THE TRANSITION FROM DIFFUSION TO BLOW-UP FOR A NON LINEAR SCHRÖDINGER EQUATION IN DIMENSION ONE

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Abstract. We consider the time-dependent one-dimensional non linear Schrödinger equation with pointwise singular potential. We prove that if the strength of the nonlinear term is small enough, then the solution is well defined for any time, regardless of the choice of initial data; in contrast, if the nonlinearity power is larger than a critical value, for some initial data a blow-up phenomenon occurs in finite time. In particular, if the system is initially prepared in the ground state of the linear part of the Hamiltonian, then we obtain an explicit condition on the parameters for the occurrence of the blow-up.

1. Introduction

In this paper we study the dynamics of a one-dimensional quantum particle subject to a linear point interaction and to a nonlinearity of the power type. The time evolution of the system is then ruled by the time dependent non linear Schrödinger (NLS) equation

$$\left\{ \begin{array}{l} i\dot{\psi}_t = H\psi_t, \quad H\psi = -\psi'' + V\psi \\ \psi_t(x)|_{t=t_0} = \psi_0(x) \end{array} \right. , \quad \dot{\psi}_t = \frac{\partial \psi_t}{\partial t} \ \text{ and } \ \psi_t' = \frac{\partial \psi_t}{\partial x} \tag{1}$$

where ψ is a square integrable function on the real line and $V = V(|\psi|, x)$ is a non linear potential. In particular, we restrict ourselves to the case

$$V\psi = \gamma\psi\delta_0 + \rho|\psi|^{2\mu}\psi, \ \ \mu \geq 0, \ \ \gamma, \ \rho \in \mathbb{R}, \tag{2}$$

where δ_0 is the Dirac's delta pointwise interaction at $x = 0$.

First introduced to describe the propagation of laser beams (see e.g. [18]), NLS equations have revealed themselves to be a quite flexible and far reaching tool. In particular, the one-dimensional NLS with cubic nonlinearity (namely $\mu = 1$) is currently used to model the dynamics of quasi one-dimensional Bose-Einstein condensates (see e.g. [15, 13]) and the propagation of laser pulses in non linear media. Besides, it is worth mentioning that the case $\mu = 1/2$ is also significant, at least in experiments (see e.g. [12]).

On the other hand, Dirac's delta potentials provide a general and idealized model for short-range interactions. Introduced by Fermi in three dimensions in order to investigate the scattering of slow neutrons by atoms ([10]), such potentials were later recognized to provide the simplest example of exactly solvable quantum models and have been widely employed in toy-models.

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The model (1)-(2) was introduced by Witthaut, Mossmann and Korsch ([19]) in order to describe at a phenomenological level the effect of short-range obstacles in models of nonlinear transport. The main topics of such investigation were the properties of the stationary states. Our contribution is complementary to their results, and focuses on well-posedness, conservation laws, and study of the blow-up phenomenon.

The spirit of this paper is the same of [1], where the authors introduced a one dimensional model with nonlinear point interactions and investigated the related blow-up phenomenon, and of [2, 3], where the results were extended to a three-dimensional setting. On the other hand, it is worth recalling that the dynamics of a quantum particle under the action of a quadratic potential (attractive or repulsive) has been extensively studied in a series of works by Carles (see e.g. [6, 7]) with peculiar care on the effects of the external potential to the occurrence of blow-up and to the blow-up time. In this respect our investigation permits to say that, if the non linear term is attractive, the nonlinearity power greater than 4, its strength greater or equal to a critical value, and the energy negative, then a linear point interaction cannot prevent the blow-up.

The basic observation is that the time propagator of the free Laplacian with a delta interaction restricted to the space associated to its absolutely continuous spectrum satisfies some dispersive estimates and consequently some inequality of the Strichartz's type. Moreover, the energy space associated to a delta potential coincides with $H^1(\mathbb{R})$. Let us stress that this last feature fails in the three dimensional setting, due to the emergence of a singularity of the type $1/|x|$, and this prevents us from extending our model to three dimensions in a trivial way.

The paper is organized as follows: in Sec. 2 we introduce the model, in Sec. 3 we prove the well-posedness of the problem in the energy space and show the conservation laws of the L^2-norm and of the energy. Sec. 4 is devoted to the study of the blow-up phenomenon and finally Sec. 5 collects some concluding remarks.

We quoted some formal computations also, in order to exhibit the simple structure of some results that would risk to remain hidden by the technicalities of the rigorous proofs, and to show that to some extent delta-potential is well behaved, at least when supported on manifolds of codimension one.

Concerning notation, we shall write $\|\cdot\|_p$ for the norm in the space $L^p(\mathbb{R})$. In the case $p=2$ the index in the symbol of the norm will be omitted.

2. NLS equation with a pointwise interaction

Consider the problem defined in (1), (2). It is well known (see e.g. [5]) that when the nonlinear term is absent, that is $\rho = 0$, the operator

$$H_\gamma = -\psi'' + \gamma\psi\delta_0$$

is self-adjoint on the domain $H^2(\mathbb{R}\backslash\{0\})$ with boundary conditions

$$\psi(0+0) = \psi(0-0) \tag{3}$$

and

$$\psi'(x+0) - \psi'(x-0) = \gamma\psi(0+0). \tag{4}$$

Namely

$$D(H_\gamma) = \left\{\psi \in H^2(\mathbb{R}\backslash\{0\}) \ : \ \psi \text{ satisfies } (3) \ \text{ and } \ (4)\right\}.$$

Remark 1. *Notice that, due to (3), the function $\psi \in D(H_\gamma)$ is continuous in $x = 0$ and therefore $D(H_\gamma)$ is a subspace of $H^1(\mathbb{R})$. Thus, in the following we denote by $\psi(0)$ the limit (3).*

Let us recall some basic properties of the spectrum of H_γ. For details see [5, 4].

The essential spectrum of H_γ is purely absolutely continuous and coincides with the positive real axis:

$$\sigma_{\rm ess}(H_\gamma) = \sigma_{\rm ac}(H_\gamma) = [0, +\infty).$$

Moreover,

- If $\gamma \geq 0$ then the discrete spectrum of H_γ is empty.
- If $\gamma < 0$ then the discrete spectrum of H_γ is given by just one simple eigenvalue

$$\lambda = -\frac{1}{4}\gamma^2$$

with associated normalized eigenvector

$$\phi_\gamma(x) = \sqrt{\frac{|\gamma|}{2}} e^{-|\gamma|\cdot|x|/2}$$

Besides, we shall make use of the explicit expression of the time evolution generated by H_γ, that is an integral operator whose kernel reads ([17], [4])

$$U^t_\gamma(x,y) = U^t_0(x-y) + \\ + \begin{cases} -\dfrac{\gamma}{2}\int_0^{+\infty} du\ e^{-\frac{\gamma}{2}u}\ U_0^t(u+|x|+|y|)\,, & \gamma > 0 \\ 0, & \gamma = 0 \\ +e^{i\frac{\gamma^2}{4}t}\phi_\gamma(x)\phi_\gamma(y) + \frac{\gamma}{2}\int_0^{+\infty} du\ e^{\frac{\gamma}{2}u} U_0^t(u-|x|-|y|) & \gamma < 0 \end{cases} \tag{5}$$

where U_0^t is the integral kernel associated to the free Laplacian, namely

$$U_0^t(\zeta) = \frac{1}{\sqrt{4\pi i t}} \exp\left(-\frac{|\zeta|^2}{4it}\right).$$

3. Local existence and conservation laws

Theorem 1. *If the initial data ψ_0 belongs to $H^1(\mathbb{R})$ then the Cauchy problem (1) with potential (2) admits a unique local solution*

$$\psi \in \mathcal{H} = C\left((-T_{\min}, T_{\max}), H^1(\mathbb{R})\right) \cap C^1\left((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R})\right)$$

for some $T_{\min}, T_{\max} > 0$, satisfying the boundary conditions (3) and (4) for almost all t. Furthermore, the following conservation laws hold.

- *Conservation of the norm*

$$\mathcal{N}[\psi_t] = \mathcal{N}[\psi_0] = \|\psi_0\|$$

- *Conservation of the energy*

$$\mathcal{E}[\psi_t] = \mathcal{E}_{lin}[\psi_t] + \frac{\rho}{\mu+1}\|\psi_t\|^{2\mu+2}_{2\mu+2}, \quad \mathcal{E}_{lin}[\psi] = \|\psi'\|^2 + \gamma|\psi(0)|^2 \tag{6}$$

Proof. We split the proof of the theorem in three steps.

3.1. **Strichartz-type Estimate.** As a first step we remark that the Cauchy problem (1) is equivalent to the Duhamel formula

$$\psi_t = e^{-itH_\gamma}\psi_0 - i\rho \int_0^t e^{-i(t-s)H_\gamma}|\psi_s|^{2\mu}\psi_s ds \tag{7}$$

and that the evolution operator can be written as

$$e^{-iH_\gamma t}\psi = \begin{cases} e^{i\gamma^2 t/4}\langle\psi,\phi_\gamma\rangle\phi_\gamma + e^{-iH_\gamma t}P_c\psi & \text{if } \gamma < 0 \\ e^{-iH_\gamma t}P_c\psi & \text{if } \gamma \geq 0 \end{cases}$$

where P_c is the orthogonal projection in $L^2(\mathbb{R})$ onto the subspace orthogonal to ϕ_γ.

We are now ready to prove the basic dispersive estimate.

Lemma 1. *For any $\varphi \in L^1(\mathbb{R})$ the following estimate holds*

$$\|e^{-itH_\gamma}P_c\varphi\|_\infty \leq (\pi t)^{-1/2}\|\varphi\|_1. \tag{8}$$

Proof. ¿From the expression (5) of the time evolution generated by H_γ we have that the operator $e^{-iH_\gamma t}P_c$ is represented by the integral kernel

$$[e^{-iH_\gamma t}P_c](x,y) = U_0^t(x-y) - \frac{|\gamma|}{2}\int_0^{+\infty} du\, e^{-\frac{|\gamma|}{2}u}\, U_0^t(u \pm |x| \pm |y|)$$

where the "+" signs hold if $\gamma > 0$ and the "-" signs hold if $\gamma < 0$.

Then, given $\varphi \in L^1(\mathbb{R})$ we conclude

$$\begin{aligned} |[e^{-iH_\gamma t}P_c\varphi](x)| &\leq \left| \int_{\mathbb{R}} \frac{\exp\left(\frac{|x-y|^2}{4it}\right)}{\sqrt{4\pi it}}\varphi(y)\, dy \right| \\ &\quad + \frac{|\gamma|}{2}\left| \int_{\mathbb{R}} dy \int_0^{+\infty} du\, e^{-\frac{|\gamma|}{2}u} \frac{\exp\left(\frac{(u\pm|x|\pm|y|)^2}{4it}\right)}{\sqrt{4\pi it}}\varphi(y) \right| \\ &\leq \frac{1}{\sqrt{4\pi t}}\|\varphi\|_1 + \frac{|\gamma|}{4\sqrt{\pi t}}\int_{\mathbb{R}}|\varphi(y)|dy \int_0^{+\infty} e^{-\frac{|\gamma|}{2}u}\, du \\ &\leq \frac{1}{\sqrt{\pi t}}\|\varphi\|_1 \end{aligned}$$

□

Remark 2. *Estimate (8) together with the obvious inequality*

$$\|e^{-itH_\gamma}P_c\varphi\| \leq \|\varphi\|$$

and the Riesz-Thorin interpolation theorem, yields

$$\left\|e^{-iH_\gamma t}P_c\varphi\right\|_p \leq (\pi t)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\varphi\|_{p'}. \tag{9}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $2 \leq p \leq \infty$. Moreover, (9) gives the following Strichartz-type inequality: for any pair (q,r) with $r \geq 2$ and $q = \frac{4r}{r-2}$, $\varphi \in L^2(\mathbb{R})$ and $T > 0$ there exists a positive constant C such that

$$\left\|e^{-iH_\gamma t}P_c\varphi\right\|_{L^q((-T,T),L^r(\mathbb{R}))} \leq C\|\varphi\|. \tag{10}$$

Indeed, from the spectral resolution outlined in Section 2, we know that H_γ is a self-adjoint operator bounded from below (see e.g. [5, 16] *for details). Then theorem 2.7.1 in* [8] *applies and yields (10).*

3.2. **Local existence and uniqueness.** Let us define

$$K_\gamma = \begin{cases} H_\gamma & \text{if } \gamma \geq 0 \\ H_\gamma + \frac{\gamma^2}{4} & \text{if } \gamma < 0 \end{cases}$$

Clearly, $K_\gamma \geq 0$.
We prove the local existence and uniqueness for the solution to the equation

$$\psi_t = e^{-itK_\gamma}\psi_0 - i\rho \int_0^t e^{-i(t-s)K_\gamma}|\psi_s|^{2\mu}\psi_s ds$$

from which local existence and uniqueness for the solution to equation (7) immediately follow. Uniqueness is guaranteed by the Strichartz-type estimate (10), as one can check following for instance the proof of proposition 4.2.1. in [8]. On the other hand, in order to prove existence we apply theorem 3.7.1. in [8] (see also §3.7 and §4.12 in the same reference), and therefore we just have to prove the following lemma.

Lemma 2. *The self-adjoint operator K_γ satisfies the following conditions*

i. $X_K \hookrightarrow L^p$, *for any* $p \in [2, +\infty]$, *where* X_K *is the closure of the domain* $D(K_\gamma) = D(H_\gamma)$ *with respect to the norm induced by*

$$\|u\|^2_{K_\gamma} = \begin{cases} \|u'\|^2 + \|u\|^2 + \gamma|u(0)|^2 & \text{if } \gamma \geq 0 \\ \|u'\|^2 + \left(1 + \frac{\gamma^2}{4}\right)\|u\|^2 + \gamma|u(0)|^2 & \text{if } \gamma < 0 \end{cases}$$

ii. *The resolvent*

$$[1 - \epsilon K_\gamma]^{-1} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$$

is continuous for any $\epsilon < 0$ *and any* $p \in [2, +\infty]$ *and fulfils*

$$\left\| [1 - \epsilon K_\gamma]^{-1} u \right\|_p \leq C\|u\|_p \tag{11}$$

for some C *independent of* $\epsilon < 0$.

Proof. Condition i. immediately follow from estimate (26)

$$|u(0)|^2 \leq \|u\|^2_\infty \leq \|u'\| \cdot \|u\|, \quad \forall u \in D(K_\gamma)$$

Indeed, such estimate implies the equivalence of the norm $\| \cdot \|_{K_\gamma}$ and the ordinary norm in $H^1(\mathbb{R})$; hence, $X_K = H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ for any $p \in [2, +\infty]$. Condition ii. follows by a direct computation, making use of the fact that the resolvent operator $[H_\gamma - \zeta]^{-1}$ is an integral operator

$$\left([H_\gamma - \zeta]^{-1}\phi\right)(x) = \int_{\mathbb{R}} \mathcal{G}(x, y; k)\phi(y)dy, \quad \zeta = k^2, \ \Im k \geq 0,$$

with kernel (see e.g. [5])

$$\mathcal{G}(x, y; k) = \frac{i}{2k}e^{ik|x-y|} + \frac{1}{2k}\frac{\gamma}{i\gamma + 2k}e^{ik(|x|+|y|)}.$$

To prove (11) we define

$$\zeta = \begin{cases} \epsilon^{-1} & \text{if } \gamma \geq 0 \\ \epsilon^{-1} - \frac{\gamma^2}{4} & \text{if } \gamma < 0 \end{cases}, \ \zeta < 0,$$

and

$$k = \sqrt{\zeta} = i\lambda, \ \lambda > \begin{cases} 0 & \text{if } \gamma \geq 0 \\ -\frac{\gamma}{2} & \text{if } \gamma < 0 \end{cases}.$$

Then

$$\begin{aligned}\left([1-\epsilon K_\gamma]^{-1}\phi\right)(x) &= -\epsilon^{-1}\left([H_\gamma-\zeta]^{-1}\phi\right)(x) = -\epsilon^{-1}\int_{\mathbb{R}}\frac{i}{2k}e^{ik|x-y|}\phi(y)dy + \\ & -\epsilon^{-1}\int_{\mathbb{R}}\frac{1}{2k}\frac{\gamma}{i\gamma+2k}e^{ik(|x|+|y|)}\phi(y)dy = I + II\end{aligned}$$

where

$$\begin{aligned} I &= -\frac{1}{2\lambda\epsilon}\int_{\mathbb{R}} e^{-\lambda|x-y|}\phi(y)dy \\ II &= \frac{1}{2\lambda\epsilon}\frac{e^{-\lambda|x|}\gamma}{\gamma+2\lambda}\int_{\mathbb{R}} e^{-\lambda|y|}\phi(y)dy\end{aligned}$$

Thus

$$\begin{aligned}\|II\|_p &= \frac{1}{2\lambda|\epsilon|}\frac{\|e^{-\lambda|\cdot|}\|_p|\gamma|}{|\gamma+2\lambda|}\left|\int_{\mathbb{R}} e^{-\lambda|y|}\phi(y)dy\right| \\ &\le \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma+2\lambda|}\left\|e^{-\lambda|\cdot|}\phi\right\|_1 \\ &\le \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma+2\lambda|}\left\|e^{-\lambda|\cdot|}\right\|_{p'}\|\phi\|_p \\ &\le \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma+2\lambda|}\left[\frac{2}{\lambda p'}\right]^{1/p'}\|\phi\|_p \\ &\le \frac{|\gamma|}{p^{1/p}p'^{1/p'}}\frac{1}{\lambda^2|\epsilon||\gamma+2\lambda|}\|\phi\|_p\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and

$$\lambda^2|\epsilon||\gamma+2\lambda| = \begin{cases} |\gamma+2\lambda|>\gamma & \text{if } \gamma\ge 0 \\ \frac{4+|\epsilon|\gamma^2}{|\gamma|\sqrt{|\epsilon|}\left[\sqrt{4+|\epsilon|\gamma^2}+|\gamma|\sqrt{|\epsilon|}\right]}\frac{|\gamma|}{4}\ge\frac{|\gamma|}{8} & \text{if } \gamma<0\end{cases},$$

Hence

$$\|II\|_p \le C_{II}\|\phi\|_p,\quad C_{II}=\frac{8}{p^{1/p}p'^{1/p'}}$$

The integral term I, corresponding to the free Laplacian, is similarly treated:

$$\begin{aligned}\|I\|_p &\le \frac{1}{2\lambda|\epsilon|}\|e^{-\lambda|\cdot|}\star\phi\|_p \\ &\le \frac{1}{2\lambda|\epsilon|}\|e^{-\lambda|\cdot|}\|_1\|\phi\|_p \le \frac{1}{\lambda^2|\epsilon|}\|\phi\|_p\end{aligned}$$

where $\lambda^2|\epsilon|\ge 1$ for any ϵ. □

3.3. **Conservation laws.** Following 3.7.1 in [8], conservation of the L^2-norm and of the energy (6) follow from points i. and ii., so the proof is complete. □

Remark 3. *Our result on conservation laws can be* formally *extended to the case of a* nonlinear *pointwise interaction (see e.g.* [1]*) with potential*

$$V_\sigma\psi = \gamma|\psi|^{2\sigma}\psi\delta_0 + \rho|\psi|^{2\mu}\psi$$

In such a case the energy is defined as

$$\mathcal{E}_\sigma[\psi] = \|\psi'\|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} + \frac{\gamma}{\sigma+1}|\psi(0)|^{2\sigma+2} \tag{12}$$

Since a formal proof of conservation of the norm is fairly easy, we deal with conservation of the energy only. Let ψ be regular enough, then (for the sake of simplicity let us denote ψ_t by ψ):

$$\begin{aligned}
\frac{d\|\psi'\|^2}{dt} &= 2\Re\int_{\mathbb{R}} \bar{\psi}'(\dot{\psi})'dx = 2\Re\left[i\int_{\mathbb{R}} \bar{\psi}''(i\dot{\psi})dx\right] \\
&= 2\Re\left[i\int_{\mathbb{R}} \bar{\psi}''\left[-\psi'' + \rho|\psi|^{2\mu}\psi + \gamma|\psi|^{2\sigma}\delta_0\psi\right]dx\right] \\
&= -2\rho\Im\left[\int_{\mathbb{R}} \bar{\psi}''|\psi|^{2\mu}\psi dx\right] - 2\gamma\Im\left[\int_{\mathbb{R}} \bar{\psi}''\psi|\psi|^{2\sigma}\delta_0 dx\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\right) &= 2\frac{\rho}{\mu+1}\Re\int_{\mathbb{R}}\left(\frac{d}{dt}\psi^{\mu+1}\right)\bar{\psi}^{\mu+1}dx \\
&= 2\rho\Re\left[-i\int_{\mathbb{R}} \psi^{\mu}\bar{\psi}^{\mu+1}(i\dot{\psi})dx\right] \\
&= 2\rho\Im\left[\int_{\mathbb{R}} |\psi|^{2\mu}\left(-\bar{\psi}\psi'' + \rho|\psi|^{2\mu+2} + \gamma|\psi|^{2\sigma+2}\delta_0\right)dx\right] \\
&= -2\rho\Im\int_{\mathbb{R}} |\psi|^{2\mu}\bar{\psi}\psi'' dx = 2\rho\Im\int_{\mathbb{R}} |\psi|^{2\mu}\psi\bar{\psi}'' dx
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\gamma}{\sigma+1}|\psi(0)|^{2\sigma+2}\right) &= \frac{\gamma}{\sigma+1}\int_{\mathbb{R}} \delta_0 2\Re\left[\bar{\psi}^{\sigma+1}\left(\frac{d}{dt}\psi^{\sigma+1}\right)dx\right] \\
&= 2\gamma\Im\left[\int_{\mathbb{R}} \delta_0|\psi|^{2\sigma}\bar{\psi}''\psi dx\right]
\end{aligned}$$

Collecting all these facts we finally obtain that $\frac{d\mathcal{E}_\sigma}{dt} = 0$. In particular, for $\sigma = 0$, we obtain again the conservation of the energy (6).

Remark 4. *As in the linear case, the solution ψ_t satisfies the continuity equation*

$$\partial_t|\psi_t|^2 + \partial_x[2\Im\overline{\psi_t}\partial_x\psi_t] = 0 \tag{13}$$

The proof is elementary. From (13) one can give another proof of the conservation of the L^2-norm.

4. Analysis of the blow-up

The phenomenon of the blow-up has been investigated extensively in the case of the standard nonlinear Schrödinger equation.The most advanced results in this topics are due to Merle and Raphael ([14]).

In our problem the definition of blow-up is the same as in the standard case.

Definition 1. *Let $\psi \in C\left((-T_{\min}, T_{\max}), H^1(\mathbb{R})\right) \cap C^1\left((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R})\right)$ be the unique maximal solution of the Cauchy problem (1) with a nonlinear potential*

V like in (2) and initial data $\psi_0 \in H^1(\mathbb{R})$. We call ψ a blow-up solution *and say that ψ* blows up in finite time *if there exists a finite $T_\star$ such that*

$$\lim_{t \to T_\star} \|\psi_t\|_{H^1} = +\infty$$

If $T_\star > 0$ then we say that ψ blows up forward *in time, if $T_\star < 0$ then we say that it blows up* backward *in time.*

Remark 5. *As in the standard case, we have a blow-up alternative, namely, either the solution is global in time, or it blows up in finite time. This can be proven following theorem 3.3.9., step 2. in* [8]. *As a consequence, if ψ blows up forward (backward) in time, then $T_\star = T_{\max}$ ($T_{\min}$).*

We follow the method of Glassey ([11]), based on the computation of the moment of inertia of the solution. To this end, let us define the set

$$\mathcal{K} = H^1(\mathbb{R}) \cap \{\psi \in L^2(\mathbb{R}) \,:\, x\psi \in L^2(\mathbb{R})\} \cap \{\psi \in L^2(\mathbb{R}) \,:\, \|\psi\| = 1\}.$$

The result we prove reads as follows.

Theorem 2. *Let $\psi_0 \in H^1(\mathbb{R})$ be the initial data for the Cauchy problem (1) with a non linear potential V like in (2). Let $\psi \in (-T_{\min}, T_{\max})$ be the solution to such problem. We define the variance $I(t)$ (also called moment of inertia) of the solution ψ as follows*

$$I = I(t) = \int_{\mathbb{R}} x^2 |\psi_t(x)|^2 dx, \quad t \in (-T_{\min}, T_{\max}).$$

Then, $I \in C^2(-T_{min}, T_{max})$ and

$$\dot{I} \;=\; 4\Im \int_{\mathbb{R}} x\psi_t'(x)\bar{\psi}_t(x)dx, \;\; \psi \in \mathcal{H} \tag{14}$$

$$\ddot{I} \;=\; 8\mathcal{E}[\psi_t] + 4\rho\frac{\mu-2}{\mu+1}\|\psi_t\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi_t(0)|^2 \tag{15}$$

Proof. In order to prove that $x\psi_t \in L^2(\mathbb{R})$ and equality (14), standard regularization arguments apply. In particular, we set

$$I^\epsilon(t) = I_-^\epsilon(t) + I_+^\epsilon(t)$$

where $\epsilon > 0$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, $\theta_\epsilon(x) = e^{-\epsilon x^2}$ and

$$I_-^\epsilon(t) = \int_{\mathbb{R}_-} x^2\theta_\epsilon(x)|\psi_t(x)|^2 dx, \quad I_+(t) = \int_{\mathbb{R}_+} \theta_\epsilon(x) x^2 |\psi_t(x)|^2 dx.$$

Hereafter, for the sake of simplicity, let us drop out the explicit dependence on t where this does not cause misunderstanding. By standard computations one finds

$$\begin{aligned}
\dot{I}_+^\epsilon &= 2\Re \int_{\mathbb{R}_+} \theta_\epsilon(x) x^2 \dot{\psi}\bar{\psi}dx = 2\Im \int_{\mathbb{R}_+} \theta_\epsilon(x) x^2 [-\psi'' + \rho|\psi|^{2\mu}\psi]\bar{\psi}dx \\
&= -2\Im \int_{\mathbb{R}_+} x^2\theta_\epsilon(x)\psi''\bar{\psi}dx \\
&= 2\Im \int_{\mathbb{R}_+} \theta_\epsilon(x) x^2 \psi'\bar{\psi}' dx + 4\Im \int_{\mathbb{R}_+} \theta_\epsilon(x) x\psi'\bar{\psi}dx + 2\Im \int_{\mathbb{R}_+} 2\epsilon x^3 \theta_\epsilon(x)\psi'\bar{\psi}dx \\
&= 4\Im \int_{\mathbb{R}_+} \theta_\epsilon(x) x\psi'\bar{\psi}dx - 4\Im \int_{\mathbb{R}_+} \epsilon x^3\theta_\epsilon(x)\psi'\bar{\psi}dx
\end{aligned}$$

Since a similar result holds for $\dot{I}^\epsilon_-$, we can conclude

$$\dot{I}^\epsilon = 4J^\epsilon - 4\Im \int_{\mathbb{R}} \epsilon x^3 \theta_\epsilon(x)\psi'\bar{\psi}dx, \quad J^\epsilon = \Im \int_{\mathbb{R}} \theta_\epsilon(x) x\psi'\bar{\psi}dx.$$

Therefore, for any fixed t the variance I_ϵ is bounded as a function of $\epsilon > 0$ (for details see the proof of Lemma 6.5.2 in [8]). Hence $x\psi_t \in L^2(\mathbb{R})$ and the limit $\epsilon \to 0$ gives (14). Now, in order to prove that $I \in C^2$ and the validity of formula (15) let us write again

$$J^\epsilon = J^\epsilon_+ + J^\epsilon_-$$

where, for any fixed t,

$$\begin{aligned} \dot{J}^\epsilon_\pm &= -\Im \int_{\mathbb{R}_\pm} \dot{\psi}\left[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta'_\epsilon)\bar{\psi}\right] dx = F^\epsilon_\pm(\psi'', \psi', \psi) \\ F^\epsilon_\pm(\psi'', \psi', \psi) &= -\Re \int_{\mathbb{R}_\pm} \left(\psi'' - \rho|\psi|^{2\mu}\psi\right)\left[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta'_\epsilon)\bar{\psi}\right] dx \qquad (16) \end{aligned}$$

We have defined $F_\pm$ as functionals on $D(H_\gamma)$, but they can be extended to the whole space $H^1(\mathbb{R})$. Indeed, for any function $u \in H^2(\mathbb{R})$ satisfying the boundary conditions (3) and (4), an integration by parts gives

$$F^\epsilon_+(u'', u', u) + F^\epsilon_-(u'', u', u) = G^\epsilon(u', u) + \gamma|u(0)|^2 \qquad (17)$$

where the right hand side is well defined for any $u \in H^1(\mathbb{R})$. To see that, let us further integrate by parts and obtain

$$\int_{\mathbb{R}_+} \psi''\theta_\epsilon x\bar{\psi}'dx = -\int_{\mathbb{R}_+} \psi'\theta_\epsilon x\bar{\psi}''dx - \int_{\mathbb{R}_+} |\psi'|^2(\theta_\epsilon x)'dx.$$

Therefore,

$$\Re\left[\int_{\mathbb{R}_+} \psi'' 2\theta_\epsilon x\bar{\psi}'dx\right] = \frac{1}{2}\Re\left[-\int_{\mathbb{R}_+} |\psi'|^2(2\theta_\epsilon x)'dx\right]. \qquad (18)$$

Furthermore,

$$\int_{\mathbb{R}_+} \psi''\left(\theta_\epsilon + x\theta'_\epsilon\right)\bar{\psi}dx = -\psi'(0+0)\bar{\psi}(0+0) - \int_{\mathbb{R}_+} \psi'\left[\left(\theta_\epsilon + x\theta'_\epsilon\right)\bar{\psi}\right]' dx \qquad (19)$$

Collecting (16), (18) and (19) we can conclude that

$$\begin{aligned} F^\epsilon_\pm &= \pm\Re\left[\psi'(0\pm 0)\bar{\psi}(0\pm 0)\right] + G^\epsilon_\pm \\ G^\epsilon_\pm &= \frac{1}{2}\int_{\mathbb{R}_\pm} |\psi'|^2(2\theta_\epsilon x)'dx + \Re\left[\int_{\mathbb{R}_\pm} \psi'\left[\left(\theta_\epsilon + x\theta'_\epsilon\right)\bar{\psi}\right]' dx\right] \\ &\quad + \Re\int_{\mathbb{R}_\pm} \rho|\psi|^{2\mu}\psi\left[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta'_\epsilon)\bar{\psi}\right] dx \end{aligned}$$

and formula (17) follows since $G^\epsilon = G^\epsilon_+ + G^\epsilon_-$. Thus, by means of a continuity argument we obtain that

$$\dot{J}^\epsilon = G^\epsilon(\psi'_t, \psi_t) + \gamma|\psi_t(0)|^2$$

is well defined since $\psi_t \in H^1(\mathbb{R})$ for any t. Applying the dominate convergence theorem we compute the limit $\epsilon \to 0$ and conclude the proof.

Such a technical procedure justifies the following formal computation, too.

$$\begin{aligned}
\ddot{I} &= 4\Im\int_{\mathbb{R}}\left[x\psi'\dot{\bar{\psi}}+x\dot{\psi}'\bar{\psi}\right]dx = 4\Im\int_{\mathbb{R}}\left[x\psi'\dot{\bar{\psi}}-\dot{\psi}\bar{\psi}-x\dot{\psi}\bar{\psi}'\right]dx \\
&= 8\Im\int_{\mathbb{R}} x\psi'\dot{\bar{\psi}}dx - 4\Im\left(-i\int_{\mathbb{R}} i\dot{\psi}\bar{\psi}dx\right) \\
&= -8\Re\int_{\mathbb{R}} x\psi'\bar{\psi}''dx + 8\Re\int_{\mathbb{R}} x\psi'\bar{V}\bar{\psi}dx + 4\Re\int_{\mathbb{R}}(H\psi)\bar{\psi}dx \\
&= 4\Re\int_{\mathbb{R}}|\psi'|^2dx + 4\Re\int_{\mathbb{R}} x(\psi''\bar{\psi}'-\psi'\bar{\psi}'')dx + 8\Re\int_{\mathbb{R}} x\psi'\bar{V}\bar{\psi}dx + 4\Re\int_{\mathbb{R}}(H\psi)\bar{\psi}dx \\
&= 4\int_{\mathbb{R}}|\psi'|^2dx + 8\Re\int_{\mathbb{R}} x\psi'\bar{V}\bar{\psi}dx + 4\Re\int_{\mathbb{R}}(H\psi)\bar{\psi}dx \\
&= 8\int_{\mathbb{R}}|\psi'|^2dx + 4\int_{\mathbb{R}}\bar{\psi}V\psi dx + 8\Re\int_{\mathbb{R}} x\psi'\bar{V}\bar{\psi}dx
\end{aligned}$$

where $H\psi = -\psi'' + V\psi$, the potential V being given by (2). Hence:

$$\begin{aligned}
\ddot{I} &= 8\int_{\mathbb{R}}|\psi'|^2dx + 8\rho\Re\int_{\mathbb{R}} x\psi'\bar{\psi}|\psi|^{2\mu}dx + 4\rho\int_{\mathbb{R}}|\psi|^{2\mu+2}dx + 4\gamma\int_{\mathbb{R}}|\psi|^2\delta_0 dx \\
&= 8\left[\mathcal{E}[\psi] - \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - \gamma|\psi(0)|^2\right] + \\
&\qquad +8\rho\Re\int_{\mathbb{R}} x\psi'\bar{\psi}|\psi|^{2\mu}dx + 4\rho\|\psi\|_{2\mu+2}^{2\mu+2} + 4\gamma|\psi(0)|^2 \\
&= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-1}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi(0)|^2 - \frac{4\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\
&= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi(0)|^2
\end{aligned}$$

since

$$2(\mu+1)\Re\int_{\mathbb{R}} x\psi'\bar{\psi}|\psi|^{2\mu}dx = -\|\psi\|_{2\mu+2}^{2\mu+2}.$$

□

Now, we are ready to carry out the analysis of the blow-up for the NLS equation with potential (2).

Theorem 3. *Let ψ_t be the solution of the Cauchy problem (1), (2), with initial data $\psi_0 \in \mathcal{K}$. If one of the following conditions*

i. $\rho \geq 0$*;*
ii. $\rho < 0$ *and* $\mu < 2$*;*
iii. $-\frac{3}{\tilde{C}} < \rho < 0$ *and* $\mu = 2$*, where* $\tilde{C}$ *is the positive constant appearing in estimate (27);*
iv. $\rho_1 < \rho < 0$*, for some* $\rho_1 < 0$ *depending on* ψ_0*, and* $\mu > 2$*;*

is satisfied, then there is no blow-up and the solution ψ exists globally in time. In contrast, if

v. $\mathcal{E}[\psi_0] < 0$*,* $\rho < \rho_2$ *for some* $\rho_2 < 0$*, and* $\mu > 2$*,*

then the solution ψ blows up in finite time.

Proof. Let C denote any positive constant and, for the sake of simplicity, let us drop out in notation the explicit dependence on t (that is let us write ψ instead of ψ_t, I instead of $I(t)$, ...). From the conservation laws of the energy and of the norm and from inequality (26) it follows that

$$\begin{aligned}\mathcal{E}[\psi_0] &= \|\psi'\|^2 + \gamma|\psi(0)|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\ &\geq \|\psi'\|^2 - |\gamma||\psi(0)|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\ &\geq \|\psi'\|^2 - |\gamma|\|\psi\|_\infty^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\ &\geq \|\psi'\|^2 - |\gamma|\|\psi'\| + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\ &\geq \left(\|\psi'\| - \frac{1}{2}|\gamma|\right)^2 - \frac{1}{4}\gamma^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\end{aligned} \tag{20}$$

from which, if $\rho \geq 0$, it immediately follows

$$\|\psi'\| \leq C, \quad \forall t \geq 0,$$

and then i. is proved. On the other hand, if $\rho < 0$ then inequality (20) takes the form

$$\begin{aligned}\mathcal{E}[\psi_0] &\geq \|\psi'\|^2 - |\gamma|\|\psi\|_\infty^2 - \frac{|\rho|}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} \\ &\geq \|\psi'\|^2 - |\gamma|\|\psi'\| - \tilde{C}\frac{|\rho|}{\mu+1}\|\psi'\|^\mu\end{aligned} \tag{21}$$

where we make use of inequality (27). Therefore $\|\psi'\| \leq C$, $\forall t \geq 0$, provided that $\mu < 2$, so ii. is proven. For $\mu = 2$ the above inequality gives

$$\mathcal{E}[\psi_0] \geq \left(1 - \frac{1}{3}\tilde{C}|\rho|\right)\|\psi'\|^2 - |\gamma|\|\psi'\|$$

Since in the case iii. $1 - \frac{1}{3}\tilde{C}|\rho| > 0$, hence $\|\psi'\| \leq C$, $\forall t \geq 0$. In order to consider case iv., in which $\mu > 2$, we remark that (21) can be written as

$$\mathcal{E}[\psi_0] \geq z^2 - |\gamma|z - \tilde{C}\frac{|\rho|}{\mu+1}z^\mu, \; z = \|\psi\| \in \mathbb{R}^+, \; \mu > 2$$

Let $\mathcal{S}$ be the set of solutions of such inequality. If ρ is small enough then there exists $c_1, \tilde{C}$, $0 < c_1 < c_2$, such that $\|{\psi_0}'\| \leq c_1$ and $\mathcal{S} \cap [c_1, c_2] = \emptyset$. Hence, $\|\psi'\| \leq c_1$, $\forall t \geq 0$, proving iv.. Finally, in order to consider the case v., we set

$$\xi = \|\psi'\|^2, \quad \eta = \|\psi\|_{2\mu+2}^{2\mu+2}, \quad \zeta = |\psi(0)|^2$$

then the energy and the second derivative of the variance take the form

$$\mathcal{E}[\psi] = \xi + \frac{\rho}{\mu+1}\eta + \gamma\zeta \tag{22}$$

$$\ddot{I} = 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\eta - 4\gamma\zeta \tag{23}$$

If $\gamma > 0$, then at any time $\ddot{I} < 8\mathcal{E}(\psi_0) < 0$. Following [11], blow up is proven. Conversely, if $\gamma < 0$, then from equation (22) it follows that

$$\frac{\rho}{\mu+1}\eta = \mathcal{E}[\psi] - \xi - \gamma\zeta$$

Plugging this equation into (23) and using inequality (26) we obtain

$$\begin{aligned}
\ddot{I} &= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\eta - 4\gamma\zeta \\
&= 8\mathcal{E}[\psi] + 4(\mu-2)\left[\mathcal{E}[\psi] - \xi - \gamma\zeta\right] - 4\gamma\zeta \\
&= -4(\mu-2)\xi + 4\mu\mathcal{E}[\psi] + 4(1-\mu)\gamma\zeta \\
&\leq -4(\mu-2)\xi + 4\mu\mathcal{E}[\psi] + 4(1-\mu)\gamma\sqrt{\xi}
\end{aligned}$$

Now, setting

$$a = 4(\mu-2) > 0, \quad b = 4(1-\mu)\gamma > 0$$

and

$$c = -4\mu\mathcal{E}[\psi] > 0 \quad \text{since} \quad \mathcal{E}[\psi] < 0$$

then we obtain the inequality

$$\ddot{I} \leq -a\xi + b\sqrt{\xi} - c$$

Hence, if a, b and c satisfy

$$b^2 < 4ac \tag{24}$$

then there exists $C > 0$ such that

$$\ddot{I} < -C, \quad \forall t \geq 0$$

Condition (24) implies blow-up. That is: if $\mu > 2$, $\gamma \in \mathbb{R}$ and $\rho > 0$ are such that

$$\frac{(\mu-1)^2}{4\mu(\mu-2)} < -\frac{\mathcal{E}[\psi]}{\gamma^2} \tag{25}$$

then we have blow-up in finite time. In particular, from (6) we have blow-up if $|\rho|$ is large enough. □

5. Concluding remarks

Remark 6. *¿From Theorem 3 it follows that no blow-up can occur for the physically most important cases $\mu = \frac{1}{2}$ and $\mu = 1$. Furthermore, we emphasize that, as for the free model (i.e. $\gamma = 0$), blow-up could occur only in the critical ($\mu = 2$) and hyper-critical ($\mu > 2$) cases and for attractive nonlinearity large enough.*

Remark 7. *In the case $\gamma < 0$, the results iv. and v. discussed in theorem 3 can be investigated in more detail when the initial data ψ_0 coincides with the ground state ϕ_γ for the Hamiltonian H_γ, namely*

$$\psi_0(x) = \sqrt{\frac{|\gamma|}{2}} e^{-|\gamma|\cdot|x|/2} \in \mathcal{K}$$

with associated eigenvalues

$$\mathcal{E}_{lin}[\psi_0] = -\frac{1}{4}\gamma^2$$

An explicit computation gives

$$\begin{aligned}
\mathcal{E}[\psi] &= \mathcal{E}[\psi_0] = -\frac{1}{4}\gamma^2 - |\rho|\frac{|\gamma|^\mu}{2^\mu(\mu+1)^2} < 0 \\
I(0) &= \int_{\mathbb{R}} x^2(\psi_0(x))^2 dx = \frac{2}{\gamma^2} \\
\dot{I}(0) &= 4\Im \int_{\mathbb{R}} x\bar{\psi}_0(x){\psi_0}'(x)dx = 0 \\
\ddot{I}(0) &= -4|\rho|\frac{\mu|\gamma|^\mu}{2^\mu(\mu+1)^2} < 0
\end{aligned}$$

Then, inequality (25) implies that if the parameters $\mu > 2$ and $\rho < 0$ satisfy the following condition

$$\frac{(\mu-1)^2}{4\mu(\mu-2)} < \frac{1}{4} + |\rho|\frac{|\gamma|^{\mu-2}}{2^\mu(\mu+1)^2}$$

then the solution ψ blows up in finite time.

Remark 8. *By means of the same ideas it is possible to consider the occurrence of blow-up phenomena for the case with* nonlinear *pointwise interaction; that is with Hamiltonian*

$$H\psi = -\psi'' + V\psi, \quad V_\nu\psi = \gamma|\psi|^{2\nu}\psi\delta_0 + \rho|\psi|^{2\mu}\psi,\ \nu,\mu \geq 0,\ \gamma,\rho \in \mathbb{R}$$

If we assume existence and uniqueness for the solution ψ_t to the associated Cauchy problem, then the conservation laws of L^2-norm and energy (12) imply that there is no blow-up provided that one the following conditions is satisfied:

- $\gamma,\ \rho \geq 0$;
- $\gamma \geq 0$ *and* $\rho < 0$ *and* $\mu < 2$;
- $\rho \geq 0$ *and* $\gamma < 0$ *and* $\nu < 1$;
- $\gamma, \rho < 0$ *and* $\nu < 1$ *and* $\mu < 2$.

Indeed, in such cases inequality (20) takes the form

$$\mathcal{E}[\psi] \geq \|\psi'\|^2 - \gamma_-\|\psi'\|^{\nu+1} - \tilde{C}\rho_-\|\psi'\|^\mu$$

where

$$\gamma_- = \max(0, -\gamma),\ \rho_- = \max(0, -\rho),$$

and an a priori estimate for the norm $\|\psi'\|$ follows. In contrast, the same arguments of theorem 2 give that the formal derivation of the second derivative of the variance reads

$$\ddot{I} = 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} + 4\gamma\frac{\nu-1}{\nu+1}|\psi(0)|^{2\nu+2}$$

Hence, blow-up occurs when $\mathcal{E}[\psi] < 0$, $\mu > 2$, and $\nu > 1$.

Appendix A. Inequalities

We recall the following inequalities. Let $\psi \in L^2(\mathbb{R}) \cap H^1(\mathbb{R})$, then

$$\|\psi\|_\infty^2 \leq \|\psi\| \cdot \|\psi'\|. \tag{26}$$

If $\psi \in L^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,

$$\|\psi\|_{2\chi+2}^{2\chi+2} \leq \tilde{C}\|\nabla\psi\|^{\chi d}\|\psi\|^{2+\chi(2-d)} \tag{27}$$

for some positive constant $\tilde{C} = \tilde{C}(\chi, d)$, where

$$\chi \in \begin{cases} [0, +\infty] & \text{if } \ d = 1 \\ [0, +\infty) & \text{if } \ d = 2 \\ [0, 2/(d-2)) & \text{if } \ d > 2 \end{cases}$$

and $\tilde{C} = 4\pi^{-2}$ for $d = 1$.

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