Necklaces with interacting beads: isoperimetric problems

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ABSTRACT. We discuss a pair of isoperimetric problems which at a glance seem to be unrelated. The first one is classical: one places N identical point charges at a closed curve Γ at the same arc-length distances and asks about the energy minimum, i.e. which shape does the loop take if left by itself. The second problem comes from quantum mechanics: we take a Schrödinger operator in $L^2(\mathbb{R}^d)$, $d = 2, 3$, with N identical point interaction placed at a loop in the described way, and ask about the configuration which *maximizes* the ground state energy. We reduce both of them to geometric inequalities which involve chords of Γ; it will be shown that a sharp local extremum is in both cases reached by Γ in the form of a regular (planar) polygon and that such a Γ solves the two problems also globally.

1. Introduction

Isoperimetric problems are certainly one of the topics which keeps appearing in mathematical physics regularly. In the present paper we are going to discuss two new examples of this kind. It may seem that these problems differ mathematically and also have a different physical background; we will show nevertheless that they can be reduced to the same geometric question. A common feature is that they both concern extremal properties of interaction between $N \geq 2$ points placed at identical arc-length distances along a closed curve Γ of a fixed length $L > 0$.

The first problem comes from classical electrostatics and concerns *charged necklaces.* Let $\Gamma : [0, L] \to \mathbb{R}^3$ be such a loop and suppose that N identical charges are placed at the points $\Gamma(kL/N)$, $k = 0, 1, \dots, N-1$. We ask about the shape which this constrained family of point sources will take in the absence of external forces, i.e. about *minimum* of the potential energy of the Coulombic repulsion.

The other problem comes from quantum mechanics. Having again a loop Γ, now in two or three dimensions, we consider a class of singular Schrödinger operators in $L^2(\mathbb{R}^d)$, $d = 2, 3$, which are given formally by the expression

$$H^N_{\alpha,\Gamma} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\left(x - \Gamma\left(\frac{jL}{N}\right)\right) . \tag{1.1}$$

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We will recall below how they can be defined properly, see [**AGHH**]; following the terminology of this monograph we can label the problem as a *polymer loop.* This time we are interested in the shape of Γ which *maximizes* the ground state energy, of course, provided the discrete spectrum of $H^N_{\alpha,\Gamma}$ is non-empty.

Our goal in this paper is to show that the two problems reduce essentially to the same geometric question and that a sharp local extremum is in both cases reached by the shape with a maximum symmetry, in other words by a *regular planar polygon* with N vertices denoted in the following as $\tilde{\mathcal{P}}_N$. Furthermore, we will show that the regular polygon represents also a *global* solution to the problem by reducing the task to a norm estimate of a particular operator on $\ell^2(\mathbb{Z})$.

The indicated quantum-mechanical isoperimetric problem was formulated first in the paper [**Ex1**], to which we refer for a deeper motivation, in the particular case when the curve Γ was an equilateral polygon. The problem was restated there in purely geometric terms and the existence of the maximizer was proved locally. In [**Ex2**] a similar local result was derived for a continuous analogue of this problem in which the interaction was supported by the entire curve Γ and the maximizer is a circle. The technical improvement in the last named paper was that the geometric problem was viewed there from a general perspective in terms of inequalities for chords of the loop, more specifically, ℓ^p norms related to the functions $\Gamma(\cdot+u)-\Gamma(\cdot)$. In the subsequent work [**EHL**] a simple Fourier analysis was used to show that in the "continuous" quantum-mechanical problem the circle is a *global* maximizer.

Our first aim here is to show first that the local proof of [**Ex1**] can be extended to the more general class of curves. Then we will discuss how the argument of [**EHL**] can be modified to the present "discrete" situation. It appears that the task is more involved than in the "continuous" case, however, we will be able to reduce the question whether a regular polygon $\tilde{\mathcal{P}}_N$ represents a global extremum in both the isoperimetric problems described above to a well-defined operator problem.

2. Point interactions on a loop

To begin with, let us make more precise our requirements on the curve regularity. In what follows, we will suppose that for a fixed dimension $d \ge 2$

(ℓ) $\Gamma:\ [0,L] \to \mathbb{R}^d$ is a continuous, piecewise C^1 function such that $\Gamma(0)=\Gamma(L)$ and $|\dot{\Gamma}(s)|=1$ holds for any $s\in[0,L]$ for which $\dot{\Gamma}(s)$ exists.

The arc-length parametrization means in fact that we regard the curve as a map $\mathbb{R}\to\mathbb{R}^d \pmod{L}$. A shift in the argument is a trivial reparametrization, so without loss of generality we may assume that the point interactions are placed at

$$y_j := \Gamma\left(\frac{jL}{N}\right), \quad j=0,1,\dots,N-1\,; \tag{2.1}$$

the indices can be again regarded as integers, $y_j = y_{j (\mathrm{mod}\, N)}$. A distinguished element of the described class is a *regular polygon* for which the points y_j lie in a plane $\subset \mathbb{R}^d$ (this is trivial if $d=2$) at a circle of radius $\frac{L}{N}\left(2\sin\frac{\pi}{N}\right)^{-1}$.

Let $Y_\Gamma := \{y_j\,:\, j=0,\dots,N-1\}$ be the interaction support. The object of our interest will the Hamiltonian $-\Delta_{\alpha,Y_\Gamma}$ in $L^2(\mathbb{R}^d)$ with N point interactions, all of the same coupling constant $\alpha\in\mathbb{R}$. It is defined conventionally through boundary conditions which relate the generalized boundary values at each site y_k, the coefficient at the singularity (logarithmic for $d=2$, pole for $d=3$) and the next term in the expansion – see [**AGHH**] for a detailed discussion and recall that the construction of a point-interaction Hamiltonian does not work for $d \geq 4$. Recall also that α differs from the formal coupling constant $\tilde{\alpha}$ in (1.1); it is sufficient to realize that the absence of a point interaction corresponds to $\alpha = \infty$.

It is obvious that that spectral properties of $-\Delta_{\alpha,Y_\Gamma}$ and $-\Delta_{\alpha,Y_{\Gamma'}}$ corresponding to a pair of loops Γ and Γ' related mutually by Euclidean transformations of $\mathbb{R}^d$ are the same. This defines an equivalence relation on the set of all loops; with a terminological abuse we will speak about curves having in mind such equivalence classes. We will suppose that the operator $-\Delta_{\alpha,Y_\Gamma}$ has a non-empty discrete spectrum,

$$\epsilon_1 \equiv \epsilon_1(\alpha, Y_\Gamma) := \inf \sigma\left(-\Delta_{\alpha,Y_\Gamma}\right) < 0\,, \tag{2.2}$$

which is true for any $\alpha \in \mathbb{R}$ if $d = 2$, while in the case $d = 3$ it is a nontrivial assumption satisfied below a certain critical value of α – cf. [**AGHH**, Sec. II.1].

One of the main results of this paper referring to the second of the problems mentioned in the introduction, the polymer loop, can be formulated as follows:

THEOREM 2.1. *Assume (ℓ) and (2.2); then $\epsilon_1(\alpha, Y_\Gamma)$ is for a fixed α and $L > 0$ locally sharply maximized by a regular polygon, $\Gamma = \tilde{\mathcal{P}}_N$.*

The proof will be done in several steps; in this section we will demonstrated that the task can be reduced to a purely geometric problem. With the usual notation, $k = i\kappa$ with $\kappa > 0$, we find the eigenvalues $-\kappa^2$ from the spectral condition,

$$\det Q_k = 0 \qquad \text{with} \qquad (Q_k)_{ij} := (\alpha - \xi^k)\delta_{ij} - (1 - \delta_{ij})g^k_{ij}\,,$$

where $g^k_{ij} := G_k(y_i - y_j)$, or equivalently

$$g^k_{ij} = \begin{cases} \frac{1}{2\pi}K_0(\kappa|y_i - y_j|) & \dots \quad d = 2 \\ \frac{e^{-\kappa|y_i-y_j|}}{4\pi|y_i-y_j|} & \dots \quad d = 3 \end{cases} \tag{2.3}$$

and the regularized Green's function at the interaction site is

$$\xi^k = \begin{cases} -\frac{1}{2\pi}\left(\ln\frac{\kappa}{2} + \gamma_{\mathrm{E}}\right) & \dots \quad d = 2 \\ -\frac{\kappa}{4\pi} & \dots \quad d = 3 \end{cases}$$

where γ_{E} is the Euler number. The matrix $Q_{i\kappa}$ has N eigenvalues counting multiplicity which are decreasing in $(-\infty, 0)$ as functions of κ by [**KL, AGHH**]. The spectral threshold $\epsilon_1(\alpha, \Gamma)$ corresponds to the point κ where the lowest of the indicated eigenvalues vanishes. Consequently, we have to check that

$$\min \sigma(Q_{i\tilde{\kappa}_1}) < \min \sigma(\tilde{Q}_{i\tilde{\kappa}_1}) \tag{2.4}$$

holds locally for $\Gamma \neq \tilde{\mathcal{P}}_N$. Here and in the following the tilded quantities correspond always to regular polygon, $\Gamma = \tilde{\mathcal{P}}_N$, in particular, $-\tilde{\kappa}_1^2 = \epsilon_1(\alpha, \tilde{\mathcal{P}}_N)$.

Next we use the fact that the lowest eigenvalue of $\tilde{Q}_{i\tilde{\kappa}_1}$ corresponds to the eigenvector $\tilde{\phi}_1 = N^{-1/2}(1, \dots, 1)$, because by [**AGHH**] there is a one-to-one correspondence between an eigenfunction $c = (c_1, \dots, c_N)$ of $Q_{i\kappa}$ at the point, where the corresponding eigenvalue vanishes, and the corresponding eigenfunction of $-\Delta_{\alpha,Y_\Gamma}$ given by $c \leftrightarrow \sum_{j=1}^N c_j G_{i\kappa}(\cdot - y_j)$, up to a normalization. Again by [**AGHH**], the principal eigenvalue of $-\Delta_{\alpha,Y_\Gamma}$ is simple, so it has to be associated with a one-dimensional representation of the corresponding discrete symmetry group of $\tilde{\mathcal{P}}_N$; it follows that the coefficients are the same, $c_1 = \cdots = c_N$. This yields the expression

$$\min \sigma(\tilde{Q}_{i\tilde{\kappa}_1}) = (\tilde{\phi}_1, \tilde{Q}_{i\tilde{\kappa}_1}\tilde{\phi}_1) = \alpha - \xi^{i\tilde{\kappa}_1} - \frac{2}{N}\sum_{i<j} \tilde{g}_{ij}^{i\tilde{\kappa}_1} . \tag{2.5}$$

On the other hand, for the left-hand side of (2.4) we have a variational estimate,

$$\min \sigma(Q_{i\tilde{\kappa}_1}) \le (\tilde{\phi}_1, Q_{i\tilde{\kappa}_1}\tilde{\phi}_1) = \alpha - \xi^{i\tilde{\kappa}_1} - \frac{2}{N}\sum_{i<j} g_{ij}^{i\tilde{\kappa}_1} ,$$

which shows that it is sufficient to check validity of the inequality

$$\sum_{i<j} G_{i\kappa}(y_i - y_j) > \sum_{i<j} G_{i\kappa}(\tilde{y}_i - \tilde{y}_j) \tag{2.6}$$

for all $\kappa > 0$ and $\Gamma \neq \tilde{\mathcal{P}}_N$. Let us introduce the symbol ℓ_{ij} for the chord length $|y_i - y_j|$ and $\tilde{\ell}_{ij} := |\tilde{y}_i - \tilde{y}_j|$, and define the function $F:\, (\mathbb{R}_+)^{N(N-3)/2} \to \mathbb{R}$ by

$$F(\{\ell_{ij}\}) := \sum_{m=2}^{[N/2]} \sum_{|i-j|=m} \Big[G_{i\kappa}(\ell_{ij}) - G_{i\kappa}(\tilde{\ell}_{ij}) \Big] \,;$$

we want to show that $F(\{\ell_{ij}\}) > 0$ except if $\{\ell_{ij}\} = \{\tilde{\ell}_{ij}\}$. The function $G_{i\kappa}(\cdot)$ is by (2.3) *convex* for a fixed $\kappa > 0$ and $d = 2, 3$, thus by Jensen's inequality we have

$$F(\{\ell_{ij}\}) \ge \sum_{m=2}^{[N/2]} \nu_m \left[G_{i\kappa}\left(\frac{1}{\nu_m} \sum_{|i-j|=m} \ell_{ij} \right) - G_{i\kappa}(\tilde{\ell}_{1,1+m}) \right] ,$$

where ν_n is the number of the appropriate chords,

$$\nu_m := \begin{cases} N & \dots \quad m = 1, \dots, \left[\frac{1}{2}(N-1)\right] \\ \frac{1}{2}N & \dots \quad m = \frac{1}{2}N \quad \text{for } N \text{ even} \end{cases}$$

At the same time, $G_{i\kappa}(\cdot)$ is monotonously decreasing in $(0, \infty)$, so the sought claim would follow if we demonstrate the inequality

$$\tilde{\ell}_{1,m+1} \ge \frac{1}{\nu_n} \sum_{|i-j|=m} \ell_{ij} \tag{2.7}$$

and show that it is sharp for at least one value of m if $\Gamma \neq \tilde{\mathcal{P}}_N$; in this way we have accomplished the goal to reformulate the problem in geometric terms.

3. Inequalities for chord sums

It is useful to discuss the geometric problem without dimensional restrictions, i.e. for any $d \ge 2$. Denoting again for a given Γ by $y_j \in \mathbb{R}^d$ the points on the loop defined by (2.1), we will study the following family of inequalities

$$D_{L,N}^p(m): \quad \textstyle\sum_{n=1}^N |y_{n+m} - y_n|^p \le \frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}} , \qquad p > 0 , \tag{3.1}$$

$$D_{L,N}^{-p}(m): \quad \textstyle\sum_{n=1}^N |y_{n+m} - y_n|^{-p} \ge \frac{N^{1+p} \sin^p \frac{\pi}{N}}{L^p \sin^p \frac{\pi m}{N}} , \qquad p > 0 , \tag{3.2}$$

for any $m = 1, \dots, [\frac{1}{2}N]$, where $[\cdot]$ denotes as usual the entire part; they contain as a particular case the inequalities for equilateral polygons discussed in [**Ex2**].

The inequality (2.7) is nothing else than $D_{L,N}^1(m)$. In the next section we are going to demonstrate that it holds locally, i.e. in the vicinity of $\tilde{\mathcal{P}}_N$, proving thus

Theorem 2.1. Furthermore, in Section 5 we will show that a stronger claim can be made, namely that the inequalities (3.1) and (3.2) hold generally as long as $p \le 2$. Without imposing a restriction on N this seems to be an optimal result, because the example of a rhomboid shows that $D^p_{L,4}(2)$ cannot be valid for $p > 2$.

Notice that to establish the above described property one needs in fact to prove the inequality $D^2_{L,N}(m)$ only as the following simple result shows.

LEMMA 3.1. *$D^p_{L,N}(m)$ implies $D^{p'}_{L,N}(m)$ if $p > p' > 0$. Similarly, $D^p_{L,N}(m)$ implies $D^{-p}_{L,N}(m)$ for any $p > 0$.*

PROOF. The first claim follows from convexity of $x \mapsto x^\alpha$ in $(0, \infty)$ for $\alpha > 1$,

$$\frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}} \ge \sum_{n=1}^{N} \left(|y_{n+m} - y_n|^{p'} \right)^{p/p'} \ge N \left(\frac{1}{N} \sum_{n=1}^{N} |y_{n+m} - y_n|^{p'} \right)^{p/p'} ;$$

it is then sufficient to take both sides to the power p'/p. On the other hand,

$$\sum_{n=1}^{N} |y_{n+m} - y_n|^{-p} \ge \frac{N^2}{\sum_{n=1}^{N} |y_{n+m} - y_n|^p} \ge \frac{N^{1+p} \sin^p \frac{\pi}{N}}{L^p \sin^p \frac{\pi m}{N}}$$

holds by Schwarz inequality; it gives the second claim and completes the proof. □

4. Local extrema

First we are going to prove Theorem 2.1. As we have said one has to establish that the inequality (2.7), or $D^1_{L,N}(m)$, holds locally. Since the argument represents an extension of the proof of Theorem 4.1 in [**Ex1**] to a more general class of constraints, we will skip details in repeating the latter. We are looking for a local maximum of the function of Nd variables,

$$f_m : \; f_m(y_1, \dots, y_N) = \frac{1}{N} \sum_{i=1}^{N} |y_i - y_{i+m}| \, ,$$

under the inequality-type constraints $g_i(y_1, \dots, y_n) \ge 0$, where

$$g_i(y_1, \dots, y_n) := \frac{L}{N} - |y_i - y_{i+1}| \, , \quad i = 1, \dots, N \, ;$$

the true number of independent variables is $(N-2)(d-1) - 1$ because $2d - 1$ parameters are related to Euclidean transformations and can be fixed.

Following the convention for inequality-type constraints we introduce slack variables z_r, $r = 1, \dots, N$, and Lagrange multipliers λ_r, $r = 1, \dots, N$, which determine

$$K_m(y_1, \dots, y_N, z_1, \dots, z_N) := f_m(y_1, \dots, y_N) + \sum_{r=1}^{N} \lambda_r \left(g_r(y_1, \dots, y_n) - z_r^2 \right) . \tag{4.1}$$

The first thing to compute are the derivatives $\partial_{y_j} K_m$ which are equal to

$$\frac{1}{N} \left\{ \frac{y_j - y_{j+m}}{|y_j - y_{j+m}|} + \frac{y_j - y_{j-m}}{|y_j - y_{j-m}|} \right\} - \lambda_j N \frac{y_j - y_{j+1}}{L} - \lambda_{j-1} N \frac{y_j - y_{j-1}}{L} \, .$$

Without loss of generality we may consider a regular polygon in the plane spanned by the first two coordinate axes and to parametrize its vertices in the following way,

$$\tilde{y}_{\pm m} = \frac{L}{N} \left(\pm \sum_{n=0}^{m-1} \cos \frac{\pi}{N}(2n+1), \sum_{n=0}^{m-1} \sin \frac{\pi}{N}(2n+1) \right) ,$$

so that

$$|\tilde y_j-\tilde y_{j\pm m}| = \frac{L}{N}\left[\left(\sum_{n=0}^{m-1}\cos\frac{\pi}{N}(2n+1)\right)^2+\left(\sum_{n=0}^{m-1}\sin\frac{\pi}{N}(2n+1)\right)^2\right] =: \frac{L\Upsilon_m}{N}\,.$$

Hence the gradient components $\partial_{y_j}K_m$ will vanish for $j=1,\dots,N$ provided we choose all the Lagrange multipliers in (4.1) equal to

$$\lambda=\frac{\sigma_m}{N\Upsilon_m} \qquad \text{with} \qquad \sigma_m := \frac{\sum_{n=0}^{m-1}\sin\frac{\pi}{N}(2n+1)}{\sin\frac{\pi}{N}} = \frac{\sin^2\frac{\pi m}{N}}{\sin^2\frac{\pi}{N}}\,; \tag{4.2}$$

notice that this quantity is always nonzero. At the same time, one has to require vanishing of the derivatives

$$\partial_{z_j}K_m = 2\lambda_j z_j\,, \quad j=1,\dots,N,$$

which means that at the extremum all the slack variables vanish, $z_j=0$. This is not surprising; one naturally expects critical points of the function f_m to be reached under given constraints with the neighbor distances maximal, i.e. for a polygon.

¿From this point on the argument proceeds as in [**Ex1**]. Evaluating the Hessian at the stationary point, one can reduce the question about its negative definiteness to verification of the inequalities

$$\sin\frac{\pi m}{N}\sin\frac{\pi r}{N} > \left|\sin\frac{\pi}{N}\sin\frac{\pi m r}{N}\right|\,, \qquad 2\le r<m\le\left[\frac12 N\right]\,, \tag{4.3}$$

or equivalently, the inequalities $U_{m-1}\left(\cos\frac{\pi}{N}\right) > \left|U_{m-1}\left(\cos\frac{\pi r}{N}\right)\right|$ for Chebyshev polynomials of the second kind, which can be done directly.

The obtained result provides also a local solution to our electrostatic problem.

Theorem 4.1. *Under the assumption* (ℓ) *the Coulomb energy of a charged necklace is locally sharply minimized by a regular planar polygon,* $\Gamma=\tilde{\mathcal{P}}_N$.

Proof. For a given nonzero charge q the potential energy equals

$$q^2\sum_{j\ne k}|y_j-y_k|^{-1} = q^2\sum_{m=1}^{\left[\frac12 N\right]}\frac{\nu_m}{N}\sum_{n=1}^{N}|y_{n+m}-y_n|^{-1},$$

and since by Lemma 3.1 the inequality $D^1_{L,N}(m)$ implies $D^{-1}_{L,N}(m)$, the sum of all repulsion-energy terms is locally sharply minimized by $\tilde{\mathcal{P}}_N$. □

5. Global validity of mean-chord inequalities

Let us look now what is needed to prove global validity of the inequalities (3.1) and (3.2), in particular, to see whether a Fourier analysis in the spirit of [**EHL**] could help; in view of Lemma 3.1 it is enough to consider the case $p=2$ only. With the natural scaling properties in mind we may without loss of generality put $L=2\pi$ and to express the function Γ through its Fourier series,

$$\Gamma(s)=\sum_{0\ne n\in\mathbb{Z}} c_n\,\mathrm{e}^{ins} \tag{5.1}$$

with $c_n\in\mathbb{C}^d$; since $\Gamma(s)\in\mathbb{R}^d$ the coefficients have to satisfy the condition

$$c_{-n}=\bar c_n\,.$$

The absence of the coefficient c_0 means naturally no restriction; it can be always achieved by a choice of the coordinate system.

It is convenient to impose slightly stronger regularity requirements on Γ assuming that it is of the C^2 class; recall that validity of $D^2_{L,N}(m)$ can be extended from such a family of loops to those satisfying the hypothesis *(ℓ)* by means of the Weierstrass theorem and continuity of the functionals involved. In such a case [**KF**, Sec. VIII.1.2] the derivative of Γ is a sum of the uniformly convergent Fourier series

$$\dot\Gamma(s) = i \sum_{0\neq n\in\mathbb{Z}} nc_n\, \mathrm{e}^{ins}\,. \tag{5.2}$$

The assumed arc-length parametrization means $|\dot\Gamma(s)| = 1$ giving thus the relation

$$2\pi = \int_0^{2\pi} |\dot\Gamma(s)|^2\, \mathrm{d}s = \int_0^{2\pi} \sum_{0\neq l\in\mathbb{Z}} \sum_{0\neq n\in\mathbb{Z}} nl\, c_l^* \cdot c_n\, \mathrm{e}^{i(n-l)s}\, \mathrm{d}s\,,$$

where $c_l^* = (\bar c_{l,1}, \dots, \bar c_{l,d})$ and dot marks the inner product in $\mathbb{C}^d$, or equivalently

$$\sum_{0\neq n\in\mathbb{Z}} n^2|c_n|^2 = 1\,. \tag{5.3}$$

Furthermore, using (5.1) we can rewrite the left-hand side of $D^2_{2\pi,N}(m)$ as

$$\sum_{n=1}^{N} \sum_{0\neq j,k\in\mathbb{Z}} c_j^* \cdot c_k \left(\mathrm{e}^{-2\pi imj/N} - 1\right) \left(\mathrm{e}^{2\pi imk/N} - 1\right) \mathrm{e}^{2\pi in(k-j)/N}$$

Next we change the order of summation and observe that $\sum_{n=1}^N \mathrm{e}^{2\pi in(k-j)/N} = N$ if $j = k \pmod N$ and zero otherwise; this allows us to write the last expression as

$$4N \sum_{l\in\mathbb{Z}} \sum_{\substack{0\,\neq\, j,\,k\,\in\,\mathbb{Z}\\ j\,-\,k\,=\,lN}} |j|c_j^* \cdot |k|c_k \left| j^{-1} \sin\frac{\pi mj}{N}\right| \left| k^{-1} \sin\frac{\pi mk}{N}\right|\,. \tag{5.4}$$

If $D^2_{2\pi,N}(m)$ should be valid, this quantity must not exceed the right-hand side of (3.1) for $p = 2$; hence the sought inequality is equivalent to

$$\left(d, (A^{(N,m)} \otimes I)d\right) \le \left(\frac{\pi \sin\frac{\pi m}{N}}{N \sin\frac{\pi}{N}}\right)^2, \tag{5.5}$$

where the vector $d \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^d$ has the components $d_j := |j|c_j$ and $A^{(N,m)}$ is an operator on $\ell^2(\mathbb{Z})$ determined by its matrix representation as

$$A^{(N,m)}_{jk} := \begin{cases} \left|j^{-1} \sin\frac{\pi mj}{N}\right| \left|k^{-1} \sin\frac{\pi mk}{N}\right| & \text{if}\quad 0\neq j,k\in\mathbb{Z},\;\; j = k \pmod N \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

It is obvious that $A^{(N,m)}$ is bounded because its Hilbert-Schmidt norm is finite. Since d is a unit vector in view of (5.3) we arrive at the following conclusion: the inequality $D^2_{L,N}(m)$ with fixed N, m is valid provided the norm of the operator $A^{(N,m)}$ does not exceed the quantity at the right-hand side of (5.5).

This sufficient condition allows us to prove our second main result.

THEOREM 5.1. *Within the class of curves specified by the assumption* (ℓ) *the inequalities $D^2_{L,N}(m)$, and thus also (3.1) and (3.2) with $p \le 2$ for fixed values of $N = 2, 3, \dots$ and $m = 1, \dots, \left[\frac{1}{2}N\right]$, are valid.*

Proof. For a given $j \neq 0$ and $d \in \ell^2(\mathbb{Z})$ the relation (5.6) gives

$$\left(A^{(N,m)}d\right)_j = \left|j^{-1}\sin\frac{\pi mj}{N}\right| \sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j (\mathrm{mod}\, N)}} \left|k^{-1}\sin\frac{\pi mk}{N}\right| d_k\,.$$

The norm $\|A^{(N,m)}d\|$ is then easily estimated by means of Schwarz inequality,

$$\begin{aligned} \|A^{(N,m)}d\|^2 &= \sum_{0\neq j\in\mathbb{Z}} j^{-2}\sin^2\frac{\pi mj}{N}\left|\sum_{\substack{0 \neq k \in \mathbb{Z} \\ k = j (\mathrm{mod}\, N)}} \left|k^{-1}\sin\frac{\pi mk}{N}\right| d_k\right|^2 \\ &\le \sum_{n=0}^{N-1}\sin^4\frac{\pi mn}{N}\,S_n^2 \sum_{\substack{n+lN\neq 0 \\ l\in\mathbb{Z}}} |d_{n+lN}|^2, \end{aligned}$$

where we have introduced

$$S_n := \sum_{\substack{n+lN\neq 0 \\ l\in\mathbb{Z}}} \frac{1}{(n+lN)^2} = \sum_{l=1}^{\infty}\left\{\frac{1}{(lN-n)^2}+\frac{1}{(lN-N+n)^2}\right\}.$$

However, the above series is easily evaluated to be

$$S_n = \left(\frac{\pi}{N\sin\frac{\pi n}{N}}\right)^2,$$

and since $\|d\|^2 = \sum_{n=0}^{N-1}\sum_{l\in\mathbb{Z}}|d_{n+lN}|^2$, the sought inequality follows from (4.3). □

This allows us to strengthen our claims concerning the two original problems.

Corollary 5.2. Adopt the assumptions *(ℓ)* and (2.2); then $\epsilon_1(\alpha, Y_\Gamma)$ is for a fixed α and $L>0$ globally maximized by a regular polygon, $\Gamma = \tilde{\mathcal{P}}_N$.

Corollary 5.3. Under the assumption *(ℓ)* the Coulomb energy of a charged necklace is globally minimized by $\Gamma = \tilde{\mathcal{P}}_N$.

Remark 5.4. Notice that the inequality (4.3) used in the proof is sharp. That means that, in distinction to the "continuous" analogue of our problem, the extremum cannot be reached in the class of C^2 smooth functions in which we have performed the described Fourier analysis.

6. Concluding remarks

Let us first comment on relations to the "continuous" case treated in [**EHL**] where the global validity of the inequalities analogous to (3.1) and (3.2) was proved. Notice that formally that situation corresponds to $N=\infty$. The counterpart of the operator $A^{(N,m)}$ is then a multiple of the unit operator and it is only necessary to employ the inequality $\left|\sin\frac{\pi mj}{N}\right| \le \left|j\,\sin\frac{\pi m}{N}\right|$, or slightly more generally

$$|\sin jx| \le j\,\sin x \tag{6.1}$$

for any $j\in\mathbb{N}$ and $x\in(0,\frac{1}{2}\pi]$, which is checked easily by induction. In the present case with a finite N the operator has infinitely many side diagonals such a simple estimate based on (6.1) is too rough, because it yields an unbounded Toeplitz-type operator, and one has to do better using the matrix-element decay in (5.6). Fortunately it can be done as the proof of Theorem 5.1 shows.

Another aspect of the relation between the two cases is that in the continuous case the analogue of (5.4) follows from Parseval relation and the quantity is naturally invariant with respect to shifts in the arc-length parametrization. This is not true here; recall that the shift $s \to s + s_0$ is equivalent to the replacement of c_j by $c_j e^{is_0}$, which changes in general (5.4) due to the presence of the off-diagonal terms.

Let us also comment briefly on various extensions of the present problem restricting ourselves to the "discrete" situation only. A natural question concerns the existence and properties of the extrema in situations when we have no built-in symmetry, either by assuming a nonconstant sequence of coupling parameters $\{\alpha_j\}$ and/or taking their sites s_j at the loop with a non-equidistant distribution. In both cases the task becomes more difficult because we can no longer use the relation (2.5) which lead us to the geometric reformulation based on the inequality (2.6), in particular, the solutions for the charged necklace and polymer loops will be now in general different. Another extension could concern point interaction family in $\mathbb{R}^3$ placed on a closed surface. It is again not straightforward, however, because in distinction to a curve such a surface cannot be locally rectified and the answer will depend at the choice of the source sites at the surface.

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