The Initial Value Problem, Scattering and Inverse Scattering, for Non-Linear Schrödinger Equations with a Potential and a Non-Local Non-Linearity *†

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Abstract

We consider non-linear Schrödinger equations with a potential, and non-local non-linearities, that are models in mesoscopic physics, for example of a quantum capacitor, and that also are models of molecular structure. We study in detail the initial value problem for these equations. In particular, existence and uniqueness of local and global solutions, continuous dependence on the initial data and regularity. We allow for a large class of unbounded potentials. We have no restriction on the growth at infinity of the positive part of the potential.
We also construct the scattering operator in the case of potentials that go to zero at infinity. Furthermore, we give a method for the unique reconstruction of the potential from the small amplitude limit of the scattering operator. In the case of the quantum capacitor, our method allows us to uniquely reconstruct all the physical parameters from the small amplitude limit of the scattering operator.

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1 Introduction

In recent years there is a considerable interest in non-linear Schrödinger equations with a potential (NLSP) and a non-local non-linearity that is concentrated in a bounded region of space, for example, to model physical situations that appear in mesoscopic physics. In particular, in [27, 18, 19, 20, 28] the following equation was introduced to model a quantum capacitor (in [19] more general equations are discussed).

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{d^2}{dx^2}u(x,t) + V_0\, u(x,t) + \lambda Q(u)\chi_{[b,c]}(x)\, u(x,t), \quad u(x,0) = \varphi(x), \tag{1.1}$$

with, $x, t \in \mathbb{R}$, and where we have set Planck's constant $\hbar$ equal to one and the mass equal to 1/2. For any $O \in \mathbb{R}$, χ_O is the characteristic function of O. The external potential, V_0, is a double barrier,

$$V_0(x) = \beta\left[\chi_{[a,b]}(x) + \chi_{[c,d]}(x)\right], \qquad \beta > 0,$$

where, $a < b < c < d$. Furthermore,

$$Q(u) := \int_b^c |u(x,t)|^2\, dx$$

is the a-dimensional electric charge trapped in the well $[b, c]$.

We briefly describe the physical aspects of the model given by (1.1), following [20]. As is well known [7], the interaction between electrons can play a crucial role in the electrical transport properties of mesoscopic systems. In the case considered in [27], [20] a cloud of electrons move in a double barrier heterostructure in which the well region confined between the two potential barriers acts like a quantum capacitor whose energy depends on the electron charge trapped inside it. The localization of the interaction is justified by the existence of a resonant state that allows for a long sojourn time of the electrons inside the well. This leads to an accumulation of electric charge inside the quantum capacitor. A main feature of equation (1.1) is that the non-linearity is concentrated only in the region where the resonant state is localized, i.e., within the two barriers. In [20], among other results, a one mode approximation was considered, an adiabatic condition was introduced, and numerical simulations where performed.

In this paper we study the following generalization of (1.1),

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{d^2}{dx^2}u(x,t) + V_0(x)u(x,t) + \lambda\,(V_1 u, u)\,V_2(x)\,u(x,t), \quad u(x,0) = \varphi(x), \quad (1.2)$$

$x, t \in \mathbb{R}$. By $(\cdot,\cdot)$ we denote the L^2 scalar product. The external potential V_0 is real valued, and $V_1(x), V_2(x)$ are, in general, complex valued. $\lambda \in \mathbb{C}$ is a coupling constant.

In the particular case where V_0 is a double well, $V_1 = V_2$ is real valued, and $\lambda \in \mathbb{R}$, this equation was extensively studied as a model for molecular structure [10, 11], [16, 17], [32, 33, 34] and the references quoted there. In these papers, molecular localization, the suppression of the beating effect by the non-linearity, and the semi-classical limit were studied, among other problems.

Below, we study in a detailed way the initial value problem for (1.2). We prove existence and uniqueness of local solutions in $L^2, \mathbb{H}^1_Q$, and $\mathbb{H}^2_Q$, where $\mathbb{H}^j_Q, j = 1, 2$, are Sobolev spaces, and $\mathbb{H}^2_Q \subset \mathbb{H}^1_Q \subset L^2$ (see section 2 for the definition of $\mathbb{H}^j_Q, j = 1, 2$). We also prove the continuity of the solution on the initial value and a regularity result that tells us that if the initial value belongs to $\mathbb{H}^1_Q$, the solution in $\mathbb{H}^1_Q$ can not blow-up before the solution in L^2 does. We obtain also a similar result on regularity between $\mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ solutions, when the initial data belongs to $\mathbb{H}^2_Q$. Furthermore, we prove that if V_1, V_2, are real valued and $\lambda \in \mathbb{R}$ the L^2 solutions are global, i.e., they exist for all times $t \in \mathbb{R}$. If moreover, $V_1 = V_2$ we prove that the $\mathbb{H}^j_Q, j = 1, 2$, solutions are global. We prove these results for a large class of unbounded external potentials, V_0 (see section 2). In fact, we have no restriction on the growth at infinity on the positive part of the potential V_0. These results prove that (1.2) forms a dynamical system by generating a continuous local/global flow [22]. In this sense, the spaces $L^2, \mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ are fundamental for the equation (1.2).

In [17], [34] the existence and uniqueness of global solutions in the Sobolev spaces $\mathbb{H}^j, j = 1, 2$, was proven in the case where the external potential V_0 is bounded.

Then, we consider external potentials that decay at infinity, and we construct the scattering operator for (1.2) with reference dynamics given by the self-adjoint realization in L^2 of $-\frac{d^2}{dx^2}$. Furthermore, we study the inverse scattering problem. We prove that the small amplitude limit of the scattering operator allows us to uniquely reconstruct V_0 and λ. In the particular case of the quantum capacitor, (1.1), this gives us all the physical parameters.

There is a very extensive literature on the initial value problem and in scattering for the non-linear Schrödinger equation without a potential and local non-linearities. As general references see, for example, [38, 22, 29, 15, 8] and [39]. For the case of non-linearities concentrated in a finite number of points see [1, 2, 3, 4].

For the initial value problem for the NLSP with a local non-linearity and a potential that has bounded second derivative see [22]. For the forced problem on the half-line see [49], where for local solutions there is no restriction on the growth of the positive part of the potential at infinity, and for global solutions only mild restrictions that allow, for example, for exponential growth.

For direct and inverse scattering for the NLSP with local non-linearities see [41], [44] and [45], and for the case of the non-linear Klein-Gordon equation with a potential and local non-linearities see [42] and [46]. For an expository review of these results see [47]. For the forced NLSP on the half-line see [48].

For center manifolds for the NLSP with local non-linearities see [43], and for the non-linear Schrödinger equation with a double-well potential and a cubic local non-linearity see [35].

The paper is organized as follows. In section 2 we prove our results on the initial value problem. In section 3 we construct the scattering operator (direct scattering) and we prove our results on inverse scattering.

2 The initial value problem

In this section we absorb the coupling constant λ into V_2 and we consider the initial value problem for the following NLSP,

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{d^2}{dx^2}u(x,t) + V_0(x)u(x,t) + (V_1 u, u)\ V_2\, u(x,t), \quad u(x,0) = \varphi(x). \tag{2.1}$$

By $W_{j,p}, j = 1, 2, \cdots, 1 \leq p \leq \infty$, we denote the Sobolev space [5] of all functions in L^p such that all its derivatives of order up to j are functions in L^p. By $\| \cdot \|_{j,p}$ we denote the norm in $W_{j,p}$. By $\mathbb{H}^j, j = 1, 2, \cdots$, we denote the L^2 based Sobolev spaces, $\mathbb{H}^j := W_{j,2}$.

We suppose that the potential V_0 satisfies the following assumption.

Assumption A

Assume that,

$$V_0 = V_{0,1} + V_{0,2}, \text{ with } V_{0,j} \in L^1_{\text{loc}}, V_{0,1} \geq 0, \tag{2.2}$$

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |V_{0,2}(x)|\, dx < \infty. \tag{2.3}$$

We denote , $Q := \sqrt{V_{0,1}}$ and by $D(Q)$ the domain in L^2 of the operator of multiplication by Q.

Let us denote by H_0 the self-adjoint realization of $-\frac{d^2}{dx^2}$ in L^2 with domain $\mathbb{H}^2$. Equation (2.3) implies that $V_{0,2}$ is a quadratic form bounded perturbation of H_0 with relative bound zero (this is proven, for example, as in the proof of [49, Proposition 2.1]), that is, for any $\epsilon > 0$ there is a constant K_ϵ such that,

$$|(V_{0,2}\varphi, \varphi)| \leq \epsilon \|\varphi'\|^2_{L^2} + K_\epsilon \|\varphi\|^2_{L^2}. \tag{2.4}$$

It follows that the quadratic form,

$$h(\varphi, \psi) := (\varphi', \psi') + (V_0\varphi, \psi), \text{ with domain, } D(h) := \mathbb{H}^1 \cap D(Q), \tag{2.5}$$

is closed and bounded from below. Let H be the associated bounded-below, self-adjoint operator (see, [21, 31]). Then,

$$D(H) = \left\{ \varphi \in D(h) : -\frac{d^2}{dx^2}\varphi + V\varphi \in L^2 \right\} \text{ and } H\varphi = -\frac{d^2}{dx^2}\varphi + V\varphi,\ \varphi \in D(H). \tag{2.6}$$

Let us take $N > 0$ such that $H + N > 1$ and let us denote $D := \cap_{m=-\infty}^{\infty} D((H+N)^m)$. For any $s \in \mathbb{R}$ let $\mathbb{H}^s_Q$ be the completion of D in the norm, $\|(H+N)^{s/2}\|_{L^2}$. Notice that $\mathbb{H}^0_Q = L^2, \mathbb{H}^1_Q = D(\sqrt{H+N}) = D(h)$ and that $\mathbb{H}^2_Q = D(H)$.

Observe that the following norm is equivalent to the norm of $\mathbb{H}^1_Q$,

$$\max\left[\|\varphi\|_{\mathbb{H}^1},\, \|Q\varphi\|_{L^2} \right].$$

Furthermore, if $V_{0,1} = 0$ the potential V_0 is a quadratic form bounded perturbation of H_0 with relative bound zero, and then, $\mathbb{H}^1_Q = \mathbb{H}^1$. In this case the $\mathbb{H}^1_Q$ solutions that we consider below are just $\mathbb{H}^1$ solutions.

As $\mathbb{H}^2_Q = D(H)$, the following norm is equivalent to the norm of $\mathbb{H}^2_Q$,

$$\max \left[\|\varphi\|_{\mathbb{H}^1_Q}, \left\| \left(-\frac{d^2}{dx^2} + V_0 \right) \varphi \right\|_{L^2} \right].$$

The paper [23] gives sufficient conditions in V_0 that assure that $\mathbb{H}^2_Q \subset \mathbb{H}^2$. Furthermore, if

$$\sup_{x \in \mathbb{R}} \int_x^{x+1} |V_0(x)|^2 \, dx < \infty, \tag{2.7}$$

it follows from (2.4), and as $\|V_0 \varphi\|^2_{L^2} = (V_0^2 \varphi, \varphi)$, and $(\varphi', \varphi') \leq \|H_0 \varphi'\|^2 + \|\varphi\|^2$, that V_0 is relatively bounded with respect to H_0 with relative bound zero, that is, for any $\epsilon > 0$ there is a constant K_ϵ such that,

$$\|V_0 \varphi\| \leq \epsilon \|H_0 \varphi\|_{L^2} + K_\epsilon \|\varphi\|_{L^2}. \tag{2.8}$$

In this case (see [21], [31]) $\mathbb{H}^2_Q = \mathbb{H}^2$ and the $\mathbb{H}^2_Q$ solutions that we consider below are just $\mathbb{H}^2$ solutions.

It follows from the functional calculus of self-adjoint operators that H is bounded from $\mathbb{H}^s_Q$ to $\mathbb{H}^{s-2}_Q$ and that e^{-itH} is a strongly-continuous unitary group on $\mathbb{H}^s_Q$, $s \in \mathbb{R}$. Moreover, for any $\varphi \in \mathbb{H}^s_Q$, $s \in \mathbb{R}$, $e^{-itH} \varphi \in C\left(\mathbb{R}, \mathbb{H}^s_Q\right) \cap C^1\left(\mathbb{R}, \mathbb{H}^{s-2}_Q\right)$ and,

$$i \frac{\partial}{\partial t} e^{-itH} \varphi = H e^{-itH} \varphi = e^{-itH} H \varphi, \quad \varphi \in \mathbb{H}^s_Q.$$

We introduce some further notation that we use below. Let $I =: [0, T]$, if $0 < T < \infty$ and $I = [0, \infty)$, if $T = \infty$. For any Banach space $\mathcal{X}$ we denote by $\mathcal{X}_R$ the closed ball in $\mathcal{X}$ with centre zero and radius R. If $T < \infty$ we denote by $C(I, \mathcal{X})$ the Banach space of continuous functions from I into $\mathcal{X}$ and if $T = \infty$ we denote by $C_B(I, \mathcal{X})$ the Banach space of continuous and bounded functions from I into $\mathcal{X}$. For $T < \infty$, we define, $\mathcal{N} := C(I, L^2)$ and $\mathcal{N}^j := C(I, \mathbb{H}^j_Q)$, $j = 1, 2$. For functions $u(t, x)$ defined in $\mathbb{R}^2$ we denote $u(t)$ for $u(t, \cdot)$.

We study the initial value problem (2.1) for $t \geq 0$, but changing t into $-t$ and taking the complex conjugate of the solution (time reversal) we also obtain the results for $t \leq 0$.

By a L^2 solution on I to (2.1) we mean a function $u \in C(I, L^2) \cap C^1(I, \mathbb{H}^{-2}_Q)$ that satisfies (2.1).

Multiplying both sides of (2.1) (evaluated at τ) by $e^{-i(t-\tau)H}$ and integrating in τ from zero to t we obtain that,

$$u(t) = e^{-itH}\varphi + \frac{1}{i}\,GF(u), \text{ with } F(u) := (V_1 u, u)\, V_2 u \tag{2.9}$$

and where,

$$Gu := \int_0^t e^{-i(t-\tau)H} u(\tau)\, d\tau. \tag{2.10}$$

Moreover, let $u \in C(I, L^2)$ be a solution to (2.9). Then, it follows from (2.9) that $u \in C^1(I, \mathbb{H}_Q^{-2})$. We prove that u solves (2.1) taking the derivative of both sides of (2.9). Hence, equations (2.1) and (2.9) are equivalent. We obtain our results below solving the integral equation (2.9).

In the next theorem we prove the existence of local solutions in L^2.

THEOREM 2.1. *Suppose that assumption A is satisfied and that $V_j \in L^\infty, j = 1, 2$. Then, for any $\varphi \in L^2$ there is $0 < T < \infty$, such that (2.1) has a unique solution, $u \in C(I, L^2)$ with, $u(0) = \varphi$. T depends only on $\|\varphi\|_{L^2}$.*

Proof: We define,

$$\mathcal{C}(u) : e^{-itH}\varphi + \frac{1}{i} GF(u). \tag{2.11}$$

We will prove that we can take R large enough, and T so small, that $\mathcal{C}$ is a contraction on $\mathcal{N}_R$.

It follows from Schwarz inequality that for any $u, v \in \mathcal{N}_R$,

$$\|F(u) - F(v)\|_{\mathcal{N}} \leq C\, (\|u\|^2_{\mathcal{N}} + \|v\|^2_{\mathcal{N}})\, \|u - v\|_{\mathcal{N}} \leq C\, 2\, R^2\, \|u - v\|_{\mathcal{N}}. \tag{2.12}$$

Then, as e^{-itH} is unitary on L^2,

$$\|\mathcal{C}(u)\|_{\mathcal{N}} \leq \left[\|\varphi\|_{L^2} + C\, T \|u\|^3_{\mathcal{N}}\right] \leq \left[\|\varphi\|_{L^2} + C\, T\, R^3\right], \tag{2.13}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}} \leq CT(\|u\|^2_{\mathcal{N}} + \|v\|^2_{\mathcal{N}})\, \|u - v\|_{\mathcal{N}} \leq C\, T\, 2\, R^2 \|u - v\|_{\mathcal{N}}. \tag{2.14}$$

Then, we can take R, T such that $\|\varphi\|_{L^2} + C\, T\, R^3 \leq R$ and, $d := C\, T\, 2\, R^2 < 1$, what makes $\mathcal{C}$ a contraction on $\mathcal{N}_R$. By the contraction mapping theorem [30] $\mathcal{C}$ as a unique fixed point, u, in $\mathcal{N}$ that is a solution to (2.9).

Suppose that there is another solution $v \in \mathcal{N}$. By the argument above, we have that $v(t) = u(t)$ for $t \in [0, T_0]$ for some $T_0 \leq T$. By iterating this argument we prove that $v(t) = u(t), 0 \leq t \leq T$.

□

We now prove that the solution depends continuously on the initial data.

THEOREM 2.2. *Suppose that assumption A is satisfied and that $V_j \in L^\infty, j = 1, 2$. Then, the solution $u \in C([0,T], L^2), 0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.1, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$ strongly in L^2. Then, for n large enough, the solutions $u_n \in C([0,T], L^2)$ to (2.1) with initial values φ_n exist and $u_n \to u$ in $C([0,T], L^2)$.*

Proof: We first prove a local version of the theorem with T replaced by a T_0 small enough . We define,

$$\mathcal{C}_n(u) := e^{-itH}\varphi_n + \frac{1}{i} GF(u), \quad u \in \mathcal{N}(T_0) := C([0,T_0], L^2). \tag{2.15}$$

As $e^{-itH}\varphi_n \to e^{-itH}\varphi$ in $\mathcal{N}(T_0)$, for n large enough $\mathcal{C}_n$ and $\mathcal{C}$ are contractions in $\mathcal{N}_R(T_0)$ with the same R, T_0, d. The unique fixed points, u_n, u are solutions to (2.1) in $C([0,T_0], L^2)$ with, respectively, $u_n(0) = \varphi_n, u(0) = \varphi$. Moreover, as $\mathcal{C}_n(u_n) - \mathcal{C}(u) = \mathcal{C}_n(u_n) - \mathcal{C}_n(u) + \mathcal{C}_n(u) - \mathcal{C}(u)$,

$$\|u_n - u\|_{\mathcal{N}(T_0)} = \|\mathcal{C}_n(u_n) - \mathcal{C}(u)\|_{\mathcal{N}(T_0)} \leq d\|u_n - u\|_{\mathcal{N}(T_0)} + \|\varphi_n - \varphi\|_{L^2}, \tag{2.16}$$

and as $d < 1, u_n \to u$ in $\mathcal{N}(T_0)$. As the interval of existence of the solution given in theorem 2.1 depends only on the L^2 norm of the initial value, we can extend this argument, step by step, to the whole interval $[0,T]$.

□

REMARK 2.3. let T_m be the maximal time such that the solution u given in theorem 2.1 can be extended to a solution $u \in C([0,T_m), L^2)$ with $u(0) = \varphi$. Then, if T_m is finite we must have that $\lim_{t\uparrow\infty} \|u(t)\|_{L^2} = \infty$. In other words, the solution exists for all times unless it blows up in the L^2 norm for some finite time. To prove this result suppose that $\|u(t)\|_{L^2}$ remains bounded as $t \uparrow T_m$. Then, by theorem 2.1 we can extend the solution continuously to $T_m + \epsilon$ for some $\epsilon > 0$, contradicting the definition of T_m.

Another consequence of theorem 2.1 is that (2.1) has at most one solution in $C(I, L^2)$ with $u(0) = \varphi$. Suppose, on the contrary, that there are two, u_1, u_2. Then, by theorem 2.1, $u_1(t) = u_2(t), t \in [0, T_0]$ for some $0 < T_0 < T$. Let T_m be the maximal time such that $u_1(t) = u_2(t), t \in [0, T_m)$. Consider first the case where $T < \infty$. Then, $T_m = T$, because if $T_m < T$, theorem 2.1 would imply that $u_1(t) = u_2(t)$ for $t \leq T_m + \epsilon$ for some $\epsilon > 0$, in contradiction with the definition of T_m. Then, $T_m = T$ and by continuity, $u_1(T) = u_2(T)$, completing the proof in the case $T < \infty$. A similar argument proves that if $T = \infty$, T_m can not be finite.

□

We will use the uniqueness of L^2 solutions given by theorem 2.1 and remark 2.3 in the construction of the scattering operator in theorem 3.1 in section 3.

We now study solutions in $\mathbb{H}^1_Q$.

THEOREM 2.4. *Suppose that assumption A holds, that V_1 satisfies (2.3) and that $V_2 \in W_{1,\infty}$. Then, for any $\varphi \in \mathbb{H}^1_Q$ there is a $0 < T < \infty$ such that (2.1) has a unique solution $u \in C([0,T], \mathbb{H}^1_Q)$, with $u(0) = \varphi$. T depends only on $\|\varphi\|_{\mathbb{H}^1_Q}$.*

Proof: As V_1 satisfies (2.3), it follows from (2.4) that,

$$|(V_1\varphi, \varphi)| \leq C\|\varphi\|^2_{\mathbb{H}^1}, \quad \text{for all}\, \varphi \in \mathbb{H}^1. \tag{2.17}$$

Then, defining $\mathcal{C}$ as in (2.11) and as $V_2 \in W_{1,\infty}$,

$$\|\mathcal{C}(u)\|_{\mathcal{N}^1} \leq \|\varphi\|_{\mathbb{H}^1_Q} + C\,T\|u\|^3_{\mathcal{N}^1}, \tag{2.18}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}^1} \leq C\,T\left(\|u\|^2_{\mathcal{N}^1} + \|v\|^2_{\mathcal{N}^1}\right)\|u - v\|_{\mathcal{N}^1}. \tag{2.19}$$

We take R, T such that $\|\varphi\|_{\mathbb{H}^1_Q} + C\,T\,R^3 \leq R$ and, $d := C\,T\,2\,R^2 < 1$. By (2.18) and (2.19), with this choice $\mathcal{C}$ a contraction on $\mathcal{N}^1_R$ with contraction rate d. The unique fixed point, u, is a solution to (2.1) with $u(0) = \varphi$. We prove the uniqueness of the solution in $C([0,T], \mathbb{H}^1_Q)$ as in the proof of theorem 2.1.

□

There is also continuous dependence of the solutions in $\mathbb{H}^1_Q$.

THEOREM 2.5. *Suppose that assumption A holds, that V_1 satisfies (2.3) and that $V_2 \in W_{1,\infty}$. Then, the solution $u \in C([0,T], \mathbb{H}^1_Q), 0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.4, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$, strongly in $\mathbb{H}^1_Q$. Then, for n large enough, the solutions $u_n \in C([0,T], \mathbb{H}^1_Q)$ to (2.1) with initial values, φ_n exist and $u_n \to u$ in $C([0,T], \mathbb{H}^1_Q)$.*

Proof: The theorem is proven as in the proof of theorem 2.2 replacing in the argument $\mathcal{N}(T_0)$ by $C([0,T_0], \mathbb{H}^1_Q)$.

□

REMARK 2.6. We prove as in remark 2.3 that the solution to (2.1) in $\mathbb{H}^1_Q$ exists for all $t > 0$ unless it blows up in the $\mathbb{H}^1_Q$ norm for some finite time and that theorem 2.4 implies that (2.1) has at most one solution on $C(I, \mathbb{H}^1_Q)$.

□

Let us now study solution in $\mathbb{H}^2_Q$.

THEOREM 2.7. *Suppose that assumption A holds, that V_1 satisfies (2.3) and that $V_2 \in W_{2,\infty}$. Then, for any $\varphi \in \mathbb{H}^2_Q$ there is a $0 < T < \infty$ such that (2.1) has a unique solution $u \in C([0,T], \mathbb{H}^2_Q)$, with $u(0) = \varphi$. T depends only on $\|\varphi\|_{\mathbb{H}^2_Q}$.*

Proof: As $V_2 \in W_{2,\infty}$, for some constant C,

$$\|V_2\varphi\|_{\mathbb{H}^2_Q} \leq C\|\varphi\|_{\mathbb{H}^2_Q}, \quad \text{for all}\, \varphi \in \mathbb{H}^2_Q. \tag{2.20}$$

Then, with $\mathcal{C}$ defined as in (2.11),

$$\|\mathcal{C}(u)\|_{\mathcal{N}^2} \leq \|\varphi\|_{\mathbb{H}^2_Q} + C\, T \|u\|^3_{\mathcal{N}^2}, \tag{2.21}$$

$$\|\mathcal{C}(u) - \mathcal{C}(v)\|_{\mathcal{N}^2} \leq C\, T \left(\|u\|^2_{\mathcal{N}^2} + \|v\|^2_{\mathcal{N}^2}\right) \|u - v\|_{\mathcal{N}^2}. \tag{2.22}$$

Let R, T be such that $\|\varphi\|_{\mathbb{H}^2_Q} + C\, T\, R^3 \leq R$ and, $d := C\, T\, 2\, R^2 < 1$. Hence, it follows from (2.21) and (2.22) that $\mathcal{C}$ is a contraction on $\mathcal{N}^2_R$ with contraction rate d. The unique fixed point, u, is a solution to (2.1) with $u(0) = \varphi$. We prove the uniqueness of the solution in $C([0,T], \mathbb{H}^2_Q)$ as in the proof of theorem 2.1.

□

THEOREM 2.8. *Suppose that assumption A holds, that V_1 satisfies (2.3) and that $V_2 \in W_{2,\infty}$. Then, the solution $u \in C([0,T], \mathbb{H}^2_Q), 0 < T < \infty$, to (2.1) with $u(0) = \varphi$, given by theorem 2.7, depends continuously on the initial value φ. In a precise way, let $\varphi_n \to \varphi$, strongly in $\mathbb{H}^2_Q$. Then, for n large enough, the solutions $u_n \in C([0,T], \mathbb{H}^2_Q)$ to (2.1) with initial values φ_n exist and $u_n \to u$ in $C([0,T], \mathbb{H}^2_Q)$.*

Proof: The theorem is proven as in the proof of theorem 2.2 replacing in the argument $\mathcal{N}(T_0)$ by $C([0,T_0], \mathbb{H}^2_Q)$.

□

REMARK 2.9. We prove as in remark 2.3 that the solution to (2.1) in $\mathbb{H}^2_Q$ exists for all $t > 0$ unless it blows up in the $\mathbb{H}^2_Q$ norm for some finite time and that theorem 2.7 implies that (2.1) has at most one solution on $C(I, \mathbb{H}^2_Q)$.

□

We now consider the problem of the regularity of solutions. Suppose that the conditions of theorems 2.1 and 2.4 are satisfied and that $\varphi \in \mathbb{H}^1_Q$. Then, by theorem 2.1, (2.1) has a unique L^2 solution and by theorem 2.4 a unique $\mathbb{H}^1_Q$ solution, both with initial value, φ. In the proposition below we prove that it is impossible that the $\mathbb{H}^1_Q$ solution blows-up before the L^2 solution.

PROPOSITION 2.10. *Suppose that assumption A holds that $V_1 \in L^\infty$ and that $V_2 \in W_{1,\infty}$. Let $u \in C([0,T], L^2), 0 < T < \infty$ be a solution to (2.1) with $u(0) = \varphi \in \mathbb{H}^1_Q$. Then, $u \in C([0,T], \mathbb{H}^1_Q)$.*

Proof: By theorem 2.4 there is a $0 < T_0 \leq T$ such that $u \in C([0,T_0], \mathbb{H}^1_Q)$. Let us denote, $v := \sqrt{H+N}u, 0 \leq t \leq T_0$. Multiplying both sides of (2.9) by $\sqrt{H+N}$ we obtain that,

$$v(t) = e^{-itH}(\sqrt{H+N})\varphi + \frac{1}{i}\, G(V_1 u, u)\left[(H+N)^{1/2} V_2 (H+N)^{-1/2}\right] v. \tag{2.23}$$

Note that as $V_2 \in W_{1,\infty}$, $\left[(H+N)^{1/2} V_2 (H+N)^{-1/2}\right]$ is a bounded operator in L^2.

Equation (2.23) is a linear equation for v, where u is a fixed function in $\mathcal{N}$. Solving this equation in an interval $[T_0, T_0 + \Delta]$, with Δ small enough, we prove that $v(t) \in L^2$, for

$T_0 \le t \le T_0 + \Delta$. Note that the length of Δ depends only on $\|u\|_{\mathcal{N}}$. Repeating this argument, step by step, we prove that $v \in C([0,T], L^2)$ and, in consequence, that $u \in C([0,T], \mathbb{H}^1_Q)$.

$\square$

In the following proposition we prove regularity between $\mathbb{H}^1_Q$ and $\mathbb{H}^2_Q$ solutions.

PROPOSITION 2.11. *Suppose that assumption A holds, that V_1 satisfies (2.7) and that $V_2 \in W_{2,\infty}$. Let $u \in C([0,T], \mathbb{H}^1_Q), 0 < T < \infty$, be a solution to (2.1) with $u(0) = \varphi \in \mathbb{H}^2_Q$. Then, $u \in C([0,T], \mathbb{H}^2_Q)$.*

Proof: By theorem 2.7 there is a $0 < T_0 \le T$ such that $u \in C([0,T_0], \mathbb{H}^2_Q)$ and then, by (2.1) $v := \frac{\partial}{\partial t} u(t) \in C_B([0,T_0], L^2)$. Moreover, taking the derivative in time of (2.9) we obtain that,

$$iv = e^{-itH}[H\varphi + F(\varphi)] + G\left(2\left\{\mathrm{Re}\,(v, V_1 u)\right\} V_2 u + (V_1 u, u)\, V_2 v\right). \tag{2.24}$$

Solving the real-linear equation (2.23)- where now u is a fixed function in $\mathcal{N}^1$- in an interval $[T_0, T_0+\Delta]$, with Δ small enough, we prove that $v(t) \in L^2$, for $T_0 \le t \le T_0 + \Delta$. Note that as V_1 satisfies (2.7), it follows from (2.4) that $\|V_1 u(t)\|^2_{L^2} = (V_1^2 u(t), u(t)) \le C\|u\|^2_{\mathcal{N}^1}$. In consequence, the length of Δ depends only on $\|u\|_{\mathcal{N}^1}$. Repeating this argument, step by step, we prove that $v \in C([0,T], L^2)$ and, in consequence, that $u \in C([0,T], \mathbb{H}^2_Q)$.

$\square$

Let us now consider the existence of global L^2 solutions. For this purpose we prove that the L^2 norm is constant.

LEMMA 2.12. *Suppose that assumption A holds, that $V_1 \in L^\infty$, that $V_2 \in W_{1,\infty}$, and, furthermore, that V_1, V_2 are real valued. Then, the L^2 norm of the solution to (2.1) given by theorem 2.1 and the L^2 norm of the $\mathbb{H}^1_Q$ solution given by theorem 2.4 are constant.*

Proof: We first prove the theorem for the solution $u \in C([0,T], \mathbb{H}^1_Q)$. By (2.9) $u \in C^1([0,T], \mathbb{H}^{-1}_Q)$ and then, by (2.1),

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2} = \mathrm{Re}\left(\frac{d}{dt}u(t), u(t)\right) = \mathrm{Re}\,\frac{1}{i}\left[(Hu,u) + (V_1 u, u)\,(V_2 u, u)\right] = 0, \tag{2.25}$$

and then, $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$. The result in the case of solutions $u \in C([0,T], L^2)$ follows approximating $u(0) = \varphi$ in the L^2 norm by $\varphi_n \in \mathbb{H}^1_Q$ and applying theorem 2.2.

□

THEOREM 2.13. *Suppose that assumption A holds, that $V_1 \in L^\infty$, that $V_2 \in W_{1,\infty}$, and, furthermore, that V_1, V_2 are real valued. Then, the L^2 solution, $u(t)$, given by theorem 2.1 exists for all times, and $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, t \in [0, \infty)$.*

Proof: the theorem follows from remark 2.3 and lemma 2.12.

□

For $u(t) \in C(I, \mathbb{H}^1_Q)$ we define the energy at time t as follows,

$$E(u(t)) := (u'(t), u'(t)) + (V_0 u(t), u(t)) + \frac{1}{2}(V_1 u(t), u(t))\,(V_2 u(t), u(t)). \tag{2.26}$$

LEMMA 2.14. *Suppose that assumption A holds, that $V_1 = \lambda V_2$, for some real λ, that V_2 is real valued, and that $V_2 \in W_{2,\infty}$. Then, the energy of the $\mathbb{H}^1_Q$ solution to (2.1) given by theorem 2.4 and the energy of the $\mathbb{H}^2_Q$ solution to (2.1) given by theorem 2.7 are constant in time.*

Proof: We first prove the lemma for solutions $u \in C([0,T], \mathbb{H}^2_Q)$. If follows from (2.9) that, $u \in C^1([0,T], L^2)$. As $\mathbb{H}^2_Q = D(H)$, we can write the energy as follows,

$$E(u(t)) := (u(t), Hu(t)) + \frac{\lambda}{2}(V_2 u(t), u(t))^2.$$

It follows that,

$$\frac{d}{dt}E(u(t)) = 2\,\mathrm{Re}\,[(\dot u(t), Hu(t)) + \lambda(V_2 u(t), u(t))(V_2 \dot u(t), u(t))] =$$
$$= 2\,\mathrm{Re}\,\frac{1}{i}\left[\|Hu(t)\|^2_{L^2} + \lambda(V_2 u(t), u(t))\left\{(V_2 u(t), Hu(t)) + (Hu(t), V_2 u(t))\right\}\right] = 0.$$

The result in the case of solutions $u \in C([0,T], \mathbb{H}^1_Q)$ follows approximating $u(0) = \varphi$ in the $\mathbb{H}^1_Q$ norm by $\varphi_n \in \mathbb{H}^2_Q$ and applying theorem 2.5.

□

THEOREM 2.15. *Suppose that assumption A holds, that $V_1 = \lambda V_2$, for some real λ, that V_2 is real valued, and that $V_2 \in W_{2,\infty}$. Then, the $\mathbb{H}^1_Q$ solution to (2.1) given by theorem 2.4 and the $\mathbb{H}^2_Q$ solution to (2.1) given by theorem 2.7 exist for all times and the L^2 norm and the energy of the solutions are constant in time.*

Proof: Let us first consider the solution $u \in C(I, \mathbb{H}^1_Q)$. By lemmata 2.12 and 2.14

$$(u'(t), u'(t)) + (V_0 u(t), u(t)) + N(u(t), u(t)) \leq E(u(0)) + N \|u(0)\|^2_{L^2} + \frac{|\lambda|}{2} \|V_2\|^2_{L^\infty} \|u(0)\|^4_{L^2}.$$

Then, by remark 2.6 u exists for all times, and the L^2 norm and the energy are constant in time. The theorem follows in the case of $\mathbb{H}^2_Q$ solutions by proposition 2.11.

3 Scattering

In this section we construct the small amplitude scattering operator, S, for equation (1.2) and we give a method for the unique reconstruction of the potential V_0 and the coupling constant λ, from S.

We first introduce some standard notations and some results that we need.

For any $\gamma \in \mathbb{R}$ let us denote by L^2_γ the Banach space of all complex-valued measurable functions on $\mathbb{R}$ such that,

$$\|\varphi\|_{L^1_\gamma} := \int |\varphi(x)| \, (1 + |x|)^\gamma \, dx < \infty. \tag{3.1}$$

If $V_0 \in L^1_1$ the differential expression $\tau := -\frac{d^2}{dx^2} + V_0(x)$ is essentially self-adjoint on the domain,

$$D(\tau) := \left\{ \varphi \in L^2_C : \varphi, \text{ and } \varphi' \text{ are absolutely continuous and } \tau\varphi \in L^2 \right\},$$

where L^2_C denotes the set of all functions in L^2 that have compact support. We denote by H the unique self-adjoint realization of τ. As is well known, [12], [50], H has a finite number of negative eigenvalues, it has no positive of zero eigenvalues, it has no singular-continuous spectrum, and the absolutely-continuous spectrum is $[0, \infty)$. By H_0 we denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain $\mathbb{H}^2$. The wave operators are defined as follows,

$$W_\pm := \text{s-} \lim_{t \to \pm\infty} e^{itH} \, e^{-itH_0}.$$

The limits above exist in the strong topology in L^2 and $\text{Range}\, W_\pm = \mathcal{H}_{ac}$, where $\mathcal{H}_{ac}$ denotes the space of absolute continuity of H. Moreover, the intertwining relations hold, $HW_\pm =$

$W_\pm H_0$. for these results see [36]. The linear scattering operator is defined as

$$S_L := W_+^* W_-. \tag{3.2}$$

For any pair, φ, ψ, of solutions to the stationary Schrödinger equation

$$-\frac{d^2}{dx^2}\varphi + V_0\varphi = k^2\varphi, k \in \mathbb{C}, \tag{3.3}$$

let $[\varphi, \psi]$ denote the Wronskian of φ and ψ,

$$[\varphi, \psi] := \varphi'\psi - \varphi\psi'.$$

Let $f_j(x,k), j = 1,2$ be the Jost solutions to (3.3) that satisfy, $f_1(x,k) \approx e^{ikx}, x \to \infty, f_2(x,k) \approx e^{-ikx}, x \to -\infty$, [13, 14, 12, 9]. The potential V_0 is said to be *generic* if $[f_1(x,0), f_2(x,0)] \neq 0$, and it is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. When V_0 is *exceptional* there is a bounded solution to (3.3) with $k^2 = 0$, that is called a half-bound state or a zero energy resonance. The trivial potential $V_0 = 0$ is exceptional.

Below we will always assume that $V_0 \in L^1_\gamma$, where in the *generic case* $\gamma > 3/2$ and in the *exceptional case* $\gamma > 5/2$.

In Theorem 1.1 of [40] it was proven that the operators $W_\pm$ and $W_\pm^*$ are bounded on $W_{j,p}, j = 0, 1, 1 < p < \infty$.

By Theorem 3 in page 135 of [37]

$$\|\mathcal{F}^{-1}(1+q^2)^{j/2}(\mathcal{F}f)(q)\|_{L^p}, \tag{3.4}$$

is a norm that is equivalent to the norm of $W_{j,p}, 1 < p < \infty$. In (3.4) $\mathcal{F}$ denotes the Fourier transform.

If H has no eigenvalues the $W_\pm$ are unitary operators on L^2, and it follows from the intertwining relations that

$$(1+H)^{j/2} = W_\pm\,(1+H_0)^{j/2}\,W_\pm^*,$$

and then, by (3.4),

$$\|(I+H)^{j/2} f\|_{L^p}, \tag{3.5}$$

defines a norm that is equivalent to the norm of $W_{j,p}, j = 0, 1, 1 < p < \infty$. Below we use this equivalence without further comments. Furthermore, the following $L^p - L^{p'}$ estimate holds

$$\left\| e^{-itH} \right\|_{\mathcal{B}\,(W_{1,p},\, W_{1,p'})} \leq C \frac{1}{|t|^{\frac{1}{p}-\frac{1}{2}}}, 1 \leq p \leq 2, \tag{3.6}$$

where for any pair of Banach spaces $\mathcal{X}, \mathcal{Y}, \mathcal{B}(\mathcal{X}, \mathcal{Y})$ denotes the Banach space of all bounded operators from $\mathcal{X}$ into $\mathcal{Y}$. When H has bound states estimate (3.6) is proven in [41] for the restriction of e^{-itH} to the subspace of continuity of H.

The norm (3.5) for the Sobolev spaces $W_{j,p}, 1 < p < \infty$, and the $L^p - L^{p'}$ estimate (3.6) are the basic tools that we use in order to construct the scattering operator and to solve the inverse scattering problem.

For any $1 < q < 3/2$ we denote, $p := (q+1)/(q-1)$, $r := (4q)/(2q-1)$. We designate, $P := (1/r, 1/(p+1))$ and we define,

$$L(P) := L^r \left(\mathbb{R}, L^{p+1}\right).$$

Let $\mathcal{M}$ be the following Banach space,

$$\mathcal{M} := C_B \left(\mathbb{R}, L^{p+1}\right) \cap L(P), \tag{3.7}$$

with norm,

$$\|\varphi\|_{\mathcal{M}} := max \left[\|\varphi\|_{C_B(\mathbb{R},\, L^{p+1})}, \|\varphi\|_{L(P)} \right].$$

Recall that $C_B\,(\mathbb{R}, L^{p+1})$ denotes the Banach space of all bounded and continuous functions from $\mathbb{R}$ into L^{p+1}.

In the following theorem we construct the small amplitude non-linear scattering operator.

THEOREM 3.1. *Suppose that $V_0 \in L^1_\gamma$ where in the generic case $\gamma > 3/2$ and in the exceptional case $\gamma > 5/2$ and that H has no eigenvalues. Moreover, assume that $V_j \in L^q \cap L^\infty$ for some $1 < q < 3/2$. Then, there is a $\delta > 0$ such that for every $\varphi_- \in \mathbb{H}^1$ with $\|\varphi_-\|_{\mathbb{H}^1} < \delta$*

there is a unique solution, u, to (1.2) such that $u \in \mathcal{M} \cap C_B(\mathbb{R}, L^2)$ and

$$\lim_{t\to -\infty} \|u(t) - e^{-itH}\varphi_-\|_{L^2} = 0. \tag{3.8}$$

Moreover, there is a unique $\varphi_+ \in L^2$ such that

$$\lim_{t\to \infty} \|u(t) - e^{-itH}\varphi_+\|_{L^2} = 0. \tag{3.9}$$

Furthermore, $e^{-itH}\varphi_\pm \in \mathcal{M}$ and

$$\left\|u(t) - e^{-itH}\phi_\pm\right\|_{\mathcal{M}} \leq C\left\|e^{-itH}\varphi_\pm\right\|^3_{\mathcal{M}}, \tag{3.10}$$

$$\|\varphi_+ - \varphi_-\|_{L^2} \leq C\,\|\varphi_-\|^3_{\mathbb{H}^1}. \tag{3.11}$$

The scattering operator $S_{V_0} : \varphi_- \hookrightarrow \varphi_+$ is injective.

Proof: Observe that $u \in C_B(\mathbb{R}, L^2) \cap \mathcal{M}$ is a solution to (1.2) with $\lim_{t\to -\infty} \|u(t) - e^{-itH_0}\varphi_-\|_{L^2} = 0$ if and only if it is a solution to the following integral equation (this is proven as in the proof of the equivalence of (2.1) and (2.9))

$$u = e^{-itH}\varphi_- + \frac{1}{i}\int_{-\infty}^t e^{-i(t-\tau)\,H} F(u(\tau))\,d\tau, \tag{3.12}$$

where,

$$F(u) := \lambda\,(V_1 u, u)\; V_2\, u. \tag{3.13}$$

We will prove that the integral in the right-hand side of (3.12) is absolutely convergent in $\mathcal{M}$ and in L^2.

For $u \in \mathcal{M}$ we define,

$$\mathcal{P}(u)(t) := \int_{-\infty}^t e^{-i(t-\tau)H}\, F(u(\tau))\,d\tau. \tag{3.14}$$

By (3.6) and Hölder's inequality,

$$\begin{aligned} \|(\mathcal{P}u)(t)\|_{L^{p+1}} &\leq C \int_{-\infty}^t \tfrac{1}{|t-\tau|^d}\, \|F(u(\tau))\|_{L^{\frac{p}{p+1}}} \\ &\leq C \int_{-\infty}^t \tfrac{1}{|t-\tau|^d}\, \|u(\tau)\|^3_{L^{p+1}}\, d\tau, \end{aligned} \tag{3.15}$$

where, $d := 1/(2q)$. Then,

$$\|(\mathcal{P}u)(t)\|_{L^{p+1}} \leq C\,(I_1 + I_2), \tag{3.16}$$

where,

$$I_1 := \int_{-\infty}^{t-1} \frac{1}{|t-\tau|^d}\,\|u(\tau)\|_{L^{p+1}}^3\,d\tau,$$

and

$$I_2 := \int_{t-1}^{t} \frac{1}{|t-\tau|^d}\,\|u(\tau)\|_{L^{p+1}}^3\,d\tau.$$

Let us denote by $\chi_{(1,\infty)}$ the characteristic function of $(1,\infty)$. Then, by Hölder's inequality,

$$I_1 \leq \left\|\chi_{(1,\infty)}(\tau)\frac{1}{|\tau|^d}\right\|_{L^\alpha} \|u\|_{L(P)}^3,$$

where, $\alpha := r/(r-3)$. Note that as $d\alpha > 1$, $\|\chi_{(1,\infty)}(\tau)\frac{1}{|\tau|^d}\|_{L^\alpha} < \infty$.

Moreover, as $d < 1$,

$$I_2 \leq C\|u\|_{C_B(\mathbb{R},L^{p+1})}^3.$$

It follows that,

$$\|(\mathcal{P}u)(t)\|_{L^{p+1}} \leq C\,\|u\|_{\mathcal{M}}^3. \tag{3.17}$$

We prove in a similar way that $(\mathcal{P}u)(\cdot)$ is a continuous function on $\mathbb{R}$ with values in L^{p+1}.

Furthermore, it follows from (3.15) and the generalized Young inequality [31], that,

$$\|\mathcal{P}(u)\|_{L(P)} \leq C\|u\|_{L(P)}^3. \tag{3.18}$$

Then, by (3.17) and (3.18),

$$\|\mathcal{P}u\|_{\mathcal{M}} \leq C\,\|u\|_{\mathcal{M}}^3. \tag{3.19}$$

In an analogous way we prove that,

$$\|\mathcal{P}u - \mathcal{P}v\|_{\mathcal{M}} \leq C\,(\|u\|_{\mathcal{M}}^2 + \|v\|_{\mathcal{M}}^2)\,\|u-v\|_{\mathcal{M}}. \tag{3.20}$$

Furthermore,

$$\|(\mathcal{P}u)(t)\|_{L^2}^2 = \int_{-\infty}^{t} d\tau\,\left(F(u(\tau)), \int_{-\infty}^{t} e^{-i(\tau-\tau')H}\,F(u(\tau'))\right) =$$
$$2\mathrm{Re}\int_{-\infty}^{t} (F(u(\tau)), \mathcal{P}(u)(\tau))\,d\tau.$$

We define,

$$u_t(\tau) := \chi_{(-\infty,t)}(\tau)\, u(\tau), \qquad g_t(\tau) := \int_{-\infty}^{\tau} \frac{1}{|\tau-\tau'|^d} \|u_t(\tau')\|^3_{L^{p+1}}\, d\tau.$$

Then, by (3.6) and Hölder's inequality,

$$\begin{aligned} \|(\mathcal{P}u)(t)\|^2_{L^2} &\leq \int_{-\infty}^{t} \|u_t(\tau)\|^3_{L^{p+1}}\, g_t(\tau)\, d\tau, \\ &\leq C \left[\int \|u_t(\tau)\|^{3r/(r-1)}_{L^{p+1}}\, d\tau \right]^{(r-1)/r} \|g_t\|_{L^r}. \end{aligned} \tag{3.21}$$

By the generalized Young's inequality,

$$\|g_t\|_{L^r} \leq C \|u_t\|^3_{L(P)}.$$

Furthermore,

$$\left[\int \|u_t(\tau)\|^{3r/(r-1)}_{L^{p+1}}\, d\tau \right]^{(r-1)/r} \leq \|u\|^{4-r}_{\mathcal{M}}\, \|u_t\|^{r-1}_{L(P)}.$$

Hence, by (3.21)

$$\|(\mathcal{P}u)(t)\|^2_{L^2} \leq \|u\|^{4-r}_{\mathcal{M}} \|u_t\|^{r+2}_{\mathcal{M}}. \tag{3.22}$$

We prove in a similar way that the function $t \in \mathbb{R} \to (\mathcal{P}u)(t)$ with values in L^2 is continuous.

We first prove that equation (3.12) has at most one solution in $\mathcal{M}$ and then, we prove the existence of a solution for φ small.

Suppose that there are two solutions in $\mathcal{M}$ to (3.12), u, v, and denote, $u_T := \chi_{(-\infty,T)} u, v_T := \chi_{(-\infty,T)} v$. Then,

$$u_T(t) - v_T(t) = (\mathcal{P}u_T)(t) - (\mathcal{P}v_T)(t), \text{ for } t \leq T. \tag{3.23}$$

Arguing as in the proof of (3.18), and as $u_T(t) = v_T(t) = 0$ for $t \geq T$, we prove that,

$$\|u_T - v_T\|_{L(P)} \leq C\, (\|u_T\|^2_{L(P)} + \|v_T\|^2_{L(P)})\, \|u_T - v_T\|_{L(P)}. \tag{3.24}$$

As $\lim_{T\to-\infty} (\|u_T\|^2_{L(P)} + \|v\|^2_{L(P)}) = 0$ we can take T so negative that,

$$C\, (\|u_T\|^2_{L(P)} + \|v\|^2_{L(P)}) < 1/2,$$

where C is the constant in (3.24). Then, for such T equation (3.24) implies that,

$$\|u_T - v_T\|_{L(P)} < \frac{1}{2} \|u_T - v_T\|_{L(P)},$$

and then, $u_T(t) = v_T(t), t \leq T$ and by the uniqueness of the initial value problem at finite time (see theorem 2.1), $u(t) = v(t), t \in \mathbb{R}$.

We now prove that $e^{-itH} \in \mathcal{B}(\mathbb{H}^1, \mathcal{M})$.

Since e^{-itH} is a strongly continuous unitary group on L^2 that commutes with $(1+H)^{1/2}$ we have that, $e^{-itH} \in \mathcal{B}(\mathbb{H}^1, C_B(\mathbb{R}, \mathbb{H}^1))$. Furthermore, as by Sobolev's theorem [5] and interpolation [31] $\mathbb{H}^1$ is continuously imbedded in L^{p+1}, it follows that, $e^{-itH} \in \mathcal{B}(\mathbb{H}^1, C_B(\mathbb{R}, L^{p+1}))$.

Moreover, by (3.6) and Lemma 3.1 of [22] it follows that (in [22] the operator e^{-itH_0} is considered, but the same proof applies in our case) $e^{-itH} \in \mathcal{B}\left(L^2, L^r(\mathbb{R}, L^l)\right)$ with, $l = 2r/(r-2)$, and then, we have that, $e^{-itH} \in \mathcal{B}(\mathbb{H}^1, L^r(\mathbb{R}, W_{1,l}))$. Furthermore, as by Sobolev's theorem and interpolation [31], $W_{1,l}$ is continuously imbedded in L^{p+1}, we have that, $e^{-itH} \in \mathcal{B}(\mathbb{H}^1, L^r(\mathbb{R}, L^{p+1}))$. Then,

$$e^{-itH} \in \mathcal{B}\left(\mathbb{H}^1, \mathcal{M}\right). \tag{3.25}$$

As in section 2, for $R > 0$ we denote,

$$\mathcal{M}_R := \{u \in \mathcal{M} : \|u\|_{\mathcal{M}} \leq R\}.$$

We now take R so small that $C \max[R^3, (2R)^2] < 1/2$, where C is the biggest of the constants in (3.19) and (3.20), and δ so small that

$$\|e^{-itH}\varphi\|_{\mathcal{M}} \leq R/4, \text{ for } \|\varphi\|_{\mathbb{H}^1} < \delta.$$

Then if $\|\varphi\|_{\mathbb{H}^1} < \delta$ the operator,

$$\mathcal{C}(u) := e^{-itH}\varphi + \mathcal{P}(u) \tag{3.26}$$

is a contraction on $\mathcal{M}_R$. By the contraction mapping theorem [30] $\mathcal{C}$ has a unique fixed point in $\mathcal{M}_R$ that is a solution to (3.12), and moreover,

$$\|u\|_{\mathcal{M}} \leq \|e^{-itH}\varphi\|_{\mathcal{M}} + \frac{1}{2}\|u\|_{\mathcal{M}}, \tag{3.27}$$

and hence,

$$\|u\|_{\mathcal{M}} \leq 2\|e^{-itH}\varphi_-\|_{\mathcal{M}}. \tag{3.28}$$

By (3.12) and (3.22), $u \in C_B(\mathbb{R}, L^2)$ and (3.8) holds. Equations (3.12), (3.19) and (3.28) imply that (3.10) holds for φ_-. We define,

$$\varphi_+ := \varphi_- + \frac{1}{i}\int_{-\infty}^{\infty} e^{i\tau\, H} F(u(\tau))\, d\tau. \tag{3.29}$$

Estimating as in the proof of (3.17) we prove that $\varphi_+ \in L^{p+1}$, and arguing as in the proof of (3.22) it follows that $\varphi_+ \in L^2$ and that

$$\|\varphi_+ - \varphi_-\|_{L^2} \leq C\|u\|_{\mathcal{M}}^3 \leq C\|e^{-itH}\varphi\|_{\mathcal{M}}^3.$$

Equation (3.11) follows now from (3.25).

By (3.12) and (3.29),

$$u = e^{-itH}\varphi_+ - \frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)\, H} F(u(\tau))\, d\tau. \tag{3.30}$$

Equation (3.9) follows from (3.30) estimating as in the proof of (3.22). Moreover, estimating as in the proof of (3.19) we have that,

$$\left\| \int_t^{\infty} e^{-i(t-\tau)\, H} F(u(\tau))\, d\tau \right\|_{\mathcal{M}} \leq C\|u\|_{\mathcal{M}}^3. \tag{3.31}$$

Multiplying both sides of (3.29) by e^{-itH} and estimating as in the proof of (3.19) we prove that $e^{-itH}\varphi_+ \in \mathcal{M}$. By (3.30), (3.31) and arguing as in the proof of (3.28) we obtain that,

$$\|u\|_{\mathcal{M}} \leq 2\|e^{-itH}\varphi_+\|_{\mathcal{M}}. \tag{3.32}$$

At this point, (3.10) for φ_+ follows from (3.30)–(3.32). Note that the uniqueness of φ_+ is immediate from the fact that e^{-itH} is unitary on L^2.

Finally, we prove that S_{V_0} is injective. Suppose that $\varphi_+ = S_{V_0}\varphi_- = 0$. Then, by (3.30)

$$u = -\frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)\, H} F(u(\tau))\, d\tau. \tag{3.33}$$

We prove that (3.33) implies that $u = 0$ arguing as in the proof of the uniqueness of the solution to equation (3.12) and then, it follows from (3.8) that $\varphi_- = 0$.

$\square$

We now define the scattering operator that relates asymptotic states that are solutions to the free Schrödinger equation,

$$i\frac{\partial}{\partial t}u = H_0 u,$$

given by,

$$S := W_+^* \, S_{V_0} \, W_-. \tag{3.34}$$

In the following theorem we show that we can uniquely reconstruct the linear scattering operator from the small amplitude behaviour of S.

THEOREM 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then, for every $\varphi \in \mathbb{H}^1$,*

$$\frac{d}{d\epsilon} S(\epsilon\varphi) = S_L \, \varphi, \tag{3.35}$$

where the derivative exists in the strong convergence in L^2.

Proof: Since $S(0) = 0$ and the wave operators $W_\pm$ are bounded on $\mathbb{H}^1$ [40] it is sufficient to prove that,

$$\text{s-}\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[S_{V_0}(\epsilon\varphi) - \epsilon\varphi \right] = 0. \tag{3.36}$$

But (3.36) follows from (3.11) with φ_- replaced by $\epsilon\varphi$.

$\square$

COROLLARY 3.3. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then, S uniquely determines the linear potential V_0.*

Proof: By Theorem 3.2 we uniquely reconstruct S_L from S. From S_L we obtain the reflection coefficients for linear Schrödinger scattering on the line (see Section 9.7 of [26]). As H has no bound states we uniquely reconstruct V_0 from one of the reflection coefficients using any of the standard methods. See, for example, [13, 14], [12], [25], [24], [9], [6].

$\square$

Note that our proof gives a constructive method to uniquely reconstruct V_0 from S. We first compute S_L from the derivative in (3.35)and then, we obtain the reflection coefficients and we reconstruct V_0 from one of them.

The following Theorem gives us a convergent expansion at low amplitude for S_{V_0}

THEOREM 3.4. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then, for any* $\varphi \in \mathbb{H}^1$,

$$i\left((S_{V_0} - I)(\epsilon\varphi), \varphi\right) = \epsilon^3 \lambda \int_{-\infty}^{\infty} \left(V_1 e^{-itH}\varphi, e^{-itH}\varphi\right) \left(V_2 e^{-itH}\varphi, e^{-itH}\varphi\right) dt$$
$$+O(\epsilon^5), \quad \epsilon \to 0. \tag{3.37}$$

Proof: Suppose that $u_j \in \mathcal{M}, j = 1, \cdots, 4$. Then, by Hölder's inequality,

$$\begin{aligned}
&\left| \int (V_1 u_1(t), u_2(t)) \, (V_2 u_3(t), u_4(t)) \, dt \right| \\
&\leq \|V_1\|_{L^q} \|V_2\|_{L^q} \int \|u_1(t)\|_{L^{p+1}} \|u_2(t)\|_{L^{p+1}} \|u_3(t)\|_{L^{p+1}} \|u_4(t)\|_{L^{p+1}} \, dt \\
&\leq \|V_1\|_{L^q} \|V_2\|_{L^q} \|u_1\|_{L(P)} \|u_2\|_{L(P)} \|u_3\|_{L(P)} \left[\int \|u_4(t)\|_{L^{p+1}}^{r/(r-3)} \, dt \right]^{(r-3)/r} \\
&\leq \|V_1\|_{L^q} \|V_2\|_{L^q} \|u_1\|_{L(P)} \|u_2\|_{L(P)} \|u_3\|_{L(P)} \|u_4\|_{\mathcal{M}}^{4-r} \|u_4\|_{L(P)}^{r-3} \\
&\leq \|V_1\|_{L^q} \|V_2\|_{L^q} \Pi_{j=1}^4 \|u_j\|_{\mathcal{M}}.
\end{aligned} \tag{3.38}$$

As $e^{-itH}\varphi \in \mathcal{M}$, (3.38) with $u_j = e^{-itH}\varphi, j = 1, 2, 3, 4$, proves that the integral in the right-hand side of (3.37) is absolutely convergent.

By the contraction mapping theorem, the solution u that satisfies (3.8) with $\epsilon\varphi$ instead of φ_- is given by,

$$u(t) = \lim_{j\to\infty} \mathcal{C}^j(\epsilon \, e^{-itH}\varphi) = \epsilon \, e^{-itH}\varphi + v(t), \text{ where } v(t) = \sum_{j=1}^{\infty} \mathcal{P}^j(\epsilon e^{-itH}\varphi). \tag{3.39}$$

Moreover, it follows from (3.19) that if ϵ is small enough,

$$\|v\|_{\mathcal{M}} \leq C \, \epsilon^3. \tag{3.40}$$

Then, (3.37) follows by (3.29) with $\epsilon\varphi$ instead of φ_- and (3.38)-(3.40).

□

COROLLARY 3.5. *Suppose that the conditions of Theorem 3.1 are satisfied and that* V_1, V_2, *are real-valued functions that are not identically zero. Moreover, assume either that* $V_1 = V_2$ *or that* $V_j, j = 1, 2$, *do not change sign and* $V_1 V_2 \neq 0$ *in a set of positive measure. Then, the scattering operator,* S, *and* $V_j, j = 1, 2$, *determine uniquely* λ.

Proof:

By (3.37)

$$\lambda = \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \frac{i \left((S_{V_0} - I)(\epsilon\varphi), \varphi \right)}{\int_{-\infty}^{\infty} (V_1 e^{-itH}\varphi, e^{-itH}\varphi)\ (V_2 e^{-itH}\varphi, e^{-itH}\varphi)\ dt}. \tag{3.41}$$

By corollary 3.3 V_0 is known, and then $H := H_0 + V_0$ is known. Then, $W_\pm$ are known, and S uniquely determines S_{V_0} (see (3.34)). Hence, the right-hand side of (3.41) is uniquely determined by our data. Moreover, under our conditions we can always find a $\varphi \in \mathbb{H}^1$ such that the denominator of the right-hand side of(3.41) is not zero.

□

Note that (3.41) gives us a formula for the reconstruction of λ.

Let us now go back to the quantum capacitor (1.1) where we take a slightly more general external potential V_0, namely,

$$V_0(x) = \left[\beta_1 \chi_{[a,b]}(x) + \beta_2 \chi_{[c,d]}(x) \right],$$

where $\beta_1, \beta_2 \in \mathbb{R}$. By corollary 3.3 we uniquely reconstruct V_0 from S. Then, $\beta_1, \beta_2, a, b, c$ and d are uniquely reconstructed. Moreover, by corollary 3.5 we uniquely reconstruct λ. Hence, from S we uniquely reconstruct all the physical parameters of the quantum capacitor.

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