A new branch of mountain pass solutions for the choreographical *3*-body problem

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Abstract. We prove the existence of a new branch of solutions of Mountain Pass type for the periodic 3–body problem with choreographical constraint. At first we describe the variational structure of the action functional associated to the choreographical three body problem in $\mathbb{R}^3$. In the second part, using on a bisection algorithm, we provide a numerical non-rigorous solution of Mountain Pass type for this problem in a rotating frame with angular velocity 1.5. The last step consists in the rigorous computer-assisted proof of the existence of a full branch of solutions for the problem starting from the Mountain Pass solution detected numerically.

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1 Introduction

In recent years the study of periodic solutions of the n-body problem has received a new boost from the application of variational methods in spaces of symmetric loops (see [23, 16, 8, 6, 15]). The discovery of the eight-shaped solution by Chenciner and Montgomery is emblematic of this renewed interest. Another reason of this attention is provided by the development of new techniques for computer–assisted proofs which may be applied in order to prove that close to a numerical solution, possibly obtained by variational methods, there exists a true solution (see [3, 4, 19]). In this paper we combine these two techniques to show the existence of a new branch of spatial periodic solutions for the 3-body problem.

We are interested in trajectories for the Newtonian 3–body problem which are periodic in a rotating frame. After rescaling, the period can always be set to 2π; denoting $\boldsymbol{\omega}= (0,0,-\omega)$, $\omega \in \mathbb{R}$ the angular velocity and $\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3$ the positions, the resulting system writes

$$(P)_\omega \qquad \begin{cases} \ddot{\mathbf{q}}^i = -2\mathbb{J}\omega\dot{\mathbf{q}}^i - \omega^2\mathbb{J}^2\mathbf{q}^i + \dfrac{\partial V}{\partial \mathbf{q}^i} \\ \mathbf{q}^i(t+2\pi) = \mathbf{q}^i(t), \quad \forall t \in \mathbb{R}, \\ i = 1,2,3, \end{cases}$$

where

$$V(\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3) := \frac{1}{|\mathbf{q}^1 - \mathbf{q}^2|} + \frac{1}{|\mathbf{q}^1 - \mathbf{q}^3|} + \frac{1}{|\mathbf{q}^2 - \mathbf{q}^3|}, \tag{1}$$

is the keplerian potential and the linear operator $\mathbb{J} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as $\mathbb{J}((x,y,z)) = (-y,x,0)$. We are concerned with a special class of trajectories, called *simple choreographies* (see [12], [8]); this constraint forces the bodies to move on the same curve, exchanging their positions after a fixed period of time $\tau = 2\pi/3$.

Let us consider the Hilbert loop space

$$X := H^1_{2\pi}(\mathbb{R}, \mathbb{R}^3)$$

and the open subset

$$\mathcal{X} := \{\mathbf{q} \in X : \mathbf{q}(t) \neq \mathbf{q}(t+\tau), \forall t \in \mathbb{R}\}. \tag{2}$$

Taking into account the singular set of the potential V and the simple choreography constraint, we are lead to the search of solutions of $(P)_\omega$ such that $\mathbf{q}^i(t) = \mathbf{q}(t + \tau(i-1))$, $i = 1,2,3$ for some $\mathbf{q} \in \mathcal{X}$. These solutions, thanks to Palais principle of symmetrical criticality (see [23]), can be found as critical points in the loop space $\mathcal{X}$ of the action functional

$$\mathcal{A}^\omega(\mathbf{q}) = \frac{1}{2}\int_0^{2\pi} |\dot{\mathbf{q}}(t) + \mathbb{J}\omega\mathbf{q}(t)|^2 dt + \int_0^{2\pi} \frac{dt}{|\mathbf{q}(t) - \mathbf{q}(t+\tau)|}. \tag{3}$$

Next we investigate the variational structure of the functional $\mathcal{A}^\omega$. At first we remark that the simple choreography constraint makes the action functional $\mathcal{A}^\omega$ coercive for every $\omega \notin \mathbb{Z}\backslash 3\mathbb{Z}$ (Proposition 1). However, as proved in Proposition 2 (and for any number of bodies in [8]), the bare minimization of $\mathcal{A}^\omega$ over $\mathcal{X}$ provides an uninteresting existence result; indeed, if k is the integer closest to ω, then global minimizers of $\mathcal{A}^\omega$ are uniform circular motions (*Lagrange motions*) with minimal period $2\pi/k$ and radius depending on ω. Figure 3 represents the values of the action functional $\mathcal{A}^\omega$ on the branches of circular orbits L^ω_k.

On the other hand, a deeper analysis of this picture suggests the presence of critical points different from the Lagrange motions. Indeed, let us take the angular velocity $\omega = 1.5$: in this case there are two distinct global minimizers, the uniform circular motions with minimal period 2π and π, lying in the plane orthogonal to the rotation direction. This is a well known structure in Critical Point Theory, known as the mountain pass geometry which gives the existence of a third critical point, provided the Palais-Smale condition is fullfilled, with an additional information on the Morse index. Theorem A follows from the application of the Mountain Pass Theorem (Theorem 1) to the action functional $\mathcal{A}^{3/2}$:

Theorem A. There exists a (possibly collision) critical point for the action functional $\mathcal{A}^{3/2}$ with Morse index smaller than 1 and distinct from any Lagrange motion.

Once the existence of a mountain pass critical point is established, we wish to study its main properties. To this aim we apply the bisection algorithm proposed in [7] to approximate the maximal of a locally optimal path joining the two strict global minimizers. Of course, there is no proof that the numerical solution found by applying the bisection algorithm is close to the the mountain pass solution, whose existence is ensured by Theorem A. On the other hand, we have strong evidence that this is the case, since we can prove the existence of a true solution very close to the numerical output of the mountain pass algorithm. This argument is based upon a fixed point principle and involves a rigorous computer assisted proof. As a consequence, we obtain the existence of a new solution for the spatial 3-body problem (see Figures 1). Once we have defined a suitable space of symmetric loops $(\mathcal{X}_\rho, \|\cdot\|_\rho)$, $\mathcal{X}_\rho \subset \mathcal{X}$, (see (28) in Section 5) we can state the following result:

Theorem B. There exists a loop $\tilde{\mathbf{q}} \in \mathcal{X}_\rho$ such that for $i = 1, 2, 3$ the trajectories $\tilde{\mathbf{q}}^i(t) = \tilde{\mathbf{q}}\left(t + \frac{2\pi}{3}(i-1)\right)$ are solutions of the dynamical system $(P)_\omega$ when $\omega = 1.5$ and $\|\tilde{\mathbf{q}} - \mathbf{q_0}\|_\rho \leq 10^{-6}$ where $\mathbf{q_0} \in \mathcal{X}_\rho$ is defined in [2].

In [2], $\mathbf{q_0}$ is given as the sum of its 60 Fourier coefficients, the first nonzero (truncated) Fourier coefficients are listed in Table 4.2 (at page 17). From Theorem B we can deduce some relevant features of the new solution: the orbit is not planar, its winding number with respect, for instance, to the line $x = -0.2$, $y = 0$ is 2 and it does not intersect itself. Moreover, the numerical computation of its Morse index indicates that the new orbit cannot be a minimizer and that it is of mountain pass type. The computer assisted method, introduced in [3] and relying on the Fixed Point Theorem, can be adapted to the present setting and gives the existence of a unique solution of the dynamical system $(P)_\omega$ in a small neighborhood of the numerical mountain pass solution. A natural question is whether this solution can be continued as a function of the parameter ω. By applying a technique developed in [4], we are able to prove the existence of a full branch of solutions, when ω varies in $[1, 2]$.

Theorem C. There exists a smooth map $B : [1, 2] \to \mathcal{X}_\rho$ such that $B(\omega)$ is a locally unique solution of the dynamical system $(P)_\omega$, for all $\omega \in [1, 2]$.

Surprisingly enough, it turns out from further numerical computation that this branch does not bifurcate from a Lagrange motion; apparently, it starts from the branch of P_{12} solutions described by Marchal in [22]. The details of the bifurcation diagram is depicted in Figure 6.

2 Variational properties of the Lagrange motions

Let $\mathcal{A}^\omega$ the lagrangian action functional in (3) defined and C^2 on the open subset $\mathcal{X}$ of the Hilbert space $H^1_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ introduced in (2), corresponding to the lagrangian action for the 3-body problem with simple choreography constraint in

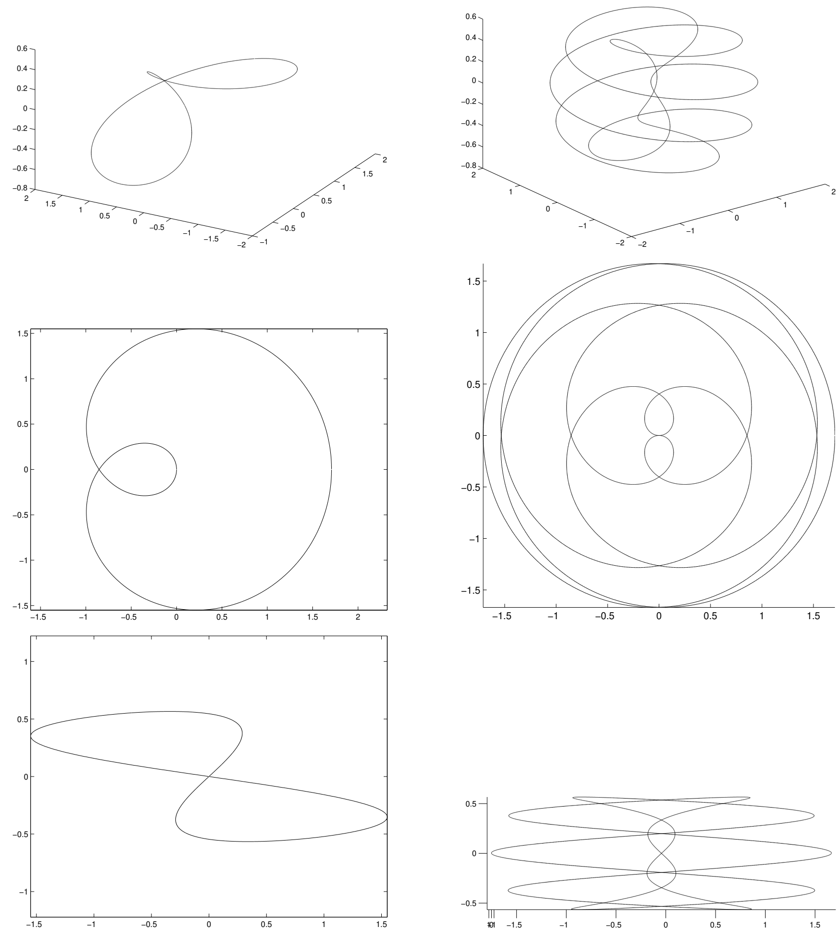

Figure 1: The numerical mountain pass solution for the 3-body problem whose first Fourier coefficients are listed in Table 4.2. The pictures in the left column refer to the 2π-periodic orbit in the rotating frame, on the right the corresponding 4π-periodic solution in the inertial frame. In the first line we see the 3-dimensional trajectories; in the second and third line the projections of the orbit on the rotating plane and on a plane containing the angular velocity vector respectively.

a rotating frame with fixed rotation direction. As shown in [16] the choreography symmetry is a natural constraint for the action functional. This fact, together with the condition $\mathbf{q}(t) \neq \mathbf{q}(t+\tau)$, $\forall t \in \mathbb{R}$, for every $\mathbf{q} \in \mathcal{X}$, implies that critical points for the functional $\mathcal{A}^\omega$ on the open set $\mathcal{X}$ are 2π-periodic C^2 solutions of its associated Euler-Lagrange equations in $(P)_\omega$ (*Palais Principle of symmetric criticality*, [23]).

Remark 1. The definition of the functional $\mathcal{A}^\omega$ can be extended to the whole space X; we remark that $\mathcal{A}^\omega(x)$ may or may not be finite when $x \in \partial\mathcal{X}$, depending on the singularity of the potential function.

Remark 2. The study of the lagrangian action functional $\mathcal{A}^\omega$ when ω ranges in $[0, 3/2]$ is exhaustive. To prove this assertion, consider the square matrix associ-

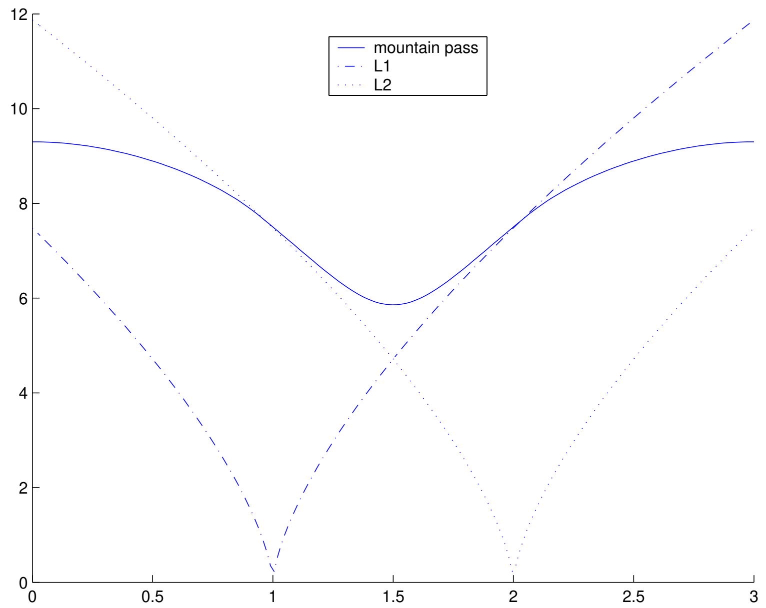

Figure 2: Action levels for the Lagrange and the mountain pass solutions in the 3-body problem. On the x-axes the angular velocity varies in the interval $[0, 3)$.

ated to the linear operator $\mathbb{J}$, we still name it $\mathbb{J}$, and its associated exponential matrix

$$\mathbb{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{\mathbb{J}t} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{q} \in \mathcal{X}$ be a solution for $(P)_\omega$, then $\mathbf{q}^-$ defined as

$$\mathbf{q}^-(t) := \mathbf{q}(-t), \qquad \forall t \in \mathbb{R},$$

is still 2π-periodic and solves $(P)_{-\omega}$. Moreover the values of the corresponding action functional on the two loops are the same that is

$$\mathcal{A}^\omega(\mathbf{q}) = \mathcal{A}^{-\omega}(\mathbf{q}^-).$$

For every $n \in \mathbb{Z}$ we now define the 2π-periodic loop $\mathbf{q}^n$ as

$$\mathbf{q}^n(t) := e^{\mathbb{J}nt}\mathbf{q}(t), \qquad \forall t \in \mathbb{R}.$$

Provided that 3 divides n, we have that at every time $t \in \mathbb{R}$

$$V(\mathbf{q}^n(t)) = V(\mathbf{q}(t))$$

and

$$\frac{\partial V(\mathbf{q}^n(t))}{\partial \mathbf{q}_i^n} = e^{\mathbb{J}nt}\frac{\partial V(\mathbf{q}(t))}{\partial \mathbf{q}_i}, \quad i = 1, 2, 3.$$

When $\mathbf{q}$ solves $(P)_\omega$ the first two derivatives of $\mathbf{q}_i$ in terms of $\mathbf{q}_i^n$, $i = 1, 2, 3$, are

$$\begin{aligned} \dot{\mathbf{q}}_i &= e^{-\mathbb{J}nt}\left(-\mathbb{J}n\mathbf{q}_i^n + \dot{\mathbf{q}}_i^n\right), \\ \ddot{\mathbf{q}}_i &= e^{-\mathbb{J}nt}\left(\mathbb{J}^2 n^2 \mathbf{q}_i^n - 2\mathbb{J}n\dot{\mathbf{q}}_i^n + \ddot{\mathbf{q}}_i^n\right), \end{aligned} \tag{4}$$

hence, replacing (4) in $(P)_\omega$, we obtain

$$\ddot{\mathbf{q}}_i^n = -2\mathbb{J}(\omega - n)\dot{\mathbf{q}}_i^n - \mathbb{J}^2(\omega - n)^2\mathbf{q}_i^n + \frac{\partial V(\mathbf{q}^n(t))}{\partial \mathbf{q}_i^n}.$$

We can then conclude that $\mathbf{q}^n$ solves $(P)_{\omega-n}$. We moreover easily compute

$$\mathcal{A}^\omega(\mathbf{q}) = \mathcal{A}^{\omega-n}(\mathbf{q}^n).$$

The following proposition is a consequence of a more general result in [8]; for the reader convenience we prove it in the special setting of this paper.

Proposition 1. *Let $\mathcal{A}^\omega$ be the functional defined in* (3)*; then $\mathcal{A}^\omega$ is coercive on the space $\mathcal{X}$ if and only if $\omega \in [0, 3) \setminus \{1, 2\}$.*

Proof. For every $\omega \in [0, 3) \setminus \{1, 2\}$ the kinetic part of the action functional is strictly positive on the set $\mathcal{X}$ and diverges to $+\infty$ as $\|\mathbf{q}\|_{H^1} \to +\infty$.

When $\omega = k$, $k = 1, 2$, we consider the loops $\mathbf{q}_k^\nu(t) = R_\nu e^{-Jkt}$, where $R_\nu \to +\infty$ as $\nu \to +\infty$; since $(\mathbf{q}_k^\nu)_\nu$ form a non converging minimizing sequence for $\mathcal{A}^k$, $k = 1, 2$, then the lagrangian action functional is not coercive. □

We define the quadratic form

$$Q^\omega(\mathbf{q}) = \frac{1}{2}q^\omega(\mathbf{q}, \mathbf{q}) := \frac{1}{2\pi}\int_0^{2\pi} |\dot{\mathbf{q}}(t) + \mathbb{J}\omega\mathbf{q}(t)|^2 dt \tag{5}$$

and the function

$$\bar{V}(\mathbf{q}(t)) := \frac{1}{|\mathbf{q}(t) - \mathbf{q}(t + \tau)|}, \tag{6}$$

to obtain the following expression for the action functional

$$\mathcal{A}^\omega(\mathbf{q}) = \pi Q^\omega(\mathbf{q}) + \int_0^{2\pi} \bar{V}(\mathbf{q}(t))dt. \tag{7}$$

Remark that

$$\int_0^{2\pi} \bar{V}(\mathbf{q}(t))dt = \frac{1}{3}\int_0^{2\pi} V(\mathbf{q}(t))dt,$$

where $V(\mathbf{q}) = V(\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3)$ has been defined in (1). For every loop $\mathbf{q} \in X$ we can define at every time t the *moment of inertia* associated to the 3-body choreography on $\mathbf{q}$, that is $I(\mathbf{q}(t)) = |\mathbf{q}^1(t)|^2 + |\mathbf{q}^2(t)|^2 + |\mathbf{q}^3(t)|^2$.

Proposition 2. *The global minimum of $\mathcal{A}^\omega$ on $\mathcal{X}$, when it exists, is achieved on uniform circular orbits with radius depending on ω and minimal period 2π or π, when $\omega \in [0, 3/2] \setminus \{1\}$ or $\omega \in [3/2, 3) \setminus \{2\}$ respectively.*

Proof. The idea of the proof comes from the proof of a similar proposition in [10].

For every loop $\mathbf{q} \in \mathcal{X}$ the following estimate on the kinetic part of the action functional holds

$$\pi Q^{\omega}(\mathbf{q}) \geq \frac{c(\omega)}{6} \int_0^{2\pi} I(\mathbf{q}(t))dt \tag{8}$$

where

$$c(\omega) = \min_{n \in \mathbb{Z}} (n+\omega)^2.$$

Concerning the potential part of the action functional we have the following inequality

$$\int_0^{2\pi} \bar{V}(\mathbf{q}(t))dt \geq \frac{1}{3} V_0 \int_0^{2\pi} I(\mathbf{q}(t))^{-1/2} \tag{9}$$

where $V_0 = 3$ is the minimal value that the potential function assume on the ellipsoid $I = 1$. From (8) and (9) we can then conclude that

$$\mathcal{A}^{\omega}(\mathbf{q}) \geq \frac{1}{3} \left\{ \int_0^{2\pi} \frac{c(\omega)}{2} I(\mathbf{q}(t)) + \frac{V_0}{\sqrt{I(\mathbf{q}(t))}} \right\} \tag{10}$$

and that the equality in (10) is achieved when $\mathbf{q}(t)$ is a circle lying in the rotating plane with minimal period $2\pi/k$ where $k = 1$ or 2 is such that $-k$ minimizes the quantity $c(\omega)$, that is k is the integer closest to ω. R^{ω} being the radius of such circular motion, we have that the function at the right hand side of (10) is

$$f(R^{\omega}) = 2\pi \left\{ \frac{c(\omega)}{2} (R^{\omega})^2 + \frac{1}{\sqrt{3} R^{\omega}} \right\}$$

that assume its minimal value when

$$R^{\omega} = \left(\sqrt{3} c(\omega) \right)^{-1/3}.$$

The minimum of the action functional then is

$$\mathcal{A}^{\omega}_{min} = \frac{\sqrt{3}\pi}{R^{\omega}}.$$

□

Corollary 1. *When $\omega = 1.5$, the 3-body problem in $\mathbb{R}^3$ with simple choreography constraint admits exactly two distinct global minimizers, the Lagrange solutions with minimal period π and 2π lieing in the rotating plane with radius $R^{3/2} = (4\sqrt{3}/3)^{1/3}$.*

Let $\bar{\mathbf{q}}$ be the Lagrange central configuration, (a regular triangle) with edge of length 1; we term $\bar{\mathbf{q}}_l = l\bar{\mathbf{q}}$, $l > 0$, the regular triangular configuration with edge of length l, radius $l/\sqrt{3}$ and moment of inertia l^2. Then the following proposition holds

Proposition 3. *Fixed $k \in \mathbb{N} \setminus \{n \equiv 0 \mod 3\}$, the uniform circular motion with minimal period $2\pi/k$ minimizes $\mathcal{A}^\omega$ on the set of planar loops $\{\mathbf{q}_l(t) = e^{-Jkt}\bar{\mathbf{q}}_l : l > 0\}$ when l verifies the following relation*

$$(k-\omega)^2 I(\bar{\mathbf{q}}_l) = V(\bar{\mathbf{q}}_l), \tag{11}$$

that is $l = \left[(k-\omega)^2/3\right]^{-1/3}$.

Proof. We cal easily compute the action functional at $\mathbf{q}_l$

$$\frac{1}{2\pi}\mathcal{A}^\omega(\mathbf{q}_l) = \frac{1}{2}(k-\omega)^2\frac{l^2}{3} + \frac{1}{l} = \frac{1}{2}(k-\omega)^2\frac{I(\bar{\mathbf{q}}_l)}{3} + \frac{V(\bar{\mathbf{q}}_l)}{3}.$$

The minimal value of $\mathcal{A}^\omega(\mathbf{q}_l)$, $l > 0$, is then achieved when l verifies relation (11). □

Remark 3. In Propositions 2, 3 and in Corollary 1 we actually prove the existence of connected components of minimizers since the action functional is $SO(2)$-invariant. In what follows we will identify two orbits when they differ by a rotation on the rotating plane, with this meaning we give the next definition.

Definition 1. For every $k \in \mathbb{N} \setminus \{n \equiv 0 \mod 3\}$, we name L_k^ω the circular orbits with minimal period $2\pi/k$ associated to the regular triangular configuration $\bar{\mathbf{q}}_l$ verifying (11).

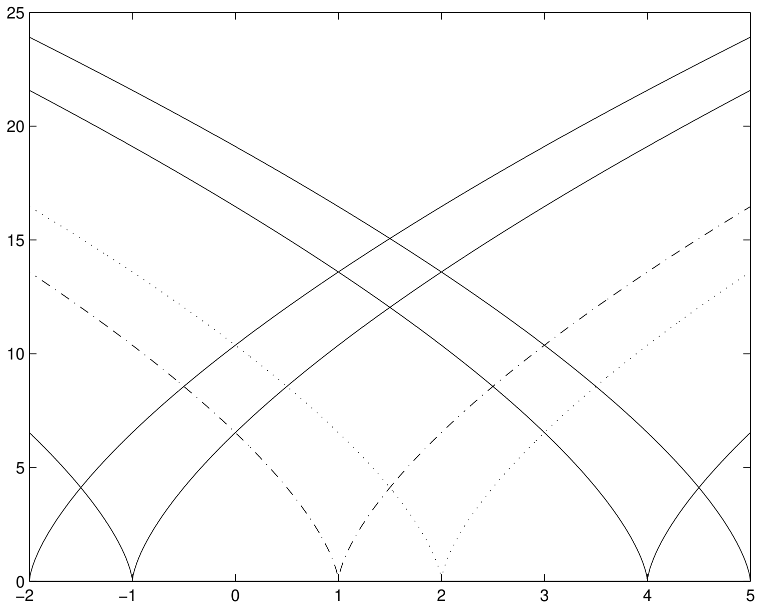

Figure 3: The graphs in the picture represent the levels of the action functional evaluated at the minimal lagrange motion L_k^ω, $k = -2, -1, 1, 2, 4, 5$. On the x-axes, the angular velocity varies in the interval $[-2, 5]$.

Figure 3 represents the values of the action functional $\mathcal{A}^\omega$ on the branches of circular orbits L_k^ω. From (11) we can indeed compute that

$$\mathcal{A}^\omega\left(L_k^\omega\right) = \pi V\left(L_k^\omega\right) = 3\pi\sqrt[3]{\frac{(k-\omega)^2}{3}}$$

We remark that (as Corollary 1 states) $\mathcal{A}^{3/2}(L_1^{3/2}) = \mathcal{A}^{3/2}(L_2^{3/2})$; in the next section we will prove the existence of a critical point for $\mathcal{A}^{3/2}$ distinct from any Lagrange motion as a mountain pass point between the two strict global minimizers $L_1^{3/2}$ and $L_2^{3/2}$.

3 Mountain pass solutions for the 3-body problem

In this section we prove the existence of a solution of mountain pass type for the 3-body problem with a simple choreography constraint working in a rotating frame with intensity of the angular velocity $\omega = 1.5$. We want to stress that even if we deal with the Keplerian potential, the following results still hold when we consider homogeneous potentials of degree $-\alpha$, $\alpha > 0$.

3.1 The Mountain Pass Theorem

Let X be an Hilbert space and $\Omega \subset X$ an open subset such that $\bar{\Omega} = X$; we consider a functional f on X, $f \in C^2(\Omega)$ and we recall the following definitions

Definition 2. For a given $c \in \mathbb{R}$, we define the *c-sublevel* and the *set of critical points* of f respectively as

$$f^c := \{x \in X : f(x) < c\} \quad \text{and} \quad \mathrm{Crit}(f) := \{x \in \Omega : \nabla f(x) = 0\}.$$

Definition 3. Let $x_0 \in \mathrm{Crit}(f)$ be a critical point of f. We define the *Morse Index* of x_0 (if it exists) as the maximal positive integer m such that the Hessian of f at x_0 is negative definite on a m-dimensional subspace of X.

Definition 4. A sequence $(x_m)_m \subset \Omega$ is called a *Palais-Smale sequence in the interval* $[a, b]$ *for the functional* f if

$$a \leq f(x_m) \leq b, \forall m \in \mathbb{N} \quad \text{and} \quad \nabla f(x_m) \to 0 \text{ as } m \to +\infty.$$

The functional f *satisfies the Palais-Smale condition in the interval* $[a, b]$ if every Palais-Smale sequence in the interval $[a, b]$ for the functional f, $(x_m)_m$, has a converging subsequence $x_{m_k} \to x_0 \in X$. Similarly, a sequence $(x_m)_m \subset X$ is a *Palais-Smale sequence at level c for the functional f* if

$$f(x_m) \to c \quad \text{and} \quad \nabla f(x_m) \to 0 \text{ as } m \to +\infty.$$

The functional f *satisfies the Palais-Smale condition at level c* if every Palais-Smale sequence at level c for f has a converging subsequence.

Definition 5. When every Palais-Smale sequence at level c for f entirely contained in Ω, $(x_m)_m$, has a converging subsequence $(x_{m_k})_{m_k}$ such that $x_{m_k} \to \bar{x}$ in Ω, then we say that the functional f *satisfies the Palais-Smale condition at level c in the open set Ω*, $(\mathrm{PS})_{c,\Omega}$.

Definition 6. The operator f is Fredholm of index zero at $x_0 \in \Omega$ if the dimension of the kernel and the codimention of the range of the linearized operator $\nabla f(x_0)$ are finite and equal.

We are now ready to state the following version of the Mountain Pass Theorem (see [18, 21, 25])

Theorem 1 (Mountain Pass Theorem). *Let X be an Hilbert space, $\Omega \subset X$ an open and dense subset of X and let f be a C^2 functional on Ω. Let $x_1, x_2 \in \Omega$, let Γ_{x_1,x_2} be the set of paths*

$$\Gamma_{x_1,x_2} := \{\gamma \in C([0,1],\Omega) : \gamma(0) = x_1, \gamma(1) = x_2\} \tag{12}$$

and c_0 the level

$$c_0 := \inf_{\gamma \in \Gamma_{x_1,x_2}} \sup_{s\in[0,1]} f(\gamma(s)). \tag{13}$$

such that

$$c_0 > \max\{f(x_1), f(x_2)\}. \tag{14}$$

If the functional f satisfies (PS)$_{c_0,\Omega}$, then there exists a critical point in Ω for the functional f at level c_0. Moreover, if ∇f is a Fredholm operator of index zero at every critical point $\bar{x} \in f^{-1}(c_0) \cap \Omega$, then at least one critical point has Morse index $m \leq 1$.

3.2 Existence of a mountain pass solution for the 3-body problem

Our aim now is to apply Theorem 1 to the action functional associated to the three body problem and to show that it implies the existence of a solution which does not coincide with and Lagrange motion.

Definition 7. By L_k we denote the Lagrange motions $L_k^{3/2}$ introduced in Definition 1 when $\omega = 1.5$.

Proposition 4. *The functional $\mathcal{A}^{3/2}$ defined in* (3) *satisfies the Palais-Smale condition at every level $c \geq 0$.*

Proof. Let $(\mathbf{q}_\nu)_\nu \subset X$ be a Palais-Smale sequence for the functional $\mathcal{A}^{3/2}$ at level $c \geq 0$. Our aim is to find an element $\tilde{\mathbf{q}} \in X$, such that $\mathbf{q}_\nu \to \tilde{\mathbf{q}}$, as $\nu \to +\infty$. Since $\mathcal{A}^{3/2}$ is coercive, the sequence $(\mathbf{q}_\nu)_\nu$ in bounded in X and then, up to subsequences, weakly converging to $\tilde{\mathbf{q}}$.

The weak convergence in $H^1_{2\pi}$ implies the strong convergence in C^0, hence

$$\lim_{\nu\to+\infty} \int_0^{2\pi} \nabla\bar{V}(\mathbf{q}_\nu)(\mathbf{q}_\nu - \tilde{\mathbf{q}}) = 0. \tag{15}$$

From (15) and from the Palais-Smale condition

$$\lim_{\nu\to+\infty} \nabla\mathcal{A}^{3/2}(\mathbf{q}_\nu)(\mathbf{q}_\nu - \tilde{\mathbf{q}}) =$$
$$\pi\left[q^{3/2}(\mathbf{q}_\nu,\mathbf{q}_\nu) - q^{3/2}(\mathbf{q}_\nu,\tilde{\mathbf{q}})\right] + \int_0^{2\pi} \nabla\bar{V}(\mathbf{q}_\nu)(\mathbf{q}_\nu - \tilde{\mathbf{q}}) = 0,$$

we deduce that

$$\lim_{\nu\to+\infty} Q^{3/2}(\mathbf{q}_\nu) = Q^{3/2}(\tilde{\mathbf{q}}).$$

Since the quadratic form $Q^{3/2}(\cdot)$ is an equivalent norm in X, we conclude the strong convergence to $\tilde{\mathbf{q}}$ of the sequence $(\mathbf{q}_\nu)_\nu$. □

Remark 4. We observe that the action functional $\mathcal{A}^{3/2}$ verifies the Palais-Smale condition on the whole space X, but not on the open subset $\mathcal{X}$. For this reason we define the following regularized functional

$$\mathcal{A}_\epsilon^{3/2}(\mathbf{q}) = \frac{1}{2}\int_0^{2\pi}\left|\dot{\mathbf{q}}(t) + \frac{3}{2}\mathbb{J}\mathbf{q}(t)\right|^2 dt + \int_0^{2\pi}\frac{1}{(|\mathbf{q}(t)-\mathbf{q}(t+\tau)|^2+\epsilon^2)^{\frac{1}{2}}}dt\,, \quad (16)$$

where $\epsilon > 0$ is a suitable small number. The functional $\mathcal{A}_\epsilon^{3/2}$ is smooth, more precisely C^2, on the whole Hilbert space of periodic loops $X = H^1_{2\pi}(\mathbb{R},\mathbb{R}^3)$, where it satisfies the Palais-Smale condition.

The following definition generalizes the classical notion of critical point for the action to $\mathcal{A}^{3/2}$ to collision orbits $\mathbf{q} \in \partial\mathcal{X}$:

Definition 8. We say that $\mathbf{q}$ is a *generalized critical point* for $\mathcal{A}^{3/2}$ if:

1. there exists a sequence $(\mathbf{q}_\epsilon)_\epsilon \subset X$ such that $\mathbf{q}_\epsilon$ is a critical point for the functional $\mathcal{A}_\epsilon^{3/2}$, for every ϵ;
2. there exists a constant C such that for every $\mathbf{q}_\epsilon$ we have $|\mathcal{A}_\epsilon^{3/2}(\mathbf{q}_\epsilon)| < C$;
3. the sequence $(\mathbf{q}_\epsilon)_\epsilon$ strongly converges to $\mathbf{q}$ as $\epsilon \to 0$.

Remark 5. We observe that from Definition 8 it follows that if $\mathbf{q}$ is a generalized critical point for $\mathcal{A}^{3/2}$, then $(\mathbf{q}^1,\mathbf{q}^2,\mathbf{q}^3)$, $\mathbf{q}^i(t) = \mathbf{q}\left(t+\tau(i-1)\right)$, $i = 1,2,3$ solves the dymamical system $(P)_{3/2}$ at every t such that $\mathbf{q}(t) \neq \mathbf{q}(t+\tau l)$, $l = 1,2$. Moreover the energy and the moment of inertia of a generalized critical point are bounded.

Lemma 1. *For every $k \in \mathbb{N}\backslash 3\mathbb{N}$, $k \geq 4$, the Morse Index of the action functional $\mathcal{A}^{3/2}$ at L_k is at least 2.*

Proof. Let $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ an orthonormal basis of $\mathbb{R}^3$ such that the angular velocity vector ω is proportional to $\mathbf{e}_3$. Take $\varepsilon > 0$ and a pair of periodic functions $\mathbf{q},\varphi \in \mathcal{X}$ such that $\mathbf{q}(t)\cdot\mathbf{e}_3 = 0$ and $\varphi(t)\cdot\mathbf{e}_3 = \varphi(t)$ for every $t \in \mathbb{R}$; following [15] we can compute

$$\frac{d^2\mathcal{A}^{3/2}((\mathbf{q},\varepsilon\varphi))}{d\varepsilon^2}|_{\varepsilon=0} = \int_0^{2\pi} |\dot{\varphi}(t)|^2 - \frac{(\varphi(t)-\varphi(t+\tau))^2}{|\mathbf{q}(t)-\mathbf{q}(t+\tau)|^3}dt. \quad (17)$$

Choosing $\mathbf{q} \equiv L_k$ and $\varphi(t) = \mathbf{q}(t/k) \cdot \mathbf{e}_1$, from (17) we obtain

$$\frac{d^2\mathcal{A}^{3/2}((\mathbf{q}, \varepsilon\varphi))}{d\varepsilon^2}|_{\varepsilon=0} = \frac{\pi}{3}\left(I(\bar{\mathbf{q}}_R) - V(\bar{\mathbf{q}}_R)\right) = \frac{\pi}{3}\left[1 - \left(k - \frac{3}{2}\right)^2\right] I(\bar{\mathbf{q}}_R), \quad (18)$$

since $\bar{\mathbf{q}}_R$ verifies condition (11). The right hand side of (18) is negative if we take $k \geq 4$. We now consider the vertical variation $\phi \in \mathcal{X}$, $\phi(t) \cdot \mathbf{e}_3 = \phi(t)$, for every $t \in \mathbb{R}$, defined as $\phi(t) = \mathbf{q}(2t/k) \cdot \mathbf{e}_1$; from (17) we now obtain

$$\frac{d^2\mathcal{A}^{3/2}((\mathbf{q}, \varepsilon\phi))}{d\varepsilon^2}|_{\varepsilon=0} = \frac{\pi}{3}\left[4 - \left(k - \frac{3}{2}\right)^2\right] I(\bar{\mathbf{q}}_R) \quad (19)$$

and the right hand side of (19) is negative when $k \geq 4$.

We conclude the proof with the following computation

$$\begin{aligned}\frac{d^2\mathcal{A}^{3/2}((\mathbf{q}, \varepsilon(\lambda\varphi + \mu\phi)))}{d\varepsilon^2}|_{\varepsilon=0} &= \frac{\pi}{3}\left[(\lambda^2 + 4\mu^2) - \left(k - \frac{3}{2}\right)^2 (\lambda^2 + \mu^2)\right] I(\bar{\mathbf{q}}_R) \\ &= \lambda^2 \frac{d^2\mathcal{A}^{3/2}((\mathbf{q}, \varepsilon\varphi))}{d\varepsilon^2}|_{\varepsilon=0} + \mu^2 \frac{d^2\mathcal{A}^{3/2}((\mathbf{q}, \varepsilon\phi))}{d\varepsilon^2}|_{\varepsilon=0} < 0,\end{aligned}$$

for every $\lambda, \mu \in \mathbb{R}^2 \setminus \{(0,0)\}$. □

Theorem 2. *Let*

$$c_0 := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} \mathcal{A}^{3/2}(\gamma(s)), \quad (20)$$

where Γ is the set of paths in the open set $\mathcal{X}$ joining the relative equilibrium motions L_1 and L_2. Then there exists a generalized critical point for the action functional $\mathcal{A}^{3/2}$ at level c_0, lying in the closure of the set $\mathcal{X}$, which does not coincide with any relative equilibrium motion L_k, $k \in \mathbb{N} \setminus \{n \equiv 0 \mod 3\}$.

Proof. Since the action functional $\mathcal{A}^{3/2}$ does not verify the Palais-Smale condition in the open set $\mathcal{X}$, but just on the whole space X, we need to consider a regularization $\mathcal{A}^{3/2}_\varepsilon$ with $\varepsilon > 0$, defined in (16), in order to apply the Mountain Pass Theorem. For ε small enough, $\mathcal{A}^{3/2}_\varepsilon$ has two strict minimizers, $L_{1,\varepsilon}$, $L_{2,\varepsilon}$ which strongly converge to L_1, L_2 respectively, as $\varepsilon \to 0$. Moreover, for every $\varepsilon > 0$, the functional $\mathcal{A}^{3/2}_\varepsilon$ verifies the assumptions of the Mountain Pass Theorem; we then deduce the existence of a critical point for $\mathcal{A}^{3/2}_\varepsilon$ at level $c_{0,\varepsilon}$, $\mathbf{m}_\varepsilon$, distinct from $L_{1,\varepsilon}$ and $L_{2,\varepsilon}$, with Morse Index smaller then 1. The sequence $(\mathbf{m}_\varepsilon)_\varepsilon$ will then converge, as $\varepsilon \to 0$, to $\mathbf{m} \in X$ such that $\mathcal{A}^{3/2}(\mathbf{m}) = c_0$, where c_0 is defined in (20). When $\mathbf{m} \in \mathcal{X}$ then we conclude using the lower-semi continuity of the Morse Index and Lemma 1. When $\mathbf{m} \in \partial\mathcal{X}$, then we have a generalized collision critical point for $\mathcal{A}^{3/2}$ that cannot coincide with any L_k. □

4 Determination of a numerical solution for the 3-body problem

In this section we explain how we can numerically detect a new critical point for the action functional $\mathcal{A}^{3/2}$. First, we provide a description of an algorithm that can be used to find a numerical solution whenever the functional has two strict minimizers. Then we apply it in Paragraph 4.2 to the action functional associated to the 3-body problem.

4.1 A bisection algorithm

The algorithm described in this paragraph provides a constructive proof for the existence of critical points of mountain pass type and has the main advantage that it can be easily implemented numerically. The reader can find the proofs concerning this subject in [7].

To explain this method, we need some preliminaries about the steepest descent flow associated to the functional f still of class C^2 on the open subset Ω of the Hilbert space X.

Definition 9. The point $x_0 \in Crit(f)$ is a *local minimizer* for the functional f, if there exists $r > 0$, such that $f(x) \geq f(x_0)$, for all $x \in B_r(x_0)$; x_0 is called a *strict local minimizer* if there exists $r_0 > 0$ such that for every $r < r_0$, $\inf\limits_{x \in \partial B_r(x_0)} f(x) > f(x_0)$.

Definition 10. The *steepest descent flow* associated with the functional f is the function $\eta : \mathbb{R}_+ \times \Omega \to \Omega$ defined as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\eta(t,x) = -\dfrac{\nabla f(\eta(t,x))}{1 + \|\nabla f(\eta(t,x))\|} \\ \eta(0,x) = x \end{cases} \tag{21}$$

Definition 11. We say that a subset $\Omega_0 \subset \Omega$ is *positively invariant* for the flow η if, for every $x_0 \in \Omega_0$, $\{\eta(t,x_0), t \geq 0\} \subset \Omega_0$. The *$\omega$-limit of* $x \in \Omega$ for the flow η, is defined as the closed positively invariant set

$$\omega_x = \left\{ \lim_{t_n \to +\infty} \eta(t_n, x) : (t_n)_n \subset \mathbb{R}_+ \right\}.$$

Let η be the steepest descent flow defined in (21); let $x \in \Omega$ and $T > 0$; then one can easily prove the following inequalities (see Lemmata 2.1 and 2.2 in [5]):

$$\left| \left\{ t \in [0,T] : \frac{\|\nabla f(\eta(t,x))\|^2}{1 + \|\nabla f(\eta(t,x))\|} \geq \gamma \right\} \right| \leq \frac{f(x) - f(\eta(T,x))}{\gamma}, \quad \gamma > 0; \tag{22}$$

$$|\{t \in [0,T] : \|\nabla f(\eta(t,x))\| \geq \gamma\}| \leq \frac{f(x) - f(\eta(T,x))}{\gamma^2/2}, \quad \gamma \in (0,1]. \tag{23}$$

Let $c \in \mathbb{R}$ be such that the sublevel f^c is disconnected, we denote $(F_i^c)_i$ its disjoint connected components

$$f^c = \bigcup_i F_i^c, \quad F_i^c \cap F_j^c = \emptyset, \ \forall i \neq j.$$

For every index i, we consider the basin of attraction of the set F_i^c

$$\mathcal{F}_i^c := \{x \in \Omega : \omega_x \subset F_i^c\}\,.$$

We can now state the following results (see [7] for the proofs)

Theorem 3. *Let f^c be a disconnected sublevel for the functional f. Let F_i^c be the disjoint connected components of f^c and let $\mathcal{F}_i^c$ be their basins of attraction. Let $x_i \in F_i^c$, $i = 1, 2$, and $\gamma \in \Gamma_{x_1,x_2}$, where Γ_{x_1,x_2} is the set of path defined in* (12). *Then there exists $\bar{x} \in \gamma([0, 1]) \cap \partial\mathcal{F}_1^c$.*

Corollary 2. *In the same conditions of Theorem 3, let $\bar{x} \in \gamma([0, 1]) \cap \partial\mathcal{F}_1^c$, then $f(\omega_{\bar{x}}) \geq c$ and there exists a sequence $(x_n)_n \subset \partial\mathcal{F}_1^c$, $x_n = \eta(t_n, \bar{x})$, such that*

$$\lim_{n\to+\infty} \nabla f(x_n) = 0 \text{ and } \lim_{n\to+\infty} f(x_n) = f(\omega_{\bar{x}}).$$

In particular there exists a sequence $(\tilde{y}_n)_n \subset \Omega$, such that

$$\lim_{n\to+\infty} \nabla f(\tilde{y}_n) = 0, \text{ and } c \leq f(\tilde{y}_n) \leq f(\bar{x}),\ \forall n \in \mathbb{N}.$$

The following algorithm was proposed in [7] to obtain the N-th element of the sequence $(\tilde{y}_n)_n$, as described in Corollary 2, starting from a path γ joining the two strict minimizers x_1 and x_2. Notice that the algorithm converges to a critical point in Ω whenever the Palais-Smale in Ω condition is fulfilled.

Algorithm 1. $(x_1, x_2, N; \tilde{y}_N)$

`Step 0.` $s_1^0 = 0$, $s_2^0 = 1$, $s_m^0 = \dfrac{s_1^0 + s_2^0}{2}$,
$x_1^0 = x_1$, $x_2^0 = x_2$, $x_m^0 = \gamma(s_m^0)$

`Step i.` if $\omega_{x_m^{i-1}} \subset F_1^{c_0}$, $s_1^i = s_m^{i-1}$, $s_2^i = s_2^{i-1}$
else $s_1^i = s_1^{i-1}$, $s_2^i = s_m^{i-1}$
$x_1^i = \gamma(s_1^i)$, $x_2^i = \gamma(s_2^i)$, $s_m^i = \dfrac{s_1^i + s_2^i}{2}$, $x_m^i = \gamma(s_m^i)$
$T_i := \inf\left\{t \geq 0 : f(\eta(t, x_1^i)) \leq c\right\}$
$\tilde{T}_i \in [0, T_i]$ such that $\|\nabla f(\eta(T_i, x_1^i))\| \leq \|\nabla f(\eta(t, x_1^i))\|$, $\forall t \in [0, T_i]$,
$\tilde{y}_i := \eta(\tilde{T}_i, x_1^i)$.

The maximum number of steps N iterated in Algorithm 1 may depend on the distance between the starting points x_1, x_2 and on the possible strong sharpness of the graph of f. Of course this may cause numerical errors in the integration method. Fix $\varepsilon, \delta > 0$. The following algorithm allows us to approximate a locally optimal path joining (by juxtaposition of a finite number of locally optimal paths) the starting points x_1 and x_2.

Algorithm 2. (x_1, x_2, N_{max})

`Step 0.` $N_{max}^0 = N_{max}$;
Algorithm 1 $(x_1, x_2, N_{max}^0; \tilde{y}_{N_{max}^0})$

`Step k.` if $\|\nabla f(\tilde{y}_{N_{max}^{k-1}})\| < \delta$, STOP

else $T_{def} := \inf\left\{t > 0 : \text{dist}\left(\eta(t, x_1^{N_{max}^{k-1}}), \eta(t, x_2^{N_{max}^{k-1}})\right) \geq \frac{\varepsilon}{2^{k-1}}\right\}$;

$x_j := \eta(T_{def}, x_j^{N_{max}^{k-1}})$, $j = 1, 2$;

$N_{max}^k := N_{max}^{k-1} - 1$;

Algorithm 1 $(x_1, x_2, N_{max}^k; \tilde{y}_{N_{max}^k})$

4.2 The numerical algorithm and its implementation

We now describe how Algorithm 2 can be used to detect numerical critical points that are not strict local minimizers for the 3-body problem with the simple choreography constraint. To avoid numerical collision solutions, we consider the perturbation of $\mathcal{A}^{3/2}$ defined in (16). Starting from a point in X, we use a steepest descent method to reach a local minimizer for (16). The steepest descent direction at $\mathbf{q} \in X$ is the one $\mathcal{A}_\varepsilon^{3/2}$ decreases most rapidly per unit distance traveled in the functional space $H^1_{2\pi}(\mathbb{R}, \mathbb{R}^3)$ (see Definition 10).

To compute the action functional $\mathcal{A}_\varepsilon^{3/2}$ and its derivative, we use the finite real Fourier representation of the elements of X; for every $\mathbf{q} \in X$ we consider the approximation

$$\mathbf{q}(t) \approx \left(\sum_{k=1}^{M} \mathbf{q}_k \sin(kx) + \sum_{k=0}^{M} \mathbf{q}'_k \cos(kx)\right). \tag{24}$$

In our investigations we take $M = 60$ and we consider $\mathbf{q}_0 = \mathbf{0}$; the second condition means that we actually work in a space of zero mean loops; this is not a restriction since minimizers of the $\mathcal{A}^{3/2}$ have zero mean.

Remark 6. To reduce the number of Fourier coefficients in the sum (24), we suppose that every $\mathbf{q} = (x, y, z) \in X$ satisfies the following natural symmetry constraint

$$x(t) = x(-t),\ y(t) = -y(-t),\ z(t) = -z(-t). \tag{25}$$

Even if condition (25) is not necessary in this setting, it allows us to deal with *strict* local minimizers; in fact when we consider the action functional on X with the choreography constraint, without any additional symmetry, if there exists a minimizer we necessarily have a continuum of minimizers generated by the groups $SO(2)$ or $SO(3)$, when we are in a rotating or in an inertial frame, respectively.

We now turn to the numerical computation of the mountain pass critical point whose existence is proved in Theorem 2. Since L_1 and L_2 are strict global minimizers (see Proposition 2), there exists $\bar{\varepsilon} > 0$ such that the sublevel $(\mathcal{A}_\varepsilon^{3/2})^{c_L + \bar{\varepsilon}}$ is disconnected. Proposition 4 ensures that the sequence defined in Algorithm 2 converges to a critical point that is not a strict minimizer and whose action level exceeds c_L. In order to implement the algorithm, we need to initialize some variables:

1) We fix $\varepsilon_L < \bar{\varepsilon}$ that we use as a threshold in order to decide whether $\mathbf{q} \in X$ coincides with a global minimizer or not. In terms of the Fourier sum, since $\mathbf{q}$ is approximated as in (24), we assume that

- $\mathbf{q}$ coincides with L_1 if $\sum_{k\neq 1} |\mathbf{q}_k| + |\mathbf{q}'_k| < \varepsilon_L$;
- $\mathbf{q}$ coincides with L_2 if $\sum_{k\neq 2} |\mathbf{q}_k| + |\mathbf{q}'_k| < \varepsilon_L$.

2) We fix an $\varepsilon_g > 0$ such that if the norm of the gradient of the action functional evaluated at a point $\mathbf{q}$ is smaller then ε_g, then x is considered a critical point for the functional.

3) Since we seek non–planar solutions, we define the path $\gamma : [0,1] \to X$ joining L_1 and L_2 in such a way that $\gamma(s)$ does not entirely lie in the rotating plane for every s. With this aim, we consider a pair of non planar perturbations m_1, m_2 respectively of L_1 and L_2 such that $\omega_{m_i} = L_i$, $i = 1,2$ (in the sense explained in 1)). We term $\bar\gamma$ the linear path joining m_1 to m_2, and we define the path γ as the juxtaposition $-\alpha_{m_1} \circ \bar\gamma \circ \alpha_{m_2}$, where

$$\alpha_{m_i} := (\gamma^1_{m_i} \circ \gamma^2_{m_i}), \quad i = 1,2$$

and

$$\gamma^1_{m_i}(\lambda) := \eta(T_{m_i}\lambda, m_i), \quad \gamma^2_{m_i}(\lambda) := (1-\lambda)\eta(T_{m_i}, m_i) + \lambda L_i, \quad i = 1,2,$$

where T_{m_i} is the smallest $t > 0$ such that $\eta(t, m_i)$ lies in the connected component of $(\mathcal{A}^{3/2}_\varepsilon)^{c_L+\bar\varepsilon}$ containing L_i.

4) We consider the maximal value of the action functional along the path γ, $M_\gamma := \max_\tau \mathcal{A}^{3/2}_\varepsilon(\gamma(\tau))$, and we define the time

$$T_\gamma \geq \frac{2(M_\gamma - c_L)}{\varepsilon_g^2}$$

in such a way that, for every $\tau \in [0,1]$, there exists $t_\tau \in [0, T_\gamma]$ such that $\|\nabla \mathcal{A}^{3/2}_\varepsilon(\eta(t_\tau, \gamma(\tau)))\| < \varepsilon_g$ (see inequality (23)).

In order to use Algorithm 2, we still have to give an appropriate meaning to the sentence

$$\texttt{if } \omega_{x^i_m} \subset F_1^{c_0} \texttt{ ...else ... } .$$

The condition `if` $\omega_{x^i_m} \subset F_1^{c_0}$ has to be interpreted (in the sense specified in 1) at page 15) as `if` $\eta(T_\gamma, x^i_m)$ `coincides with` L_1; if this requirement is not verified three possible situations can occur

(i) $\eta(T_\gamma, x^i_m)$ coincides with L_2;

(ii) $\eta(T_\gamma, x^i_m)$ does not coincide with L_2, but $\|\nabla \mathcal{A}^{3/2}_\varepsilon(\eta(T, x^i_m))\| < \varepsilon_g$;

(iii) $\|\nabla \mathcal{A}^{3/2}_\varepsilon(\eta(T, x^i_m))\| \geq \varepsilon_g$

When (i) or (ii) happens, then $\eta(T_\gamma, x_m^i)$ is a critical point different from L_1. In particular if (ii) occurs, then we have found a new approximate critical point for $\mathcal{A}_\varepsilon^{3/2}$. If (iii) is verified we have to replace T_γ with $T'_\gamma > T_\gamma$ such that $\eta(T'_\gamma, x_m^i)$ verifies (i) or (ii).

Taking the above precautions, Algorithm 2 was used to determine the numerical simple choreography for the 3-body problem in $\mathbb{R}^3$ whose first non-zero truncated Fourier coefficients are written in Table 4.2. As the reader can see in Figure 1, this solution it is not planar, it does not intersect itself and it is clearly different from the well known choreographies for the 3-body problem in $\mathbb{R}^3$.

$\hat{x}'_1 = 0.849736$	$\hat{y}_1 = 0.889862$	$\hat{z}_1 = -0.535402$
$\hat{x}'_2 = 0.874442$	$\hat{y}_2 = 0.865156$	$\hat{z}_2 = 0.088436$
$\hat{x}'_3 = 0$	$\hat{y}_3 = 0$	$\hat{z}_3 = 0$
$\hat{x}'_4 = -0.020397$	$\hat{y}_4 = -0.019728$	$\hat{z}_4 = -0.004747$
$\hat{x}'_5 = 0.004740$	$\hat{y}_5 = 0.004545$	$\hat{z}_5 = 0.001273$
$\hat{x}'_6 = 0$	$\hat{y}_6 = 0$	$\hat{z}_6 = 0$
$\hat{x}'_7 = 0.000343$	$\hat{y}_7 = -0.000325$	$\hat{z}_7 = -0.000107$
$\hat{x}'_8 = 0.000100$	$\hat{y}_8 = 0.000094$	$\hat{z}_8 = 0.0000327.$

Table 1: First Fourier (truncated) coefficients of the numerical trajectory determined using Algorithm 2.

4.3 More numerical non rigorous results

We conclude this section exposing some numerical results we obtained concerning the existence of a branch of solutions starting from the mountain pass orbit numerically detected in the previous paragraph. We would like to stress that the results in this section, as well as those of Section 4.2, are not rigorous, but they provide some hints towards the computer assisted proofs of the following section.

Figure 2 represents the action levels of some solutions for the 3-body problem with the choreography constraint when $\omega \in [0, 3)$. This graph has been obtained by using a continuation method: we start from a known solution, $\mathbf{q}_{\bar{\omega}}$, for $\omega = \bar{\omega}$, then we modify the value of the angular velocity of a fixed quantity $\varepsilon > 0$, and we start a Newton's method from $\mathbf{q}_{\bar{\omega}}$ to find a critical point $\mathbf{q}_{\bar{\omega}+\varepsilon}$ for the functional $\mathcal{A}^{\bar{\omega}+\varepsilon}$ (respectively $\mathbf{q}_{\bar{\omega}-\varepsilon}$ for the functional $\mathcal{A}^{\bar{\omega}-\varepsilon}$). Our starting points are the two Lagrange solutions, L_1 and L_2 with periods respectively 2π and π and the mountain pass solution in Figure 1 found with the numerical algorithm when $\omega = 1.5$. The existence of the two branches of Lagrange solutions (corresponding to L_1 and L_2) follows from Proposition 2, while the existence of the branch in the interval $[1, 2]$ starting from the mountain pass solution at $\omega = 1.5$ will be rigorously proved in the next section. Figure 4 and 5 shows the orbits on the branch starting from the mountain pass solution when $\omega = 1.25$ and $\omega = 1$ respectively.

As shown in [22] and in [9], in the interval $\omega \in [0, 1)$ the Lagrange solution with minimal period π, L_2, is no more a minimizer and, from the graph of its action

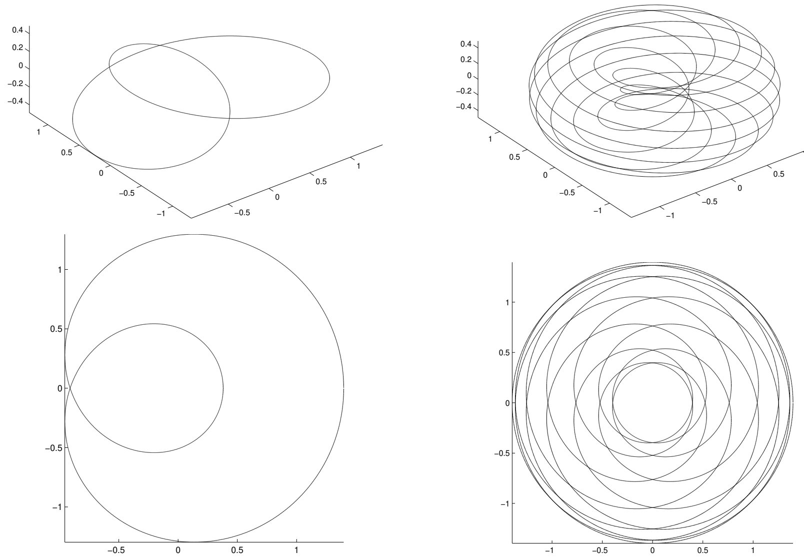

Figure 4: Orbit obtained when $\omega = 1.25$ using a continuation method from the mountain pass solution at $\omega = 1.5$. The left pictures show the orbit in the rotating frame, the right the trajectory in the inertial one.

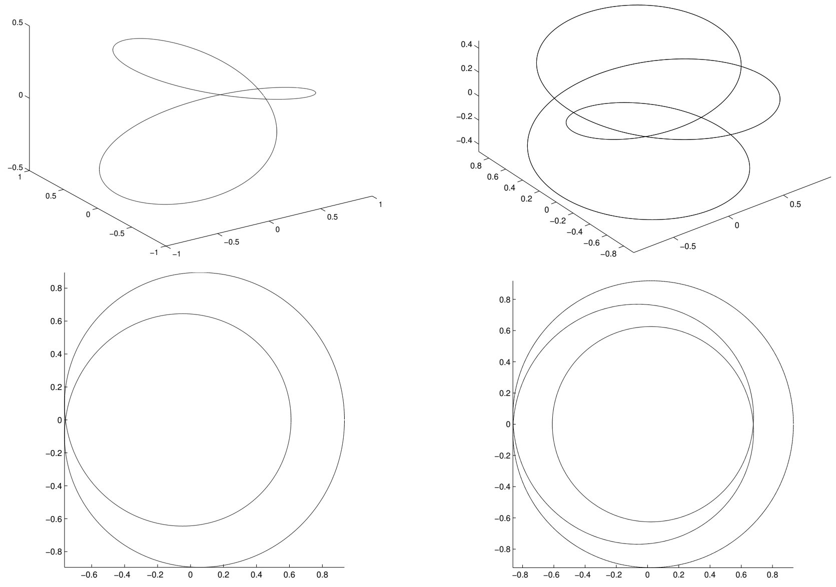

Figure 5: Orbit obtained when $\omega = 1$ starting from the mountain pass solution at $\omega = 1.5$. In the left pictures the orbit in the rotating frame, at the right the trajectory in the inertial one.

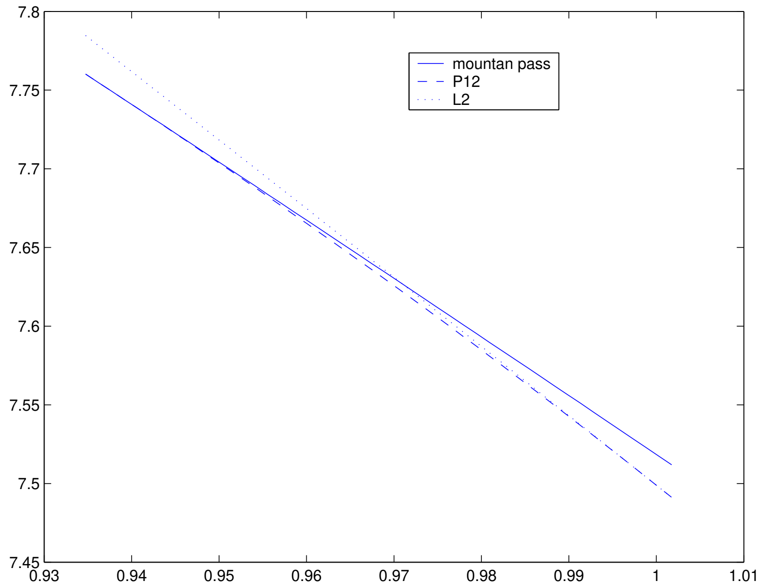

Figure 6: Action levels for the Lagrange L_2, the mountain pass solution and the P_{12}-symmetry solution when the angular velocity is close to 1.

level, bifurcates the one of the P_{12}-symmetry solution which ends, as $\omega = 0$ in the eight-shaped solution introduced in [12]. In Figure 6, where we focus our attention on the values of ω close to 1, we see that the graph of the action level of the new mountain pass solution bifurcates for the one of the P_{12}-symmetry solution when the angular velocity is approximately 0.94; in particular this numerical result agrees with the one obtained in [11].

5 The computer–assisted proof

The proofs of Theorems B and C are based on a computer assisted method exploited in the context of the Fermi–Pasta–Ulam model and of the Kuramoto–Sivashinski equation in [3] and [4]. This procedure is based on the method introduced by Koch in [20]. We detail here the main novelties with respect to the arguments in the above mentioned papers, to which we refer for the full proof.

Given $\rho > 0$, let $\mathcal{D}_\rho = \{\xi \in \mathbb{C}\colon |\mathrm{Im}(\xi)| < \rho\}$, and denote by $\mathcal{C}_\rho$ the space of all functions $f : \mathcal{D}_\rho \to \mathbb{C}$,

$$f(\xi) = \sum_{k=1}^{\infty} f_k \sin(k\xi) + \sum_{k=0}^{\infty} f'_k \cos(k\xi)\,, \qquad \xi \in \mathcal{D}_\rho\,, \tag{26}$$

which take real values when restricted to $\mathbb{R}$ and for which the norm

$$\|f\|_\rho = \sup_{\xi \in \mathcal{D}_\rho} |f(\xi)| \tag{27}$$

is finite.

We point out that the computer assisted technique that we use to prove the existence of a solution, requires such a solution to be isolated. It is therefore necessary to break the $SO(2)$ symmetry of the problem. We achieve the isolatedness by restricting our search for solutions to the space of symmetric loops $\mathcal{X}_\rho \subset \left(\mathcal{C}_\rho\right)^3$ defined as

$$\begin{aligned}\mathcal{X}_\rho = \Big\{\mathbf{q} :\ \forall t \in \mathbb{R}\ \ \mathbf{q}(-t) = R_x\mathbf{q}(t)\ \text{ and } \\ \mathbf{q}(t) + \mathbf{q}(t+2\pi/3) + \mathbf{q}(t+4\pi/3) = \mathbf{0}\Big\}\end{aligned} \tag{28}$$

where R_x is the linear operator associated to the reflection with respect to x-axis, that is

$$R_x(x,y,z) = (x,-y,-z), \qquad \forall (x,y,z) \in \mathbb{R}^3.$$

On $\mathcal{X}_\rho$, we define the norm

$$\|\mathbf{q}\|_\rho = \max\{\|x\|_\rho, \|y\|_\rho, \|z\|_\rho\}\,.$$

Remark 7. We observe for every 2π-periodic function $f \in \mathcal{C}_\rho$ with Fourier expansion (26), we have that condition $f(t)+f(t+2\pi/3)+f(t+4\pi/3) = 0$ is equivalent to impose that its $3n$-th Fourier coefficients are zero, that is $f_{3n} = f'_{3n} = 0$, for every $n \in \mathbb{N}$. This fact follows easily from the identities

$$\begin{aligned}&\sin(2\pi k/3) + \sin(4\pi k/3) = 0 &&\forall k \in \mathbb{N},\\ &\cos(2\pi k/3) + \cos(4\pi k/3) = -1 &&\forall k \in \mathbb{N} \text{ such that } 3 \nmid k.\end{aligned}$$

Then the constraint $\mathbf{q}(t) + \mathbf{q}(t+2\pi/3) + \mathbf{q}(t+4\pi/3) = \mathbf{0}$, verified by the loops in $\mathcal{X}_\rho$, can be replaced by condition

$$P_k(\mathbf{q}) = \mathbf{0}, \qquad \text{whenever } k = 3n,\ n \in \mathbb{N}$$

where P_k indicates the projection in the k-th component.

Given a $2\pi-$periodic function $f : \mathbb{R} \to \mathbb{R}$, we define f_r and f_a as

$$f_r(t) = f(t) - f(t+2\pi/3) \qquad \text{and} \qquad f_a(t) = f(t) - f(t-2\pi/3).$$

Given a $2\pi-$periodic function in $\mathbb{R}^3$ $(x(t), y(t), z(t))$, let $R_r(t)$ be defined by $R_r(t) = \sqrt{x_r(t)^2 + y_r(t)^2 + z_r(t)^2}$ and analogously define $R_a(t)$. Moreover, by P_o and P_e we denote the projection of $\mathcal{C}_\rho$ on its subspaces of odd and even functions respectively.

Proposition 5. *Let F be the operator defined on $\mathcal{C}_\rho$ as*

$$\begin{cases} F_1(x,y,z) = \omega\partial^{-1}y + \partial^{-2}\left(\omega^2 x + x_r R_r^{-3} + x_a R_a^{-3}\right), \\ F_2(x,y,z) = -\omega\partial^{-1}x + \partial^{-2}\left(\omega^2 y + y_r R_r^{-3} + y_a R_a^{-3}\right), \\ F_3(x,y,z) = \partial^{-2}\left(z_r R_r^{-3} + z_a R_a^{-3}\right), \end{cases} \tag{29}$$

where ∂^{-1} denotes the antiderivative operator on the space of continuous 2π-periodic functions with average zero. Then $F(\mathcal{X}_\rho) \subset \mathcal{X}_\rho$.

Proof. A short computation shows that, if $x(t)$ is even and $y(t)$ is odd, then $x_a(t) = x_r(-t)$ and $y_a(t) = -y_r(-t)$. It follows that, if $\mathbf{q} = (x, y, z) \in \mathcal{X}_\rho$, then $R_a(t) = R_r(-t)$ and therefore $x_r R_r^{-3} + x_a R_a^{-3} = 2P_e\left(x_a R_a^{-3}\right) = 2P_e\left(x_r R_r^{-3}\right)$, $y_r R_r^{-3} + y_a R_a^{-3} = 2P_o\left(y_a R_a^{-3}\right) = 2P_o\left(y_r R_r^{-3}\right)$ and similarly for the z component. We can then conclude that $F\mathbf{q}(-t) = R_x\left(F\mathbf{q}(t)\right)$, for every $t \in \mathbb{R}$. We are left to prove that whenever $\mathbf{q} \in \mathcal{X}_\rho$ then $F\mathbf{q}(t) + F\mathbf{q}(t + 2\pi/3) + F\mathbf{q}(t + 4\pi/3) = \mathbf{0}$. With this aim we simply observe that for every $(x, y, z) \in \mathcal{X}_\rho$

$$\begin{cases} x_r(t+2\pi/3) = -x_a(t+4\pi/3) \\ x_r(t+4\pi/3) = -x_a(t) \\ x_r(t) = -x_a(t+2\pi/3) \end{cases} \quad \text{and} \quad \begin{cases} R_r(t+2\pi/3) = R_a(t+4\pi/3) \\ R_r(t+4\pi/3) = R_a(t) \\ R_r(t) = R_a(t+2\pi/3)\,. \end{cases}$$

It follows that, if $X(t) = x_r(t)R_r^{-3}(t) + x_a(t)R_a^{-3}(t)$, then $X(t) + X(t + 2\pi/3) + X(t + 4\pi/3) = 0$. With similar computations on the y and z components, we conclude that $\left(P_e\left(x_a R_a^{-3}\right), P_o\left(y_a R_a^{-3}\right), P_o\left(z_a R_a^{-3}\right)\right) \in \mathcal{X}_\rho$. $\square$

The dynamical system associated to the 2π-periodic choreographic 3-body problem in a 3-dimensional space with angular velocity $(0, 0, \omega)$ (see $(P)_\omega$) is then equivalent to

$$(P)'_\omega \qquad \begin{cases} \ddot{x} - 2\omega\dot{y} - \omega^2 x = x_r R_r^{-3} + x_a R_a^{-3} \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = y_r R_r^{-3} + y_a R_a^{-3} \\ \ddot{z} = z_r R_r^{-3} + z_a R_a^{-3} \end{cases}$$

where $(x, y, z) \in \mathcal{X}_\rho$. As a straightforward consequence of Proposition 5 we have the following result.

Proposition 6. *Fixed points of the function F in $\mathcal{X}_\rho$ are solutions to $(P)_\omega$.*

We note that $\mathcal{C}_\rho$ is a Banach algebra, that is, $\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho$, for all f and g in $\mathcal{C}_\rho$. Furthermore, ∂^{-1} acts as a compact linear operator on $\mathcal{X}_\rho$. This shows that equation (29) defines a differentiable map F on $\mathcal{X}_\rho$ with compact derivatives $DF(\mathbf{q})$. Thus, F can be well approximated locally by its restriction to a suitable finite dimensional subspace of $\mathcal{X}_\rho$. This property makes it ideal for a computer-assisted analysis.

Our goal is to find fixed points for F by using a Newton like iteration, starting from the initial guess $\mathbf{q}_0$. The standard Newton map $\mathbb{N}$ associated with F is given by $\mathbb{N}(\mathbf{q}) = F(\mathbf{q}) - \mathbb{M}(q)[F(\mathbf{q}) - \mathbf{q}]$, with $\mathbb{M}(\mathbf{q}) = [DF(\mathbf{q}) - \mathbb{I}]^{-1} + \mathbb{I}$. If the spectrum of $DF(\mathbf{q})$ is bounded away from 1 and $\mathbf{q}_0$ is sufficiently close to a fixed point of F, then $\mathbb{N}$ is a contraction in some neighborhood of $\mathbf{q}_0$. Due to the compactness of $DF(\mathbf{q})$, this contraction property is preserved if we replace $\mathbb{M}(\mathbf{q})$ by a suitably fixed linear operator M close to $\mathbb{M}(\mathbf{q}_0)$. This leads us to consider the new map $\mathcal{C}$, defined by

$$\mathcal{C}(\mathbf{q}) = F(\mathbf{q}) - M[F(\mathbf{q}) - \mathbf{q}]\,, \qquad \mathbf{q} \in \mathcal{X}_\rho\,. \tag{30}$$

To be more specific, M will be chosen to be a finite dimensional matrix, in the sense that $M = \mathbb{P}_\ell M \mathbb{P}_\ell$ for some $\ell > 0$, where $\mathbb{P}_\ell$ denotes the canonical projection in $\mathcal{X}_\rho$ onto Fourier polynomials of degree $k \le \ell$. We also verify that $M - \mathbb{I}$ is invertible, so that $\mathcal{C}$ and F have the same set of fixed points. For the reasons mentioned above, we expect $\mathcal{C}$ to be a contraction on some close ball $B(\mathbf{q}_0, r)$ in $\mathcal{X}_\rho$ of radius $r > 0$, centered at $\mathbf{q}_0$. More precisely, we use the following modification of the Contraction Principle, whose proof is straightforward:

Lemma 2. *Let $F : \mathcal{X}_\rho \to \mathcal{X}_\rho$ be a differentiable map. Let M be a bounded linear operator on $\mathcal{X}_\rho$, such that $M - \mathbb{I}$ has a bounded inverse. Let $\mathcal{C}$ be defined as in (30). Consider a pair $(\mathbf{q}_0, r)$, $r > 0$ and $\mathbf{q}_0 \in \mathcal{X}_\rho$. If there exist $\varepsilon, K \in (0, 1)$ such that*

$$\|\mathcal{C}(\mathbf{q}_0) - \mathbf{q}_0\|_\rho < \varepsilon\,, \qquad \|D\mathcal{C}(\mathbf{q})\| < K\,, \qquad \varepsilon + Kr < r\,, \tag{31}$$

for all functions $\mathbf{q}$ in a closed ball $B(\mathbf{q}_0, r) \subset \mathcal{X}_\rho$, then F has a unique fixed point in $B(\mathbf{q}_0, r)$.

The following lemma is proved by computer assisted methods, see [2] for the details of the proof, and it yields directly the proof of Theorem B.

Lemma 3. *Let $\rho = 2^{-27}$, $\varepsilon = 1.66 \cdot 10^{-8}$, $r = 10^{-6}$ and $K = 0.156$. There exists a Fourier polynomial $\mathbf{q}_0$ such that $\|\mathcal{C}(\mathbf{q}_0) - \mathbf{q}_0\|_\rho < \varepsilon$ and $\|D\mathcal{C}(\mathbf{q})\| < K$ for all $\mathbf{q} \in B(\mathbf{q}_0, r)$. The first components of $\mathbf{q_0}$ are given in Table 4.2.*

As a next step we wish to extend the existence of a solution to the existence of a full branch of solutions, depending on the angular velocity ω. To start with, keeping in mind the dependence of the nonlinear operator on the angular velocity, we apply a suitable version of the local implicit function theorem, which easily follows from Lemma 2 (see also [4]).

Proposition 7. *Consider a triple $(I, \mathbf{q}_0, r)$, where I is a subinterval of $\mathbb{R}$, $\mathbf{q}_0$ a function in $\mathcal{X}_\rho$, and r a positive real number. Assume that there exists a bounded linear operator M on $\mathcal{X}_\rho$, and constants $\varepsilon, K > 0$, such that $M - \mathbb{I}$ has a bounded inverse, and such that the bounds (31) hold, uniformly with respect to ω in an open neighborhood of I, and for all functions $\mathbf{q}$ in a closed ball B in $\mathcal{X}_\rho$ of radius r, centered at $\mathbf{q}_0$. Then for every $\omega \in I$, the function F defined in* (29) *has a unique fixed point $\mathbf{q}_\omega \in B$ and the map $\omega \mapsto \mathbf{q}_\omega$ is smooth.*

Next, we consider the problem of gluing such local solution curves together, into a unique branch.

Definition 12. We say that a pair of triples $(\omega_i, \mathbf{q}_i, r_i)$, $i = 0, 1$, is *admissible* if both triples satisfies the hypotheses of Lemma 2 and if there exists a third element $\bar{\mathbf{q}} \in \mathcal{X}_\rho$, and a real number $R \geq \max_i(\|\mathbf{q}_i - \bar{\mathbf{q}}\| + r_i)$, such that $(I, \bar{\mathbf{q}}, R)$ satisfies the hypotheses of Proposition 7, where $I = [\omega_0, \omega_1]$.

Notice that, due to the uniqueness statement in Proposition 7, the solution curve associated with $(I, \bar{\mathbf{q}}, R)$ has to pass through the two solutions associated with the triples $(\omega_i, \mathbf{q}_i, r_i)$. Thus, such pairs can be linked together to form a chain which "shadows" a unique solution curve.

The following lemma is proved by computer assisted methods, see [2] for the details of the proof, and taking into account Remark 2 it yields directly the proof of Theorem C.

Lemma 4. *Let $\rho = 2^{-27}$. There exists a monotone sequence of real numbers $\{\omega_i\}_{i=1}^n$, $\omega_1 = 1$, $\omega_n = 3/2$ and a sequence $\{(\mathbf{q}_i, r_i)\}_{i=1}^n$ in $\mathcal{X}_\rho \times \mathbb{R}_+$, such that the pair*

$$\{(\omega_i, \mathbf{q}_i, r_i), (\omega_{i+1}, \mathbf{q}_{i+1}, r_{i+1})\}$$

is admissible for each positive $i < n$.

The methods developed in [3, 4] work for polynomial nonlinearities. A major difficulty to face in the present problem is that the Kepler potential is not polynomial, though analytic in the mutual distances. Hence we have to chose a suitable polynomial approximation: all the estimates carried with computer assistence will have to take into account of this extra error, in the manner described hereafter. We chose to approximate the functions $x \mapsto x^{-3/2}$ and $x \mapsto x^{-5/2}$ with Chebyshev's polynomials in a suitable domain. We describe in detail the first case, the other being similar. First we compute rigorous bounds $0 < a < b$ and $c > 0$ such that, for all $\mathbf{q}$ in a closed ball $B(\mathbf{q}_0, r)$ in $\mathcal{X}_\rho$ as given in Lemmas 2 and 4, if R_r is defined as above, then $R_r(\mathcal{D}_\rho) \subset [a, b] \times [-c, c]$. In order compute such bounds, we added to the algorithms described in [20] the bounds of the functions sine and cosine, and we use Lagrange's theorem.

Then we compose the function R_r with the linear map $T(x) = \frac{2x}{b-a} + \frac{a+b}{a-b}$, so that the real part of $T(R_r)(\mathcal{D}_\rho)$ lies in $[-1, 1]$. We approximate the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \left(T^{-1}(x)\right)^{-3/2}$ with a polynomial P of order n. in order to do so, let $P_n(x)$ be Chebyshev's approximation of the function f of order n. Our choice is motivated both by computational reasons (indeed Chebyshev's polynomials are easy to compute) and by our need to have good estimates of the errors in L^∞; indeed as it is well known, they provide an almost optimal approximation in $[-1, 1]$ (see [17]). We recall that Chebyshev's polynomials are defined recursively as follows: $T_0(x) = 1$, $T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. The polynomial $T_n(x)$ has n zeros at the points $x_k = \cos\left(\frac{\pi\left(k-\frac{1}{2}\right)}{n}\right)$, $k = 1 \ldots, n$. Chebyshev's approximation of order n of the function f is defined as

$$P_n(x) = -\frac{1}{2}c_0 + \sum_{k=0}^{n} c_k T_k(x) \,,$$

where

$$c_j = \frac{2}{n+1} \sum_{k=0}^{n} f\left(\cos\left(\frac{\pi\left(k+\frac{1}{2}\right)}{n+1}\right)\right) \cos\left(\frac{\pi j\left(k+\frac{1}{2}\right)}{n+1}\right) .$$

It is well known that $P_n(x_k) = f(x_k)$ for all $x_k = \cos\left(\frac{\pi\left(k+\frac{1}{2}\right)}{n+1}\right)$, $k = 0, \ldots, n$.

Note that, due to round–off errors, we cannot use exactly Chebyshev's polynomials, but only their representable counterpart. Of course, we have to estimate the distance between the actual polynomial approximation that we use and the original function. More precisely, we choose a degree n and then we need to compute a rigorous bound

$$E_n \geq \sup_{t \in [a,b] \times [-c,c]} |f - P_n| \,.$$

This estimate is also obtained with computer assistance, using the Taylor expansion of f. Then we can compute the representation of R_r^{-3} by the algorithms described above. We can take the approximation error E_n into account by adding it to the V_0^+ component. We observed that this choice of approximation provides good bounds for the function F, which needs to be computed at a point in order to satisfy the assumptions of Lemma 3. For this computation, depending on the value of ω, we used n varying in the interval [22, 33]. A major difficulty arises in the computation of $R_r^{-3}(\mathbf{q}_0 + h)$ and $R_r^{-5}(\mathbf{q}_0 + h)$, where h is an arbitrary function of norm less than r. This is a crucial step in controlling the Lipschitz constant of our candidate contraction $\mathcal{C}$. In this case, the procedure described above does not provide good bounds, because when computing the representation of a polynomial of high order of the representation of a ball, the errors grow too fast and it is not possible to obtain useful estimates. On the other hand, given a polynomial P of degree N, the following trivial inequality holds:

$$P(\mathbf{q}_0 + h) = \sum_{k=0}^{N} P^{(k)}(\mathbf{q}_0)\frac{h^k}{k!} ,$$

where $P^{(k)}$ is the k-th derivative of P, therefore it is possible to compute $R_r^{-3}(\mathbf{q}_0+h)$ and $R_r^{-5}(\mathbf{q}_0 + h)$ without computing directly the polynomial on the ball. This computation turns out to be much more efficient.

The precise definition of all these bounds, down to the level of inequalities between (sums and products of) representable numbers, has been written in the programming language Ada95. A computer (Intel Pentium class PC) was then used to translate these definitions to machine code (with the public version 3.15p of the GNAT compiler) and to verify the actual inequalities. The computer programs and input data are available at [2], together with an explanatory file.

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