On a boundary value problem for composite type third order equations with non-local boundary conditions

Nehan Aliyev and Akhmedali Aliyev

Institute of Applied Mathematics, Baku State University, Z.Khalilov-23, Az1148, Baku, Azerbaijan

N.Aliyev. 104 S.Rahimov, Baku AZ1009, Azerbaijan
Tel: (994 12) 955228. Email: .jeff@azdata.net

A.Aliyev. 47Ataturk, apt.6, Baku AZ1069, Azerbaijan
Tel: (994 12) 625142. Email: .ahmadm19@yahoo.com

The paper is devoted to the investigation of solution of a boundary value problem of the odd order partial equation where the boundary of the considered domain holds boundary conditions of this problem.

The objective is to clarify the linear non-local boundary conditions with which the boundary value problem is of Fredholm type. The investigation of the problem is based on fundamental solution [12] of the considered equation. The necessary conditions similar to the ones in [5]-[10] are obtained. After regularizing these necessary conditions, and joining them with given boundary conditions, the condition is found for the given problem to belong to Fredholm type. A similar problem for Schrodinger equation and for parabolic equation of first order is considered in [5,7,8].

The stated problem is devoted to the investigation of solution for a boundary value problem of the odd order partial equation when all the boundary of the considered domain is a support of boundary conditions of this problem.

It is known that boundary value problems may have no solutions owing to equations, or the considered domain, or boundary conditions. The problem on the existence of solutions for boundary value problems owing for to boundary conditions were was considered in our investigations [5]-[10]. The study of these problems led us to necessary conditions related to which are connected only with the domain where the boundary value problem is studied, and with to the considered equation.

Is should be noted that these necessary conditions don't depend on boundary conditions. They dictate determine us the kind of boundary conditions of a concrete specific problem. A.A. Dezin [1] obtained these conditions for an ordinary linear differential equation. For Laplace equations in bounded domain they were obtained by A.V. Bitzadze [2]. and they don't yield to generalization.

A.V. BitzadzeHe named then the conditions that could not be generalized as "necessary and sufficient" conditions.

As it follows from A.N. Tikhonov [3] and M.M. Lavrentiev's [4] inveatigations investigations, mathematically ill-ill-posedposed problems may occur more and morefrequently. Then there arises a question: how should be the statement of the is the boundary value problem stated for at random chosenan arbitrary equation? The scheme of derivation of the abovementioned necessary conditions answers to the this question.

These necessary conditions lead us to non-local boundary condition (sometimes boundary conditions may contain even global addendsaddends), in which the number of boundary conditions coincide with the highest order of derivative on spatial variable contained in the equations of the considered problem (this regularity was observed for in ordinary linear differential equations). Many boundary value problems, for instance [5]-[10], were investigated by this method, e.g. [5]-[10].

These problems were stated for the first and second order equations. A boundary value problem for two-dimensional first order mixed type equation was considered in [5]. A problem for the Laplace equation with non-local and global addends addends in boundary conditions was stated in [6]. In [7] a mixed problem for a Schrodinger equation with non-local boundary conditions was investigated. In [8] a mixed problem was considered for a parabolic equation whose boundary conditions contain both non-local and global addendsaddends. Here the results on the whole of the mixed problem is are stated briefly. In [9] the Cauchy problem (on the whole of space) was considered for Navier-Stokes equations system. Paper [10] is mainly devoted to the Fredholm boundary value problem obtained by means of Laplace transformation for a parabolic equation with non-local and global boundary conditions. The Nnecessary conditions were investigated in detail.

Thus, consider the following problem:

$$\frac{\partial^3 u(x)}{\partial x_2^3} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2} = 0, \qquad x \in D \subset R^2 \tag{1}$$

$$\left.\frac{\partial^k u(x)}{\partial x_1^k}\right|_{x_2=\gamma_1(x_1)} = \left.\frac{\partial^k u(x)}{\partial x_2^k}\right|_{x_2=\gamma_2(x_1)} + \varphi_k(x_1) k = 0,1,2 x_1 \in [a,b] \tag{2}$$

where $\gamma_k(x_1)$ $(k = 1,2)-$ are the equations of open lines Γ_k $(\Gamma_1 \cup \Gamma_2 = \Gamma)$ obtained from the boundary Γ of the domain D, by means of orthogonal transformation of the domain on the axis x_1, $[a_1, b_1] = {}_{x_1}\Gamma_1 = {}_{x_1}\Gamma_2$ while $\gamma_1(x_1) < \gamma_2(x_1)$, $x_1 \in (a_1, b_1)$.

Proceeding from Fourier transformation [12], [13] we obtain for equation (1) a fundamental solution in the form of integral on the plane, which contains supersingularity. Regularizing it by means of bilateral Hermander stairs [11] and performing some calculations, we getthe following is obtained:

$$U(x-\xi) = \frac{x_2-\xi_2}{2\pi}\left[\ln\sqrt{|x_1-\xi_1|^2 + (x_2-\xi_2)^2} - 1\right] + \frac{|x_1-\xi_1|}{2\pi} arctg \frac{x_2-\xi_2}{|x_1-\xi_1|} \tag{3}$$

Then, by applying the method of papers [5]-[10] to solve equation (1) and its derivative up to the second order inclusively we get necessary conditions both for $x_2 = \gamma_1(x_1)$ and $x_2 = \gamma_2(x_1)$. Note that only six from of these twelve expressions (i.e. necessary conditions connected with related to the second order of derivative) contain singular integrals.

To reduce these singular addends addends, from fundamental solution (3) and allowing for

$$\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\cdot\frac{\gamma_p'(\sigma_p(x_1,\xi_1))}{(x_1-\xi_1)\left(1+\gamma_p'^2(\tau_p)\right)}, \tag{4}$$

$$\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi}\frac{1}{(x_1-\xi_1)\left(1+\gamma_p'^2(\tau_p)\right)}, \tag{5}$$

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_P(\xi_1)}} = -\frac{1}{2\pi}\cdot\frac{\gamma_p'(\tau_p)}{(x_1-\xi_1)\left(1+\gamma_p'^2(\tau_p)\right)}, \tag{6}$$

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_q(\xi_1)}} = \delta(x_1-\xi_1)e(\gamma_p(x_1)-\gamma_q(\xi_1)) - \frac{1}{2\pi}\cdot\frac{\gamma_p'(x_1)-\gamma_q(\xi_1)}{(x_1-\xi_1)^2+\left(\gamma_p(x_1)-\gamma_q(\xi_1)\right)^2}, \tag{7}$$

havethe following is derived:

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)} = \\ = -\frac{1}{\pi}\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \frac{dx_1}{x_1-\xi_1} + ... \tag{8}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)} - \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_1(\xi_1)} = \\ = -\frac{1}{\pi}\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} \frac{dx_1}{x_1-\xi_1} + ... \tag{9}$$

$$\left.\frac{\partial^2 u\left(\xi\right)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_k(\xi_1)} = \frac{(-1)^{k-1}}{\pi}\int_{a_1}^{b_1} \left.\frac{\partial^2 u\left(x\right)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_k(x_1)} \frac{dx_1}{x_1-\xi_1} + ... \qquad k=1,2 \tag{10}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} = \frac{(-1)^{k-1}}{\pi}\int_{a_1}^{b_1} \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_k(x_1)} \frac{dx_1}{x_1-\xi_1} + ...k=1,2 \tag{11}$$

Thus the following statement is established.

Theorem 1. Let D be a bounded, convex in the direction x_2, plane domain with the Liapunov Γ-line boundary. Then boundary values of each solution of equations (1) determined on domain D satisfy necessary conditions, a part of which has no singularities, and the remaining part are connected withrelated to the boundary value of the second derivative and contain singularity represented in the form (8)-(11). There $\gamma_k\left(x_1\right)$ $k=1,2;$ $x_1 \in (a_1, b_1)$ are the equations of the part Γ_k $(k=1,2)$ of the boundary Γ of the domain D which are obtained by orthogonaly projecting this domain on the axis x_1

$$\gamma_1\left(x_1\right) < \gamma_2\left(x_1\right) \qquad x_1\left(a_1, b_1\right)$$

Remark Comment 1. Integrating (1) with respect to the variable x_2 from $\gamma_k\left(x_1\right)$ to x_2 we have the following necessary conditions

$$\left.\Delta u\left(x\right)\right|_{x_2=\gamma_1(x_1)} = \left.\Delta u\left(x\right)\right|_{x_2=\gamma_2(x_1),} \qquad x_1 \in [a_1, b_1] \tag{12}$$

which are also the corollaries result of the abovementioned conditions.

Remark Comment 2. Taking into account the remark comment made above it is easy to see that (8) and (9) are reduced to necessary condition (10).

Differentiating the given boundary conditions (2) 2-k times with respect to the variable x_1 and allowing for (12), we havethe following is derived:

$$\begin{aligned} \left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_1(x_1)} &= \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} + \varphi_2\left(x_1\right) \\ \left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)} &= \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} + \varphi_2\left(x_1\right) \qquad x_1 \in [a_1, b_1] \end{aligned} \tag{13}$$

and

$$\begin{cases} \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} \left[1-\gamma_1'^2(x_1)\right] - 2\left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \gamma_1'(x_1) - \\ - \left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left[1-\gamma_2'^2(x_1)\right] + 2\left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} \gamma_2'(x_1) = -\varphi_0''(x_1) + \\ + \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} \cdot \gamma_1''(x_1) - \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} \cdot \gamma_2''(x_1), \\ \left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} \left[1-\gamma_2'^2(x_1)\right] + \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \gamma_1'(x_1) - \\ - \left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} = \varphi_1'(x_1) - \varphi_2(x_1), \qquad x_1 \in [a_1, b_1] \end{cases} \tag{14}$$

in which (13) determines $\left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_k(x_1)}$ if $\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_k(x_1)}$ are known, and (14) gives usproduces two relations between four unknowns

$\left.\frac{\partial^2 u(x)}{\partial x_2}\right|_{x_2=\gamma_k(x_1)}$ and $\left.\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_k(x_1)}$ $(k=1,2)$

One more pair of regular relations (regular) with respect to these unknowns are obtained by the scheme method of papers [5]-[10] allowing for (14) proceeding from necessary conditions (10) and (11).

$$\begin{aligned} &-\sum_{k=1}^{2} 2\gamma_k'(\xi_1) \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_k(\xi_1)} - \sum_{k=1}^{2} \left[1-\gamma_k'^2(\xi_1)\right] \left.\frac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} = \\ &= \frac{1}{\pi}\int_{a_1}^{b_1} \left[-\varphi_0''(x_1) + \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)} \gamma_1''(x_1) - \left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)} \gamma_2''(x_1)\right] \frac{dx_1}{x_1-\xi_1} + \ldots \end{aligned} \tag{15}$$

$$\begin{aligned} &\gamma_1'(x_1) \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_1(\xi_1)} + \left.\frac{\partial^2 u(\xi)}{\partial \xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)} + [1-\gamma_2'(\xi_1)] \left.\frac{\partial^2 u(\xi)}{\partial \xi_1 \partial \xi_2}\right|_{\xi_2=\gamma_2(\xi_1)} = \\ &= \frac{1}{\pi}\int_{a_1}^{b_1} [\varphi_1'(x_1) - \varphi_2(x_1)] \frac{dx_1}{x_1-\xi_1} + \ldots \end{aligned} \tag{16}$$

Taking into account that the necessary conditions obtained for $\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_k(x_1)}$ contain under the integral sign the normal derivatives from $\left.\frac{\partial U(x-\xi)}{\partial x_2}\right|_{\substack{x_2=\gamma_k(x_1)\\ \xi_2=\gamma_2(\xi_1)}}$ which are regular for Lyapunov line [1], we havethe following is stated:

Theorem 2. By the conditions of Theorem 1, if $\varphi_k(x_1) \in C^{3-k}(a_1, b_1)$ and $\varphi_k^{(2-k)}(a_1) = \varphi_k^{(2-k)}(b_1) = 0$, $k = \overline{0,2}$, the relations (15) and (16) are regular.

Thus, we get Hence the following statement:

Theorem 3. Onder the conditions of the theorem 2 the condition of Fredholm property of the boundary value problem (1)-(2) is equvivalent to the fact the certain matrix has non-zero determinant.

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