Constructive Quantum Field Theory

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1. Euclidean Quantum Fields

The construction of a relativistic quantum field is still an open problem for fields in space-time dimension $d \ge 4$. The conceptual difficulty that sometimes led to fear an incompatibility between nontrivial quantum systems and special relativity has however been solved in the case of dimension $d = 2, 3$ although, so far, has not influenced the corresponding debate on the foundations of quantum mechanics, still much alive.

It began in the early 1960's with Wightman's work on the axioms and the attempts at understanding the mathematical aspects of renormalization theory and with Hepps' renormalization theory for scalar fields. The breakthrough idea was, perhaps, Nelson's realization that the problem could really be studied in *Euclidean form*. A solution in dimensions $d = 2, 3$ has been obtained in the 1960's and 1970's through a remarkable series of papers by Nelson, Glimm, Jaffe, Guerra. While the works of Nelson and Guerra relied on the "Euclidean approach" (see below) and on $d = 2$ the early works of Glimm and Jaffe dealt with $d = 3$ making use of the "Minkowskian approach" (based on second quantization) but making already use of a *multiscale analysis* technique. The latter received great impulsion and systematization by the adoption of Wilson's views and methods on renormalization: in Physics terminology *renormalization group* methods; a point of view taken here following the Euclidean approach. The solution dealt initially with *scalar fields* but it has been subsequently considerably extended.

The *Euclidean approach* studies quantum fields through the following problems

(1) existence of the functional integrals defining the generating functions of the probability distribution of the interacting fields in finite volume: the *ultraviolet stability problem*,
(2) existence of the infinite volume limit of the generating functions: the *infrared problem*,
(3) check that the infinite volume generating functions satisfy the axioms needed to pass from the Euclidean, probabilitstic, formulation to a Minkowskian formulation guaranteeing existence of the Hamiltonian operator, relativistic covariance, Ruelle–Haag scattering theory: the *reconstruction problem*.

The characteristic problem for the construction of quantum fields is (1) and here attention will be confined to it with the further restriction to the paradigmatic massive scalar fields cases. The dimension d of the space-time will be $d = 2, 3$ unless specified otherwise.

Given a cube Λ of side L, $\Lambda \subset \mathbb{R}^d$, consider the following *functional integral* on the space of the fields on Λ, *i.e.* on functions $\varphi^{(\le N)}_{\boldsymbol\xi}$ defined for $\boldsymbol\xi \in \Lambda$,

$$Z_N(\Lambda, f) = \int e^{-\int_\Lambda (\lambda_N \varphi^{(\le N)\,4}_{\boldsymbol\xi} + \mu_N \varphi^{(\le N)\,2}_{\boldsymbol\xi} + \nu_N + f_{\boldsymbol\xi} \varphi^{(\le N)}_{\boldsymbol\xi})\, d\boldsymbol\xi}\, P_N(d\varphi^{(\le N)}) \tag{1.1}$$

The fields $\varphi^{(\le N)}_{\boldsymbol\xi}$ are called "Euclidean" fields with *ultraviolet cut–off* $N > 0$, $f_{\boldsymbol\xi}$ is a smooth function with compact support bounded by $|f_{\boldsymbol\xi}| \le 1$ (for definiteness), the constants $\lambda_N > 0$, μ_N, ν_N are called *bare couplings*, and P_N is a Gaussian probability distribution defining the *free field distribution* with *mass* m and ultraviolet cut–off N; the probability distribution P_N is determined by its "covariance" $C^{(\le N)}_{\boldsymbol\xi,\boldsymbol\eta} \overset{def}{=} \int \varphi^{(\le N)}_{\boldsymbol\xi} \varphi^{(\le N)}_{\boldsymbol\eta}\, dP_N$, which in the Physics literature is called a *propagator*, given by

$$C^{(\le N)}_{\boldsymbol\xi,\boldsymbol\eta} = \frac{1}{(2\pi)^d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \int \frac{e^{i\mathbf{p}\cdot(\boldsymbol\xi-\boldsymbol\eta+\mathbf{n}L)}}{\mathbf{p}^2+m^2}\, \chi_N(|\mathbf{p}|)\, d^d\mathbf{p} \tag{1.2}$$

The sum over the integers $\mathbf{n}\in\mathbb{Z}^d$ is introduced so that the field $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ is *periodic* over the box Λ: this is not really necessary as in the limit $L\to\infty$ either translation invariance would be recovered or lack of it properly understood, but it makes the problem more symmetric and generates a few technical simplifications; here $\chi_N(z)$ is a *regularizer* and a standard choice is $\chi_N(|\mathbf{p}|)=\frac{m^2\,(\gamma^{2N}-1)}{\mathbf{p}^2+\gamma^{2N}m^2}$ with $\gamma>1$, which is such that

$$\frac{\chi_N(|\mathbf{p}|)}{\mathbf{p}^2+m^2}\equiv\frac{1}{\mathbf{p}^2+m^2}-\frac{1}{\mathbf{p}^2+\gamma^{2N}m^2}\equiv\sum_{h=1}^{N}\big(\frac{1}{\mathbf{p}^2+\gamma^{2(h-1)}m^2}-\frac{1}{\mathbf{p}^2+\gamma^{2h}m^2}\big)\tag{1.3}$$

here $\gamma>1$ can be chosen arbitrarily: so $\gamma=2$. If $d>3$ the above regularization will not be sufficient and a χ_N decayng faster than $\mathbf{p}^{-2}$ would be needed.

A simple estimate yields, if $\varepsilon\in(0,1)$ is fixed and c is suitably chosen,

$$|C^{(\le N)}_{\boldsymbol{\xi},\boldsymbol{\eta}}|\le\,c\,\gamma^{(d-2)N}e^{-m|\boldsymbol{\xi}-\boldsymbol{\eta}|}\qquad|C^{(\le N)}_{\boldsymbol{\xi},\boldsymbol{\eta}}-C^{(\le N)}_{\boldsymbol{\xi},\boldsymbol{\eta}'}|\le\,c\,\gamma^{(d-2)N}\,(\gamma^N m\,|\boldsymbol{\eta}-\boldsymbol{\eta}'|)^{\varepsilon}\tag{1.4}$$

with $\gamma^{(d-2)N}$ interpreted as N if $d=2$.

The $\zeta(f)=\log\frac{Z_N(\Lambda,f)}{Z_N(\Lambda,0)}$ defines a "generating function" of a probability distribution P_{int} over the fields on Λ which will be called the "distribution with φ^4-interaction" regularized on Λ and at length scale $m^{-1}\gamma^{-N}$: the integral, in (1.1),

$$V_N(\varphi^{(\le N)})\stackrel{def}{=}-\int_\Lambda(\lambda_N\varphi^{(\le N)\,4}_{\boldsymbol{\xi}}+\mu_N\varphi^{(\le N)\,2}_{\boldsymbol{\xi}}+\nu_N+f_{\boldsymbol{\xi}}\varphi^{(\le N)}_{\boldsymbol{\xi}})\,d^d\boldsymbol{\xi}\tag{1.5}$$

will be called the *interaction potential* with external field f. The regularization is introduced to guarantee that the integral (1.1), $\int e^{V_N}dP_N$, is well defined if $\lambda_N>0$. The momenta of P_{int} are the functional derivatives of $\zeta(f)$: they are called *Schwinger functions*.

The problem (1) can now be made precise: it is to show existence of λ_N,μ_N,ν_N so that the limit $\lim_{N\to\infty}\frac{Z_N(\Lambda,f)}{Z_N(\Lambda,0)}$ exists for all f and *is not Gaussian*, *i.e.* it is not the exponential of a quadratic form in f: which would be the case if $\lambda_N,\mu_N\to0$ fast enough: the last requirement is of course essential because the Gaussian case describes, in the physical interpretation, free fields and non interacting particles *i.e.* it is *trivial*. Note that ν_N does not play a role: its introduction is useful to be able to study separately the numerator and the denominator of the fraction $\frac{Z_N(\Lambda,f)}{Z_N(\Lambda,0)}$.

References: [WG65],[SW64],[Ne66],[OS73],[Si74].

2. The regularized free field

Since the propagator decays exponentially over a scale m^{-1} and is smooth over a scale $m^{-1}\gamma^{-N}$ the fields $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ sampled with distribution P_N are rather singular objects. Their properties cannot be described by a single length scale: they are extremely large for large N, take independent values only beyond distances of order m^{-1} but, at the same time, they look smooth only on the much smaller scale $m^{-1}\gamma^{-N}$. Their essential feature is that fixed $\varepsilon<1$, *e.g.* $\varepsilon=\frac12$, with P_N-probability 1 there is $B>0$ such that (interpreting $\gamma^{\frac{d-2}{2}N}$ as N if $d=2$)

$$|\varphi^{(\le N)}_{\boldsymbol{\xi}}|\le B\gamma^{\frac{d-2}{2}N},\quad|\varphi^{(\le N)}_{\boldsymbol{\xi}}-\varphi^{(\le N)}_{\boldsymbol{\eta}}|<B\gamma^{\frac{d-2}{2}N}(\gamma^N m|\boldsymbol{\xi}-\boldsymbol{\eta}|)^{\frac{\varepsilon}{2}}\tag{2.1}$$

and furthermore the probability of the relations in (2.1) will be N-independent, *i.e.* $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ are bounded and roughly of size $\gamma^{\frac{d-2}{2}N}$ as $N\to\infty$ and, on a very small length scale $m^{-1}\gamma^{-N}$, almost constant.

Substantial control on the field $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ statistically sampled with distribution P_N can be obtained by decomposing it, through (1.3), into "components of various scales": *i.e.* as a sum of statistically mutually *independent* fields whose properties are entirely characterized by a *single scale of length.*

This means that they have size of order 1 and are *independent and smooth on the same length scale*.

Assuming the side of Λ to be an integer multiple of m^{-1}, let $\mathcal{Q}_h$ be a *pavement* of Λ into boxes of side $m^{-1}\gamma^{-h}$, imagined *hierarchically arranged* so that the boxes of $\mathcal{Q}_h$ are exactly paved by those of $\mathcal{Q}_{h+1}$.

Define $z^{(h)}_{\boldsymbol{\xi}}$ to be the random field with propagator $C^{(h)}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ defined as the Fourier transform of $\sum_{\mathbf{n}\in\mathbb{Z}^d}\big(\frac{1}{\mathbf{p}^2+\gamma^{-2}m^2}-\frac{1}{\mathbf{p}^2+m^2}\big)e^{i\mathbf{n}\cdot\mathbf{p}\,L\,\gamma^h}$: so that $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ and its propagator $C^{(\le N)}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ can be represented, see (1.2),(1.3), as

$$\varphi^{(\le N)}_{\boldsymbol{\xi}}\equiv\sum_{h=1}^{N}\gamma^{\frac{d-2}{2}h}z^{(h)}_{\gamma^h\boldsymbol{\xi}},\qquad C^{(\le N)}_{\boldsymbol{\xi},\boldsymbol{\eta}}=\sum_{h=1}^{N}\gamma^{(d-2)h}C^{(h)}_{\gamma^h\boldsymbol{\xi},\gamma^h\boldsymbol{\eta}}\tag{2.2}$$

where the fields $z^{(h)}$ are independently distributed Gaussian fields. Note that the fields $z^{(h)}$ are *also almost identically distributed* because their propagator is obtained by periodizing over the period $\gamma^h L$ the *same* function $\overline{C}^{(0)}_{\boldsymbol{\xi},\boldsymbol{\eta}}\stackrel{def}{=}\int\frac{e^{i\mathbf{p}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})}d\mathbf{p}}{(2\pi)^d}\big(\frac{1}{\mathbf{p}^2+\gamma^{-2}m^2}-\frac{1}{\mathbf{p}^2+m^2}\big)$: *i.e.* their propagator is $C^{(h)}_{\boldsymbol{\xi},\boldsymbol{\eta}}=\sum_{\mathbf{n}\in\mathbb{Z}^d}\overline{C}^{(0)}_{\boldsymbol{\xi},\boldsymbol{\eta}+\gamma^h\mathbf{n}L}$. The reason why they are not exactly equally distributed is that the field $z^{(h)}_{\boldsymbol{\xi}}$ is periodic with period $\gamma^h L$ rather than L. But proceeding with care the sum over $\mathbf{n}$ in the above expressions can be essentially ignored: this is a little price to pay if one wants translation invariance built in the analysis since the beginning.

The representation (2.2) defines a *multiscale representation* of the field $\varphi^{(\le N)}_{\boldsymbol{\xi}}$. Smoothness properties for the field $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ can be read from those of its "components" $z^{(h)}$. Define, for $\Delta\in\mathcal{Q}_0$,

$$||z^{(h)}||_\Delta=\max_{\substack{\boldsymbol{\xi}\in\Delta,\boldsymbol{\eta}\in\Lambda\\|\boldsymbol{\xi}-\boldsymbol{\eta}|\le m^{-1}}}\big(|z^{(h)}_{\boldsymbol{\xi}}|+\tau\frac{|z^{(h)}_{\boldsymbol{\xi}}-z^{(h)}_{\boldsymbol{\eta}}|}{|\boldsymbol{\xi}-\boldsymbol{\eta}|^{\frac14}}\big)\tag{2.3}$$

and τ will be chosen $\tau=0$ or $\tau=1$ as needed (in practice $\tau=0$ if $d=2$ and $\tau=1$ if $d=3$): $\tau=1$ will allow to discuss some smoothness properties of the fields which will be necessary (*e.g.* if $d=3$). Then the size $||z||_\Delta$ *of any field* $z^{(h)}$, for all $h\ge1$, is estimated by

$$P(\max_{\Delta\subset\mathcal{Q}_0}||z||_\Delta\le B)\ge e^{-c\,e^{-c'\,B^2}\,|\Lambda|},\qquad P(||z||_\Delta\ge B_\Delta,\forall\Delta\in\mathcal{D})\le\prod_{\Delta\in\mathcal{D}}c\,e^{-c'\,B^2_\Delta}\tag{2.4}$$

where P is the Gaussian probability distribution of z, $\mathcal{D}$ is any collection of boxes $\Delta\in\mathcal{Q}_0$ and $c,c'>0$ are suitable constants. The (2.4) imply in particular (2.1). The estimates (2.4) follow from the Markovian nature of the Gaussian field $z^{(h)}$, *i.e.* from the fact that the propagator is the Green's function of an elliptic operator (of fourth order, see the first of (1.3)), with constant coefficients which implies also the inequalities (fixing $\varepsilon\in(0,1)$)

$$|C^{(h)}_{\boldsymbol{\xi},\boldsymbol{\eta}}|\equiv\big|\int z_{\boldsymbol{\xi}}z_{\boldsymbol{\eta}}\,P(dz)\big|\le c\,e^{-m\,c'\,|\boldsymbol{\xi}-\boldsymbol{\eta}|},\qquad |C^{(h)}_{\boldsymbol{\xi},\boldsymbol{\eta}}-C^{(h)}_{\boldsymbol{\xi},\boldsymbol{\eta}'}|\le c\,(m|\boldsymbol{\eta}-\boldsymbol{\eta}'|)^\varepsilon,\tag{2.5}$$

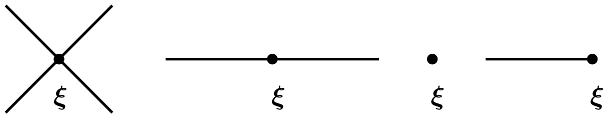
where $|\boldsymbol{\xi}-\boldsymbol{\eta}|$ is *reinterpreted* as the distance between $\boldsymbol{\xi},\boldsymbol{\eta}$ measured over the periodic box $\gamma^h\Lambda$ (hence $|\boldsymbol{\xi}-\boldsymbol{\eta}|$ differs from the ordinary distance only if the latter is of the order of $\gamma^h L$). The interpretation of (2.5) is that $z^{(h)}_{\boldsymbol{\xi}}$ are essentially *bounded* variables which, on scale $\sim m^{-1}$, are essentially *constant* and furthermore beyond length $\sim m^{-1}$ are essentially *independently distributed*.

References: [Wi72],[Ga81],[Ga85].

3. Perturbation theory

The naive approach to the problem is to fix $\lambda_N \equiv \lambda > 0$ and to develop $Z_N(\Lambda, f)$ or, more conveniently and equivalently, $\frac{1}{|\Lambda|}\log Z_N(\Lambda, f)$ in powers of λ. If one fixes *a priori* μ_N, ν_N independent of N, however, even a formal power series is not possible: this is trivially due to the divergence of the coefficients of the power series, *already to second order* for generic f in the limit $N \to \infty$. *Nevertheless* it is possible to determine $\mu_N(\lambda), \nu_N(\lambda)$ as functions of N and λ so that a formal power series exists (to all orders in λ): this is the key result of *renormalization theory.*

To find the perturbative expansion the simplest is to use a graphical representation of the coefficients of the power expansion in λ, μ_N, ν_N, f and the Gaussian integration rules which yield (after a classical computation) that the coefficient of $\lambda^n \mu_N^p f_{\boldsymbol{\xi}_1} \ldots f_{\boldsymbol{\xi}_r}$ is obtained by considering the following *graph elements*



(1)

where the segments will be called *half lines* and the graph elements will be called, respectively, *coupling* or φ^4*-vertex*, *mass vertex*, *vacuum vertex* and *external vertex.*

The half lines of the graph elements are considered *distinct* (*i.e.* imagine a label attached to distinguish them). Then consider all possible *connected* graphs G obtained by first drawing, respectively, n, p, r graph elements in Fig.1, which are not vacuum vertices, with their nodes marked by points in Λ named $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n, \boldsymbol{\xi}_{n+1}, \ldots, \boldsymbol{\xi}_{n+p+r}$; and form all possible graphs obtained by attaching pairs of halph lines emerging from the vertices of the graph elements. These are the "nontrivial graphs". Furthermore consider also the single "trvial" graph formed just by the third graph element and consisting of a single point. All graphs obtained in this way are particular *Feynman graphs.*

Given a nontrivial graph G (there are many of them) we define its *value* to be the product

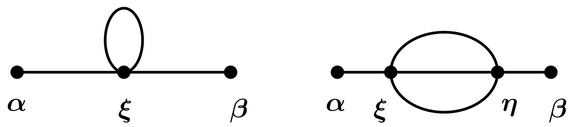
$$W_G(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n, \boldsymbol{\xi}_{n+1}, \ldots, \boldsymbol{\xi}_{n+p+r}) = (-1)^{n+p+r} \frac{\lambda^n \mu_N^p \prod f_{\boldsymbol{\xi}_{n+p+j}}}{n!p!r!} \prod_\ell C^{(\le N)}_{\boldsymbol{\xi}_\ell, \boldsymbol{\eta}_\ell} \tag{3.1}$$

where the last product runs over all pairs $\ell = (\boldsymbol{\xi}_\ell, \boldsymbol{\eta}_\ell)$ of half lines of G that are joined and connect two vertices labeled by points $\boldsymbol{\xi}_\ell, \boldsymbol{\eta}_\ell$: call *line* of G any such pair. If the graph consists of the single vacuum vertex its value will be ν_N. The series for $\frac{1}{|\Lambda|}\log Z_N(\Lambda, f)$ is then

$$-\nu_N + \frac{1}{|\Lambda|} \sum_G \int W_G(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n+p+r}) \prod_{j=1}^{n+p+r} d\boldsymbol{\xi}_j \tag{3.2}$$

and the integral will be called the *integrated graph value.*

Suppose first that $\mu_N = \nu_N = 0$. Then if a graph G contains subgraphs like



(2)

the corresponding respective contribution to the integral in (3.2) (considering only the integrals over $\boldsymbol{\eta}$ and suitably taking care of the combinatorial factors) is a factor obtained by integrating over $\boldsymbol{\xi}$ the quantities

$$-6\,\lambda\, C^{(\le N)}_{\boldsymbol{\alpha\xi}} C^{(\le N)}_{\boldsymbol{\xi\xi}} C^{(\le N)}_{\boldsymbol{\xi\beta}} \quad \text{or} \quad \frac{4^2 \cdot 3!}{2!} \lambda^2\, C^{(\le N)}_{\boldsymbol{\alpha\xi}} \int C^{(\le N)\,3}_{\boldsymbol{\xi\eta}}\, C^{(\le N)}_{\boldsymbol{\eta\beta}}\, d\boldsymbol{\eta} \tag{3.3}$$

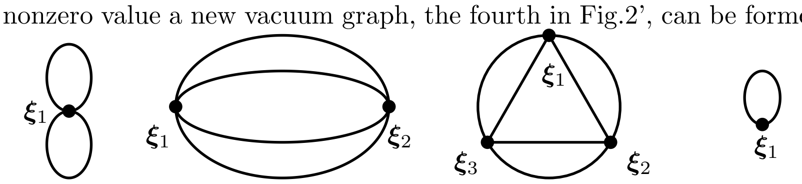
which if $d = 3$ diverge as $N \to \infty$ as γ^N or, respectively, as N; the second factor does not diverge in dimension $d = 2$ while the first still diverges as N. The divergences arise from the fact that as $\boldsymbol{\xi} - \boldsymbol{\eta} \to \mathbf{0}$ the propagator behaves as $|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-N}$ if $d = 3$ or as $-\log|\boldsymbol{\xi} - \boldsymbol{\eta}|$ if $d = 2$, all the way until saturation occurs at distance $|\boldsymbol{\xi}-\boldsymbol{\eta}|\simeq m^{-1}\gamma^{-N}$: for this reason the latter divergences are called *ultraviolet divergences.*

However if we set $\mu_N\ne 0$ then for every graph containing a subgraph like those in Fig.2 there is another one identical except that the points $\boldsymbol{\alpha},\boldsymbol{\beta}$ are connected via a mass vertex, see Fig.1, with the vertex in $\boldsymbol{\xi}$, by a line $\boldsymbol{\alpha\xi}$ and a line $\boldsymbol{\xi\beta}$; the new graph value receives a contribution from the mass vertex inserted in $\boldsymbol{\xi}$ between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ simply given by a factor $-\mu_N$. Therefore if we fix, for $d=3$,

$$\mu_N=-6\,\lambda\,C^{(\le N)}_{\boldsymbol{\xi\xi}}+\frac{4^2\cdot 3!}{2}\,\lambda^2\int_\Lambda C^{(\le N)\,3}_{\boldsymbol{\xi\eta}}\,d\boldsymbol{\eta}\stackrel{def}{=}-6\,\lambda\,C^{(\le N)}_{\boldsymbol{\xi\xi}}+\delta\mu_N \tag{3.4}$$

we can simply consider graphs which *do not contain any mass graph element and in which there are no subgraphs like the first in Fig.2* while the subgraphs like the second in Fig.2 do not contribute a factor $\int C^{(\le N)}_{\boldsymbol{\alpha\xi}}C^{(\le N)\,3}_{\boldsymbol{\xi\eta}}C^{(\le N)}_{\boldsymbol{\eta\beta}}\,d\boldsymbol{\eta}$ but a *renormalized* factor $\int C^{(\le N)}_{\boldsymbol{\alpha\xi}}C^{(\le N)\,3}_{\boldsymbol{\xi\eta}}\big(C^{(\le N)}_{\boldsymbol{\eta\beta}}-C^{(\le N)}_{\boldsymbol{\xi\beta}}\big)\,d\boldsymbol{\eta}$. If $d=2$ we only need to define μ_N as the first term in the *r.h.s.* of (3.4) and we can leave the subgraphs like the second in Fig.2 as they are (without any renormalization).

Graphs without external lines are called *vacuum graphs* and there are a few such graphs which are divergent. Namely, if $d=3$, they are the first three drawn in Fig.2'; furthermore if μ_N is set to the above nonzero value a new vacuum graph, the fourth in Fig.2', can be formed. Such graphs



(2')

contribute to the graph value, respectively, the addends in the sum

$$-3\,\lambda\,C^{(\le N)\,2}_{\boldsymbol{\xi}_1,\boldsymbol{\xi}_1}+\frac{4!}{2}\,\lambda^2\int C^{(\le N)\,4}_{\boldsymbol{\xi}_1\boldsymbol{\xi}_2}\,d\boldsymbol{\xi}_2-\frac{2^3\cdot 3!^3}{3!}\lambda^3\int C^{(\le N)\,2}_{\boldsymbol{\xi}_1\boldsymbol{\xi}_2}C^{(\le N)\,2}_{\boldsymbol{\xi}_2\boldsymbol{\xi}_3}C^{(\le N)\,2}_{\boldsymbol{\xi}_3\boldsymbol{\xi}_1}\,d\boldsymbol{\xi}_2\,d\boldsymbol{\xi}_3-\mu_N C^{(\le N)}_{\boldsymbol{\xi}_1\boldsymbol{\xi}_1} \tag{3.5}$$

and diverge, respectively, as $\gamma^{2N},\gamma^N,N,\gamma^{2N}$ if $d=3$ while, if $d=2$, only the first and the last diverge, like N^2.

Therefore if we fix ν_N as minus the quantity in (3.4) we can disregard graphs like those in Fig.2'; if $d=2$ ν_N can be defined to be the sum of the first and last terms in (3.5).

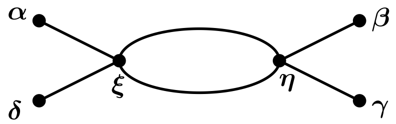
The formal series in λ and f thus obtained is called the *renormalized series* for the field φ^4 in dimension $d=2$ or, respectively, $d=3$. Note that with the given definitions and choices of μ_N,ν_N the *only* graphs G that need to be considered to construct the expansion in λ and f are formed by the first and last graph elements in Fig.1, paying attention that the grapfs in Fig.2' do not contribute and, if $d=3$, the graphs with subgraphs like the second in Fig.2 have to be computed with the modification described.

In the next section it will be shown that the above are the only sources of divergences as $N\to\infty$ and therefore the problem of studying (1.1) is solved at the level of formal power series by the *subtraction* in (3.4). This also shows that giving a meaning to the series thus obtained is likely to be much easier if $d=2$ than if $d=3$.

The coefficients of order k of the expansion in λ of $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$ can be ordered by the number $2n$ of vertices representing external fields: and have the form $\int S^{(k)}_{2n}(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{2n})\prod_{i=1}^{2n}(f_{\boldsymbol{\xi}_i}d\boldsymbol{\xi}_i)$: the kernels $S^{(k)}_{2n}$ are the Schwinger functions of order $2n$, see Sect.1.

Remark: if $d=4$ the regularization at cut-off N in (1.2) is not sufficient as in the subtraction procedure smoothness of the first derivatives of the field $\varphi^{(\le N)}$ is necessary, while the regularization (1.2) does not even imply (2.1), *i.e.* not even Hölder continuity. A higher regularization (*i.e.* using a χ_N like the square of the χ_N in (1.3)). Furthermore the subtractions discussed in the case $d=3$

are not sufficient to generate a formal power series and many more subtractions are needed: for instance graphs with a subgraph like



(3)

would give a contribution to the graph value which is a factor $\lambda^2\ell_N \stackrel{def}{=} \frac{2\cdot 6^2}{2!}\,\lambda^2\int_\Lambda C^{(\le N)\,2}_{\boldsymbol{\xi}\boldsymbol{\eta}}\,d\boldsymbol{\eta}$, also divergent as $N\to\infty$ proportionally to N. Although this divergence could be canceled by changing λ into $\lambda_N=\lambda+\lambda^2\ell_N$ the previously discussed cancellations would be affected and a change in the value of μ_N would become necessary; furthermore the subtraction in (3.4) will not be sufficient to make finite the graphs, not even to second order in λ, unless a new term $-\alpha_N\int(\partial_{\boldsymbol{\xi}}\varphi^{(\le N)}_{\boldsymbol{\xi}})^2\,d\boldsymbol{\xi}$ with $\alpha_N=\frac12\lambda^2\int\partial_{\boldsymbol{\eta}}C^{(\le N)\,3}_{\boldsymbol{\xi}\boldsymbol{\eta}}(\boldsymbol{\xi}-\boldsymbol{\eta})^2$ is added in the exponential in (1.1). But all this will not be enough and still new divergences, proportional to λ^3, will appear.

And so on indefinitely: the consequence being that it will be necessary to define $\lambda_N,\mu_N,\alpha_N,\nu_N$ as formal power series in λ (with coefficients diverging as $N\to\infty$) in order to obtain a formal power series in λ for (1.1) in which all coefficients have a finite limit as $N\to\infty$. Thus the interpretation of the formal renormalized series in the case $d=4$ is substantially different and naturally harder than the cases $d=2,3$. Beyond formal perturbation expansions the case $d=$ *is still an open problem*: the most widespread conjecture is that the series cannot be given a meaning other than setting to 0 all coefficients of λ^j, $j>0$. In other words, the conjecture claims, there should be no nontrivial solution to the ultraviolet problem for scalar φ^4 fields in $d=4$. But this is far from being proved, even at a heuristic level. The situation is simpler if $d\ge5$: in such cases it is impossible to find formal power series in λ for $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$, even allowing $\lambda_N,\mu_N,\alpha_N,\nu_N$ to be formal power series in λ with divergent coefficients.

The distinctions between the cases $d=2,3,4,>4$ explain the terminology given to the φ^4-scalar field theories calling them *superrinormalizable* if $d=2,3$, renormalizable if $d=4$ and *non renormalizable* if $d>4$. Since the (divergent) coefficients in the formal power series defining $\lambda_N,\mu_N,\alpha_N,\nu_N$ are called *counterterms* the φ^4-scalar fields require finitely many counterterms (see (3.4)) in the superrenormalizable cases and infinitely many in the renormalizable case. The nonrenormalizable cases ($d>4$) cannot be treated in a way analogous to the renormalizable ones.

References: [Ga85],[Fr82].

4. Finiteness of the renormalized series, $d=2,3$: "power counting".

Checking that the renormalized series is well defined to all orders is a simple *dimensional estimate* characteristic of many multiscale arguments that in Physics have become familiar with the name of "renormalization group arguments".

Consider a graph G with $n+r$ vertices built over n graph elements with vertices $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n$ each with 4 half lines and r graph elements with vertices $\boldsymbol{\xi}_{n+1},\ldots,\boldsymbol{\xi}_{n+r}$ representing the external fields: as remarked in Sec.3 these are the only graphs to be considered to form the renormalized series.

Develop *each* propagator into a sum of propagators as in (2.2). The graph G value will, as a consequence, be represented as a sum of values of new graphs obtained from G by adding *scale labels* on its lines and the value of the graph will be computed as a product of factors in which a line joining $\boldsymbol{\xi}\boldsymbol{\eta}$ and bearing a scale label h will contribute with $C^{(h)}_{\boldsymbol{\xi}\boldsymbol{\eta}}$ replacing $C^{(\le N)}_{\boldsymbol{\xi}\boldsymbol{\eta}}$. To avoid proliferation of symbols we shall call the graphs obtained in this way, *i.e.* with the scale labels attached to each line, still G: no confusion should arise as *we shall, henceforth, only consider graphs G with each line carrying also a scale label.*

The scale labels added on the lines of the graph G allow us to organize the vertices of G into *clusters*: a cluster of *scale h* consists in a maximal set of vertices (of the graph elements in the graph) connected by lines of scale $h' \geq h$ among which one at least has scale h.

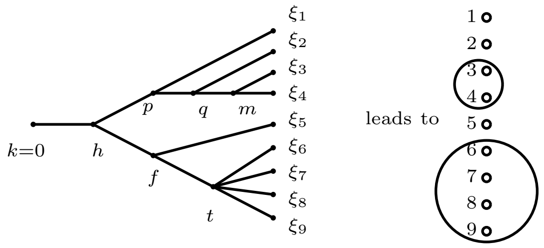
It is convenient to consider the vertices of the graph elements as " trivial" clusters of highest scale: conventionally call them clusters of scale $N+1$.

The clusters can be of "first generation" if they contain only trivial clusters, of "second generation" if they contain only clusters which are trivial or of the first generation, and so on.

Imagine to enclose in a box the vertices of graph elements inside a cluster of the first generation and then into a larger box the vertices of the clusters of the second generation and so on: the set of boxes ordered by inclusion can then be represented by a *rooted tree* graph whose nodes correspond to the clusters and whose "top points" are nodes representing the trivial clusters (*i.e.* the vertices of the graph).

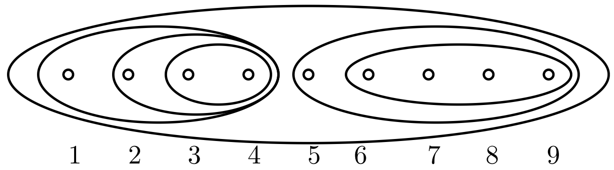
If the maximum number of nodes that have to be crossed to reach a top point of the tree starting from a node v is n_v (v included and the top nodes included) then the node v represents a cluster of the n_v–th generation. The first node before the root is a cluster containing all vertices of G and the root of the tree will not be considered a node and it can conventionally bear the scale label 0: it represents symbolically the value of the graph.

For instance in Fig.4 a tree θ is drawn: its nodes correspond to clusters whose scale is indicated next to them; in the second part of the drawing the trivial clusters as well as the clusters of the first generation are enclosed into boxes.



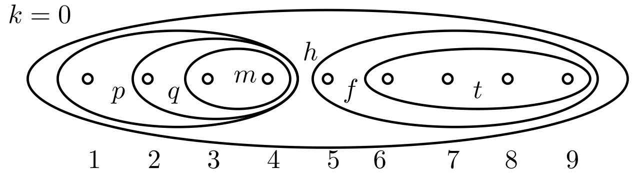
(4)

Then consider the next generation clusters, *i.e.* the clusters which only contain clusters of the first generation or trivial ones, and draw boxes enclosing all the graph vertices that can be reached from each of them by descending the tree, *etc.* Fig.5 represents all boxes (of any generation) correspondinf to the nodes of the tree in Fig.4



(5)

The representations of the clusters of a graph G by a tree or by hierarchically ordered boxes (see Fig.4 and Fig.5) are completely equivalent provided inside each box not representing a top point of the tree the scale h_v of the corresponding cluster v is marked. For instance in the case of Fig.5 one gets



(6)

By construction if two top points $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are inside the same box b_v of scale h_v but not in inner boxes then there is a path of graph lines joining $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ all of which have scales $\geq h_v$ and one at least has scale h_v.

Given a graph G fix one of its points $\boldsymbol{\xi}_1$ (say) and integrate the absolute value of the graph over the positions of the remaining points. The exponential decay of the propagators implies that if a point $\boldsymbol{\eta}$ is linked to a point $\boldsymbol{\eta}'$ by a line of scale h the integration over the position of $\boldsymbol{\eta}'$ is essentially constrained to extend only over a distance $\gamma^{-h}m^{-1}$. Furthermore the maximum size of the propagator associated with a line of scale h is bounded proportionally to $\gamma^{(d-2)h}$. Therefore, recalling that $|f_{\boldsymbol{\xi}}|$ is supposed bounded by 1, the mentioned integral can be immediately bounded by

$$\frac{\lambda^n}{n!r!}\,C^{n+r}\,I\stackrel{def}{=}\frac{\lambda^n\,C^{n+r}}{n!r!}\prod_\ell\gamma^{\frac{d-2}{2}\,h_\ell}\prod_v\gamma^{-d\,h_v\,(s_v-1)}\tag{4.1}$$

where, C being a suitable constant, the first product is over the half lines ℓ composing the graph lines and the second is over the tree nodes (*i.e.* over the clusters of the graph G), s_v is the number of subclusters contained in the cluster v but not in inner clusters; and in (4.1) the scale of a half line ℓ is h_ℓ if ℓ is paired with another half line to form a line ℓ (in the graph G) of scale label h_ℓ.

Denoting by v' the cluster immediately containing v in G, by n_v^{inner} the number of half lines in the cluster v, by n_v, r_v the numbers of graph elements of the first type or of the fourth type in Fig.1 with vertices in the cluster v, and denoting by n_v^e the number of lines which are not in the cluster v but have one extreme on a vertex in v ("lines external to v"), the identities ($k=0$)

$$\begin{aligned}&\sum_{v>root}(h_v-k)(s_v-1)\equiv\sum_{v>root}(h_v-h_{v'})(n_v+r_v-1),\\&\sum_{v>root}(h_v-k)\,n_v^{inner}\equiv\sum_{v>root}(h_v-h_{v'})\,\widetilde n_v^{inner},\qquad\text{with}\\&\widetilde n_v^{inner}\stackrel{def}{=}4n_v+r_v-n_v^e,\end{aligned}\tag{4.2}$$

hold, so that the estimate (4.1) can be elaborated into

$$I\le\prod_{v>r}\gamma^{-\rho_v\,(h_v-h_{v'})},\qquad\rho_v\stackrel{def}{=}-d+(4-d)n_v+r_v\frac{d+2}{2}+\frac{d-2}{2}n_v^e\tag{4.3}$$

where $h_{v'}=k=0$ if v is the first nontrivial node (*i.e.* $v'=root$), and an estimate of the integral of the absolute value of the graphs G with given tree structure but different scale labels is proportional to $\sum_{\{h_v\}}I<\infty$ if (and only if) $\rho_v>0,\forall v$.

But there may be clusters v with only two external lines $n_v^e=2$ and two graph vertices inside: for which $\rho_v=0$. However this can happen *only if* $d=3$ and in *only one case*: namely if the graph G contains a subgraph of the second type in Fig.2 and the three intermediate lines form a cluster v of scale h_v while the other two lines are external to it: hence on scale $h'>h$. In this case one has to remember that the subtraction in Sec.3 has led to a modification of the contribution of such a subgraph to the value of the graph (integrated over the position labels of the vertices). As discussed in Sec.3 the change amounts at replacing the propagator $C^{(h')}_{\boldsymbol{\eta},\boldsymbol{\beta}}$ by $C^{(h')}_{\boldsymbol{\eta},\boldsymbol{\beta}}-C^{(h')}_{\boldsymbol{\xi},\boldsymbol{\beta}}$.

This improves, in (4.3), the estimate of the contribution of the line joining $\boldsymbol{\eta}$ to $\boldsymbol{\beta}$ from being proportional to $\int C^{(\le h_v)\,3}_{\boldsymbol{\xi\eta}}C^{(\le h')}_{\boldsymbol{\eta\beta}}d\boldsymbol{\eta}$ to being proportional to $\int C^{(\le h_v)\,3}_{\boldsymbol{\xi\eta}}(C^{(\le h')}_{\boldsymbol{\eta\beta}}-C^{(\le h')}_{\boldsymbol{\xi\beta}})d\boldsymbol{\eta}$; and this changes the contribution of the line $\boldsymbol{\eta\beta}$ from $\gamma^{(d-2)h'}$ to $\int e^{-m\gamma^{h_v}|\boldsymbol{\xi}-\boldsymbol{\eta}|}(\gamma^{h'}|\boldsymbol{\xi}-\boldsymbol{\eta}|)^{\frac12}d\boldsymbol{\eta}$ because $C^{(h')}$ is regular on scale $\gamma^{-h'}m^{-1}$, see (2.5) with $\varepsilon=\frac12$.

Since $\boldsymbol{\xi},\boldsymbol{\eta}$ are in a cluster of higher scale h_v this means that the estimate is improved by $\gamma^{-\frac12(h_v-h')}$. In terms of the final estimate this means that ρ_v in (4.3) can be improved to $\overline\rho_v=\rho_v+\frac12$ for the clusters for which $\rho_v=0$. Hence the integrated value of the graph G (after taking also into account the integration over the initially selected vertex $\boldsymbol{\xi}_1$, trivially giving a further factor $|\Lambda|$ by translation invariance), and summed over the possible scale labels is bounded proportionally to $|\Lambda|\sum_{\{h_v\}}I<\infty$ once the estimate of I is improved as described.

Note that the graphs contributing to the perturbation series for $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$ to order λ^n are finitely many because the number r of external vertices is $r\le 2n+2$ (since graphs must be connected). Hence the perturbation series is finite to all orders in λ.

The above is the renormalizability proof of the scalar φ^4-fields in dimension $d=2,3$. The theory is renormalizable even if $d=4$ as mentioned in the remark at the end of Sec.3. The analysis would be very similar to the above: it is just a little more involved power counting argument.

References: [He66], Sect. 8 and 16 in [Ga85].

5. Asymptotic freedom ($d=2,3$). Heuristic analysis.

Finiteness to all orders of the perturbation expansions is by no means sufficient to prove the existence of the ultraviolet limit for $Z_N(\Lambda,f)$ or for $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$: and *a priori* it might not even be necessary. For this purpose the first step is to check uniform (upper and lower) boundedness of $Z_N(\Lambda,f)$ as $N\to\infty$.

The reason behind the validity of a bound $e^{|\Lambda|E_-(\lambda,f)}\le Z_N(\Lambda,f)\le e^{|\Lambda|E_+(\lambda,f)}$ with $E_\pm(\lambda,f)$ cut-off independent has been made very clear after the introduction of the renormalization group methods in field theory. The approach studies the integral $Z_N(\Lambda,f)$, recursively, decomposing the field $\varphi^{(\le N)}_{\boldsymbol{\xi}}$ into its regular components $z^{(h)}_{\boldsymbol{\xi}}$, see (2.2), and integrating first over $z^{(N)}$, then over $z^{(N-1)}$ and so on.

The idea emerges naturally if the potential V_N in (1.1), (1.4) is written in terms of the "normalized" variables $X^{(N)}_{\boldsymbol{\xi}}\overset{def}{=}\gamma^{-\frac{d-2}{2}N}\varphi^{(\le N)}_{\boldsymbol{\xi}}$, see (2.1); here if $d=2$ the factor $\gamma^{\frac{d-2}{2}N}$ is interpreted as $N^{\frac12}$.

The key remark is that *as far as the integration over the small scale component $z^{(N)}$ is concerned* the field $X^{(N)}_{\boldsymbol{\xi}}$ is a sum of two fields of size of order 1 (statistically), $X^{(N)}_{\boldsymbol{\xi}}\equiv z^{(N)}_{\gamma^N\boldsymbol{\xi}}+\gamma^{-\frac{d-2}{2}}X^{(N-1)}_{\boldsymbol{\xi}}$ (if $d=2$ this becomes $X^{(N)}_{\boldsymbol{\xi}}\equiv\frac{1}{N^{\frac12}}z^{(N)}_{\gamma^N\boldsymbol{\xi}}+\frac{(N-1)^{\frac12}}{N^{\frac12}}X^{(N-1)}_{\boldsymbol{\xi}}$) and it can be considered to be smooth on scale $m^{-1}\gamma^{-N}$ (also statistically). Hence *approximately constant* and of size of order $O(1)$ on the small cubes Δ of volume $\gamma^{-dN}m^{-d}$ of the pavement $\mathcal{Q}_N$ introduced before (2.2); at the same time it can be considered to take (statistically) *independent values* on different cubes of $\mathcal{Q}_N$. This is suggested by the inequalities (2.3),(2.4),(2.5).

Therefore it is natural to decompose the potential V_N, see (1.5), as a sum over the small cubes Δ of volume $\gamma^{-dN}m^{-d}$ of the pavement $\mathcal{Q}_N$ as (see (3.4) for the definition of μ_N,ν_N), taking henceforth $m=1$,

$$V_N(z^{(N)})\overset{def}{=}-\sum_{\Delta\in\mathcal{Q}_N}\gamma^{-Nd}\int_\Delta\Big(\lambda\gamma^{2(d-2)N}X^{(N)\,4}_{\boldsymbol{\xi}}+\mu_N\gamma^{(d-2)N}X^{(N)\,2}_{\boldsymbol{\xi}}+\nu_N+f_{\boldsymbol{\xi}}\gamma^{\frac{d-2}{2}N}X^{(N)}_{\boldsymbol{\xi}}\Big)\frac{d\boldsymbol{\xi}}{|\Delta|}\tag{5.1}$$

where $\gamma^{(d-2)N}$ is interpreted as N if $d=2$. Hence if $d=3$ it is

$$V_N(z^{(N)})\overset{def}{=}-\sum_{\Delta\in\mathcal{Q}_N}\gamma^{-N}\int_\Delta\Big(\lambda X^{(N)\,4}_{\boldsymbol{\xi}}+\overline{\mu}_NX^{(N)\,2}_{\boldsymbol{\xi}}+\overline{\nu}_N+f_{\boldsymbol{\xi}}\gamma^{-\frac32 N}X^{(N)}_{\boldsymbol{\xi}}\Big)\frac{d\boldsymbol{\xi}}{|\Delta|}\tag{5.2}$$

where $\overline{\mu}_N\overset{def}{=}(-6\lambda c_N+\lambda^2N\gamma^{-N}c'_N)$, $\overline{\nu}_N\overset{def}{=}3\lambda c_N^2+\lambda^2\gamma^{-N}b_N+\lambda^3N\gamma^{-2N}b'_N$, and c_N,c'_N,b_N,b'_N, computable from (3.5),(3.4), admit a limit as $N\to\infty$. While if $d=2$ it is

$$V_N(z^{(N)})\overset{def}{=}-\sum_{\Delta\in\mathcal{Q}_N}N^2\gamma^{-2N}\int_\Delta\Big(\lambda X^{(N)\,4}_{\boldsymbol{\xi}}+\overline{\mu}_NX^{(N)\,2}_{\boldsymbol{\xi}}+\overline{\nu}_N+f_{\boldsymbol{\xi}}N^{-\frac32}X^{(N)}_{\boldsymbol{\xi}}\Big)\frac{d\boldsymbol{\xi}}{|\Delta|}\tag{5.3}$$

where $\overline{\mu}_N\overset{def}{=}-6\lambda c_N$ and $\overline{\nu}_N=3\lambda c_N^2$ and c_N, computable from (3.3), admits a limit as $N\to\infty$.

The fields $z^{(N)}$ and $X^{(N-1)}$ can be considered constant over boxes $\Delta\in\mathcal{Q}_N$: $z^{(N)}_{\boldsymbol\xi}=s_\Delta, X^{(N-1)}_{\boldsymbol\xi}=x_\Delta$ for $\boldsymbol\xi\in\Delta$ and the s_Δ can be considered statistically independent on the scale of the lattice $\mathcal{Q}_N$.

Therefore (5.2),(5.3) show that integration over $z^{(N)}$ in the integral defining $Z_N(\Lambda,f)$ is not too different from the computation of a partition function of a lattice continuous spin model in which the "spins" are s_Δ and, most important, interact extremely weakly if N is large. In fact the coupling constants are of order of a power of $|X^{(N-1)}|$ times $O(\gamma^{-N})$ if $d=3$ ($O(N^2\gamma^{-2N})$ if $d=2$), or of order $O(\gamma^{-\frac{d+2}{2}N}\max|f_{\boldsymbol\xi}|)$, *no matter* how large λ and f.

This says that the smallest scale fields are *extremely weakly coupled*. The fields $X^{(N-1)}$ can be regarded as external fields of size that will be called B_{N-1}, of order 1 or even allowed to grow with a power of N, see (2.1). Their presence in V_N does not affect the size of the couplings, as far as the analysis of the integral over $z^{(N)}$ is concerned, because the couplings remain *exponentially small* in N, see (5.2),(5.3), being at worst multiplied by a *power* of B_{N-1}, *i.e.* changed by a factor which is a power of N.

The smallness of the coupling at small scale is a property called *asymptotic freedom*. Once fields and coordinates are "correctly scaled" the real size of the coupling becomes manifest, *i.e.* it is extremely small and the addends in V_N proportional to the "counterterms" μ_N,ν_N, which looked divergent when the fields were not properly scaled, are in fact of the same order or much smaller than the main φ^4-term.

Therefore the integration over $z^{(N)}$ can be, heuristically, performed by techniques well established in statistical mechanics (*i.e.* by straightforward perturbation expansions): *at least if the field $X^{(\le N-1)}_{\boldsymbol\xi}$ is smooth and bounded*, as prescribed by (2.1), with $B=B_{N-1}$ growing as a power of N. In this case, denoting symbolically the integration over $z^{(N)}$ by P or by $\langle\ldots\rangle$, *it can be expected* that it should give

$$\int e^{V_N}dP(z^{(N)})\equiv e^{V_{j;N-1}+\overline{\mathcal{R}}(j,N)|\Lambda|}\tag{5.4}$$

where $V_{j;N-1}$ is the Taylor expansion of $\log\int e^{V_N}dP(z^{(N)})$ in powers of λ (hence essentially in the very small parameter $\lambda\gamma^{-(4-d)N}$) truncated at order j, *i.e.*

$$\begin{aligned}V_{1;N-1}=&[\langle V_N\rangle]^{\le1},\qquad V_{2;N-1}=\Big[\langle V_N\rangle+\frac{(\langle V_N^2\rangle-\langle V_N\rangle^2)}{2!}\Big]^{\le2},\\ V_{3;N-1}=&\Big[\langle V_N\rangle+\frac{(\langle V_N^2\rangle-\langle V_N\rangle^2)}{2!}+\frac{\big(\langle V_N(\langle V_N^2\rangle-\langle V_N\rangle^2)\rangle-\langle V_N\rangle(\langle V_N^2\rangle-\langle V_N\rangle^2)\big)}{3!}\Big]^{\le3},\ \ldots\end{aligned}\tag{5.5}$$

where $[\cdot]^{\le j}$ denotes truncation to order j in λ, and $\overline{\mathcal{R}}(j,N)$ is a remainder (depending on $\varphi^{(\le N-1)}_{\boldsymbol\xi}$) which can be *expected* to be estimated by

$$|\overline{\mathcal{R}}(j,N)|\le\mathcal{R}(j,N)\stackrel{def}{=}C_jB_N^{4j}(\lambda\,N^2\,\gamma^{-(4-d)N})^{j+1}\gamma^{dN},\qquad\text{for } d=2,3\tag{5.6}$$

for suitable constants C_j, *i.e.* a remainder estimated by the $(j+1)$-th power of the coupling times the number of boxes of scale N in Λ. The relations (5.4),(5.5),(5.6) result from a naive Taylor expansion (in λ of the $\log\int e^{V_N}dP(z^{(N)})$, taking into account that, in V_N as a function of $z^{(N)}$, the $z^{(N)}$'s appear multiplied by quantities at most of size $\le\lambda\gamma^{4-d}N^2B_N^3$, by (5.2),(5.3) if $|X^{(N-1)}|\le B_{N-1}$). In a statistical mechanics model for a lattice spin system such a calculation of Z_N would lead to a mean field equation of state once the remainder was neglected.

The peculiarity of field theory is that a relation like (5.4),(5.6) has to be applied again to $V_{j;N-1}$ to perform the integration over $z^{(N-1)}$ and define $V_{j;N-2}$ and, then, again to $V_{j;N-2}$ ... Therefore it will be essential to perform the integral in (5.4) to an order (in λ) high enough so that the bound

$\mathcal{R}(j,N)$ can be summed over N: this requires (see (5.6)) an explict calculation of (5.5) pushed at least to order $j=1$ if $d=2$ or to order $j=3$ if $d=3$ and a check that the resulting $V_{j;N-1}$ can still be interpreted as low coupling spin model so that (5.4) can be iterated with $N-1$ replacing N and then with $N-2$ replacing $N-1$,....

The first necessary check towards a proof of the discussed heuristic "expectations" is that, defining recursively $V_{j;h}$ from $V_{j,h+1}$ for $h=N-1,\ldots,1,0$ by (5.5) with V_N replaced by $V_{j;h+1}$ and $V_{j;N-1}$ replaced by $V_{j;h}$, the couplings between the variables $z^{(h)}$ do not become 'worse' than those discussed in the case $h=N$. Furthermore the field $\varphi^{(\le N-1)}_{\boldsymbol{\xi}}$ has a high probability of satisfying (2.1), *but fluctuations are possible*: hence the $\mathcal{R}$-estimate has to be combined with another one dealing with the large fluctuations of $X^{(N-1)}_{\boldsymbol{\xi}}$ which has to be shown to be "not worse"..

References: [Ga78],[Ga85],[BG95].

6. Effective potentials and their scale (in)dependence.

To analyze the first problem mentioned at the end of Sec.5, *define* $V_{j;h}$ by (5.5) with V_N replaced by $V_{j;h+1}$ for $h=N-1,N-2,\ldots,0$. The quantities $V_{j;h}$, which are called *effective potentials* on scale h (and order j), turn out to be in a natural sense *scale independent*: this is a consequence of renormalizability, realized by Wilson as a much more general property which can be checked, in the very special cases considered here with $d=2,3$, at fixed j by induction, and in the superrinormalizable models considered here it requires only an elementary computation of a few Gaussian integrals as the case $j=3$ (or *even* $j=1$ if $d=2$) is already sufficient for our purposes.

It can, also, be (more easily) proved for general j by a dimensional argument parallel to the one presented in Sec.4 to check finiteness of the renormalized series. The derivation is elementary but *it should be stressed that, again, it is possible only because of the special choice of the counterterms* μ_N,ν_N. If $d=3$ the boundedness and smoothness of the fields $\varphi^{(\le h)}$ and $z^{(h)}$ expressed by the second of (2.1) and of (2.5) is essential; while if $d=2$ the smoothness is not necessary.

The structure of $V_{j;h}$ is conveniently expressed in terms of the fields $X^{(h)}_{\boldsymbol{\xi}}$, as a sum of three terms $V^{(rel)}_h$ (standing for "relevant" part), $V^{(irr)}_h$ (standing for "irrelevant" part) and a "field independent" part $E(j,h)|\Lambda|$.

The relevant part in $d=2$ is simply of the form (5.3) with h replacing N: call it $V^{(rel,1)}_h$. If $d=3$ it is given by (5.2) with h replacing N plus, for $h<N$, a second "nonlocal" term $V^{(rel,2)}_h \stackrel{def}{=} \frac{4^2\,3!}{2!\,2!}\,\lambda^2\int (C^{(\le h)\,3}_{\boldsymbol{\eta}\boldsymbol{\eta}'}-C^{(\le N)\,3}_{\boldsymbol{\eta}\boldsymbol{\eta}'})(\varphi^{(\le h)}_{\boldsymbol{\eta}}-\varphi^{(\le h)}_{\boldsymbol{\eta}'})^2 d\boldsymbol{\eta}d\boldsymbol{\eta}'$ which is conveniently expressed in terms of a "non local" field $Y^{(h)}_{\boldsymbol{\eta}\boldsymbol{\eta}'}\stackrel{def}{=}\frac{\varphi^{(\le h)}_{\boldsymbol{\eta}}-\varphi^{(\le h)}_{\boldsymbol{\eta}'}}{(\gamma^h|\boldsymbol{\eta}-\boldsymbol{\eta}'|)^{\frac14}}$ as $V^{(rel)}_h=V^{(rel,1)}_h+V^{(rel,2)}_h$ with

$$V^{(rel,2)}_h\stackrel{def}{=}-\lambda^2\gamma^{-2h}\sum_{\Delta,\Delta'\in\mathcal{Q}_h}\int_{\Delta\times\Delta'} Y^{(h)\,2}_{\boldsymbol{\eta}\boldsymbol{\eta}'}\,A^{(h)}_{\boldsymbol{\eta}\boldsymbol{\eta}'}\,e^{-c'\gamma^h|\boldsymbol{\eta}-\boldsymbol{\eta}'|}\,\frac{d\boldsymbol{\eta}d\boldsymbol{\eta}'}{|\Delta|\,|\Delta'|}\tag{6.1}$$

where $0<a\le\Big(\frac{A^{(h)}_{\boldsymbol{\eta}\boldsymbol{\eta}'}}{(\gamma^h|\boldsymbol{\eta}-\boldsymbol{\eta}'|)^{3-\frac12}}\Big)_N<a'$, with $a,a',c'>0$ and the subscript N means that the expression in parenthesis "saturates at scale N", *i.e.* it becomes $\gamma^{(3-\frac12)(h-N)}$ as $|\boldsymbol{\eta}-\boldsymbol{\eta}'|\to0$.

The expression (6.1) is not the full part of the potential $V_{j;h}$ which is of second order in the fields: there are several other contributions which are collected below as "irrelevant".

It should be stressed that *irrelevant* is a traditional technical term: by no means it should suggest "neglegibility". On the contrary it could be maintained that the whole purpose of the theory is to study the irrelevant terms. A better word to designate the irrelevant part of the potential would be *driven part* as its behavior is "controlled" by the relevant part. The Schwinger functions are simply related to the irrelevant terms.

The irrelevant part of the effective potential can be expressed as a finite sum of integrals of monomials in the fields $X^{(h)}_{\boldsymbol{\xi}}$ if $d=2$, or in the fields $X^{(h)}_{\boldsymbol{\xi}}$ *and* $Y^{(h)}_{\boldsymbol{\eta}\boldsymbol{\eta}'}$ if $d=3$, which can be written as $V_{j;h}^{(irr)}$ given by

$$\int (\prod_{k=1}^{p} X_{\boldsymbol{\xi}_k}^{(h)\, n_k} \prod_{k'=1}^{q} Y_{\boldsymbol{\eta}_{k'}\boldsymbol{\eta}'_{k'}}^{(h)\, n'_{k'}})\, e^{-\gamma^h c' d(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\eta}'_q)} \lambda^n \gamma^{-ht} W(\boldsymbol{\xi}_1 \ldots, \boldsymbol{\eta}'_q) \prod_{k=1}^{p} \frac{d\boldsymbol{\xi}_k}{|\Delta_k|} \prod_{k'=1}^{q} \frac{d\boldsymbol{\eta}_{k'} d\boldsymbol{\eta}'_{k'}}{|\Delta^1_{k'}|\,|\Delta^2_{k'}|} \tag{6.2}$$

with the integral extended to products $\Delta_1 \times \ldots \Delta_p \times \ldots \times (\Delta^1_q \times \Delta^2_q)$ of boxes $\Delta \in \mathcal{Q}_h$, and $d(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\eta}'_q)$ is the length of the shortest tree graph that connects all the $p + 2q > 0$ points, the exponents n, t are ≥ 2 and t is ≥ 3 if $q > 0$; the kernel W depends on all coordinates $\boldsymbol{\xi}_1 \ldots, \boldsymbol{\eta}'_q$ and it is bounded above by $C_j \prod_{k'=1}^{q} A_{\boldsymbol{\eta}_{k'}\boldsymbol{\eta}'_{k'}}$ for some C_j; the sums $\sum n_k + \sum n'_{k'}$ cannot exceed $4j$. The test functions f do not appear in (6.2) because by assumption they are bounded by 1: but W depends on the f's as well.

The field independent part is simply the value of $\log Z_N(\Lambda, f)$ computed by the perturbation analysis in Sec.3 *up to order j in λ but using as propagator* $(C^{(\leq N)} - C^{(\leq h)})$: thus $E(j, h)$ is a constant depending on N but uniformly bounded as $N \to \infty$ (because of the renormalizability proved in Sec.3).

If $d = 2$ there is no need to introduce the nonlocal fields $Y^{(h)}$ and in (6.2) one can simply take $q = 0$, and the relevant part also can be expressed by omitting the term $V_h^{(rel,2)}$ in (6.1): unlike the $d = 3$ case the estimate on the kernels W by an N-independent C_j holds uniformly in h without having to introduce Y. For $d = 2$ it will therefore be supposed that $V_h^{(rel,2)} \equiv 0$ in (6.1) and $q = 0$ in (6.2).

It is not necessary to have more informations on the structure of $V_{j;h}$ even though one can find simple graphical rules, closely related to the ones in Sec.3, to construct the coefficients W in full detail. The W depend, of course, on h but the uniformity of the bound on W is the only relevant property and in this sense the effective potentials are said to be (almost) "scale independent".

The above bounds on the irrelevant part can be checked by an elementary direct computation if $j \leq 3$: in spite of its "elemetary character" the uniformity in $h \leq N$ is a result ultimately playing an essential role in the theory together with the dominance of the relevant part over the irrelevant one which, once the fields are properly scaled, is "much smaller" (by a factor of order γ^{-h}, see (6.2)).

Remarks: (1) Checking scale independence for $j = 1$ is just checking that $\int P(dz^{(h)}) V_{1;h} = V_{1;h-1}$. Note that $V_{1;h} \stackrel{def}{=} \int_\Lambda \lambda(\varphi_{\boldsymbol{\xi}}^{(\leq h)\,4} - 6C_{\mathbf{00}}^{(\leq h)} \varphi_{\boldsymbol{\xi}}^{(\leq h)\,2} + 3C_{\mathbf{00}}^{(\leq h)\,2}) d\boldsymbol{\xi}$; hence calling $: \varphi_{\boldsymbol{\xi}}^{(\leq h)\,4} :$ the polynomial in the integral (*Wick's monomial of order* 4) this is an elementary Gaussian integral ("martingale property of Wick monomials"). Note the essential role of the counterterms. For $j > 1$ the computation is similar but it involves higher order polynomials (up to $4j$) and the distinction between $d = 2$ and $d = 3$ becomes important.

(2) $V_{j;0}$ contains only the field independent part $E(j, 0)|\Lambda|$ which is just a number (as there are no fields of scale 0): by the above definitions it is *identical* to the perturbative expansion truncated to j–th order in λ of $\log Z_N(\Lambda, f)$, well defined as discussed in Sec.3,4.

7. Nonperturbative renormalization: small fields

Having introduced the notion of effective potential $V_{j;h}$, of order j and scale h, satisfying the bounds (described after (6.2)) on the kernels W representing it, the problem is to estimate the remainder in (5.4) and find its relation with the value (5.6) given by the heuristic Taylor expansion. Assume $\lambda < 1$ to avoid distinguishing this case from that with $\lambda \geq 1$ which would lead to very similar estimates but to different λ-dependence on some constants.

Define $\chi_B(z^{(h)}) = 1$ if $||z^{(h)}||_\Delta \leq Bh^2$ for all $\Delta \in \mathcal{Q}_h$, see (2.3), and 0 otherwise; then the following lemma holds:

Lemma 1: *Let $||X^{(h)}||_\Delta$ be defined as (2.3) with z replaced by X and suppose $||X^{(h)}||_\Delta \le Bh^4$ for all Δ then, for all $j \ge 1$, it is*

$$\int e^{V_{j;h+1}} \chi_B(z^{(h+1)})\, dP(z^{(h+1)}) \;=\; e^{V_{j;h} \,+\, \mathcal{R}'(j,h+1)\,|\Lambda|} \tag{7.1}$$

with, for suitable constants c_-, c'_-, $|\mathcal{R}'_-(j;h+1)| \le \mathcal{R}_-(j;h+1) \stackrel{def}{=} \mathcal{R}(j;h+1) + c_- e^{-c'_- B^2 (h+1)^2}$ and $\mathcal{R}(j;h+1)$ given by (5.6) with $h+1$ in place of N.

Since $Z_N(\Lambda, f) \ge \int e^{V_N} \prod_{h=1}^N \chi_B(z^{(h)})\, P(dz^{(h)})$ *this immediately gives a lower bound on* $E = \frac{1}{|\Lambda|}\log Z_N(\Lambda, f)$: in fact if $\chi_B(||z^{(h')}||) = 1$ for $h' = 1, \ldots, h$ then $||X^{(h)}||_\Delta \le c\, B{h'}^4$ for some c so that, by recursive application of lemma 1, $Z_N(\Lambda, f) \ge e^{V_{j,0} - \sum_{h=1}^N \mathcal{R}_-(j,h)|\Lambda|}$. By the remark at the end of Sec.6, given j the lower bound on E just described agrees with the perturbation expansion of $E = \frac{1}{|\Lambda|}\log Z_N(\Lambda, f)$ truncated to order j (in λ) up to an error $\sum_{h=1}^N \mathcal{R}_-(j,h)$.

Remark: The problem solved by lemma 1 is called the *small fields problem*. The proof of the lemma is a simple Taylor expansion in $\lambda\gamma^{-h}$ if $d = 3$ or in $\lambda h^2 \gamma^{-2h}$ if $d = 2$ to order j (in λ). The constraint on $z^{(h+1)}$ makes the integrations over $z^{(h+1)}$, necessary to compute $V_{j;h}$ from $V_{j;h+1}$, not Gaussian. But the *tail estimates* (2.4), together with the Markov property of the ditributionof $z^{(h)}$ can be used to estimate the difference with respect to the Gaussian unconstrained integrations of $z^{(h+1)}$: and the result is the addition of the small "tail error" changing $\mathcal{R}$ into $\mathcal{R}_-$. The estimate of the main part of the remainder $\mathcal{R}$ would be obvious if the fields $z^{(h)}$ were independent on boxes of scale γ^{-h}: they are not independent but they are Markovian and the estimate can be done by taking into account the Markov property.

References: [Wi70],[Wi72],[Ga78],[Ga81],[BCGNPOS78],[Ga85].

8. Nonperturbative renormalization: large fields, ultraviolet stability

The small fields estimates are *not sufficient* to obtain ultraviolet stability: to control the cases in which $|X^{(h)}_{\boldsymbol\xi}| > Bh^4$ for some $\boldsymbol\xi$ or some h, or $|Y^{(h)}_{\boldsymbol\xi\boldsymbol\eta}| > Bh^4$ for some $|\boldsymbol\xi - \boldsymbol\eta| < \gamma^{-h}$, a further idea is necessary and it rests on making use of the assumption that $\lambda > 0$ which, in a sense to be determined, should suppress the contribution to the integral defining $Z_N(\Lambda, f)$ coming from very large values of the field. Assume also $\lambda < 1$ for the same reasons advanced in Sec.6.

Consider first $d = 2$. Let $\mathcal{D}_N$ be the "large field region" where $|X^{(N)}_{\boldsymbol\xi}| > BN^4$ and let $V_N(\Lambda/\mathcal{D}_N)$ be the integral defining the potential in (5.3) extended to the region $\Lambda/\mathcal{D}_N$, complement of $\mathcal{D}_N$. This region is typically *very* irregular (and random as X itself is random with distribution P_N).

An upper bound on the integral defining $Z_N(\Lambda, f)$ is obtained by simply replacing e^{V_N} by $e^{V_N(\Lambda/\mathcal{D}_N)}$ because in $\mathcal{D}_N$ the first term in the integrand in (5.3) is $\le -\lambda N^2 \gamma^{2N}(BN^4) < 0$ and it overwhelmingly dominates on the remaining terms whose value is bounded by a similar expression with a smaller power of N. Then if $\mathcal{E}^c \stackrel{def}{=} \Lambda/\mathcal{E}$ denotes the complement in Λ of a set $\mathcal{E} \subset \Lambda$:

Lemma 2: *Let $d = 2$. Define $V_h(\mathcal{D}^c_h)$,to be given by the expression (5.4) with the integrals extending over $\Delta_j/\mathcal{D}_h$ and define $\mathcal{R}(j, h+1)$ by (5.6). Then*

$$\int e^{V_{h+1}(\mathcal{D}^c_{h+1})}\, dP(z^{(h+1)}) = e^{V_h(\mathcal{D}^c_h) + \overline{\mathcal{R}}_+(j,h+1)|\Lambda|} \tag{8.1}$$

where $|\overline{\mathcal{R}}_+(j, h+1| \le \mathcal{R}_+(j, h+1 \stackrel{def}{=} \mathcal{R}(j;h+1) + c_+ e^{-c'_+ B^2 (h+1)^2}$ with suitable c_+, c'_+.

Remark: Lemma 2 is genuinely not perturbative and making essential use of the positivity of λ. Below the analysis of the proof of the lemma, which consists essentially in its reduction to Lemma

1, is described in detail. It is perhaps the most interesting part and the core of the theory of the proof that truncating the expansion in λ of $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$ to order j gives as a result an estimate *exact* to order λ^{j+1} of $\frac{1}{|\Lambda|}\log Z_N(\Lambda,f)$.

Let R_N be the cubes $\Delta\in\mathcal{Q}_N$ in which there is at least one point $\boldsymbol{\xi}$ where $|z^{(N)}_{\boldsymbol{\xi}}|\ge BN^2$. By definition, the region $\mathcal{D}_N/\mathcal{D}_{N-1}$ is covered by R_N.

Remark that in the region $\mathcal{D}_{N-1}/R_N$ the field $X^{(N-1)}$ is large but z_N is not large so that $X^{(N)}$ is still very large: this is so because the bounds set to define the regions $\mathcal{D}$ and R are quite different being BN^4 and BN^2 respectively. Hence if a point is in $\mathcal{D}_{N-1}$ and not in R_N then the field $X^{(N)}$ must be of the order $\gg BN^3$. Therefore *by positivity* of the $\lambda\varphi^{(\le N)\,4}_{\boldsymbol{\xi}}$ term (which dominates all other terms so that $V^{(N)}(\varphi^{(\le N)}_{\boldsymbol{\xi}})<0$ for $\boldsymbol{\xi}\in\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N)$) we can replace $V_N(\mathcal{D}^c_N)$ by $V((\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N))^c)$, for the purpose of obtaining an upper bound.

Furthermore modulo a suitable correction it is possible to replace $V((\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N))^c)$ by $V((\mathcal{D}_{N-1}\cup R_N)^c)$: because the integrand in V_N is bounded below by $-b\lambda\gamma^{-2N}N^2$ if $d=2$ (by $-b\lambda\gamma^{-N}$ if $d=3$), for some b, so that the points in R_N can at most lower $V((\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N))^c)$ by $-b\lambda N^2\gamma^{-(4-d)N}\,\#(R_N)$ if $\#R_N$ is the number of boxes of $\mathcal{Q}_N$ in R_N and $V(\varphi_{\boldsymbol{\xi}})$ is bounded *below by its minimum*: thus $V((\mathcal{D}_{N-1}\cup R_N)^c)+b\lambda N^2\gamma^{(4-d)N}\,\#(R_N)$ is an upper bound to $V((\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N))^c)$.

In the complement of $\mathcal{D}_{N-1}\cup R_N$ all fields are "small"; if $X^{(N-1)}$ and R_N are fixed this region is not random (as a function of $z^{(N)}$) any more. Therefore if $X^{(N-1)},R_N$ are fixed the integration over $z^{(N)}$, *conditioned to having $z^{(N)}$ fixed (and large) in the region R_N*, is performed by means of the same argument necessary to prove lemma 1 (essentially a Taylor expansion in $\lambda\gamma^{-(4-d)N}$). The large size of $z^{(N)}$ in R_N does not affect too much the result because on the boundary of R_N the field $z^{(N)}$ is $\le BN^2$ (recalling that $z^{(N)}$ is continuous) and since the variable $z^{(N)}$ is Markovian the boundary effect decays exponentially from the boundary ∂R_N: it adds a quantity that can be shown to be bounded by the number of boxes in R_N on the boundary of R_N, hence by $\#R_N$, times $b'(N-1)^2\gamma^{-(4-d)}(B(N-1)^4)^4$ for some b'.

The result of the integration over $z^{(N)}$ of $e^{V_N((\mathcal{D}_N\cup(\mathcal{D}_{N-1}/R_N))^c)}$ *conditioned to the large field values of $z^{(N)}$ in R_N* leads to an upper bound on $\int e^{V_N}P(dz^{(N)})$ as

$$\sum_{R_N} e^{V_{j;N-1}(\mathcal{D}^c_{N-1})+\mathcal{R}(j,N)|\Lambda|}\prod_{\Delta\in R_N}\Big(c\,e^{-c'(BN^2)^2}e^{+c''\lambda\gamma^{-(4-d)N}N^2(BN^4)^4}\Big)^{\#R_N}\tag{8.2}$$

where c,c',c'' are suitable constants: this is explained as follows.

(i) Taylor expansion (in λ) of the integral $e^{V_N((\mathcal{D}_{N-1}\cup R_N)^c)+b\lambda N^2\gamma^{-(4-d)N}\#(R_N)}$ (which, by construction, is an upper bound on $e^{V_N(\mathcal{D}^c_N)}$) with respect to the field $z^{(N)}$, conditioned to be fixed and large in R_N, would lead to an upper bound as $e^{V_{j;N-1}((\mathcal{D}_{N-1}\cup R_N)^c)+\mathcal{R}'(j,N)|\Lambda|+b''\lambda(BN^4)^4\gamma^{(4-d)N}\,\#(R_N)}$ with $\mathcal{R}'$ equal to (5.6) possibly with some C'_j replacing C_j. The second exponential in the *r.h.s.* of (8.2) arises partly from the above correction $b''\lambda(BN^4)^4\gamma^{-(4-d)N}\,\#(R_N)$ and partly from a contribution of similar form explained in (iii) below.
(ii) Integration over the large conditioning fields fixed in R_N is controlled by the second estimate in (2.4) (the *tail estimate*): the first factors in parenthesis is the tail estimate just mentioned, *i.e.* the probability that $z^{(N)}$ is large in the region R_N. The second factor is only partly explained in (i) above.
(iii) Without further estimates the bound (8.2) would contain $V_{j;N-1}((\mathcal{D}_{N-1}\cup R_N)^c)$ rather than $V_{j;N-1}(\mathcal{D}^c_{N-1})$. Hence there is the need to change the potential $V_{j;N-1}((\mathcal{D}_{N-1}\cup R_N)^c)$ by "reintroducing" the contribution due to the fields in $R_N/\mathcal{D}_{N-1}$ in order to reconstruct $V_{j;N-1}(\mathcal{D}^c_{N-1})$. Reintroducing this part of the potential costs a quantity like $b'\lambda N^2\gamma^{(4-d)N}(BN^4)^4\#(R_N)$ (because the reintroduction occurs in the region $R_N/\mathcal{D}_{N-1}$ which is covered by R_N and in such points the

field $X_{\boldsymbol{\xi}}^{(N-1)}$ is not large, being bounded by $B(N-1)^4$); so that their contribution to the effective potential is still dominated by the φ^4 term and therefore by $\gamma^{-(4-d)N}$ times a power of BN^4 times the volume of R_N (in units γ^{-N}, *i.e.* $\#R_N$). All this is taken care of by suitably fixing c''.

Note that the sum over R_N of (8.2) is $(1+c\,e^{-c'B^2N^4}e^{+c''\lambda\gamma^{-(4-d)N}N^2(BN^4)^4})^{\gamma^{dN}|\Lambda|}$ (because Λ contains $|\Lambda|\gamma^{dN}$ cubes of $\mathcal{Q}_N$) hence it is bounded above by $e^{c_+e^{-c'_+B^2N^2}}$ for suitably defined c_+, c'_+.

The *same argument* can be repeated for $V_{j;h}(\mathcal{D}_h^c)$ with any h if $V_{j;h}(\mathcal{D}_h^c)$ is defined by the sum over Δ's in $\mathcal{Q}_h$ of the same integrals as those in (6.1),(6.2) with $\Delta_j/\mathcal{D}_h$ replacing Δ_j in the integration domains.

Applying lemma 1 and lemma 2 recursively (with $j\ge 3$) it follows that there exist N*-independent* upper and lower bounds $E_\pm\,|\Lambda|$ on $\log Z(\Lambda,f)$ of the form $V_{j;0}\pm\sum_{h=1}^\infty(\mathcal{R}(j,h)+c_\pm e^{-c'_\pm B^2h^2})|\Lambda|$ for $c_\pm, c'_\pm>0$ suitably chosen and λ–independent for $\lambda<1$. By the remark at the end of Sec.6, given j the bounds just described agree with the perturbation expansion $E(j,0)|\Lambda|\equiv V_{j;0}$ of $\log Z(\Lambda,f)$ truncated to order j (in λ) up to the remainders $\pm\sum_{h=1}^N\mathcal{R}_\pm(j,h)$. Hence if B is chosen proportional to $\log_+\lambda^{-1}\stackrel{def}{=}\log(e+\lambda^{-1})$ the upper and lower bounds coincide to order j in λ with the value obtained by truncating to order j the perturbative series.

The latter remark is important as it implies not only that the bounds are finite (by Sec.3) but also that $\frac{1}{|\Lambda|}\log Z(\Lambda,f)$ is *not quadratic in* f: already to order 1 in λ it is quartic in f (containing a term equal to $-\lambda(\int C_{\boldsymbol{\xi},\mathbf{0}}\,f_{\boldsymbol{\xi}}\,d\boldsymbol{\xi})^4$). Thus the outline of the proof of lemma 2, which together with lemma 1 forms the core of the analysis of the ultraviolet stability for $d=2$, is completed.

If $d=3$ more care is needed because (very mild) smoothness, like the considered Hölder continuity with exponent $\frac14$, of z,X is necessary to obtain the key scale independence property discussed in Sec.6: therefore the natural measure of the size of $z^{(h)}$ and $X^{(h)}$ in a box $\Delta\in\mathcal{Q}_h$ *is no longer the maximum of* $|z_{\boldsymbol{\xi}}^{(h)}|$ *or of* $|X_{\boldsymbol{\xi}}^{(h)}|$. The region $\mathcal{D}_h$ becomes more involved as it has to consist of the points $\boldsymbol{\xi}$ where $|X_{\boldsymbol{\xi}}^{(h)}|>Bh^4$ and of the pairs $\boldsymbol{\eta},\boldsymbol{\eta}'$ where $|Y_{\boldsymbol{\eta},\boldsymbol{\eta}'}|\equiv\frac{|X_{\boldsymbol{\eta}}^{(h)}-X_{\boldsymbol{\eta}'}^{(h)}|}{(\gamma^h|\boldsymbol{\eta}-\boldsymbol{\eta}'|)^{\frac14}}>Bh^4$: *i.e.* it is not just a subset of Λ.

However, if $d=3$, the relevant part *also contains the negative term* $V^{(rel,2)}$*, see (6.1)*: and since it dominates over all other terms which contain a Y-field (because their coupling are smaller by about γ^{-h}) the argument given for $d=2$ can be adapted to the new situation. Two regions $\mathcal{D}_h^1,\mathcal{D}_h^2$ will be defined: the first consists of all the points $\boldsymbol{\xi}$ where $|X_{\boldsymbol{\xi}}^{(h)}|>Bh^4$ and the second of all the pairs $\boldsymbol{\eta},\boldsymbol{\eta}'$ where $|Y_{\boldsymbol{\eta},\boldsymbol{\eta}'}^{(h)}|>Bh^4$. The region R_h will be the collection of all $\Delta\in\mathcal{Q}_h$ where $||z^{(h)}||_\Delta>Bh^2$. Then $V(\mathcal{D}_h^c)$ will be defined as the sum of the integrals in (6.1), (6.2) with the integrals over $\boldsymbol{\xi}_i$ further restricted to $\boldsymbol{\xi}_i\notin\mathcal{D}_h^1$ and those over the pairs $\boldsymbol{\eta}_i,\boldsymbol{\eta}'_i$ are further restricted to $(\boldsymbol{\eta}_i,\boldsymbol{\eta}'_i)\notin\mathcal{D}_h^2$. With the new settings lemma 2 can be proved also for $d=3$ along the same lines as in the $d=2$ case.

References: [Wi70],[Wi72],[BCGNPOS78],[Ga81].

9. Ultraviolet limit, infrared behavior and other applications

The results on the ultraviolet stability are nonperturbative, as no assumption is made on the size of λ (the assumption $\lambda<1$ has been imposed in Sec.7,8 only to obtain simpler expressions for the λ–dependence of various constants): nevertheless the multiscale analyis has allowed us to use perturbative techniques (*i.e.* the Taylor expansion in lemmata 1,2) to find the solution. The latter procedure is the essence of the renormalization group methods: they aim at reducing a difficult multiscale problem to a sequence of simple single scale problems. Of course in most cases it is difficult to implement the approach and the scalar quantum fields in dimension $2,3$ are among the simplest examples. The analysis of the *beta function* and of the *running couplings*, which appear in essentially all renormalization group applications, does not play a role here (or, better, their role is so inessential that it has even been possible to avoid mentioning them). This makes the models somewhat special from the renormalization group viewpoint: the running couplings at length scale h, if introduced, would tend exponentially to 0 as $h \to \infty$; unlike what happens in the most interesting renormalization group applications in which they either tend to zero only as powers of h or do not tend to zero at all.

The multiscale analysis method, *i.e.* the renormalization group method, in a form close to the one discussed here has been applied very often since its introduction in Physics and it has led to the solution of several important problems. The following is a not exhaustive list together with a few open questions.

(1) The arguments just discussed imply with minor extra work that $Z_N(\Lambda, f)$ as $N \to \infty$ not only admit uniform upper and lower bounds but also that the limit as $N \to \infty$ actually exists and it is a C^∞ function of λ, f. Its λ and f–derivatives at $\lambda = 0$ and $f = 0$ are given by the formal perturbation calculation. In some cases it is even possible to show that the formal series for $Z_N(\Lambda, f)$ in powers of λ is Borel summable. An interesting question is to explore the possibility of an ultraviolet stability proof which is exclusively based on the perturbation expansion without having recourse to the probabilistic methods in the analysis.

(2) The problem of removing the infrared cut–off (*i.e.* $\Lambda \to \infty$) is in a sense more a problem of statistical mechanics. In fact it can be solved for $d = 2, 3$ by a typical technique used in statistical mechanics, the *cluster expansion*. This is not intended to mean that it is technically an easy task: understanding its connection with the low density expansions and the possibility of using such techniques has been a major achievement that is not discussed here.

(3) The third problem mentioned in the introduction: *i.e.* checking the axioms so that the theory could be interpreted as a quantum field theory is a difficult problem which required important efforts to control and which is not analyzed here. An introduction to it can be its analysis in the $d = 2$ case.

(4) Also the problem of keeping the ultraviolet cut–off and removing the infrared cut–off while the parameter m^2 in the propagator approaches 0 is a very interesting problem related to many questions in statistical mechanics at the critical point.

(5) Field theory methods can be applied to various statistical mechanics problems away from criticality: particularly interesting is the theory of the neutral Coulomb gas and of the dipole gas in two dimensions.

(6) The methods can be applied to Fermi systems in field theory as well as in equilibrium statistical mechanics. The understanding of the ground state in not exactly soluble models of spinless fermions in 1 dimension at small coupling is one of the results. And via the trasfer matrix theory it has led to the understanding of nontrivial critical behavior in 2-dimensional models that are not exactly soluble (like Ising next nearest neighbor or Ashkin–Teller model). Fermi systems are of particular interest also because in their analysis the large fields problem is absent, but this great technical advantage is somewhat offset by the anticommutation properties of the Fermionic fields: which do not allow us to employ probabilistic techniques in the estimates.

(7) An outstanding open problem is whether the scalar φ^4-theory is possible and nontrivial in dimension $d = 4$: this is a case of a renormalizable not asymptotically free theory. The conjecture that many support is that the theory is necessarily trivial (*i.e.* the function $Z_N(\Lambda, f)$ becomes necessarily a Gaussian in the limit $N \to \infty$).

(8) Very interesting problems can be found in the study of highly symmetric quantum fields: gauge invariance presents serious difficulties to be studied (rigorously or even heuristically) because in its naive forms it is incompatible with regularizations. Rigorous treatments have been in some cases possible and in few cases it has been shown that the naive treatment is not only not rigorous but it leads to incorrect results.

(9) In connection with item (8) an outstanding problem is to understand relativistic pure gauge

Higgs-fields in dimension $d = 4$: the latter have been shown to be ultraviolet stable but the result has not been followed by the study of the infrared limit.

(10) The classical gauge theory problem is *quantum electrodynamics*, QED, in dimension 4: it is a renormalizable theory (taking into account gauge invariance) and its perturbative series truncated after the first few orders give results that can be directly confronted with experience, giving very accurate predictions. Nevertheless the model is widely believed to be incomplete: in the sense that, if treated rigorously, the result would be a field describing free non interacting assemblies of photons and electrons. It is believed that QED can make sense only if embedded in a model with more fields, representing other particles (*e.g.* the *standard model*), which would influence the behavior of the electromagnetic field by providing an effective ultraviolet cut-off high enough for not alterig the predictions on the observations on the time and energy scales on which present (and, possibly, future over a long time span) experiments are performed. In dimension $d = 3$ QED is superrenormalizable, once the gauge symmetry is properly taken into account, and it can be studied with the techniques described above for the scalar fields in the corresponding dimension.

In general constructive quantum field theory seems to be deep in a crisis: the few solutions that have been found concern very special problems and are very demanding technically; the results obtained have often not been considered to contribute appreciably to any "progress". And many consider that the work dedicated to the subject is not worth the results that one can even hope to obtain. Therefore in recent years attempts have been made to follow other paths: an attitude that in the past usually did not lead to great achievements but that is always tempting and worth pursuing because the rare major progresses made in Physics resulted precisely by such changes of attitude, leaving aside developments requiring work which was too technical and possibly hopeless: just to mention an important case one can recall quantum mechanics which disposed of all attempts at understanding the observed atomic levels quantization on the basis of refined developments of classical electromagnetism.

References: [Gu72],[GJS73],[Si74], [BCGNPOS78],[GJ81],[Ai82],[Fr82],[GK83], [GK85],[GK85b], [Ba83], [BGM04],[GM05].

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