Inequalities for means of chords, with application to isoperimetric problems

Pavel Exner[a,b], Evans M. Harrell[c], Michael Loss[c]

a) Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague, Czechia; exner@ujf.cas.cz
b) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague
c) School of Mathematics, Georgia Institute of Technology, Atlanta, Ga 30332, U.S.A.; harrell@math.gatech.edu, loss@math.gatech.edu

We consider a pair of isoperimetric problems arising in physics. The first concerns a Schrödinger operator in $L^2(\mathbb{R}^2)$ with an attractive interaction supported on a closed curve Γ, formally given by $-\Delta - \alpha\delta(x-\Gamma)$; we ask which curve of a given length maximizes the ground state energy. In the second problem we have a loop-shaped thread Γ in $\mathbb{R}^3$, homogeneously charged but not conducting, and we ask about the (renormalized) potential-energy minimizer. Both problems reduce to purely geometric questions about inequalities for mean values of chords of Γ. We prove an isoperimetric theorem for p-means of chords of curves when $p \leq 2$, which implies in particular that the global extrema for the physical problems are always attained when Γ is a circle. The article finishes with a discussion of the p–means of chords when $p > 2$.

1 Introduction

Isoperimetric problems are a trademark topic in mathematical physics. In this note we consider two of them, which at first glance are completely unrelated, and show that their solution can be reduced to the same geometric question. The first problem concerns a class of operators in $L^2(\mathbb{R}^2)$ which are given formally by the expression

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \tag{1.1}$$

with an attractive interaction, $\alpha > 0$, where Γ is a smooth loop in the plane of a fixed length $L > 0$; for a proper definition of $H_{\alpha,\Gamma}$, either through the natural quadratic form or in terms of boundary conditions on Γ see [2, 7].

We are interested in the shape of Γ that maximizes the ground state energy. There are several reasons for interest in this question. On the physical side, operators of the type (1.1) have been studied as models of *leaky quantum wires*, with the aim of more realistically taking quantum tunneling into account. For references see [3, 7]; a bibliography can be found in [1]. Such operators exhibit interesting connections between spectral properties and the shape of the curve supporting the interaction, in particular, that the curvature of Γ generates an effective attractive interaction.

Another inspiration is the isoperimetric problem for the Dirichlet Laplacian on a closed loop-shaped tube, where it can be proved that the ground state is uniquely maximized by a circular annulus [6]. By a natural analogy one expects a similar result for leaky wires, with the circle again being the maximizer. This question was posed and partially solved in [5] where it was shown that a circle maximizes the principal eigenvalue locally. See also [4] for a "discrete" analogue of this result. Here we present a full solution, demonstrating that a circular Γ is the unique global maximizer for the ground state of the operator $H_{\alpha,\Gamma}$.

A second motivation is to be found in classical electrostatics rather than quantum mechanics. Suppose that a nonconducting thread with no rigidity in the form of a loop has a homogeneous charge density. We ask which shape the thread will take in the absence of other, nonelectrostatic, forces. Roughly speaking, we seek the Γ that minimizes the potential energy of the electric field. The tricky part of the question is that the potential energy is infinite. Since the divergence is due to short–distance effects independent of the shape of the thread, however, the energy can be renormalized, yielding a well-posed problem.

We shall reduce both questions to a geometric property of chords of Γ. A class of inequalities for L^p norms of the chords in any dimension $d \geq 2$ is proved, particular cases of which will help solve the two problems stated above.

2 Mean-chord inequalities

As mentioned above it is convenient to formulate the result in an arbitrary dimension $d \geq 2$. First we have to specify regularity requirements on the curve. Let Γ be a loop in $\mathbb{R}^d$ parametrized by its arc length, in other words, a piecewise differentiable function $\Gamma:\ [0,L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot\Gamma(s)| = 1$ for any $s \in [0,L]$. Let us consider all the arcs of Γ having arc length $u \in (0, \frac{1}{2}L]$. We shall be concerned with the inequalities

$$\begin{aligned} C_L^p(u): &\qquad \int_0^L |\Gamma(s+u)-\Gamma(s)|^p\, \mathrm{d}s \;\leq\; \tfrac{L^{1+p}}{\pi^p} \sin^p \tfrac{\pi u}{L}\,, \quad p>0\,, && (2.1)\\ C_L^{-p}(u): &\qquad \int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p}\, \mathrm{d}s \;\geq\; \tfrac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}\,, \qquad p>0\,. && (2.2)\end{aligned}$$

The expressions on the right sides correspond to the maximally symmetric case, i.e., the planar circle. It is clear that the inequalities are invariant under scaling, so without loss of generality we may fix the length, e.g., to $L = 2\pi$. In the case $p = 0$ the inequalities (2.1) and (2.2) turn into trivial identities.

A natural conjecture, to be proved in this section, is that they hold for any $d \geq 2$ and $p \leq 2$. In fact, it is sufficient to consider only the case $p = 2$ as the following simple result shows.

Proposition 2.1 *$C_L^p(u)$ implies $C_L^{p'}(u)$ if $p > p' > 0$. Similarly, $C_L^p(u)$ implies $C_L^{-p}(u)$ for any $p > 0$.*

Proof: The first claim is due to the convexity of $x \mapsto x^\alpha$ in $(0,\infty)$ for $\alpha > 1$: Supposing (2.1),

$$\begin{aligned} \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} \;&\geq\; \int_0^L \left(|\Gamma(s+u)-\Gamma(s)|^{p'}\right)^{p/p'} \mathrm{d}s \\ &\geq\; L \left(\frac{1}{L}\int_0^L |\Gamma(s+u)-\Gamma(s)|^{p'}\, \mathrm{d}s\right)^{p/p'}\,. \end{aligned}$$

It is then sufficient to take both sides to the power p'/p. Furthermore, the Schwarz inequality implies

$$\int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p}\, \mathrm{d}s \geq \frac{L^2}{\int_0^L |\Gamma(s+u)-\Gamma(s)|^p\, \mathrm{d}s} \geq \frac{L^2\pi^p}{L^{1+p}\sin^p \frac{\pi u}{L}}\,,$$

which completes the proof of (2.2). ∎

The claim for $p = 2$ can be proved by means of a simple Fourier analysis.

Theorem 2.2 *Suppose that Γ belongs to the described class. Then $C_L^2(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, and the inequality is strict unless Γ is a planar circle.*

Proof: Without loss of generality we put $L = 2\pi$ and write

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n \, \mathrm{e}^{ins} \tag{2.3}$$

with $c_n \in \mathbb{C}^d$; since $\Gamma(s) \in \mathbb{R}^d$ the coefficients have to satisfy the condition

$$c_{-n} = \bar{c}_n \, .$$

The absence of c_0 can be always achieved by a choice of the origin of the coordinate system. In view of the Weierstrass theorem and continuity of the functional in question, we may also suppose that Γ is of the class C^2, in which case [8, Sec. VIII.1.2] its derivative is a sum of the uniformly convergent Fourier series

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n \, \mathrm{e}^{ins} \, . \tag{2.4}$$

By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from the relation

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 \, \mathrm{d}s = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm \, c_m^* \cdot c_n \, \mathrm{e}^{i(n-m)s} \, \mathrm{d}s \, ,$$

where c_m^* denotes the row vector $(\bar{c}_{m,1}, \dots, \bar{c}_{m,d})$ and dot marks the inner product in $\mathbb{C}^d$, we infer that

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1 \, . \tag{2.5}$$

In a similar way we can rewrite the right member of $C_{2\pi}^2(u)$ using the Parseval relation as

$$\int_0^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n \, (\mathrm{e}^{inu} - 1) \, \mathrm{e}^{ins} \right|^2 \mathrm{d}s = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left(\sin \frac{nu}{2} \right)^2 \, ,$$

and thus the sought inequality is equivalent to

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 \left(\frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2 \leq 1. \tag{2.6}$$

It is therefore sufficient to prove that

$$|\sin nx| \le n\,\sin x \tag{2.7}$$

for all positive integers n and all $x \in (0, \frac{1}{2}\pi]$. We proceed by induction. The claim is obviously valid for $n = 1$. Suppose now that (2.7) holds for a given n. We have

$$(n+1)\sin x \mp \sin(n+1)x = n\sin x \mp \sin nx \cos x + \sin x(1 \mp \cos nx),$$

where the sum of the first two terms at the right side is non-negative by assumption, and the same is clearly true for the last one. In combination with the continuity argument this proves $C_L^2(u)$ for the considered class of functions.

The induction argument shows that if (2.7) is strict for n it is likewise strict for $n+1$. Since the inequality is strict for $n = 2$, equality can occur only for $n = 1$. This in turn means that the inequality (2.6) is strict unless $c_n = 0$ for $|n| \ge 2$, and, consequently, $C_{2\pi}^2(u)$ is saturated only if the projection of Γ on the jth axis equals

$$\Gamma_j(s) = 2|c_{1,j}|\cos(s + \arg c_{1,j}),$$

in which the coefficients satisfy $\sum_{j=1}^d |c_{1,j}|^2 = 1$. Furthermore, the condition $|\dot\Gamma(s)| = 1$ can be satisfied only if there is a basis in $\mathbb{R}^d$ where $c_{1,1} = ic_{1,2} = \frac{1}{2}$ and $c_{1,j} = 0$ for $j = 3, \dots, d$, in other words, if Γ is a planar circle. The latter is thus a global maximizer in the class of C^2 smooth loops.

It remains to check that the inequality cannot be saturated for a curve Γ that is not C^2, so that the sum (2.5) diverges. This would require

$$\frac{\sum_{1\le n\le N} n^2|c_n|^2 \left(\frac{\sin\frac{nu}{2}}{n\sin\frac{u}{2}}\right)^2}{\sum_{1\le n\le N} n^2|c_n|^2} \to 1 \tag{2.8}$$

as $N \to \infty$. This is impossible, however, because the sum in the numerator is bounded by $\sec^2\frac{u}{2}\sum_{1\le n\le N}|c_n|^2$ so it has a finite limit; this concludes the proof. ∎

3 An electrostatic isoperimetric problem

As a first application of the mean–chord inequalities we turn to the electrostatic problem described in the introduction. Suppose now that Γ is a closed

C^2 curve in $\mathbb{R}^3$, so that its curvature γ and torsion τ are continuous functions on $[0, L]$. By $\mathcal{C}$ we denote a planar circle; since we work only with distances of pairs of points, it can be any representative of the equivalence class determined by Euclidean transformations of $\mathbb{R}^3$. Let us define the quantity

$$\begin{aligned} \delta(\Gamma) &:= \int_0^L \int_0^L \left[|\Gamma(s) - \Gamma(s')|^{-1} - |\mathcal{C}(s) - \mathcal{C}(s')|^{-1} \right] \mathrm{d}s\, \mathrm{d}s' \\ &= 2 \int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s \, \left[|\Gamma(s+u) - \Gamma(s)|^{-1} - \frac{\pi}{L} \csc \frac{\pi u}{L} \right] . \end{aligned} \tag{3.1}$$

Obviously the energy cost of deformation is $q^2 \delta(\Gamma)$, where q is the charge density along the loop, constant by assumption. The answer to the question stated in the introduction is given by the following:

Theorem 3.1 *For any closed C^2 curve Γ, $\delta(\Gamma)$ is finite and non-negative. It is zero if and only if $\Gamma = \mathcal{C}$, up to Euclidean equivalence.*

Proof: By Theorem 2.2 and Proposition 2.1 the inequality $C_L^{-1}(u)$ is valid for any $u \in (0, \frac{1}{2}L]$, hence the inner integral in the last expression of (3.1) is non-negative. The trouble, as mentioned in the introduction, is that taken separately each part leads to a logarithmically divergent integral due to the behavior of the integrand as $u \to 0$. It is easy to see, however, that under the stated regularity requirements on Γ one has

$$|\Gamma(s+u) - \Gamma(s)|^{-1} = u^{-1} + \mathcal{O}(1),$$

with the error term dependent on γ and τ but uniform in s, hence the singularities cancel and the integral expressing $\delta(\Gamma)$ converges. ∎

4 An isoperimetric problem for singular interactions

Let us turn to the singular Schrödinger operator (1.1). The curve Γ is finite, so by [2, 3] we have $\sigma_{\mathrm{ess}}(H_{\alpha,\Gamma}) = [0, \infty)$ while the discrete spectrum is nonempty and finite. In particular,

$$\epsilon_1 \equiv \epsilon_1(\alpha, \Gamma) := \inf \sigma\left(H_{\alpha,\Gamma}\right) < 0\,.$$

As indicated in the introduction, we ask for which curve Γ the principal eigenvalue is maximal. The answer is the following.

Theorem 4.1 *Let $\Gamma:\ [0,L]\to\mathbb{R}^2$ have the properties stated in the opening of Sec. 2; then for any fixed $\alpha>0$ and $L>0$ $\epsilon_1(\alpha,\Gamma)$ is globally uniquely maximized by the circle.*

Proof: The problem reduces to checking the inequality $C_L^1(u)$ for any $u\in(0,\frac{1}{2}L]$ which is achieved by Theorem 2.2 and Proposition 2.1. The argument was given in [5]; we recall it briefly here to make the letter self-contained. One employs a generalized Birman-Schwinger principle [2] by which there is one-to-one correspondence between eigenvalues of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathcal{R}^\kappa_{\alpha,\Gamma}\phi=\phi\,,\quad \mathcal{R}^\kappa_{\alpha,\Gamma}(s,s'):=\frac{\alpha}{2\pi}K_0\big(\kappa|\Gamma(s)-\Gamma(s')|\big)\,, \tag{4.1}$$

on $L^2([0,L])$, where K_0 is the Macdonald function; by convention we write $k=i\kappa$ with $\kappa>0$, corresponding to the bound-state energy $-\kappa^2$.

To prove the claim it is sufficient to show that for any $\kappa>0$ the largest eigenvalue of the operator $\mathcal{R}^\kappa_{\alpha,\Gamma}$ given by (4.1) is minimized by a circle. This can be combined with the explicit knowledge of the principal eigenfunction of $\mathcal{R}^\kappa_{\alpha,\mathcal{C}}$ which is constant over the loop, due to the simplicity of the respective eigenvalue in combination with rotational symmetry. Using the same function to make a variational estimate in the general case we see that one has to prove the inequality

$$\int_0^L\int_0^L K_0\big(\kappa|\Gamma(s)-\Gamma(s')|\big)\,\mathrm{d}s\mathrm{d}s'\geq\int_0^L\int_0^L K_0\big(\kappa|\mathcal{C}(s)-\mathcal{C}(s')|\big)\,\mathrm{d}s\mathrm{d}s' \tag{4.2}$$

for any $\kappa>0$. Using the symmetry of the kernel and a simple change of variables we find that (4.2) is equivalent to the positivity of the functional

$$F_\kappa(\Gamma):=\int_0^{L/2}\mathrm{d}u\int_0^L\mathrm{d}s\left[K_0\big(\kappa|\Gamma(s+u)-\Gamma(s)|\big)-K_0\left(\frac{\kappa L}{\pi}\sin\frac{\pi u}{L}\right)\right]\,.$$

By Jensen's inequality,

$$\frac{1}{L}F_\kappa(\Gamma)\geq\int_0^{L/2}\left[K_0\left(\frac{\kappa}{L}\int_0^L|\Gamma(s+u)-\Gamma(s)|\mathrm{d}s\right)-K_0\left(\frac{\kappa L}{\pi}\sin\frac{\pi u}{L}\right)\right]\mathrm{d}u\,,$$

where the inequality is strict unless $\int_0^L|\Gamma(s+u)-\Gamma(s)|\mathrm{d}s$ is independent of s, because the function K_0 is strictly convex. At the same time, it is decreasing in $(0,\infty)$, and hence it is sufficient to check that the inequality

$$\int_0^L|\Gamma(s+u)-\Gamma(s)|\,\mathrm{d}s\;\leq\;\frac{L^2}{\pi}\sin\frac{\pi u}{L}$$

is valid for all arc lengths $u \in (0, \frac{1}{2}L]$ and strict unless $\Gamma = \mathcal{C}$; this is nothing else than $C^1_L(u)$. ∎

5 Mean chords when $p > 2$

Section 2 provides a complete solution to the problem of maximizing the L^p norm of the chord of a closed loop of length 2π when $p \leq 2$, but no information is obtained there for maximizing

$$I(\Gamma, p, u) := \int_0^{2\pi} |\Gamma(s+u) - \Gamma(s)|^p ds = \|\Gamma(\cdot+u) - \Gamma(\cdot)\|_p^p$$

with larger values of p.

For any $u, 0 < |u| < \pi$, it is easy to see that $\|\Gamma(s+u) - \Gamma(s)\|_\infty$ is maximized, with the value $|u|$, by any curve containing a straight interval of length $|u|$ or greater. Meanwhile, for the circle, $\|\Gamma(s+u) - \Gamma(s)\|_\infty < u$, and therefore, by continuity (2.1) is false for sufficiently large values of p and any fixed $u, 0 < |u| < \pi$. The same is true when $u = \pi$, when the requirement that $\Gamma \in C^2$ is relaxed.

Little is known, however, about the values of $p > 2, u > 0$ for which the maximizer of $I(\Gamma, p, u)$ remains circular. Furthermore, the noncircular maximizing curves for large finite p remain unidentified.

Examples show that the isoperimetric result does not extend far beyond the interval $[0, 2]$:

Example 5.1 *Let Γ consist of an interval of length π traversed first in one direction and then in the opposite direction. (This can be regarded as the limiting case of a stadium of perimeter 2π formed by joining two semicircles by parallel straight line segments.) An elementary calculation shows that for this curve, $\int_0^{2\pi} |\Gamma(s+\pi) - \Gamma(s)|^p \, \mathrm{d}s = 4 \int_0^{\frac{\pi}{2}} (2x)^p \, \mathrm{d}x = \frac{2^{2+p}}{p+1} \left(\frac{\pi}{2}\right)^{p+1}$, which exceeds the value for the standard circle when p is greater than the largest solution of $p + 1 = \left(\frac{\pi}{2}\right)^p$, i.e., numerically for $p \geq 3.15296$.*

Example 5.1 provides the smallest value, among simple curves we have checked, of p for which (2.1) fails. We note that, paradoxically, examples that are more nearly circular require larger values of p for (2.1) to fail:

Example 5.2 *Let $\Gamma_S(a)$ be a standard stadium of length 2π, for which the straight intervals are of length πa, supposed small. A calculation shows that*

$$\left.\frac{\partial I(\Gamma_S(a), p, \pi)}{\partial a}\right|_{a=0} = 0$$

and

$$\left.\frac{\partial^2 I(\Gamma_S(a), p, \pi)}{\partial a^2}\right|_{a=0} = 2^{p-1} p\pi \left(\left(\pi^2 - 12\right) + \left(\pi^2 - 8\right)\left(\frac{p}{2} - 1\right)\right),$$

which > 0 for $p > \frac{8}{\pi^2-8} = 4.27898....$

Example 5.3 *Let $\Gamma_P(2m)$ be a regular polygon with $2m$ sides, m supposed large. A calculation shows that*

$$I(\Gamma_P(2m), p, \pi) = 2^{1+p}\pi \left|\sin\frac{u}{2}\right|^p \left(1 + p\left(\frac{p-6}{5760}\right)\left(\frac{\pi}{m}\right)^4 + \mathcal{O}\left(\left(\frac{\pi}{m}\right)^6\right)\right),$$

which exceeds the value for the circle when $p > 6$ and m is sufficiently large.

The examples lead one to expect that as p increases the maximizing curve may not bifurcate continuously from the circle, but rather springs from a different geometry altogether. Indeed, the circle remains a local maximizer of the p-mean of the chord, with respect to smooth perturbations, for all finite $p < \infty$, in a sense made precise in the following theorem. For convenience the curve is placed in $\mathbb{C}$ rather than $\mathbb{R}^2$, entailing a slight shift of notation, and a certain choice of orientation is made.

Theorem 5.4 *Let $\Gamma(\gamma, s)$ be a closed curve in $\mathbb{C}$ parametrized by arc length s, of the form $(1-\gamma)e^{is} + \Theta(\gamma, s)$, where $\gamma \geq 0$. Suppose that Θ is smooth (say, C^2 in γ and s), and that for each γ, $\Theta(\gamma, s)$ is orthogonal to e^{is}. Then $\Gamma(0, s)$ is a circle of radius 1, and for any u, $0 < u < 2\pi$,*

$$\left.\frac{\partial I(\Gamma(\gamma), p, u)}{\partial \gamma}\right|_{\gamma=0} < 0.$$

Proof: The observation (2.5), which applies to any curve parametrized by arc length, implies that $\Theta(0, s) = 0$, corresponding to the case of a circle.

Since Θ is smooth, we may differentiate I under the integral sign, finding

$$\begin{aligned}\frac{\partial I(\Gamma(\gamma),p,u)}{\partial\gamma} &= \int_0^{2\pi} \frac{\partial|\Gamma(\gamma,s+u)-\Gamma(\gamma,s)|^p}{\partial\gamma}\,ds \\ &= p\int_0^{2\pi} |\Gamma(\gamma,s+u)-\Gamma(\gamma,s)|^{p-2} \\ &\quad\times \mathrm{Re}\left(\overline{(\Gamma(\gamma,s+u)-\Gamma(\gamma,s))}\left(\frac{\partial\Gamma(\gamma,s+u)}{\partial\gamma}-\frac{\partial\Gamma(\gamma,s)}{\partial\gamma}\right)\right)ds.\end{aligned}$$

When $\gamma=0$ this simplifies to

$$\begin{aligned}&p\left|e^{iu}-1\right|^{p-2}\int_0^{2\pi}\mathrm{Re}\Bigg(e^{-is}\left(e^{-iu}-1\right) \\ &\quad\times\left(-e^{is}\left(e^{iu}-1\right)+\frac{\partial\Theta(0,s+u)}{\partial\gamma}-\frac{\partial\Theta(0,s)}{\partial\gamma}\right)\Bigg)ds.\end{aligned}$$

Since the orthogonality of e^{is} to $\Theta(\gamma,s)$ implies the orthogonality of e^{is} to $\Theta(\gamma,s+u)$, $\Theta_\gamma(\gamma,s)$, and $\Theta_\gamma(\gamma,s+u)$,

$$\frac{\partial I(\Gamma(0),p,u)}{\partial\gamma} = -p\left|e^{iu}-1\right|^p\int_0^{2\pi}1ds = -p2^{1+p}\pi\left|\sin\frac{u}{2}\right|^p<0. \qquad \blacksquare$$

Acknowledgments

P.E. is grateful for the hospitality at Georgia Tech where a part of the work was done. The research was supported by ASCR and its Grant Agency within the projects IRP AV0Z10480505 and A100480501; and by US NSF grants DMS-0204059 and DMS-0300349.

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