\rho_{\Lambda_k^*}(\xi_\Lambda\omega)\,\rho_k(\xi_k x_{\Lambda_k^*}\omega)\,\widetilde{\rho}_\Lambda(x_\Lambda\omega)}{\rho_k(x_\Lambda\omega)\,\rho_{\Lambda_k^*}(x_{\Lambda_k^*}\xi_k\omega)} \tag{6.59}$$

for λ^Λ-a.a. $\xi_\Lambda\in\Omega_\Lambda$. If we integrate both sides with respect $\lambda^\Lambda(\xi_\Lambda)$, we get

$$1 \;=\; \Big(\lambda_\Lambda(\widetilde{\rho}_\Lambda)\Big)(\omega) \;=\; \frac{\widetilde{\rho}_\Lambda(x_\Lambda\omega)}{\rho_k(x_\Lambda\omega)}\,\lambda_k\left(\rho_k\,\rho^{-1}_{\Lambda_k^*}\,\lambda_{\Lambda_k^*}(\rho_{\Lambda_k^*})\right)(x_\Lambda\omega)\;. \tag{6.60}$$

Since $\lambda_{\Lambda_k^*}(\rho_{\Lambda_k^*})\equiv 1$, we obtain

$$\widetilde{\rho}_\Lambda(x_\Lambda\omega) \;=\; \frac{\rho_k(x_\Lambda\omega)}{\lambda_k\left(\rho_k\,\rho^{-1}_{\Lambda_k^*}\right)(x_\Lambda\omega)} \;=\; \rho_\Lambda(x_\Lambda\omega)\;. \tag{6.61}$$

From (6.58) and (6.61), we conclude that each $\widetilde{\rho}_\Lambda$ satisfying (6.48) is λ^Λ-a.s. uniquely determined. Since ρ_Λ itself satisfies (6.48), statement (6.49) follows.

Proof of (P4) We first remark that if $V \subset \Lambda^{c}$ we can construct some $x_\Lambda \in \bigcap_\omega b(\Lambda, V, \omega)$ simply by choosing $x_j \in \bigcap_\omega b(j, V \cup \Lambda_j^*, \omega)$ [see definition (3.9)]. Let $k \in \Lambda$ and $x_{\Lambda_k^*} \in \bigcap_\omega b(\Lambda_k^*, k, \omega)$. The inductive hypotheses (P4) implies the continuity of the functions $(\rho_k\, \rho_{\Lambda_k^*}^{-1})(\sigma_k x_{\Lambda_k^*}\,\cdot\,)$ for each $\sigma_k \in E$. These functions are uniformly bounded above by $\sup_\omega \rho_k\, \rho_{\Lambda_k^*}^{-1}(\sigma_k x_{\Lambda_k^*}\omega)$ which —by the inductive assumption (6.50)— is integrable with respect to $\lambda^k(d\sigma_k)$. The sequential continuity of the function $\lambda_k(\rho_k\, \rho_{\Lambda_k^*}^{-1})(x_{\Lambda_k^*}\,\cdot\,)$ follows, then, from the dominated convergence theorem. This function is strictly positive because of the choice of $x_{\Lambda_k^*}$. These continuity and non-nullness, plus the inductive continuity hypothesis, imply that

$$\rho_\Lambda(\,\cdot\,) \;\triangleq\; \frac{\rho_{\Lambda_k^*}(\,\cdot\,)\, \rho_k(x_{\Lambda_k^*}\,\cdot\,)}{\rho_{\Lambda_k^*}(x_{\Lambda_k^*}\,\cdot\,)\, \lambda_k(\rho_k\, \rho_{\Lambda_k^*}^{-1})(x_{\Lambda_k^*}\,\cdot\,)} \tag{6.62}$$

is a continuous function.

Finally we prove (6.50). The existence of some $x_\Lambda \in \bigcap_\omega b(\Lambda, i, \omega)$ yields the identity

$$\big(\rho_i\, \rho_\Lambda^{-1}\big)\,(\sigma_i x_\Lambda \omega) \;=\; \big(\rho_i\, \rho_k^{-1}\big)\,(\sigma_i x_\Lambda \omega) \times \int \Big(\rho_k\, \rho_{\Lambda_k^*}^{-1}\Big)\,(\sigma_k \sigma_i x_\Lambda \omega)\, \lambda^k(d\sigma_k)\;, \tag{6.63}$$

valid for *all* $\omega \in \Omega_{(\Lambda\cup\{i\})^{c}}$, each $k \in \Lambda$ and each $\sigma_i \in \Omega_i$. We take supremum over ω and integrate with respect to λ^i to obtain

$$\begin{aligned}
&\int \sup_\omega \big(\rho_i\, \rho_\Lambda^{-1}\big)\,(\sigma_i x_\Lambda \omega)\, \lambda^i(d\sigma_i)\\
&\quad \le \int \sup_\omega \Big(\rho_i\, \rho_k^{-1}\Big)(\sigma_i x_\Lambda \omega)\, \lambda^i(d\sigma_i) \times \int \sup_\omega \Big(\rho_k\, \rho_{\Lambda_k^*}^{-1}\Big)\,(\sigma_k x_{\Lambda_k^*} \omega)\, \lambda^k(d\sigma_k)\;.
\end{aligned} \tag{6.64}$$

Both integrals in the RHS are finite by the inductive assumption (P4).

Proof of (P5) Fix $i \in \Lambda$. Since $\mu \in \mathcal{N}$ satisfies singleton consistency (6.42),

$$\begin{aligned}
\mu\big((\rho_\Lambda\, \lambda_\Lambda)(h)\big) \;&=\; \mu\left(\lambda_\Lambda\left(\mathbb{1}_{B_\Lambda}\, \rho_\Lambda\, h\right)\right)\\
&=\; \mu\left(\lambda_\Lambda\left(\mathbb{1}_{B_\Lambda} \frac{\rho_i}{\lambda_i\big(\rho_i\, \rho_{\Lambda_i^*}^{-1}\big)}\, h\right)\right)\;.
\end{aligned} \tag{6.65}$$

where the first and second identities come respectively from parts Lemma (3)(b) and (1) of Lemma 6.39. We write $B_\Lambda = \bigcap_{k\in\Lambda} B(k, \Lambda_k^*)$ and decompose

$\lambda_\Lambda = \lambda_{\Lambda_i^*}\,\lambda_i$. Using the measurability and the support property of parts (2) and (3)(b) of Lemma 6.39, we see that

$$\begin{aligned}
\mu\big((\rho_\Lambda\,\lambda_\Lambda)(h)\big) &= \mu\left(\lambda_{\Lambda_i^*}\left(\mathbb{1}\Big\{\bigcap_{j\in\Lambda_i^*} B(j,\Lambda_j^*)\Big\}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\right)\\
&= \mu\left(\mathbb{1}_{B(i,\Lambda_i^*)}\lambda_{\Lambda_i^*}\left(\mathbb{1}\Big\{\bigcap_{j\in\Lambda_i^*} B(j,\Lambda_j^*)\Big\}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\right)\\
&= \mu\left(\lambda_{\Lambda_i^*}\left(\mathbb{1}_{B_\Lambda}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\right)\;. \qquad (6.66)
\end{aligned}$$

The inductive hypotheses (P5) implies that $\mu(\rho_{\Lambda_i^*}\,\lambda_{\Lambda_i^*}) = \mu$. Hence

$$\begin{aligned}
\mu\big((\rho_\Lambda\,\lambda_\Lambda)(h)\big) &= \mu\left(\frac{\mathbb{1}_{B_\Lambda}}{\rho_{\Lambda_i^*}}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\\
&= \mu\left(\frac{1}{\rho_{\Lambda_i^*}}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\;, \qquad (6.67)
\end{aligned}$$

where the second line comes from support property of Lemma 6.39 (3)(d). By singleton consistency (6.42)

$$\begin{aligned}
\mu\big((\rho_\Lambda\,\lambda_\Lambda)(h)\big) &= \mu\,\lambda_i\left(\frac{\rho_i}{\rho_{\Lambda_i^*}}\frac{\lambda_i\big(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h\big)}{\lambda_i\big(\rho_i\,\rho_{\Lambda_i^*}^{-1}\big)}\right)\\
&= \mu\big(\lambda_i(\mathbb{1}_{B(i,\Lambda_i^*)}\,\rho_i\,h)\big)\;. \qquad (6.68)
\end{aligned}$$

Therefore

$$\mu\big((\rho_\Lambda\,\lambda_\Lambda)(h)\big) \;=\; \mu\big(\lambda_i(\rho_i\,h)\big) \;=\; \mu(h)\;, \qquad (6.69)$$

where once again we use Lemma 6.39 (3)(b) and singleton consistency. □

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