ON THE ASYMPTOTIC DYNAMICS OF A QUANTUM SYSTEM COMPOSED BY HEAVY AND LIGHT PARTICLES

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Abstract. We consider a non relativistic quantum system consisting of K heavy and N light particles in dimension three, where each heavy particle interacts with the light ones via a two-body potential αV. No interaction is assumed among particles of the same kind.
Choosing an initial state in a product form and assuming α sufficiently small we characterize the asymptotic dynamics of the system in the limit of small mass ratio, with an explicit control of the error. In the case $K = 1$ the result is extended to arbitrary α.
The proof relies on a perturbative analysis and exploits a generalized version of the standard dispersive estimates for the Schrödinger group.
Exploiting the asymptotic formula, it is also outlined an application to the problem of the decoherence effect produced on a heavy particle by the interaction with the light ones.

1. Introduction

The study of the dynamics of a non relativistic quantum system composed by heavy and light particles is of interest in different contexts and, in particular, the search for asymptotic formulae for the wave function of the system in the small mass ratio limit is particularly relevant in many applications.
In this paper we consider the case of K heavy and N light particles in dimension three, where the heavy particles interact with the light ones via a two-body potential. To simplify the analysis we assume that light particles are not interacting among themselves and that the same is true for the heavy ones.
We are interested in the dynamics of the system when the initial state is in a product form, i.e. no correlation among the heavy and light particles is assumed at time zero. Moreover we consider the regime where only scattering processes between light and heavy particles can occur and no other reaction channel is possible.
We remark that the situation is qualitatively different from the usual case studied in molecular physics where the light particles, at time zero, are assumed to be in a bound state corresponding to some energy level $E_n(R_1, \ldots, R_K)$ produced by the interaction potential with the heavy ones considered in the fixed positions $R_1, \ldots, R_K$.

In that case it is well known that the standard Born-Oppenheimer approximation applies and one finds that, for small values of the mass ratio, the rapid motion of the light particles produces a persistent effect on the slow (semiclassical) motion of the heavy ones, described by the effective potential $E_n(R_1, \dots, R_K)$ (see e.g. [H], [HJ] and references therein).
The main physical motivation at the root of our work is the attempt to understand in a quantitative way the loss of quantum coherence induced on a heavy particle by the interaction with the light ones. This problem has attracted much interest among physicists in the last years (see e.g. [JZ], [GF], [HS], [HUBHAZ], [GJKKSZ], [BGJKS] and references therein). In particular in ([HS], [HUBHAZ]) the authors performed a very accurate analysis of the possible sources of collisional decoherence in experiments of matter wave interpherometry. We consider the results presented in the final section of this paper a rigorous version of some of their results.
At a qualitative level, the process has been clearly described in [JZ], where the starting point is the analysis of the two-body problem involving one heavy and one light particle.
For a small value of the mass ratio, it is reasonable to expect a separation of two characteristic time scales, a slow one for the dynamics of the heavy particle and a fast one for the light particle. Therefore, for an initial state of the form $\phi(R)\chi(r)$, where ϕ and χ are the initial wave functions of the heavy and the light particle respectively, the evolution of the system is assumed to be given by the instantaneous transition

$$\phi(R)\chi(r) \to \phi(R)\left(S(R)\chi\right)(r) \tag{1.1}$$

where $S(R)$ is the scattering operator corresponding to the heavy particle fixed at the position R.
The transition (1.1) simply means that the final state is computed in a zero-th order adiabatic approximation, with the light particle instantaneously scattered far away by the heavy one considered as a fixed scattering center.
Notice that in (1.1) the evolution in time of the system is completely neglected, in the sense that time zero for the heavy particle corresponds to infinite time for the light one.
In [JZ] the authors start from formula (1.1) to investigate the effect of multiple scattering events. They assume the existence of collision times and a free dynamics of the heavy particle in between. In this way they restore, by hand, a time evolution of the system.
Our aim in this paper is to give a mathematical analysis of this kind of process in the more general situation of many heavy and light particles.
Starting from the Schrödinger equation of the system we shall derive the asymptotic form of the wave function for small values of the mass ratio and give an estimate of the error.
The result can be considered as a rigorous derivation of formula (1.1), generalized to the many particle case and modified taking into account the internal motion of the heavy particles.
Furthermore, we shall exploit the asymptotic form of the wave function to briefly outline how the decoherence effect produced on the heavy particles can be explicitely computed.
At this stage our analysis leaves untouched the question of the derivation of a master equation for the heavy particles in presence of an environment consisting of a rarefied gas of light particles

(see e.g. [JZ], [HS]). Such derivation involves the more delicate question of the control of the limit $N \to \infty$ and requires a non trivial extension of the techniques used here.
The analysis presented in this paper generalizes previous results for the two-body case obtained in [DFT], where a one-dimensional system of two particles interacting via a zero-range potential was considered, and in [AFFT], where the result is generalized to dimension three with a generic interaction potential (se also [CCF] for the case of a three-dimensional two-body system with zero-range interaction).

We now give a more precise formulation of the model. Let us consider the following Hamiltonian

$$H = \sum_{l=1}^{K} \left(-\frac{\hbar^2}{2M} \Delta_{R_l} + U_l(R_l) \right) + \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \Delta_{r_j} + \alpha_0 \sum_{l=1}^{K} V(r_j - R_l) \right) \tag{1.2}$$

acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3(K+N)}) = L^2(\mathbb{R}^{3K}) \otimes L^2(\mathbb{R}^{3N})$.
The Hamiltonian (1.2) describes the dynamics of a quantum system composed by a sub-system of K particles with position coordinates denoted by $R = (R_1, \ldots, R_K) \in \mathbb{R}^{3K}$, each of mass M and subject to the one-body interaction potential U_l, plus a sub-system of N particles with position coordinates denoted by $r = (r_1, \ldots, r_N) \in \mathbb{R}^{3N}$, each of mass m. The interaction among the particles of the two sub-systems is described by the two-body potential $\alpha_0 V$, where $\alpha_0 > 0$.
The potentials U_l, V are assumed to be smooth and rapidly decreasing at infinity.
In order to simplify the notation we fix $\hbar = M = 1$ and denote $m = \varepsilon$; moreover the coupling constant will be rescaled according to $\alpha = \varepsilon \alpha_0$, with α fixed. Then the Hamiltonian takes the form

$$H(\varepsilon) = X + \frac{1}{\varepsilon} \sum_{j=1}^{N} \left(h_{0j} + \alpha \sum_{l=1}^{K} V(r_j - R_l) \right) \tag{1.3}$$

where

$$X = \sum_{l=1}^{K} \left(-\frac{1}{2} \Delta_{R_l} + U_l(R_l) \right) \tag{1.4}$$

$$h_{0j} = -\frac{1}{2} \Delta_{r_j} \tag{1.5}$$

We are interested in the following Cauchy problem

$$\begin{cases} i\frac{\partial}{\partial t}\Psi^{\varepsilon}(t) = H(\varepsilon)\Psi^{\varepsilon}(t) \\ \\ \Psi^{\varepsilon}(0;R,r) = \phi(R)\prod_{j=1}^{N}\chi_j(r_j) \equiv \phi(R)\chi(r) \end{cases} \tag{1.6}$$

where ϕ, χ_j are sufficiently smooth given elements of $L^2(\mathbb{R}^{3K})$ and $L^2(\mathbb{R}^3)$ respectively.
Our aim is the characterization of the asymptotic behaviour of the solution $\Psi^{\varepsilon}(t)$ for $\varepsilon \to 0$, with a control of the error.
Under suitable assumptions on the potentials and the initial state, we find that the asymptotic form $\Psi_a^{\varepsilon}(t)$ of the wave function $\Psi^{\varepsilon}(t)$ for $\varepsilon \to 0$ is explicitely given by

$$\Psi_a^{\varepsilon}(t;R,r) = \int dR' e^{-itX}(R,R')\phi(R')\prod_{j=1}^{N}\left(e^{-i\frac{t}{\varepsilon}h_{0j}}\Omega_+(R')^{-1}\chi_j\right)(r_j) \tag{1.7}$$

where, for any fixed $R \in \mathbb{R}^{3K}$, we have defined the following wave operator acting in the one-particle space $L^2(\mathbb{R}^3)$ of the j-th light particle

$$\Omega_+(R)\chi_j = \lim_{\tau\to+\infty} e^{i\tau h_j(R)}\, e^{-i\tau h_{0j}}\chi_j \tag{1.8}$$

and in (1.8) we have denoted $h_j(R) = h_{0j} + \alpha\sum_{l=1}^{K} V(r_j - R_l)$.
It should be remarked that (1.7) reduces to (1.1) if we formally set $t = 0$ and assume that $\Omega_+(R')^{-1}\chi_j$ can be replaced by $S(R')\chi_j$, which is approximately true for suitably chosen state χ_j (see e.g. [HS]).
It is important to notice that the asymptotic evolution defined by (1.7) is not factorized, due to the parametric dependence on the configuration of the heavy particles of the wave operator acting on each light particle state.
Then the asymptotic wave function describes an entangled state for the whole system of heavy and light particles. In turn this implies a loss of quantum coherence for the heavy particles as a consequence of the interaction with the light ones.
The precise formulation of the approximation result will be given in the next section. Here we only mention that in the case of an arbitrary number K of heavy particles our result holds for α sufficiently small, while in the simpler case $K = 1$ we can prove the result for any α.

The plan of the paper is the following.
In section 2 we introduce some notation and formulate our main results, which are summarized in theorems 1, 1′.
In section 3 we give the main steps of the proof of theorems 1,1′.
In section 4 we prove some estimates for the unitary group generated by the Hamiltonian of the light particles $h_j(R)$, parametrically dependent on the position R of the heavy particles, uniform with respect to the parameter R.

In section 5 we collect some other technical lemmas concerning the unitary group generated by the Hamiltonian of the heavy particles X.
In section 6 we briefly discuss a possible application of the asymptotic formula for the computation of the decoherence effect induced on a heavy particle.

2. Results and notation

Our main result is given in theorem 1 below and concerns the general case $K \geq 1$. In the special case $K = 1$ we find a stronger result, summarized in theorem 1'.
The reason is that for the first case we follow and adapt to our situation the approach to dispersive estimates valid for small potentials as given in [RS], while for the second one we can prove the result for any α exploiting the approach to dispersive estimates via wave operators developed in [Y].
As a consequence we shall introduce two sets of different assumptions on the potential V and on the initial state χ of the light particles.
Let us denote by $W^{m,p}(\mathbb{R}^d)$, $H^m(\mathbb{R}^d)$ the standard Sobolev spaces and by $W^{m,p}_n(\mathbb{R}^d)$, $H^m_n(\mathbb{R}^d)$ the corresponding weighted Sobolev spaces, with $m, n, d \in \mathbb{N}$, $1 \leq p \leq \infty$.
Then we introduce the following assumptions

(A-1) $U_l \in W^{4,\infty}_2(\mathbb{R}^3)$, for $l = 1, \dots, K$;

(A-2) $\phi \in H^4_2(\mathbb{R}^{3K})$ and $\|\phi\|_{L^2(\mathbb{R}^{3K})} = 1$;

and, moreover, for the case $K \geq 1$

(A-3) $V \in W^{4,1}(\mathbb{R}^3) \cap H^4(\mathbb{R}^3)$;

(A-4) $\chi \in L^1(\mathbb{R}^{3N}) \cap L^2(\mathbb{R}^{3N})$, $\chi(r) = \prod_{j=1}^N \chi_j(r_j)$, and $\|\chi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, \dots N$.

while for the case $K = 1$

(A-5) $V \in W^{4,\infty}_\delta(\mathbb{R}^3)$, $\delta > 5$, and $V \geq 0$;

(A-6) $\chi \in W^{4,1}(\mathbb{R}^{3N}) \cap H^4(\mathbb{R}^{3N})$, $\chi(r) = \prod_{j=1}^N \chi_j(r_j)$, and $\|\chi_j\|_{L^2(\mathbb{R}^3)} = 1$ for $j = 1, \dots N$.

We notice that, under the above assumptions, the Hamiltonian (1.3) is self-adjoint and bounded from below in $\mathcal{H}$, the wave operator introduced in (1.8) exists and moreover the expression for the asymptotic wave function (1.7) makes sense.

We now state our main result. Denoting by $\| \cdot \|$ the norm in $\mathcal{H}$, for the case $K \geq 1$ we have

Theorem 1. *Let $K \geq 1$ and let us assume that U_l, ϕ, V, χ satisfy assumptions* (A-1),(A-2),(A-3),(A-4); *moreover let us fix T, $0 < T < \infty$, and define*

$$\alpha^* = \frac{\pi^{2/3}}{24K} \|V\|_{W^{4,1}}^{-1/3} \|V\|_{H^4}^{-2/3} \tag{2.1}$$

Then for any $t \in (0, T]$ and $\alpha < \alpha^*$ we have

$$\|\Psi^\varepsilon(t) - \Psi_a^\varepsilon(t)\| \leq C \sqrt{\frac{\varepsilon}{t}} \tag{2.2}$$

where C is a positive constant depending on the interaction, the initial state and T.

On the other hand, for the case $K = 1$ we prove

Theorem 1$'$. *Let $K = 1$ and let us assume that U, ϕ, V, χ satisfy assumptions* (A-1),(A-2),(A-5),(A-6); *moreover let us fix T, $0 < T < \infty$. Then for any $t \in (0, T]$ the estimate* (2.2) *holds, with a positive constant C' depending on the interaction, the initial state and T.*

Let us briefly comment on the results stated in theorems 1, 1$'$.
The estimate (2.2) clearly fails for $t \to 0$ and this fact is intrinsic in the expression of $\Psi_a^\varepsilon(t)$, which doesn't approach $\Psi^\varepsilon(0)$ for $t \to 0$.
Another remark concerns the estimate of the error in (2.2), which is probably not optimal. Indeed in the simpler two-body case analysed in [DFT], where the explicit form of the unitary group is available, the error found is $O(\varepsilon)$.
We also notice that the knowledge of the explicit dependence of the constant C on the interaction, the initial state and T is clearly interesting and will be given during the proof. We shall find that C grows with T, which is rather unnatural from the physical point of view and is a consequence of the specific method of the proof. In the two-body case studied in [AFFT] it is shown that the constant C is bounded for T large.
Concerning the method of the proof, we observe that the approach is perturbative and it is essentially based on Duhamel's formula. The main technical ingredient for the estimates is a generalized version of the dispersive estimates for Schrödinger groups.
In fact, during the proof we shall consider the one-particle Hamiltonian for the j-th light particle $h_j(R)$, parametrically dependent on the positions $R \in \mathbb{R}^{3K}$ of the heavy ones.
In particular, we shall need estimates (uniform with respect to R) for the L^∞-norm of derivatives with respect to R of the unitary evolution $e^{-i\tau h_j(R)}\chi_j$.
Apparently, such kind of estimates haven't been considered in the literature (see e.g. [RS], [Sc], [Y]) and then we exhibit a proof (see section 4) for $K \geq 1$ and small potential, following the approach of [RS], and also for $K = 1$ and arbitrary potential, following [Y].

We conclude this section collecting some notation which will be frequently used throughout the paper.
- For any $l = 1, \ldots, K$ we denote

$$X_{0,l} = -\frac{1}{2}\Delta_{R_l} \tag{2.3}$$

and

$$X_0 = \sum_{l=1}^{K} X_{0,l} \tag{2.4}$$

- U_l, U are multiplication operator by $U_l(R_l)$ and $U(R) = \sum_{l=1}^{K} U_l(R_l)$.
- For any fixed $R \in \mathbb{R}^{3K}$

$$h(R) = \sum_{j=1}^{N} h_j(R) = \sum_{j=1}^{N}\left(h_{0j} + \alpha \sum_{l=1}^{K} V(r_j - R_l)\right) \tag{2.5}$$

denotes an operator in the Hilbert space $L^2(\mathbb{R}^{3N})$, while $h_j(R)$ and h_{0j} act in the one-particle space $L^2(\mathbb{R}^3)$ of the j-th light particle.
- For any $t > 0$

$$\xi(t; R, r) = \phi(R)\left(e^{-ith(R)}\chi\right)(r) = \phi(R)\prod_{j=1}^{N}\left(e^{-ith_j(R)}\chi_j\right)(r_j) \tag{2.6}$$

$$\zeta^{\varepsilon}(t; R, r) = \left[e^{-itX}\xi(\varepsilon^{-1}t)\right](R, r) \tag{2.7}$$

defines two vectors $\xi(t)$, $\zeta^{\varepsilon}(t) \in \mathcal{H}$.
- V_R is the function in $\mathbb{R}^3$ defined by $V_R(x) = \sum_{l=1}^{K} V(x - R_l)$, for any fixed $R \in \mathbb{R}^{3K}$.
- V_{jl} denotes the multiplication operator by $V(r_j - R_l)$.
- $<x>$ is the multiplication operator by $(1 + x^2)^{1/2}$, for $x \in \mathbb{R}^d$, $d \in \mathbb{N}$.
- $d\hat{r}_j = dr_1, \dots, dr_{j-1}dr_{j+1}, \dots, dr_N$ and $d\hat{R}_l = dR_1, \dots, dR_{l-1}dR_{l+1}, \dots, dR_K$ denote two Lebesgue measures in $\mathbb{R}^{3(N-1)}$ and $\mathbb{R}^{3(K-1)}$ respectively.
- The derivative of order γ with respect to s-th component of R_m is denoted by

$$D^{\gamma}_{m,s} = \frac{\partial^{\gamma}}{\partial R^{\gamma}_{m,s}}\,, \qquad \gamma \in \mathbb{N}, \quad m = 1, \dots, K, \quad s = 1, 2, 3 \tag{2.8}$$

with $D^1_{m,s} = D_{m,s}$.
- As already mentioned, the norm in $\mathcal{H}$ is indicated by $\|\cdot\|$; the norm in $L^p(\mathbb{R}^3)$, in the Sobolev spaces $W^{m,p}(\mathbb{R}^3)$, $H^m(\mathbb{R}^3)$ and in the weighted Sobolev spaces $W^{m,p}_n(\mathbb{R}^3)$, $H^m_n(\mathbb{R}^3)$, $1 \le p \le \infty$, $m, n \in \mathbb{N}$, will be denoted by $\|\cdot\|_{L^p}$, $\|\cdot\|_{W^{m,p}}$, $\|\cdot\|_{H^m}$, $\|\cdot\|_{W^{m,p}_n}$, $\|\cdot\|_{H^m_n}$ respectively.
- We find convenient to introduce also a sort of slightly modified weighted Sobolev spaces, where both the weight and the derivatives concern the coordinates associated with only one of the heavy particles. More precisely, the weighted Sobolev space related to the l-th heavy particle, with indices $m, n \in \mathbb{N}$, $1 \le p \le \infty$, is defined as follows

$$\begin{aligned} W^{m,p}_{l,n}(\mathbb{R}^{3K}) &= \big\{f : \mathbb{R}^{3K} \to \mathbb{C},\ < R_l >^n D^{\gamma_1}_{l,1} D^{\gamma_2}_{l,2} D^{\gamma_3}_{l,3} f \in L^p(\mathbb{R}^{3K}) \\ &\quad \text{for any } \ (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}^3,\ \gamma_1+\gamma_2+\gamma_3 \le m\big\} \end{aligned} \tag{2.9}$$

The space $W^{m,p}_{l,n}(\mathbb{R}^{3K})$ is a Banach space with the norm

$$\|f\|_{W^{m,p}_{l,n}(\mathbb{R}^{3K})} = \sum_{\gamma_1=0}^{m} \sum_{\gamma_2=0}^{m-\gamma_1} \sum_{\gamma_3=0}^{m-\gamma_1-\gamma_2} \| < R_l >^n D^{\gamma_1}_{l,1} D^{\gamma_2}_{l,2} D^{\gamma_3}_{l,3} f\|_{L^p(\mathbb{R}^{3K})} \tag{2.10}$$

It is clear that for $f \in W^{m,p}_{n}(\mathbb{R}^{3K})$ the quantity $\sum_{l=1}^{K} \|f\|_{W^{m,p}_{l,n}}$ defines a norm equivalent to the standard one.
Moreover, we shall denote the space $W^{m,2}_{l,n}(\mathbb{R}^{3K})$ by $H^m_{l,n}(\mathbb{R}^{3K})$.
- The operator norm of $A\ :\ E \to F$, where $E,\ F$ are Banach spaces, is denoted by $\|A\|_{\mathcal{L}(E,F)}$.
- Finally, the symbol c will denote a generic, positive, numerical constant.

3. Proof of theorems 1,1′

We give here the proof of our main result making repeated use of some estimates which will be proved in sections 4, 5.
We start with the proof of theorem 1 and then we assume $\alpha < \alpha^*$. This condition guarantees the validity of a key technical ingredient, i.e. the uniform dispersive estimate

$$\sup_R \left\| \left(\prod_{i=1}^{n} D^{\gamma_i}_{m_i,s_i} \right) e^{-ith_j(R)} \right\|_{\mathcal{L}(L^1,L^\infty)} \le \frac{C_\gamma}{t^{3/2}} \tag{3.1}$$

where $\gamma = \sum_{i=1}^{n} \gamma_i$ and the constant C_γ is explicitely given (see (4.32) in section 4).
The estimate (3.1) is valid for any string of integers γ_i (including zero), $m = 1, \ldots, K$, $s = 1, 2, 3$ and $\alpha < \alpha^*$.
In the proof we also make use of the following uniform L^2 estimate

$$\sup_R \left\| \left(\prod_{i=1}^{n} D^{\gamma_i}_{m_i,s_i} \right) \prod_{k=1,k\neq j}^{N} e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3(N-1)})} \le \hat{C}_\gamma \tag{3.2}$$

where the constant $\hat{C}_\gamma$ is explicitely given too (see (4.37) in section 4). Notice that both C_γ and $\hat{C}_\gamma$ are increasing with respect to γ.
The proofs of (3.1) and (3.2) are postponed to section 4.
The first step is to show that $\zeta^\varepsilon(t)$ is a good approximation of $\Psi^\varepsilon_a(t)$ and this is a direct consequence of the existence of the wave operator (1.8).

Indeed, from (1.7) and (2.7) we have

$$
\begin{aligned}
&\|\Psi_a^\varepsilon(t) - \zeta^\varepsilon(t)\| \\
&= \left(\int dR |\phi(R)|^2 \left\| \prod_{j=1}^N e^{-i\frac{t}{\varepsilon}h_{0j}} \Omega_+(R)^{-1}\chi_j - \prod_{j=1}^N e^{-i\frac{t}{\varepsilon}h_j(R)}\chi_j \right\|^2_{L^2(\mathbb{R}^{3N})} \right)^{1/2} \\
&\le \sup_R \left\| \prod_{j=1}^N e^{-i\frac{t}{\varepsilon}h_{0j}} \Omega_+(R)^{-1}\chi_j - \prod_{j=1}^N e^{-i\frac{t}{\varepsilon}h_j(R)}\chi_j \right\|_{L^2(\mathbb{R}^{3N})} \\
&\le \sup_R \sum_{n=1}^N \left\| e^{-i\frac{t}{\varepsilon}h_1(R)}\chi_1 \cdots e^{-i\frac{t}{\varepsilon}h_{n-1}(R)}\chi_{n-1} \left(e^{-i\frac{t}{\varepsilon}h_{0n}} \Omega_+(R)^{-1}\chi_n - e^{-i\frac{t}{\varepsilon}h_n(R)}\chi_n \right) \right. \\
&\left. e^{-i\frac{t}{\varepsilon}h_{0n+1}} \Omega_+(R)^{-1}\chi_{n+1} \cdots e^{-i\frac{t}{\varepsilon}h_{0N}} \Omega_+(R)^{-1}\chi_N \right\|_{L^2(\mathbb{R}^{3N})} \\
&\le \sup_R \sum_{n=1}^N \left\| e^{i\frac{t}{\varepsilon}h_{0n}} e^{-i\frac{t}{\varepsilon}h_n(R)}\chi_n - \Omega_+(R)^{-1}\chi_n \right\|_{L^2}
\end{aligned}
\tag{3.3}
$$

Let us recall that for any $\tau > 0$

$$
e^{i\tau h_{0n}} e^{-i\tau h_n(R)}\chi_n - \Omega_+(R)^{-1}\chi_n = i\alpha \int_\tau^\infty ds \;\; e^{ish_{0n}} V_R e^{-ish_n(R)}\chi_n \tag{3.4}
$$

Then using the dispersive estimate (3.1) we conclude

$$
\begin{aligned}
\|\Psi_a^\varepsilon(t) - \zeta^\varepsilon(t)\| &\le \alpha \sup_R \left(\|V_R\|_{L^2} \sum_{n=1}^N \int_{t/\varepsilon}^\infty ds \left\| e^{-ish_n(R)}\chi_n \right\|_{L^\infty} \right) \\
&\le \frac{\sqrt{\varepsilon}}{\sqrt{t}}\, C_0 \alpha K \|V\|_{L^2} \left(\sum_{n=1}^N \|\chi_n\|_{L^1} \right)
\end{aligned}
\tag{3.5}
$$

The next and more delicate step is to show that $\zeta^\varepsilon(t)$ approximates the solution $\Psi^\varepsilon(t)$. By a direct computation one has

$$
i\frac{\partial}{\partial t}\zeta^\varepsilon(t) = H(\varepsilon)\zeta^\varepsilon(t) + \mathcal{R}^\varepsilon(t) \tag{3.6}
$$

where

$$
\mathcal{R}^\varepsilon(t) = \frac{\alpha}{\varepsilon} \sum_{j=1}^N \sum_{l=1}^K [e^{-itX}, V_{jl}]\xi(\varepsilon^{-1}t) \tag{3.7}
$$

Using Duhamel's formula and writing

$$
[e^{-itX}, V_{jl}] = (e^{-itX} - I)V_{jl} - V_{jl}(e^{-itX} - I) \tag{3.8}
$$

we have

$$
\begin{aligned}
\|\Psi^\varepsilon(t)-\zeta^\varepsilon(t)\| &\le \int_0^t ds \|\mathcal{R}^\varepsilon(s)\| \\
&\le \frac{\alpha}{\varepsilon}\sum_{l=1}^K\sum_{j=1}^N \int_0^t ds \left[\|(e^{-isX}-I)V_{jl}\xi(\varepsilon^{-1}s)\| + \|V_{jl}(e^{-isX}-I)\xi(\varepsilon^{-1}s)\|\right] \\
&= \alpha \sum_{l=1}^K\sum_{j=1}^N \int_0^{\varepsilon^{-1}t} d\sigma \left(\mathcal{A}^\varepsilon_{jl}(\sigma)+\mathcal{B}^\varepsilon_{jl}(\sigma)\right)
\end{aligned}
\tag{3.9}
$$

where we have defined

$$
\mathcal{A}^\varepsilon_{jl}(\sigma) = \|(e^{-i\varepsilon\sigma X}-I)V_{jl}\xi(\sigma)\| \tag{3.10}
$$

$$
\mathcal{B}^\varepsilon_{jl}(\sigma) = \|V_{jl}(e^{-i\varepsilon\sigma X}-I)\xi(\sigma)\| \tag{3.11}
$$

The problem is then reduced to the estimate of the two terms (3.10) and (3.11).
The basic idea is that both terms are controlled by $e^{-i\varepsilon\sigma X}-I$ for σ small with respect to $\varepsilon^{-1}t$ and by the dispersive character of the unitary group $e^{-i\sigma h(R)}$ for σ of the order $\varepsilon^{-1}t$.
It turns out that such strategy is easily implemented for (3.10) while for (3.11) the estimate is a bit more involved.

3.a *Estimate of* $\mathcal{A}^\varepsilon_{jl}(\sigma)$

For the estimate of $\mathcal{A}^\varepsilon_{jl}(\sigma)$, using the spectral theorem we have

$$
\begin{aligned}
\mathcal{A}^\varepsilon_{jl}(\sigma) &\le \varepsilon\sigma\|XV_{jl}\xi(\sigma)\| \\
&\le \varepsilon\sigma\sum_{m=1}^K \|X_{0,m}(V_{jl}\xi(\sigma))\| + \varepsilon\sigma\|UV_{jl}\xi(\sigma)\| \\
&\le \frac{\varepsilon\sigma}{2}\sum_{m=1}^K\sum_{s=1}^3 \|D^2_{m,s}(V_{jl}\xi(\sigma))\| + \varepsilon\sigma\|UV_{jl}\xi(\sigma)\| \\
&\le \frac{\varepsilon\sigma}{2}\sum_{m=1}^K\sum_{s=1}^3\sum_{\gamma=0}^2 \binom{2}{\gamma}\left\| D^{2-\gamma}_{m,s}\left(V_{jl}\phi\prod_{k=1,k\neq j}^N e^{-i\sigma h_k(\cdot)}\chi_k\right)\cdot D^\gamma_{m,s}\left(e^{-i\sigma h_j(\cdot)}\chi_j\right)\right\| \\
&+\varepsilon\sigma\left\|UV_{jl}\phi e^{-i\sigma h(\cdot)}\chi\right\|
\end{aligned}
\tag{3.12}
$$

where we have used the definition (2.6) and the Leibniz's rule.

Using (3.1), the Leibniz's rule again and (3.2), we have

$$\sum_{m=1}^{K}\sum_{s=1}^{3}\left\| D_{m,s}^{2-\gamma}\left(V_{jl}\phi\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right)\cdot D_{m,s}^{\gamma}\left(e^{-i\sigma h_j(\cdot)}\chi_j\right)\right\|$$

$$\leq \sum_{m=1}^{K}\sum_{s=1}^{3}\sup_R\left\|D_{m,s}^{\gamma}\,e^{-i\sigma h_j(R)}\chi_j\right\|_{L^\infty}\left\|D_{m,s}^{2-\gamma}\left(V_{jl}\phi\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right)\right\|$$

$$\leq \frac{C_\gamma}{\sigma^{3/2}}\|\chi_j\|_{L^1}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\delta=0}^{2-\gamma}\binom{2-\gamma}{\delta}\sup_R\left\|D_{m,s}^{2-\gamma-\delta}\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(R)}\chi_k\right\|_{L^2(\mathbb{R}^{3(N-1)})}$$

$$\cdot\left\|D_{m,s}^{\delta}(V_{jl}\phi)\right\|_{L^2(\mathbb{R}^{3(K+1)})}$$

$$\leq \frac{C_\gamma}{\sigma^{3/2}}\hat{C}_2\|\chi_j\|_{L^1}\ \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\delta=0}^{2-\gamma}\binom{2-\gamma}{\delta}\left\|D_{m,s}^{\delta}(V_{jl}\phi)\right\|_{L^2(\mathbb{R}^{3(K+1)})} \tag{3.13}$$

Moreover

$$\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\delta=0}^{2-\gamma}\binom{2-\gamma}{\delta}\left\|D_{m,s}^{\delta}(V_{jl}\phi)\right\|_{L^2(\mathbb{R}^{3(K+1)})}\leq 4\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\delta=0}^{2-\gamma}\sum_{\lambda=0}^{\delta}\|D_{m,s}^{\lambda}V_{jl}\|_{L^2}\|D_{m,s}^{\delta-\lambda}\phi\|_{L^2(\mathbb{R}^{3K})}$$

$$\leq c\,\|V\|_{H^2}\|\phi\|_{H^2(\mathbb{R}^{3K})} \tag{3.14}$$

Then, using (3.14) in (3.13), we obtain

$$\sum_{m=1}^{K}\sum_{s=1}^{3}\left\| D_{m,s}^{2-\gamma}\left(V_{jl}\phi\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right)\cdot D_{m,s}^{\gamma}\left(e^{-i\sigma h_j(\cdot)}\chi_j\right)\right\|$$

$$\leq \frac{c}{\sigma^{3/2}}C_\gamma\hat{C}_2\|\chi_j\|_{L^1}\|V\|_{H^2}\|\phi\|_{H^2(\mathbb{R}^{3K})} \tag{3.15}$$

Concerning the last term in (3.12), we use the uniform dispersive estimate (3.1) again

$$\|UV_{jl}\phi e^{-i\sigma h(\cdot)}\chi\| = \left(\int dR\int dr_j|V(r_j-R_l)|^2|U(R)\phi(R)|^2\int d\hat{r}_j\left|\left(e^{-i\sigma h(R)}\chi\right)(r)\right|^2\right)^{1/2}$$

$$= \left(\int dR\int dr_j|V(r_j-R_l)|^2|U(R)\phi(R)|^2\left|\left(e^{-i\sigma h_j(R)}\chi_j\right)(r_j)\right|^2\right)^{1/2}$$

$$= \sup_R\left\|e^{-i\sigma h_j(R)}\chi_j\right\|_{L^\infty}\|V\|_{L^2}\|U\phi\|_{L^2(\mathbb{R}^{3K})}$$

$$\leq \frac{C_0}{\sigma^{3/2}}\|\chi_j\|_{L^1}\|V\|_{L^2}\|U\|_{L^\infty(\mathbb{R}^{3K})} \tag{3.16}$$

Using (3.15), (3.16) in (3.12) we find

$$\mathcal{A}_{jl}^{\varepsilon}(\sigma)\leq c\frac{\varepsilon}{\sqrt{\sigma}}\|\chi_j\|_{L^1}\Big(C_2\,\hat{C}_2\|V\|_{H^2}\|\phi\|_{H^2(\mathbb{R}^{3K})}+C_0\|V\|_{L^2}\|U\|_{L^\infty(\mathbb{R}^{3K})}\Big) \tag{3.17}$$

and then

$$\alpha \sum_{l=1}^{K} \sum_{j=1}^{N} \int_0^{\varepsilon^{-1}t} d\sigma \mathcal{A}_{jl}^{\varepsilon}(\sigma)$$

$$\leq c\sqrt{\varepsilon}\sqrt{t} K\alpha \|V\|_{H^2} C_2 \hat{C}_2 \left(1 + \|U\|_{L^\infty(\mathbb{R}^{3K})}\right) \left(\sum_{j=1}^{N} \|\chi_j\|_{L^1} \right) \|\phi\|_{H^2(\mathbb{R}^{3K})} \tag{3.18}$$

3.b *Estimate of* $\mathcal{B}_{jl}^{\varepsilon}(\sigma)$

For the estimate of (3.11) we first introduce a convergence factor which makes finite the integral with respect to the variable R_l. In fact we write

$$\begin{aligned}
&\left(\mathcal{B}_{jl}^{\varepsilon}(\sigma)\right) \\
&= \left[\int dr_j \int dR_l \frac{|V(r_j - R_l)|^2}{<R_l>^4} \int d\hat{r}_j \int d\hat{R}_l \left| <R_l>^2 \left[\left(e^{-i\varepsilon\sigma X} - I \right) \xi(\sigma) \right] (R, r) \right|^2 \right]^{1/2} \\
&\leq \left[\int dx |V(x)|^2 \int dy \frac{1}{(y^2+1)^2} \sup_{r_j} \sup_{R_l} \int d\hat{r}_j \int d\hat{R}_l \left| <R_l>^2 \left[\left(e^{-i\varepsilon\sigma X} - I \right) \xi(\sigma) \right] (R, r) \right|^2 \right]^{1/2} \\
&= \pi \|V\|_{L^2} \left[\sup_{r_j} \sup_{R_l} \int d\hat{r}_j \int d\hat{R}_l \left| <R_l>^2 \left[\left(e^{-i\varepsilon\sigma X} - I \right) \xi(\sigma) \right] (R, r) \right|^2 \right]^{1/2} \\
&\leq \pi \|V\|_{L^2} \left[\sup_{r_j} \int d\hat{r}_j \int d\hat{R}_l \sup_{R_l} \left| <R_l>^2 \left[\left(e^{-i\varepsilon\sigma X} - I \right) \xi(\sigma) \right] (R, r) \right|^2 \right]^{1/2} \\
&\leq \pi \|V\|_{L^2} \sup_{r_j} \left[\int d\hat{r}_j \left\| (X_{0,l} + I) <R_l>^2 \left(e^{-i\varepsilon\sigma X} - I \right) \xi(\sigma; \cdot, r) \right\|_{L^2(\mathbb{R}^{3K})}^2 \right]^{1/2}
\end{aligned} \tag{3.19}$$

where we have exploited the estimate for a.e. $y \in \mathbb{R}^q$ and $x \in \mathbb{R}^3$

$$|F(x,y)| \leq \left\{ \int dx \left| \left[(-\Delta_x + I) F \right] (x,y) \right|^2 \right\}^{1/2} \tag{3.20}$$

which holds for any function $F \in L^2(\mathbb{R}^{3+q})$, $q \in \mathbb{N}$, such that $F(\cdot, y) \in H^2(\mathbb{R}^3)$ for a.e. $y \in \mathbb{R}^q$. Notice that the proof of (3.20) is simply obtained taking the Fourier transform of F with respect to the variable x.

It is convenient to introduce the abridged notation

$$\|f\|_{L_j^\infty L^2} = \sup_{r_j} \left[\int d\hat{r}_j \|f(\cdot, r)\|^2_{L^2(\mathbb{R}^{3K})} \right]^{1/2} \tag{3.21}$$

where $f : \mathbb{R}^{3K} \times \mathbb{R}^{3N} \to \mathbb{C}$. Then using the formula

$$<R_l>^2 \left(e^{-itX} - I\right) = \left(e^{-itX} - I\right) <R_l>^2 + \left[R_l^2, e^{-itX}\right] \tag{3.22}$$

we have

$$\begin{aligned} &\left(\mathcal{B}^\varepsilon_{jl}(\sigma)\right) \leq \pi \|V\|_{L^2} \left\|(X_{0,l} + I) \left(e^{-i\varepsilon\sigma X} - I\right) <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &+ \pi \|V\|_{L^2} \left\|(X_{0,l} + I)[R_l^2, e^{-i\varepsilon\sigma X}]\xi(\sigma)\right\|_{L_j^\infty L^2} \\ &\equiv (I) + (II) \end{aligned} \tag{3.23}$$

Writing

$$(X_{0,l} + I)(e^{-itX} - I) = (e^{-itX} - I)(X_{0,l} + I) + \left[X_{0,l}, e^{-itX}\right] \tag{3.24}$$

and using the spectral theorem for the estimate of $e^{-i\varepsilon\sigma X} - I$, we have

$$\begin{aligned} &(I) \leq \pi \|V\|_{L^2} \left\|\left(e^{-i\varepsilon\sigma X} - I\right) (X_{0,l} + I) <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &+\pi \|V\|_{L^2} \left\|\left[X_{0,l}, e^{-i\varepsilon\sigma X}\right] <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &\leq \pi\varepsilon\sigma \|V\|_{L^2} \left\|X_0 \left(X_{0,l} + I\right) <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} + \pi\varepsilon\sigma \|V\|_{L^2} \left\|U(X_{0,l} + I) <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &+\pi \|V\|_{L^2} \left\|\left[X_{0,l}, e^{-i\varepsilon\sigma X}\right] <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &\equiv (III) + (IV) + (V) \end{aligned} \tag{3.25}$$

We decompose the term (III) as follows

$$\begin{aligned} &(III) \leq \pi\varepsilon\sigma \|V\|_{L^2} \left\|X_0 <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} + \pi\varepsilon\sigma \|V\|_{L^2} \left\|X_0 X_{0,l} <R_l>^2 \xi(\sigma)\right\|_{L_j^\infty L^2} \\ &\equiv (IIIa) + (IIIb) \end{aligned} \tag{3.26}$$

To estimate $(IIIa)$ we take into account the definition (2.6) and the Leibniz's rule

$$
\begin{aligned}
(IIIa) &\leq c\,\varepsilon\sigma\|V\|_{L^2}\sum_{m=1}^{K}\sum_{s=1}^{3}\left\|D^2_{m,s} <R_l>^2 \phi\prod_{k=1}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right\|_{L^\infty_j L^2} \\
&\leq c\,\varepsilon\sigma\|V\|_{L^2}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\gamma=0}^{2}\binom{2}{\gamma}\left\|\left(D^\gamma_{m,s}e^{-i\sigma h_j(\cdot)}\chi_j\right)\left(D^{2-\gamma}_{m,s} <R_l>^2 \phi\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right)\right\|_{L^\infty_j L^2} \\
&\leq c\,\varepsilon\sigma\|V\|_{L^2}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\gamma=0}^{2}\sum_{\lambda=0}^{2-\gamma}\binom{2}{\gamma}\binom{2-\gamma}{\lambda} \\
&\cdot\left\|\left(D^\gamma_{m,s}e^{-i\sigma h_j(\cdot)}\chi_j\right)\left(D^{2-\gamma-\lambda}_{m,s} <R_l>^2 \phi\right)\left(D^\lambda_{m,s}\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(\cdot)}\chi_k\right)\right\|_{L^\infty_j L^2}
\end{aligned}
\tag{3.27}
$$

From (3.1) and (3.2) we have

$$
\begin{aligned}
(IIIa) &\leq c\,\varepsilon\sigma\|V\|_{L^2}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\gamma=0}^{2}\sum_{\lambda=0}^{2-\gamma}\binom{2}{\gamma}\binom{2-\gamma}{\lambda}\sup_R\left\|D^\gamma_{m,s}e^{-i\sigma h_j(R)}\chi_j\right\|_{L^\infty} \\
&\cdot\left\|D^{2-\gamma-\lambda}_{m,s} <R_l>^2 \phi\right\|_{L^2(\mathbb{R}^{3K})}\sup_R\left\|D^\lambda_{m,s}\prod_{k=1,k\neq j}^{N}e^{-i\sigma h_k(R)}\chi_k\right\|_{L^2(\mathbb{R}^{3(N-1)})} \\
&\leq c\,C_2\hat{C}_2\,\frac{\varepsilon}{\sqrt{\sigma}}\|V\|_{L^2}\|\chi_j\|_{L^1}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\delta=0}^{2}\| <R_l>^2 D^\delta_{m,s}\phi\|_{L^2(\mathbb{R}^{3K})}
\end{aligned}
\tag{3.28}
$$

Analogously,

$$
\begin{aligned}
(IIIb) &\le c\,\varepsilon\sigma \|V\|_{L^2} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3} \left\| D^2_{m,s} D^2_{l,s'} <R_l>^2 \phi \prod_{k=1}^{N} e^{-i\sigma h_k(\cdot)}\chi_k \right\|_{L^\infty_j L^2} \\
&\le c\,\varepsilon\sigma \|V\|_{L^2} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{\gamma=0}^{2}\sum_{\gamma'=0}^{2} \binom{2}{\gamma}\binom{2}{\gamma'} \\
&\cdot \left\| \left(D^{\gamma}_{m,s} D^{\gamma'}_{l,s'} e^{-i\sigma h_j(\cdot)}\chi_j \right) \left(D^{2-\gamma}_{m,s} D^{2-\gamma'}_{l,s'} <R_l>^2 \phi \prod_{k=1,k\neq j}^{N} e^{-i\sigma h_k(\cdot)}\chi_k \right) \right\|_{L^\infty_j L^2} \\
&\le c\,\varepsilon\sigma \|V\|_{L^2} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3}\sum_{\gamma=0}^{2}\sum_{\gamma'=0}^{2}\sum_{\lambda=0}^{2-\gamma}\sum_{\lambda'=0}^{2-\gamma'} \binom{2}{\gamma}\binom{2-\gamma}{\lambda}\binom{2}{\gamma'}\binom{2-\gamma'}{\lambda'} \\
&\cdot \left\| \left(D^{\gamma}_{m,s} D^{\gamma'}_{l,s'} e^{-i\sigma h_j(\cdot)}\chi_j \right)\left(D^{2-\gamma-\lambda}_{m,s} D^{2-\gamma'-\lambda'}_{l,s'} <R_l>^2 \phi \right)\left(D^{\lambda}_{m,s} D^{\lambda'}_{l,s'} \prod_{k=1,k\neq j}^{N} e^{-i\sigma h_k(\cdot)}\chi_k \right) \right\|_{L^\infty_j L^2} \qquad (3.29)
\end{aligned}
$$

Using again (3.1), (3.2) we have

$$
\begin{aligned}
(IIIb) &\le c\,\varepsilon\sigma \|V\|_{L^2} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3}\sum_{\gamma=0}^{2}\sum_{\gamma'=0}^{2}\sum_{\lambda=0}^{2-\gamma}\sum_{\lambda'=0}^{2-\gamma'} \binom{2}{\gamma}\binom{2}{\gamma'}\binom{2-\gamma}{\lambda}\binom{2-\gamma'}{\lambda'} \\
&\cdot \sup_R \left\| D^{\gamma}_{m,s} D^{\gamma'}_{l,s'} e^{-i\sigma h_j(R)}\chi_j \right\|_{L^\infty} \left\| D^{2-\gamma-\lambda}_{m,s} D^{2-\gamma'-\lambda'}_{l,s'} <R_l>^2 \phi \right\|_{L^2(\mathbb{R}^{3K})} \\
&\cdot \sup_R \left\| D^{\lambda}_{m,s} D^{\lambda'}_{l,s'} \prod_{k=1,k\neq j}^{N} e^{-i\sigma h_k(R)}\chi_k \right\|_{L^2(\mathbb{R}^{3(N-1)})} \\
&\le c\,\frac{\varepsilon}{\sqrt{\sigma}} C_4 \hat{C}_4 \|V\|_{L^2} \|\chi_j\|_{L^1} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3}\sum_{\delta=0}^{2}\sum_{\mu=0}^{2} \| <R_l>^2\, D^{\delta}_{m,s} D^{\mu}_{l,s'}\phi \|_{L^2(\mathbb{R}^{3K})} \qquad (3.30)
\end{aligned}
$$

From (3.28) and (3.30) we obtain

$$
(III) \ \le\ c\,\frac{\varepsilon}{\sqrt{\sigma}} C_4 \hat{C}_4 \|V\|_{L^2} \|\chi_j\|_{L^1} \sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3}\sum_{\delta=0}^{2}\sum_{\mu=0}^{2} \| <R_l>^2\, D^{\delta}_{m,s} D^{\mu}_{l,s'}\phi \|_{L^2(\mathbb{R}^{3K})} \qquad (3.31)
$$

Following the same line, the estimate of (IV) is easily obtained. In fact one has

$$
\begin{aligned}
(IV) &\le \pi\varepsilon\sigma \|V\|_{L^2} \|U\|_{L^\infty(\mathbb{R}^{3K})} \left\| (X_{0,l}+I) <R_l>^2 \xi(\sigma) \right\|_{L^\infty_j L^2} \\
&\le c\,\frac{\varepsilon}{\sqrt{\sigma}} C_2 \hat{C}_2 \|V\|_{L^2} \|U\|_{L^\infty(\mathbb{R}^{3K})} \|\chi_j\|_{L^1} \sum_{s=1}^{3}\sum_{\mu=0}^{2} \| <R_l>^2\, D^{\mu}_{l,s}\phi \|_{L^2(\mathbb{R}^{3K})} \qquad (3.32)
\end{aligned}
$$

We now consider the term (V).
In section 5 we shall prove the following commutator estimate for any $f \in H^1_{l,0}(\mathbb{R}^{3K})$ and $t \in [0,T]$

$$\| \left[X_{0,l}, e^{-itX}\right] f\|_{L^2(\mathbb{R}^{3K})} \leq t\,\tilde{C}\|f\|_{H^1_{l,0}(\mathbb{R}^{3K})} \tag{3.33}$$

where the costant $\tilde{C}$ can be explicitely computed (see (5.7) in section 5). Then, using (3.33), we have

$$\begin{aligned}
(V) &\leq \pi\varepsilon\sigma\,\tilde{C}\|V\|_{L^2} \sup_{r_j} \left[\int d\hat{r}_j \| <R_l>^2 \xi(\sigma;\cdot,r)\|^2_{H^1_{l,0}(\mathbb{R}^{3K})}\right]^{1/2} \\
&\leq \pi\varepsilon\sigma\tilde{C}\|V\|_{L^2} \left\{ \left\| \left(e^{-i\sigma h_j(\cdot)}\chi_j\right) <R_l>^2 \phi \prod_{k=1,k\neq j}^{N} \left(e^{-i\sigma h_k(\cdot)}\chi_k\right) \right\|_{L^\infty_j L^2} \right. \\
&+ \sum_{s=1}^{3} \left\| \left(D_{l,s} e^{-i\sigma h_j(\cdot)}\chi_j\right) <R_l>^2 \phi \prod_{k=1,k\neq j}^{N} \left(e^{-i\sigma h_k(\cdot)}\chi_k\right) \right\|_{L^\infty_j L^2} \\
&+ \sum_{s=1}^{3} \left\| (e^{-i\sigma h_j(\cdot)}\chi_j)(D_{l,s} <R_l>^2 \phi) \prod_{k=1,k\neq j}^{N} \left(e^{-i\sigma h_k(\cdot)}\chi_k\right) \right\|_{L^\infty_j L^2} \\
&+ \left. \sum_{s=1}^{3} \left\| (e^{-i\sigma h_j(\cdot)}\chi_j) <R_l>^2 \phi D_{l,s} \left(\prod_{k=1,k\neq j}^{N} \left(e^{-i\sigma h_k(\cdot)}\chi_k\right)\right) \right\|_{L^\infty_j L^2} \right\}
\end{aligned} \tag{3.34}$$

Exploiting the estimates (3.1), (3.2) we find

$$\begin{aligned}
(V) &\leq \pi \frac{\varepsilon}{\sqrt{\sigma}} \tilde{C} C_1 \|V\|_{L^2} \|\chi_j\|_{L^1} \left[4\| <R_l>^2 \phi\|_{L^2(\mathbb{R}^{3K})} \right. \\
&+ \left. \sum_{s=1}^{3} \|D_{l,s}(<R_l>^2 \phi)\|_{L^2(\mathbb{R}^{3K})} + 3\hat{C}_1 \| <R_l>^2 \phi\|_{L^2(\mathbb{R}^{3K})} \right] \\
&\leq c \frac{\varepsilon}{\sqrt{\sigma}} \tilde{C} C_1 \hat{C}_1 \|V\|_{L^2} \|\chi_j\|_{L^1} \sum_{s=1}^{3} \sum_{\mu=0}^{1} \| <R_l>^2 D^{\mu}_{l,s}\phi\|_{L^2(\mathbb{R}^{3K})}
\end{aligned} \tag{3.35}$$

Let us consider the term (II) in (3.23). In section 5 we shall prove the estimate for any $f \in H^4_{l,2}(\mathbb{R}^{3K})$ and $t \in [0,T]$

$$\left\|(X_{0,l} + I)[R_l^2, e^{-itX}]f\right\|_{L^2(\mathbb{R}^{3K})} \leq t\,\bar{C}\|f\|_{H^4_{l,2}(\mathbb{R}^{3K})} \tag{3.36}$$

where the constant $\bar{C}$ can be explicitely computed (see (5.28) in section 5).

Exploiting (3.36), we have

$$
\begin{aligned}
&(II) \leq \pi \varepsilon \sigma \bar{C} \|V\|_{L^2} \sup_{r_j} \left[\int d\hat{r}_j \|\xi(\sigma; \cdot, r)\|^2_{H^4_{l,2}(\mathbb{R}^{3K})} \right]^{1/2} \\
&\leq \varepsilon \sigma \bar{C} \|V\|_{L^2} \sum_{\gamma_1=0}^{4} \sum_{\gamma_2=0}^{4-\gamma_1} \sum_{\gamma_3=0}^{4-\gamma_1-\gamma_2} \left\| < R_l >^2 D^{\gamma_1}_{l,1} D^{\gamma_2}_{l,2} D^{\gamma_3}_{l,3} \left(\phi \prod_{k=1}^{N} (e^{-i\sigma h_k(\cdot)} \chi_k) \right) \right\|_{L^\infty_j L^2} \\
&\leq \varepsilon \sigma \bar{C} \|V\|_{L^2} \sum_{\gamma_1=0}^{4} \sum_{\gamma_2=0}^{4-\gamma_1} \sum_{\gamma_3=0}^{4-\gamma_1-\gamma_2} \sum_{\beta_1=0}^{\gamma_1} \sum_{\beta_2=0}^{\gamma_2} \sum_{\beta_3=0}^{\gamma_3} \binom{\gamma_1}{\beta_1} \binom{\gamma_2}{\beta_2} \binom{\gamma_3}{\beta_3} \\
&\cdot \left\| < R_l >^2 D^{\gamma_1-\beta_1}_{l,1} D^{\gamma_2-\beta_2}_{l,2} D^{\gamma_3-\beta_3}_{l,3} \left(\phi \prod_{k=1, k\neq j}^{N} (e^{-i\sigma h_k(\cdot)} \chi_k) \right) D^{\beta_1}_{l,1} D^{\beta_2}_{l,2} D^{\beta_3}_{l,3} (e^{-i\sigma h_j(\cdot)} \chi_j) \right\|_{L^\infty_j L^2} \\
&\leq c \frac{\varepsilon}{\sqrt{\sigma}} \bar{C} C_4 \|V\|_{L^2} \|\chi_j\|_{L^1} \sum_{\delta_1=0}^{4} \sum_{\delta_2=0}^{4-\delta_1} \sum_{\delta_3=0}^{4-\delta_1-\delta_2} \left\| < R_l >^2 D^{\delta_1}_{l,1} D^{\delta_2}_{l,2} D^{\delta_3}_{l,3} \left(\phi \prod_{k=1, k\neq j}^{N} (e^{-i\sigma h_k(\cdot)} \chi_k) \right) \right\|_{L^\infty_j L^2} \\
&\leq c \frac{\varepsilon}{\sqrt{\sigma}} \bar{C} C_4 \hat{C}_4 \|V\|_{L^2} \|\chi_j\|_{L^1} \sum_{\mu_1=0}^{4} \sum_{\mu_2=0}^{4-\mu_1} \sum_{\mu_3=0}^{4-\mu_1-\mu_2} \| < R_l >^2 D^{\mu_1}_{l,1} D^{\mu_2}_{l,2} D^{\mu_3}_{l,3} \phi \|_{L^2(\mathbb{R}^{3K})}
\end{aligned} \tag{3.37}
$$

where we have repeatedly used the Leibniz's rule and estimates (3.1), (3.2).
Taking into account (3.23), (3.25), (3.31), (3.32), (3.35), (3.37) we obtain

$$
\begin{aligned}
&\mathcal{B}^{\varepsilon}_{jl}(\sigma) \leq c \frac{\varepsilon}{\sqrt{\sigma}} C_4 \hat{C}_4 \|V\|_{L^2} \|\chi_j\|_{L^1} \Bigg[\sum_{m=1}^{K} \sum_{s=1}^{3} \sum_{s'=1}^{3} \sum_{\delta=0}^{2} \sum_{\mu=0}^{2} \| < R_l >^2 D^{\delta}_{m,s} D^{\mu}_{l,s'} \phi \|_{L^2(\mathbb{R}^{3K})} \\
&+ \bar{C} \sum_{\mu_1=0}^{4} \sum_{\mu_2=0}^{4-\mu_1} \sum_{\mu_3=0}^{4-\mu_1-\mu_2} \| < R_l >^2 D^{\mu_1}_{l,1} D^{\mu_2}_{l,2} D^{\mu_3}_{l,3} \phi \|_{L^2(\mathbb{R}^{3K})} \\
&+ (\tilde{C} + \|U\|_{L^\infty(\mathbb{R}^{3K})}) \sum_{s=1}^{3} \sum_{\mu=0}^{2} \| < R_l >^2 D^{\mu}_{l,s} \phi \|_{L^2(\mathbb{R}^{3K})} \Bigg]
\end{aligned} \tag{3.38}
$$

and then

$$\begin{aligned}
&\alpha \sum_{l=1}^{K} \sum_{j=1}^{N} \int_0^{\varepsilon^{-1}t} d\sigma \mathcal{B}^{\varepsilon}_{jl}(\sigma) \\
&\leq c\sqrt{\varepsilon}\sqrt{t}\alpha \|V\|_{L^2} C_4 \hat{C}_4 \left(\sum_{j=1}^{N} \|\chi_j\|_{L^1} \right) \\
&\cdot \Bigg[(1+\tilde{C}+\|U\|_{L^\infty(\mathbb{R}^{3K})}) \sum_{l=1}^{K}\sum_{m=1}^{K}\sum_{s=1}^{3}\sum_{s'=1}^{3}\sum_{\delta=0}^{2}\sum_{\mu=0}^{2} \| \langle R_l \rangle^2 \, D^{\delta}_{m,s} D^{\mu}_{l,s'} \phi \|_{L^2(\mathbb{R}^{3K})} \\
&+\bar{C} \sum_{l=1}^{K} \sum_{\mu_1=0}^{4} \sum_{\mu_2=0}^{4-\mu_1} \sum_{\mu_3=0}^{4-\mu_1-\mu_2} \| \langle R_l \rangle^2 \, D^{\mu_1}_{l,1} D^{\mu_2}_{l,2} D^{\mu_3}_{l,3} \phi \|_{L^2(\mathbb{R}^{3K})} \Bigg]
\end{aligned} \tag{3.39}$$

Notice that assumption (A-2) guarantees that the norms in (3.39) involving ϕ are finite. Finally, using (3.18), (3.39) in (3.9) and taking into account (3.5) we conclude the proof of theorem 1.

□

The proof of theorem $1'$ is obtained following exactly the same line of the previous one with only slight modifications. Then we shall limit ourselves to show the points to be modified. We fix $K = 1$ and assume (A-1), (A-2), (A-5), (A-6); moreover we make use of the following uniform estimates which hold for any value of α

$$\sup_R \left\| D^{\gamma_1}_{s_1} D^{\gamma_2}_{s_2} D^{\gamma_3}_{s_3} e^{-ith_k(R)} \chi_k \right\|_{L^\infty} \leq \frac{B_\gamma}{t^{3/2}} \|\chi_k\|_{W^{\gamma,1}} \tag{3.40}$$

$$\sup_R \left\| D^{\gamma_1}_{s_1} D^{\gamma_2}_{s_2} D^{\gamma_3}_{s_3} \prod_{k=1}^{N} e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3N})} \leq \hat{B}_\gamma \tag{3.41}$$

where $\gamma = \sum_{i=1}^{3} \gamma_i$ and B_γ and $\hat{B}_\gamma$ are positive constants, increasing with γ.
The estimates (3.40), (3.41) replace, in the case $K = 1$, the uniform estimates (3.1), (3.2), which hold for $\alpha < \alpha^*$ in the general case $K \geq 1$, and will be proved in section 4.
Proceeding for $K = 1$ as in the proof of theorem 1, we see that estimate (3.5) holds with C_0 replaced by B_0. Moreover we denote by $\mathcal{A}^{\varepsilon}_j(\sigma)$, $\mathcal{B}^{\varepsilon}_j(\sigma)$ the analogues of (3.10) and (3.11) in the case $K = 1$. Then it is easily seen that (3.18) is replaced by

$$\alpha \sum_{j=1}^{N} \int_0^{\varepsilon^{-1}t} d\sigma \, \mathcal{A}^{\varepsilon}_j(\sigma) \leq c\sqrt{\varepsilon}\sqrt{t}\alpha \|V\|_{H^2} B_2 \hat{B}_2 \left(1+\|U\|_{L^\infty}\right) \left(\sum_{j=1}^{N} \|\chi_j\|_{W^{2,1}} \right) \|\phi\|_{H^2} \tag{3.42}$$

Analogously, (3.39) is replaced by

$$\alpha \sum_{j=1}^{N} \int_0^{\varepsilon^{-1}t} d\sigma\, \mathcal{B}_j^{\varepsilon}(\sigma) \leq c\sqrt{\varepsilon}\sqrt{t}\alpha \|V\|_{L^2} B_4 \hat{B}_4 \left(1 + \|U\|_{L^\infty} + \tilde{C} + \bar{C}\right) \left(\sum_{j=1}^{N} \|\chi_j\|_{W^{4,1}}\right) \|\phi\|_{H_2^4} \tag{3.43}$$

and then the proof of theorem $1'$ is complete.

$\square$

4. Uniform estimates for the unitary group $e^{-ith(R)}$

In this section we shall prove some results concerning the unitary group of the light particles and its derivatives with respect to R, which plays here the role of a parameter. In particular we shall find estimates uniform with respect to R.
We denote the one-particle Hamiltonian in $L^2(\mathbb{R}^3)$ of the generic light particle for any fixed $R \in \mathbb{R}^{3K}$ as follows

$$\hat{h}(R) = h_0 + \alpha V_R \tag{4.1}$$

$$h_0 = -\frac{1}{2}\Delta \tag{4.2}$$

$$V_R(x) = \sum_{l=1}^{K} V(x - R_l), \qquad x \in \mathbb{R}^3 \tag{4.3}$$

Moreover $\mathcal{R}_0(z) = (h_0 - z)^{-1}$ and $\mathcal{R}_R(z) = (\hat{h}(R) - z)^{-1}$, $z \in \mathbb{C}$, denote the resolvent of h_0 and $\hat{h}(R)$ respectively.
Let us first recall some known results which will be used in the following of this section.
The potential V is a Rollnik potential if $\|V\|_{\mathcal{R}} < \infty$, where $\|V\|_{\mathcal{R}}$ is given by

$$\|V\|_{\mathcal{R}} = \left(\int_{\mathbb{R}^6} dx\, dy \frac{|V(x)||V(y)|}{|x-y|^2}\right)^{1/2} \tag{4.4}$$

It is well known (see e.g. th. I.4 in [Si]) that if $V \in L^1 \cap L^2$ then V is a Rollnik potential and

$$\|V\|_{\mathcal{R}} \leq c_1 \|V\|_{L^1}^{1/3} \|V\|_{L^2}^{2/3}, \qquad c_1 = \sqrt{3}\,(2\pi)^{1/3} \tag{4.5}$$

Furthermore, following the line of the proof in [Si], it is easy to see that

$$\left(\int_{\mathbb{R}^6} dx\,dy \frac{|V_1(x)||V_2(y)|}{|x-y|^2}\right)^{1/2} \leq c_1 \left(\|V_1\|_{L^1}^{1/3}\|V_1\|_{L^2}^{2/3}\|V_2\|_{L^1}^{1/3}\|V_2\|_{L^2}^{2/3}\right)^{1/2} \tag{4.6}$$

The estimate (4.6) is useful in perturbation theory when one considers operators like

$$K(z) = |V_1|^{1/2}\mathcal{R}_0(z)|V_2|^{1/2} \tag{4.7}$$

where $z \in \mathbb{C}$. In fact the Hilbert-Schmidt norm of $K(z)$, for $\Im\sqrt{z} \geq 0$, satisfies

$$\|K(z)\|_{\mathcal{L}(L^2,L^2)} \leq \|K(z)\|_{HS} \leq \frac{1}{2\pi}\left(\int_{\mathbb{R}^6} dx\,dy \frac{|V_1(x)||V_2(y)|}{|x-y|^2}\right)^{1/2} \tag{4.8}$$

If the potential V belongs to $L^{3/2}$ then V is also Kato smoothing (see e.g. [Y], [KY]), i.e. for any $f \in L^2$ and $\lambda \geq 0$ we have

$$\sup_{\varepsilon>0} \||V|^{1/2}\mathcal{R}_0(\lambda \pm i\varepsilon)f\|_{L^2(d\lambda)L^2(dx)} = \||V|^{1/2}\mathcal{R}_0(\lambda \pm i0)f\|_{L^2(d\lambda)L^2(dx)} \leq c\|V\|_{L^{3/2}}^{1/2}\|f\|_{L^2} \tag{4.9}$$

The potential V is a Kato potential if $\|V\|_{\mathcal{K}} < \infty$, where

$$\|V\|_{\mathcal{K}} = \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3} dy \frac{|V(y)|}{|x-y|} \tag{4.10}$$

It is straightforward to prove that if $V \in L^1 \cap L^2$ then V is a Kato potential and the following estimate holds

$$\|V\|_{\mathcal{K}} \leq c_2\|V\|_{L^1}^{1/3}\|V\|_{L^2}^{2/3}, \qquad c_2 = 3\,\pi^{1/3} \tag{4.11}$$

Notice that $c_2 > c_1$.
In the rest of this section we shall assume V sufficiently smooth in order to guarantee the validity of (4.5), (4.9), (4.11).
The first result shows that, for α sufficiently small, the usual dispersive estimate holds uniformly with respect to R.

Proposition 4.1. *Let us assume $V \in L^1 \cap L^2$ and let*

$$\alpha_0^* = \frac{2\pi^{2/3}}{3K}\,\|V\|_{L^1}^{-1/3}\|V\|_{L^2}^{-2/3} \tag{4.12}$$

Then for any $\alpha < \alpha_0^*$ there exists a constant C_0 such that

$$\sup_R \left\| e^{-it\hat{h}(R)} \right\|_{\mathcal{L}(L^1,L^\infty)} \leq \frac{C_0}{t^{3/2}} \tag{4.13}$$

Proof
The proof closely follows the proof of th. 2.6 in [RS] and it is outlined here only to highlight the uniformity with respect to R. Let us fix $\alpha < \alpha_0^*$. Taking into account (4.5) and the fact that $c_2 > c_1$ we have

$$\frac{1}{2\pi}\|\alpha V_R\|_{\mathcal{R}} \leq \frac{\alpha K}{2\pi}\|V\|_{\mathcal{R}} \leq \frac{\alpha K}{2\pi} c_2 \|V\|_{L^1}^{1/3}\|V\|_{L^2}^{2/3} \leq \frac{\alpha}{\alpha_0^*} < 1 \tag{4.14}$$

It follows that the Born series for the boundary value of the resolvent converges, that is for any real $f, g \in C_0^\infty$ we have

$$< \mathcal{R}_R(\lambda + i0)f, g > - < \mathcal{R}_0(\lambda + i0)f, g >= \sum_{l=1}^{\infty}(-\alpha)^l < \mathcal{R}_0(\lambda + i0)\left(V_R\mathcal{R}_0(\lambda + i0)\right)^l f, g > \tag{4.15}$$

It is easily seen that the r.h.s of (4.15) is an absolutely convergent series which defines an element of $L^1(d\lambda)$ and its norm is uniformly bounded with respect to R.
Using the spectral theorem and (4.15) we have

$$\begin{aligned}
&| < e^{-it\hat{h}(R)}f, g > | \leq \sup_{L\geq 1}\left| \int_0^\infty d\lambda\, e^{it\lambda}\eta\left(\frac{\sqrt{\lambda}}{L}\right)\Im < \mathcal{R}_R(\lambda + i0)f, g > \right| \\
&\leq \sum_{l=0}^{\infty}\alpha^l \int_{\mathbb{R}^6} dx_0\, dx_{l+1}|f(x_0)||g(x_{l+1})| \int_{\mathbb{R}^{3l}} dx_1 \ldots dx_l \frac{\prod_{j=1}^{l}|V_R(x_j)|}{(2\pi)^{l+1}\prod_{j=0}^{l}|x_j - x_{j+1}|} \\
&\cdot \sup_{L\geq 1}\left| \int_0^\infty d\lambda\, e^{it\lambda}\eta\left(\frac{\sqrt{\lambda}}{L}\right)\sin\left(\sqrt{\lambda}\sum_{j=0}^{l}|x_j - x_{j+1}|\right)\right| \\
&\leq \frac{c_\eta}{t^{3/2}}\sum_{l=0}^{\infty}\alpha^l \int_{\mathbb{R}^6} dx_0\, dx_{l+1}|f(x_0)||g(x_{l+1})| \\
&\cdot \int_{\mathbb{R}^{3l}} dx_1 \ldots dx_l \frac{\prod_{j=1}^{l}|V_R(x_j)|}{(2\pi)^{l+1}\prod_{j=1}^{l}|x_j - x_{j+1}|}\sum_{j=0}^{l}|x_j - x_{j+1}|
\end{aligned} \tag{4.16}$$

where η is a cut-off function, i.e. a function $\eta : \mathbb{R}^+ \to \mathbb{R}$ such that $\eta \in C_0^\infty(\mathbb{R}^+)$, $\eta(x) = 1$ for $0 < x < 1$, $\eta(x) = 0$ for $x > 2$. In (4.16) we have used the estimate (lemma 2.4 in [RS])

$$\sup_{L\geq 1}\left| \int_0^\infty d\lambda\, e^{it\lambda}\eta\left(\frac{\sqrt{\lambda}}{L}\right)\sin\left(\sqrt{\lambda}\sum_{j=0}^{l}|x_j - x_{j+1}|\right)\right| \leq \frac{c_\eta}{t^{3/2}}\sum_{j=0}^{l}|x_j - x_{j+1}| \tag{4.17}$$

where c_η only depends on η. The last integral in (4.16) can be estimated using the Kato norm of the potential (lemma 2.5 in [RS]); moreover using (4.11) we have $(2\pi)^{-1}\alpha\|V_R\|_{\mathcal{K}} < 1$. Then

$$\begin{aligned} |< e^{-it\hat{h}(R)} f, g > | &\le \frac{c_\eta}{2\pi t^{3/2}} \sum_{l=0}^{\infty} \alpha^l \int_{\mathbb{R}^6} dx_0\, dx_{l+1} |f(x_0)||g(x_{l+1})|(l+1) \left(\frac{\|V_R\|_{\mathcal{K}}}{2\pi} \right)^l \\ &\le \frac{C_0}{t^{3/2}} \|f\|_{L^1} \|g\|_{L^1} \end{aligned} \tag{4.18}$$

where

$$C_0 = \frac{c_\eta}{2\pi} \sum_{l=0}^{\infty} (l+1) \left(\frac{\alpha \|V\|_{\mathcal{K}}}{2\pi} \right)^l \tag{4.19}$$

and this concludes the proof.

□

We shall now prove the uniform dispersive estimate for the derivatives of $e^{-it\hat{h}(R)}$ with respect to the parameter R.
For the proof of theorem 1, we only need derivatives up to order four but it is easy to extend the result to derivatives of any order.

Proposition 4.2. *Let us assume* $V \in W^{\gamma,1} \cap H^\gamma$, $\gamma_1, \dots \gamma_n \in \mathbb{N}$, $\sum_{i=1}^n \gamma_i = \gamma$, $m_1, \dots m_n \in \{1, \dots K\}$, $s_1, \dots s_n \in \{1, 2, 3\}$, *and let*

$$\alpha^*_\gamma = \frac{\pi^{2/3}}{3 \cdot 2^{\gamma-1} K} \|V\|^{-1/3}_{W^{\gamma,1}} \|V\|^{-2/3}_{H^\gamma} \tag{4.20}$$

Then for any $\alpha < \alpha^*_\gamma$ *there exists a constant* $C_\gamma > 0$ *such that*

$$\sup_R \|(\prod_{i=1}^n D^{\gamma_i}_{m_i,s_i}) e^{-it\hat{h}(R)}\|_{\mathcal{L}(L^1,L^\infty)} \le \frac{C_\gamma}{t^{3/2}} \tag{4.21}$$

Proof
The proof is a slight modification of the proof of proposition 3.1. In order to avoid a cumbersome notation, we limit the proof to the case $n = 2$. The general case can be proven in the same way.
However, we stress that in the present paper we use inequality (4.21) in the cases $n = 1$ and $n = 2$ only.
The first step is to show that the Born series of the resolvent can be differentiated term by term, i.e.

$$
\begin{aligned}
&< D^{\gamma_1}_{m_1,s_1} D^{\gamma_2}_{m_2,s_2} \mathcal{R}_R(\lambda+i0) f, g > \\
&= \sum_{l=1}^{\infty} (-\alpha)^l \sum_{\gamma_{1,1}=0}^{\gamma_1} \sum_{\gamma_{1,2}=0}^{\gamma_{1,1}} \cdots \sum_{\gamma_{1,l-1}=0}^{\gamma_{1,l-2}} \binom{\gamma_1}{\gamma_{1,1}} \binom{\gamma_{1,1}}{\gamma_{1,2}} \cdots \binom{\gamma_{1,l-2}}{\gamma_{1,l-1}} \\
&\sum_{\gamma_{2,1}=0}^{\gamma_2} \sum_{\gamma_{2,2}=0}^{\gamma_{2,1}} \cdots \sum_{\gamma_{2,l-1}=0}^{\gamma_{2,l-2}} \binom{\gamma_2}{\gamma_{2,1}} \binom{\gamma_{2,1}}{\gamma_{2,2}} \cdots \binom{\gamma_{2,l-2}}{\gamma_{2,l-1}} \\
&\cdot < \mathcal{R}_0(\lambda+i0) \big(D^{\gamma_1-\gamma_{1,1}}_{m_1,s_1} D^{\gamma_2-\gamma_{2,1}}_{m_2,s_2} V_R \big) \mathcal{R}_0(\lambda+i0) \ldots \big(D^{\gamma_{1,l-1}}_{m_1,s_1} D^{\gamma_{2,l-1}}_{m_2,s_2} V_R \big) \mathcal{R}_0(\lambda+i0) f, g > \\
&= \sum_{l=1}^{\infty} (-\alpha)^l \sum_{\substack{j_{1,1}\cdots j_{1,l} \geq 0 \\ \sum_i j_{1,i} = \gamma_1}} \sum_{\substack{j_{2,1}\cdots j_{2,l} \geq 0 \\ \sum_i j_{2,i} = \gamma_2}} c_{j_{1,1}\cdots j_{1,l}} c_{j_{2,1}\cdots j_{2,l}} \\
&\cdot < \mathcal{R}_0(\lambda+i0) \big(D^{j_{1,1}}_{m_1,s_1} D^{j_{2,1}}_{m_2,s_2} V_R \big) \mathcal{R}_0(\lambda+i0) \cdots \Big(D^{j_{1,l}}_{m_1,s_1} D^{j_{2,l}}_{m_2,s_2} V_R \Big) \mathcal{R}_0(\lambda+i0) f, g >
\end{aligned}
\tag{4.22}
$$

where

$$
c_{j_{k,1}\cdots j_{k,l}} = \binom{\sum_{i=1}^{l} j_{k,i}}{\sum_{i=2}^{l} j_{k,i}} \binom{\sum_{i=2}^{l} j_{k,i}}{\sum_{i=3}^{l} j_{k,i}} \cdots \binom{j_{k,l-1}+j_{k,l}}{j_{k,l}}, \ \ k=1,2 \tag{4.23}
$$

and the r.h.s. of (4.22) is an absolutely convergent series and belongs to $L^1(\mathbb{R}^+, d\lambda)$. In order to prove this statement we estimate the L^1 norm of the general term of the series. Denoting $D^{j_{1,k}}_{m_1,s_1} D^{j_{2,k}}_{m_2,s_2} V_R$ by $V_R^{j_{1,k}j_{2,k}}$, we have

$$
\begin{aligned}
&\int_0^{+\infty} d\lambda \left| < \mathcal{R}_0(\lambda+i0) V_R^{j_{1,1}j_{2,1}} \mathcal{R}_0(\lambda+i0) \cdots V_R^{j_{1,l}j_{2,l}} \mathcal{R}_0(\lambda+i0) f, g > \right| \\
&= \int_0^{+\infty} d\lambda \left| < \left[\prod_{i=1}^{l-1} \operatorname{sgn}\left(V_R^{j_{1,i}j_{2,i}}\right) \left|V_R^{j_{1,i}j_{2,i}}\right|^{1/2} \mathcal{R}_0(\lambda+i0) \left|V_R^{j_{1,i+1}j_{2,i+1}}\right|^{1/2} \right] \right. \\
&\left. \cdot \operatorname{sgn}\left(V_R^{j_{1,l}j_{2,l}}\right) \left|V_R^{j_{1,l}j_{2,l}}\right|^{1/2} \mathcal{R}_0(\lambda+i0) f, \left|V_R^{j_{1,1}j_{2,1}}\right|^{1/2} \mathcal{R}_0(\lambda-i0) g > \right|
\end{aligned}
\tag{4.24}
$$

Using (4.8) and (4.6), we have

$$\left\| \prod_{i=1}^{l-1} \operatorname{sgn}\left(V_R^{j_{1,i}j_{2,i}}\right) \left|V_R^{j_{1,i}j_{2,i}}\right|^{1/2} \mathcal{R}_0(\lambda+i0) \left|V_R^{j_{1,i+1}j_{2,i+1}}\right|^{1/2} \right\|_{\mathcal{L}(L^2,L^2)}$$
$$\leq \left(\frac{c_1}{2\pi}\right)^{l-1} \prod_{i=1}^{l-1} \left(\|V_R^{j_{1,i}j_{2,i}}\|_{L^1}^{1/3} \|V_R^{j_{1,i}j_{2,i}}\|_{L^2}^{2/3} \|V_R^{j_{1,i+1}j_{2,i+1}}\|_{L^1}^{1/3} \|V_R^{j_{1,i+1}j_{2,i+1}}\|_{L^2}^{2/3} \right)^{1/2}$$
$$= \left(\frac{c_1}{2\pi}\right)^{l-1} \left(\|V_R^{j_{1,1}j_{2,1}}\|_{L^1}^{1/3} \|V_R^{j_{1,1}j_{2,1}}\|_{L^2}^{2/3} \right)^{1/2} \left(\|V_R^{j_{1,l}j_{2,l}}\|_{L^1}^{1/3} \|V_R^{j_{1,l}j_{2,l}}\|_{L^2}^{2/3} \right)^{1/2}$$
$$\prod_{i=2}^{l-1} \|V_R^{j_{1,i}j_{2,i}}\|_{L^1}^{1/3} \|V_R^{j_{1,i}j_{2,i}}\|_{L^2}^{2/3} \tag{4.25}$$

Exploiting (4.25), the Schwartz's inequality, the Kato smoothing property (4.9) and (4.5), we obtain

$$\int_0^{+\infty} d\lambda \left| < \mathcal{R}_0(\lambda+i0) V_R^{j_{1,1}j_{2,1}} \mathcal{R}_0(\lambda+i0) \dots V_R^{j_{1,l}j_{2,l}} \mathcal{R}_0(\lambda+i0) f, g > \right|$$
$$\leq \left(\frac{c_1}{2\pi}\right)^{l-1} \left(\|V_R^{j_{1,1}j_{2,1}}\|_{L^1}^{1/3} \|V_R^{j_{1,1}j_{2,1}}\|_{L^2}^{2/3} \right)^{1/2} \left(\|V_R^{j_{1,l}j_{2,l}}\|_{L^1}^{1/3} \|V_R^{j_{1,l}j_{2,l}}\|_{L^2}^{2/3} \right)^{1/2}$$
$$\prod_{i=2}^{l-1} \|V_R^{j_{1,i}j_{2,i}}\|_{L^1}^{1/3} \|V_R^{j_{1,i}j_{2,i}}\|_{L^2}^{2/3} \left(\int_0^{+\infty} d\lambda \left\| \left|V_R^{j_{1,1}j_{2,1}}\right|^{1/2} \mathcal{R}_0(\lambda-i0) g \right\|_{L^2}^2 \right)^{1/2}$$
$$\cdot \left(\int_0^{+\infty} d\lambda \left\| |V_R^{j_{1,l}j_{2,l}}|^{1/2} \mathcal{R}_0(\lambda+i0) f \right\|_{L^2}^2 \right)^{1/2}$$
$$\leq c \frac{c_1^{l+1}}{(2\pi)^{l-1}} \left(\prod_{i=1}^{l} \|V_R^{j_{1,i}j_{2,i}}\|_{L^1}^{1/3} \|V_R^{j_{1,i}j_{2,i}}\|_{L^2}^{2/3} \right) \|f\|_{L^2} \|g\|_{L^2}$$
$$\leq c\, c_1 \left(\frac{K}{2\pi} c_1 \|V\|_{W^{\gamma,1}}^{1/3} \|V\|_{H^\gamma}^{2/3} \right)^l \|f\|_{L^2} \|g\|_{L^2} \tag{4.26}$$

where in the last line we used $\|V_R^{j_{1,i}j_{2,i}}\|_{L^1} \leq K\|V\|_{W^{\gamma,1}}$ and $\|V_R^{j_{1,i}j_{2,i}}\|_{L^2} \leq K\|V\|_{H^\gamma}$.
The $L^1(\mathbb{R}^+, d\lambda)$ norm of (4.22) can be estimated as follows

$$\int_0^{+\infty} d\lambda \left| < D_{m_1,s_1}^{\gamma_1} D_{m_2,s_2}^{\gamma_2} \mathcal{R}_R(\lambda+i0) f, g > \right|$$
$$\leq c\, c_1 \|f\|_{L^2} \|g\|_{L^2} \sum_{l=1}^{\infty} \left(\alpha \frac{K}{2\pi} c_1 \|V\|_{W^{\gamma,1}}^{1/3} \|V\|_{H^\gamma}^{2/3} \right)^l \sum_{\substack{j_{1,1}\dots j_{1,l} \geq 0 \\ \sum_i j_{1,i} = \gamma_1}} \sum_{\substack{j_{2,1}\dots j_{2,l} \geq 0 \\ \sum_i j_{2,i} = \gamma_2}} c_{j_{1,1}\dots j_{1,l}} c_{j_{2,1}\dots j_{2,l}}$$
$$\leq c\, c_1 \|f\|_{L^2} \|g\|_{L^2} \sum_{l=1}^{\infty} l^\gamma \left(\alpha \frac{K}{2\pi} c_1 \|V\|_{W^{\gamma,1}}^{1/3} \|V\|_{H^\gamma}^{2/3} \right)^l \tag{4.27}$$

where we used the identity

$$\sum_{\sum_i j_{k,i}=\gamma_k} c_{j_{k,1}\dots j_{k,l}} = l^{\gamma_k}, \quad k=1,2 \tag{4.28}$$

Since the series in (4.27) converges for $\alpha < \alpha^*_\gamma$, we conclude that (4.22) holds and the r.h.s. is absolutely convergent and belongs to $L^1(\mathbb{R}^+, d\lambda)$.
Let us now consider the derivatives of the unitary group; using again the cut-off function η as in (4.16), we can write

$$| < D^{\gamma_1}_{m_1,s_1} D^{\gamma_2}_{m_2,s_2} e^{-it\hat{h}(R)} f, g > | \leq \sup_{L\geq 1} \left| \int_0^\infty d\lambda \, e^{it\lambda} \, \eta\left(\frac{\sqrt{\lambda}}{L}\right) \Im < D^{\gamma_1}_{m_1,s_1} D^{\gamma_2}_{m_2,s_2} \mathcal{R}_R(\lambda + i0) f, g > \right| \tag{4.29}$$

Using (4.22), Fubini's theorem and (4.17), we have

$$\begin{aligned}
&| < D^{\gamma_1}_{m_1,s_1} D^{\gamma_2}_{m_2,s_2} e^{-it\hat{h}(R)} f, g > | \leq \sum_{l=1}^\infty \alpha^l \sum_{\substack{j_{1,1}\dots j_{1,l}\geq 0 \\ \sum_i j_{1,i}=\gamma_1}} \sum_{\substack{j_{2,1}\dots j_{2,l}\geq 0 \\ \sum_i j_{2,i}=\gamma_2}} c_{j_{1,1}\dots j_{1,l}} c_{j_{2,1}\dots j_{2,l}} \\
&\cdot \int_{\mathbb{R}^6} dx_0 \, dx_{l+1} |f(x_0)||g(x_{l+1})| \\
&\cdot \int_{\mathbb{R}^{3l}} dx_1 \dots dx_l \frac{\prod_{i=1}^l |V_R^{j_{1,i} j_{2,i}}(x_i)|}{(2\pi)^{l+1} \prod_{i=0}^l |x_i - x_{i+1}|} \sup_{L\geq 1} \left| \int_0^\infty d\lambda \, e^{it\lambda} \, \eta\left(\frac{\sqrt{\lambda}}{L}\right) \sin\left(\sqrt{\lambda} \sum_{i=0}^l |x_i - x_{i+1}|\right) \right| \\
&\leq \frac{c_\eta}{t^{3/2}} \sum_{l=1}^\infty \alpha^l \sum_{\substack{j_{1,1}\dots j_{1,l}\geq 0 \\ \sum_i j_{1,i}=\gamma_1}} \sum_{\substack{j_{2,1}\dots j_{2,l}\geq 0 \\ \sum_i j_{2,i}=\gamma_2}} c_{j_{1,1}\dots j_{1,l}} c_{j_{2,1}\dots j_{2,l}} \int_{\mathbb{R}^6} dx_0 \, dx_{l+1} |f(x_0)||g(x_{l+1})| \\
&\cdot \int_{\mathbb{R}^{3l}} dx_1 \dots dx_l \frac{\prod_{i=1}^l |V_R^{j_{1,i} j_{2,i}}(x_i)|}{(2\pi)^{l+1} \prod_{i=0}^l |x_i - x_{i+1}|} \sum_{i=0}^l |x_i - x_{i+1}|
\end{aligned} \tag{4.30}$$

Following the line of lemma 2.5 in [RS], the last integral can be dominated uniformly in x_0 and x_{l+1} using the Kato norm of the derivatives of the potential V. In fact one obtains

$$
\begin{aligned}
&| < D^{\gamma_1}_{m_1,s_1} D^{\gamma_2}_{m_2,s_2} e^{-it\hat{h}(R)} f, g > | \\
&\leq \frac{c_\eta}{2\pi t^{3/2}} \|f\|_{L^1} \|g\|_{L^1} \sum_{l=1}^{\infty} \alpha^l \frac{l+1}{(2\pi)^l} \sum_{\substack{j_{1,1} \dots j_{1,l} \geq 0 \\ \sum_i j_{1,i} = \gamma_1}} \sum_{\substack{j_{2,1} \dots j_{2,l} \geq 0 \\ \sum_i j_{2,i} = \gamma_2}} c_{j_{1,1} \dots j_{1,l}} c_{j_{2,1} \dots j_{2,l}} \prod_{i=1}^{l} \| V_R^{j_{1,i} j_{2,i}} \|_{\mathcal{K}} \\
&\leq \frac{c_\eta}{2\pi t^{3/2}} \|f\|_{L^1} \|g\|_{L^1} \sum_{l=1}^{\infty} (l+1) \left(\alpha \frac{K}{2\pi} c_2 \|V\|^{1/3}_{W^{\gamma,1}} \|V\|^{2/3}_{H^\gamma} \right)^l \sum_{\substack{j_{1,1} \dots j_{1,l} \geq 0 \\ \sum_i j_{1,i} = \gamma_1}} \sum_{\substack{j_{2,1} \dots j_{2,l} \geq 0 \\ \sum_i j_{2,i} = \gamma_2}} c_{j_{1,1} \dots j_{1,l}} c_{j_{2,1} \dots j_{2,l}} \\
&\leq \frac{c_\eta}{2\pi t^{3/2}} \|f\|_{L^1} \|g\|_{L^1} \sum_{l=0}^{\infty} (l+1) l^\gamma \left(\frac{\alpha}{\alpha^*_\gamma} \right)^l
\end{aligned} \tag{4.31}
$$

where we used (4.11) and added the term $l = 0$ in the last sum of (4.31).
Since $\alpha < \alpha^*_\gamma$, the series in (4.31) converges and we get (4.21) with

$$
C_\gamma = \frac{c_\eta}{2\pi} \sum_{l=0}^{\infty} (l+1) l^\gamma \left(\frac{\alpha}{\alpha^*_\gamma} \right)^l \tag{4.32}
$$

□

Remark. We observe that α^*_γ is decreasing as a function of γ. Then for the proof of theorem 1 it is sufficient to choose $\alpha^* = \alpha^*_4$.
We also notice that C_γ is increasing as a function of γ, and this fact is used during the proof of theorem 1.

For the proof of theorem 1 we also need a uniform L^2 estimate of the derivatives with respect to the parameter R of the unitary group of the light particles.
For a single light particle, exploiting the spectral theorem and (4.27), we immediately get

$$
\sup_R \left\| (\prod_{i=1}^{n} D^{\gamma_i}_{m_i,s_i}) e^{-it\hat{h}(R)} f \right\|_{L^2} \leq a_\gamma \tag{4.33}
$$

for any γ integer (including zero), $\|f\|_{L^2} = 1$ and $\alpha < \alpha^*$, where $a_0 = 1$ and

$$
a_\gamma = c\, c_1 \sum_{l=1}^{\infty} l^\gamma \left(\alpha \frac{K}{2\pi} c_1 \|V\|^{1/3}_{W^{\gamma,1}} \|V\|^{2/3}_{H^\gamma} \right)^l \qquad \gamma \neq 0 \tag{4.34}
$$

For an arbitrary number N of light particles, the Leibniz's rule yields

$$\sup_R \left\| \left(\prod_{i=1}^n D^{\gamma_i}_{m_i,s_i}\right) \prod_{k=1}^N e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3N})} \leq \sum_{\substack{j_1 \dots j_N = 0 \\ \sum_i j_i = \gamma}}^{\gamma} c_{j_1 \dots j_N} a_{j_1} \cdots a_{j_N} \tag{4.35}$$

where $\|\chi_k\|_{L^2} = 1$. Notice that in the right hand side of (4.35) at most γ of the constants a_{j_i} are different from one and moreover each a_{j_i} is less or equal to a_γ. Then we obtain the uniform L^2 estimate

$$\sup_R \left\| \left(\prod_{i=1}^n D^{\gamma_i}_{m_i,s_i}\right) \prod_{k=1}^N e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3N})} \leq N^\gamma a^\gamma_\gamma \leq \hat{C}_\gamma \tag{4.36}$$

where we have defined

$$\hat{C}_\gamma = \max_{0 \leq \lambda \leq \gamma} N^\lambda a^\lambda_\lambda \tag{4.37}$$

The proof of the uniform estimates (4.13), (4.21), (4.36) are based on a perturbative analysis and this requires the assumption of a small potential. We believe that this is only a technical limitation which could be removed with a more careful analysis.
In fact, following a different approach due to Yajima ([Y]), uniform estimates can be easily proved in the simpler case $K = 1$ for an arbitrarily large potential.
In this case the crucial ingredient is the boundedness of the wave operators in the Sobolev spaces $W^{k,p}$.

Proposition 4.3. *Let $K = 1$, $\gamma \in \mathbb{N}$, $p \in [2, \infty]$ and let us assume that:*
i) $V \in W^{\gamma,\infty}_\delta$, for $\delta > 5$;
ii) $V \geq 0$;
iii) $g \in L^2 \cap W^{\gamma,q}$, $\gamma \in \mathbb{N}$, $q^{-1} = 1 - p^{-1}$.
Then there exists a constant $b_{\gamma,p,q} > 0$ such that

$$\sup_R \|D^\gamma_s e^{-it\hat{h}(R)} g\|_{L^p} \leq \frac{b_{\gamma,p,q}}{t^{3(2-q)/2q}} \|g\|_{W^{\gamma,q}} \tag{4.38}$$

Proof
Since $K = 1$, the dependence of $e^{-it\hat{h}(R)}$ on the parameter $R \in \mathbb{R}^3$ can be extracted using the unitary translation operator T_R, $(T_R f)(x) = f(x + R)$. Moreover, using the intertwining property of the wave operators one has

$$e^{-it\hat{h}(R)} = T_R \, \Omega_+ \, e^{-ith_0} \, \Omega_+^{-1} \, T_R^{-1} \tag{4.39}$$

where Ω_+ is the wave operator for the pair $(\hat{h}(0), h_0)$.
We use (4.39) to compute the derivatives with respect to the parameter R, noticing that

$$D_s\, T_R = -T_R\, d_s\,, \qquad D_s\, T_R^{-1} = T_R^{-1}\, d_s \tag{4.40}$$

where $(d_s^i f)(x) = \frac{\partial^i f}{\partial x_s^i}(x)$ and x_s is the s-th component of x.
We have

$$D_s^\gamma\, e^{-it\hat{h}(R)} g = \sum_{k=0}^{\gamma} (-1)^k \binom{\gamma}{k} T_R\, d_s^k\, \Omega_+\, e^{-ith_0}\, \Omega_+^{-1}\, d_s^{\gamma-k}\, T_R^{-1} g \tag{4.41}$$

The generic term of the sum in (4.41) can be estimated as follows

$$\begin{aligned}
&\left\| T_R\, d_s^k\, \Omega_+\, e^{-ith_0}\, \Omega_+^{-1}\, d_s^{\gamma-k}\, T_R^{-1} g \right\|_{L^p} = \left\| d_s^k\, \Omega_+\, e^{-ith_0}\, \Omega_+^{-1}\, d_s^{\gamma-k}\, T_R^{-1} g \right\|_{L^p} \\
&\le c_{k,p}(\Omega_+) \left\| e^{-ith_0}\, \Omega_+^{-1}\, d_s^{\gamma-k}\, T_R^{-1} g \right\|_{W^{k,p}} \le \frac{c}{t^{3(2-q)/2q}} c_{k,p}(\Omega_+) \left\| \Omega_+^{-1}\, d_s^{\gamma-k}\, T_R^{-1} g \right\|_{W^{k,q}} \\
&\le \frac{c}{t^{3(2-q)/2q}} c_{k,p}(\Omega_+) c_{k,q}(\Omega_+^{-1}) \left\| d_s^{\gamma-k}\, T_R^{-1} g \right\|_{W^{k,q}} \le \frac{c}{t^{3(2-q)/2q}} c_{k,p}(\Omega_+) c_{k,q}(\Omega_+^{-1}) \left\| T_R^{-1} g \right\|_{W^{\gamma,q}} \\
&= \frac{c}{t^{3(2-q)/2q}} c_{k,p}(\Omega_+) c_{k,q}(\Omega_+^{-1}) \left\| g \right\|_{W^{\gamma,q}}
\end{aligned} \tag{4.42}$$

where we have used the isometric character of T_R, the boundedness of the wave operators in $W^{k,p}$ (see [Y]), the fact that the free propagator commutes with derivatives and the standard estimate for the free Schrödinger group

$$\left\| e^{-ith_0} \right\|_{\mathcal{L}(L^q, L^p)} \le \frac{c}{t^{3(2-q)/2q}} \tag{4.43}$$

In (4.42) the symbols $c_{k,p}(\Omega_+)$, $c_{k,p}(\Omega_+^{-1})$ denote the operator norm in $W^{k,p}$ of Ω_+, Ω_+^{-1} respectively. From (4.41) and (4.42) we obtain the proof of (4.38)

□

Remark. Notice that for $p = \infty$ the estimate (4.38) reduces to the uniform dispersive estimate and, in this case, we denote

$$B_\gamma = \max_{0 \le \lambda \le \gamma} b_{\lambda,\infty,1} \tag{4.44}$$

Moreover for $p = 2$, proceeding as in (4.35), (4.36), we easily get

$$\sup_R \left\| D_s^\gamma \prod_{k=1}^N e^{-ith_k(R)} \chi_k \right\|_{L^2(\mathbb{R}^{3N})} \leq N^\gamma b_{\gamma,2,2}^\gamma \left(\max_j \|\chi_j\|_{H^\gamma} \right)^\gamma \leq \hat{B}_\gamma \tag{4.45}$$

where we have defined

$$\hat{B}_\gamma = \max_{0 \leq \lambda \leq \gamma} N^\lambda b_{\lambda,2,2}^\lambda \left(\max_j \|\chi_j\|_{H^\lambda} \right)^\lambda \tag{4.46}$$

5. Some commutator estimates involving the unitary group e^{-itX}

In this section we discuss some estimates for the commutator of the unitary group e^{-itX} with the operators X_0 and R_l^2 in the Hilbert space of the heavy particles.
Such estimates are repeatedly used in the proof of theorem 1. Since we have not found them in the literature, a simple proof is exhibited here for the convenience of the reader.
We find convenient to express the results in terms of the weighted Sobolev space related to the l-th heavy particle, which were defined in the introduction.
The first result concerns the commutator $[X_{0,l}, e^{-itX}]$.

Proposition 5.1. *Given $l \in \{1, \dots, K\}$, $f \in H^1_{l,0}(\mathbb{R}^{3K})$ and $T > 0$, there exists a constant $\tilde{C} > 0$ such that*

$$\|[X_{0,l}, e^{-itX}]f\|_{L^2(\mathbb{R}^{3K})} \leq t\, \tilde{C} \|f\|_{H^1_{l,0}(\mathbb{R}^{3K})} \tag{5.1}$$

for any $t \in [0, T]$.

Proof
For $s = 1, 2, 3$ and using the short-hand notation $\eta(t) = e^{-itX} f$, a direct computation gives

$$i \frac{\partial}{\partial t} D_{l,s} \eta(t) = X D_{l,s} \eta(t) + (D_{l,s} U) \eta(t) \tag{5.2}$$

Therefore, by Duhamel's formula

$$D_{l,s} \eta(t) = e^{-itX} D_{l,s} f - i \int_0^t d\tau e^{-i(t-\tau)X} (D_{l,s} U) \eta(\tau) \tag{5.3}$$

Iterating the procedure one finds

$$D_{l,s}^2 \eta(t) = e^{-itX} D_{l,s}^2 f - i \int_0^t d\tau e^{-i(t-\tau)X} (D_{l,s}^2 U) \eta(\tau) - 2i \int_0^t d\tau e^{-i(t-\tau)X} (D_{l,s} U) e^{-i\tau X} D_{l,s} f$$
$$-2 \int_0^t d\tau \int_0^\tau d\sigma e^{-i(t-\tau)X} (D_{l,s} U) e^{-i(\tau-\sigma)X} (D_{l,s} U) \eta(\sigma) \tag{5.4}$$

Therefore

$$X_{0,l}\eta(t) = e^{-itX}X_{0,l}f - i\int_0^t d\tau e^{-i(t-\tau)X}(X_{0,l}U)\eta(\tau) - 2i\sum_{s=1}^{3}\int_0^t d\tau e^{-i(t-\tau)X}(D_{l,s}U)e^{-i\tau X}D_{l,s}f$$
$$-2\sum_{s=1}^{3}\int_0^t d\tau\int_0^\tau d\sigma e^{-i(t-\tau)X}(D_{l,s}U)e^{-i(\tau-\sigma)X}(D_{l,s}U)\eta(\sigma) \tag{5.5}$$

Recalling the definition of $\eta(t)$, it follows

$$\begin{aligned}
&\|[X_{0,l}, e^{-itX}]f\|_{L^2(\mathbb{R}^{3K})}\\
&\leq \int_0^t d\tau\|(X_{0,l}U)e^{-i\tau X}f\|_{L^2(\mathbb{R}^{3K})} + 2\sum_{s=1}^{3}\int_0^t d\tau\|D_{l,s}U\|_{L^\infty(\mathbb{R}^{3K})}\|D_{l,s}f\|_{L^2(\mathbb{R}^{3K})}\\
&+2\sum_{s=1}^{3}\int_0^t d\tau\int_0^\tau d\sigma\|D_{l,s}U\|^2_{L^\infty(\mathbb{R}^{3K})}\|e^{-i\sigma X}f\|_{L^2(\mathbb{R}^{3K})}\\
&\leq t\|\Delta U_l\|_{L^\infty}\|f\|_{L^2(\mathbb{R}^{3K})} + 2t\|U_l\|_{W^{1,\infty}}\|f\|_{H^1_{l,0}(\mathbb{R}^{3K})} + t^2\|U_l\|^2_{W^{1,\infty}}\|f\|_{L^2(\mathbb{R}^{3K})}\\
&\leq t\,\tilde{C}\|f\|_{H^1_{l,0}(\mathbb{R}^{3K})}
\end{aligned} \tag{5.6}$$

where we used the fact that $D_{l,s}U = D_sU_l$ and defined

$$\tilde{C} = \max_l\left(\|\Delta U_l\|_{L^\infty} + 2\|U_l\|_{W^{1,\infty}} + T\|U_l\|^2_{W^{1,\infty}}\right) \tag{5.7}$$

□

Corollary 5.2. *For any $t \geq 0$ the operator e^{-itX} is continuous in $W^{m,p}_{l,0}(\mathbb{R}^{3K})$, with $l \in \{1,\dots,K\}$, $m \in \mathbb{N}$, $p \geq 1$.*

Proof

From (5.3) the following estimate is easily obtained

$$\|D_{l,s}e^{-itX}f\|_{L^p(\mathbb{R}^{3K})} \leq \|D_{l,s}f\|_{L^p(\mathbb{R}^{3K})} + t\|D_sU_l\|_{L^\infty}\|f\|_{L^p(\mathbb{R}^{3K})} \tag{5.8}$$

and continuity in $W^{1,p}_{l,0}$ immediately follows. For the case $m > 1$ the result is achieved differentiating the quantity $e^{-itX}f$.

□

The second result in this section is an estimate of the commutator $[R_l^2, e^{-itX}]$.

Proposition 5.3. *Given* $l \in \{1, \dots K\}$, $f \in H^2_{l,2}(\mathbb{R}^{3K})$ *and* $T > 0$, *there exists a constant* $\bar{C}_1 > 0$ *such that*

$$\|[R_l^2, e^{-itX}]f\|_{L^2(\mathbb{R}^{3K})} \;\leq\; t\,\bar{C}_1 \|f\|_{H^2_{l,2}(\mathbb{R}^{3K})} \tag{5.9}$$

for any $t \in [0, T]$.

Proof
First, we observe that

$$[R_l^2, e^{-itX_0}] = te^{-itX_0} S_{0,l}(t) \tag{5.10}$$

where

$$S_{0,l}(t) = -\sum_{s=1}^{3} \left(2iR_{l,s}D_{l,s} + tD^2_{l,s}\right) - 3i \tag{5.11}$$

Formula (5.10) can be easily derived integrating by parts in the explicit integral representation of the free unitary group. The action of the operator $S_{0,l}(t)$ is estimated as follows

$$\begin{aligned}
&\|S_{0,l}(t)f\|_{L^2(\mathbb{R}^{3K})} \leq 2\sum_{s=1}^{3} \|R_{l,s}D_{l,s}f\|_{L^2(\mathbb{R}^{3K})} + t\sum_{s=1}^{3} \|D^2_{l,s}f\|_{L^2(\mathbb{R}^{3K})} + 3\|f\|_{L^2(\mathbb{R}^{3K})} \\
&\leq 2\|f\|_{H^1_{l,1}(\mathbb{R}^{3K})} + t\|f\|_{H^2_{l,0}(\mathbb{R}^{3K})} + 3\|f\|_{L^2(\mathbb{R}^{3K})} \leq c\,(1+t)\|f\|_{H^2_{l,1}(\mathbb{R}^{3K})}
\end{aligned} \tag{5.12}$$

Using Duhamel's formula, the action of $[R_l^2, e^{-itX}]$ reads

$$\begin{aligned}
&[R_l^2, e^{-itX}]f = te^{-itX_0} S_{0,l}(t)f - i\int_0^t d\sigma\, \sigma e^{-i\sigma X_0} S_{0,l}(\sigma) U e^{-i(t-\sigma)X} f \\
&-i\int_0^t d\sigma\, e^{-i(t-\sigma)X_0} R_l^2 U e^{-i\sigma X} f + i\int_0^t d\sigma\, e^{-i(t-\sigma)X_0} U e^{-i\sigma X} R_l^2 f
\end{aligned} \tag{5.13}$$

Estimate (5.12) directly applies to the first term of (5.13), while for the second it gives

$$\left\|\int_0^t d\sigma\, \sigma e^{-i\sigma X_0} S_{0,l}(\sigma) U e^{-i(t-\sigma)X} f\right\|_{L^2(\mathbb{R}^{3K})} \;\leq\; c\int_0^t d\sigma\, \sigma(1+\sigma)\|Ue^{-i(t-\sigma)X} f\|_{H^2_{l,1}(\mathbb{R}^{3K})} \tag{5.14}$$

Notice that

$$
\begin{aligned}
&\|Ue^{-i(t-\sigma)X}f\|_{H^2_{l,1}(\mathbb{R}^{3K})}\\
&= \sum_{\gamma_1=0}^{2}\sum_{\gamma_2=0}^{2-\gamma_1}\sum_{\gamma_3=0}^{2-\gamma_1-\gamma_2} \| < R_l > D^{\gamma_1}_{l,1} D^{\gamma_2}_{l,2} D^{\gamma_3}_{l,3}\, U\, e^{-i(t-\sigma)X} f\|_{L^2(\mathbb{R}^{3K})}\\
&\le 8\sum_{\gamma_1=0}^{2}\sum_{\gamma_2=0}^{2-\gamma_1}\sum_{\gamma_3=0}^{2-\gamma_1-\gamma_2}\sum_{\lambda_1=0}^{\gamma_1}\sum_{\lambda_2=0}^{\gamma_2}\sum_{\lambda_3=0}^{\gamma_3} \| < R_l > D^{\lambda_1}_{l,1} D^{\lambda_2}_{l,2} D^{\lambda_3}_{l,3} U\|_{L^\infty(\mathbb{R}^{3K})}\\
&\cdot \|D^{\gamma_1-\lambda_1}_{l,1} D^{\gamma_2-\lambda_2}_{l,2} D^{\gamma_3-\lambda_3}_{l,3} e^{-i(t-\sigma)X} f\|_{L^2(\mathbb{R}^{3K})}\\
&\le c\, \|U\|_{W^{2,\infty}_{l,1}(\mathbb{R}^{3K})} \|e^{-i(t-\sigma)X} f\|_{H^2_{l,0}(\mathbb{R}^{3K})}\\
&\le c\, \|U\|_{W^{2,\infty}_{l,1}(\mathbb{R}^{3K})} \|f\|_{H^2_{l,0}(\mathbb{R}^{3K})} \qquad (5.15)
\end{aligned}
$$

where, in the last step we used Corollary 5.2.
Thus, going back to (5.14)

$$
\left\| \int_0^t d\sigma\, \sigma e^{-i\sigma X_0} S_{0,l}(\sigma) U e^{-i(t-\sigma)X} f \right\|_{L^2(\mathbb{R}^{3K})} \le c\left(\frac{t^2}{2}+\frac{t^3}{3}\right) \|U\|_{W^{2,\infty}_{l,1}(\mathbb{R}^{3K})} \|f\|_{H^2_{l,0}(\mathbb{R}^{3K})} \quad (5.16)
$$

The third term in (5.13) can be estimated as follows

$$
\left\| \int_0^t d\sigma\, e^{-i(t-\sigma)X_0} R_l^2 U e^{-i\sigma X} f \right\|_{L^2(\mathbb{R}^{3K})} \;\le\; t\, \|R_l^2 U\|_{L^\infty(\mathbb{R}^{3K})} \|f\|_{L^2(\mathbb{R}^{3K})} \qquad (5.17)
$$

Finally, the fourth term in (5.13) gives

$$
\left\| \int_0^t d\sigma\, e^{-i(t-\sigma)X_0} U e^{-i\sigma X} R_l^2 f \right\|_{L^2(\mathbb{R}^{3K})} \;\le\; t\, \|U\|_{L^\infty(\mathbb{R}^{3K})} \|f\|_{H^0_{l,2}(\mathbb{R}^{3K})} \qquad (5.18)
$$

By (5.12) , (5.16), (5.17), (5.18) we conclude

$$
\left\|[R_l^2, e^{-itX}] f\right\|_{L^2(\mathbb{R}^{3K})} \;\le\; t\, \bar{C}_1 \|f\|_{H^2_{l,2}(\mathbb{R}^{3K})} \qquad (5.19)
$$

where we defined

$$
\bar{C}_1 \;=\; c \max_l \left(1 + T + T^2 \|U\|_{W^{2,\infty}_{l,1}(\mathbb{R}^{3K})} + \|R_l^2 U\|_{L^\infty(\mathbb{R}^{3K})} + \|U\|_{L^\infty(\mathbb{R}^{3K})}\right) \qquad (5.20)
$$

□

The last estimate concerns the same commutator of the previous proposition, composed with the Laplacian with respect to R_l.

Proposition 5.4. *Given $l \in \{1, \dots K\}$, $f \in H^4_{l,2}(\mathbb{R}^{3K})$ and $T > 0$ there exists a constant $\bar{C} > 0$ such that*

$$
\|(X_{0,l} + I)[R_l^2, e^{-itX}] f\|_{L^2(\mathbb{R}^{3K})} \;\le\; t\, \bar{C} \|f\|_{H^4_{l,2}(\mathbb{R}^{3K})} \qquad (5.21)
$$

for any $t \in [0,T]$.

Proof
From (5.13) one has

$$X_{0,l}[R_l^2, e^{-itX}]f = te^{-itX_0}X_{0,l}S_{0,l}(t)f - i\int_0^t d\sigma\,\sigma e^{-i\sigma X_0}X_{0,l}S_{0,l}(\sigma)Ue^{-i(t-\sigma)X}f$$
$$-i\int_0^t d\sigma\, e^{-i(t-\sigma)X_0}X_{0,l}R_l^2 Ue^{-i\sigma X}f + i\int_0^t d\sigma\, e^{-i(t-\sigma)X_0}X_{0,l}Ue^{-i\sigma X}R_l^2 f \tag{5.22}$$

We estimate the first term in (5.22) as follows

$$\begin{aligned}
&\|te^{-itX_0}X_{0,l}S_{0,l}(t)f\|_{L^2(\mathbb{R}^{3K})}\\
&\leq t\left\|\sum_{s'=1}^{3} D_{l,s'}^2\left[\sum_{s=0}^{3}(2iR_{l,s}D_{l,s}f + tD_{l,s}^2 f) - 3if\right]\right\|_{L^2(\mathbb{R}^{3K})}\\
&\leq 2t\sum_{s'=1}^{3}\sum_{s=0}^{3}\|D_{l,s'}^2 R_{l,s}D_{l,s}f\|_{L^2(\mathbb{R}^{3K})} + t^2\sum_{s'=1}^{3}\sum_{s=0}^{3}\|D_{l,s'}^2 D_{l,s}^2 f\|_{L^2(\mathbb{R}^{3K})} + 3t\sum_{s'=1}^{3}\|D_{l,s'}^2 f\|_{L^2(\mathbb{R}^{3K})}\\
&\leq 4t\sum_{s'=1}^{3}\sum_{s=0}^{3}\|\delta_{s,s'}D_{l,s'}D_{l,s}f\|_{L^2(\mathbb{R}^{3K})} + 2t\sum_{s'=1}^{3}\sum_{s=0}^{3}\|R_{l,s}D_{l,s'}^2 D_{l,s}f\|_{L^2(\mathbb{R}^{3K})}\\
&+t^2\|f\|_{H_{l,0}^4(\mathbb{R}^{3K})} + 3t\|f\|_{H_{l,0}^2(\mathbb{R}^{3K})}\\
&\leq 7t\sum_{s=0}^{3}\|f\|_{H_{l,0}^2(\mathbb{R}^{3K})} + 2t\|f\|_{H_{l,1}^3(\mathbb{R}^{3K})} + t^2\|f\|_{H_{l,0}^4(\mathbb{R}^{3K})}\\
&\leq c\,t(1+t)\|f\|_{H_{l,1}^4(\mathbb{R}^{3K})}
\end{aligned} \tag{5.23}$$

To estimate the second term in (5.22) we exploit (5.23) and then proceed as in (5.15) obtaining

$$\left\|\int_0^t d\sigma\,\sigma e^{-i\sigma X_0}X_{0,l}S_{0,l}(\sigma)Ue^{-i(t-\sigma)X}f\right\|_{L^2(\mathbb{R}^{3K})} \leq c\int_0^t d\sigma\,\sigma(1+\sigma)\|Ue^{-i(t-\sigma)X}f\|_{H_{l,1}^4(\mathbb{R}^{3K})}$$
$$\leq c\int_0^t d\sigma\sigma(1+\sigma)\,\|U\|_{W_{l,1}^{4,\infty}(\mathbb{R}^{3K})}\|e^{-i(t-\sigma)X}f\|_{H_{l,0}^4(\mathbb{R}^{3K})} \leq ct(t+t^2)\|U\|_{W_{l,1}^{4,\infty}(\mathbb{R}^{3K})}\|f\|_{H_{l,0}^4(\mathbb{R}^{3K})} \tag{5.24}$$

For the third term in (5.22) we have

$$
\begin{aligned}
&\left\| \int_0^t d\sigma e^{-i(t-\sigma)X_0} X_{0,l} R_l^2 U e^{-i\sigma X} f \right\|_{L^2(\mathbb{R}^{3K})} \\
&\leq \int_0^t d\sigma \Bigg[\|(X_{0,l} R_l^2 U) e^{-i\sigma X} f\|_{L^2(\mathbb{R}^{3K})} + 2 \sum_{s=1}^3 \|D_{l,s}(R_l^2 U) D_{l,s} e^{-i\sigma X} f\|_{L^2(\mathbb{R}^{3K})} \\
&+ \|R_l^2 U X_{0,l} e^{-i\sigma X} f\|_{L^2(\mathbb{R}^{3K})} \Bigg] \\
&\leq t \, \|X_{0,l} R_l^2 U\|_{L^\infty(\mathbb{R}^{3K})} \|f\|_{L^2(\mathbb{R}^{3K})} + 2 \sum_{s=1}^3 \|D_{l,s}(R_l^2 U)\|_{L^\infty(\mathbb{R}^{3K})} \int_0^t d\sigma \, \|D_{l,s} e^{-i\sigma X} f\|_{L^2(\mathbb{R}^{3K})} \\
&+ \|R_l^2 U\|_{L^\infty(\mathbb{R}^{3K})} \int_0^t d\sigma \, \|X_{0,l} e^{-i\sigma X} f\|_{L^2(\mathbb{R}^{3K})} \\
&\leq c\, t \left(\|U_l\|_{W^{2,\infty}_{l,2}} \|f\|_{L^2(\mathbb{R}^{3K})} + \|U_l\|_{W^{1,\infty}_{l,2}} \|f\|_{H^1_{l,0}(\mathbb{R}^{3K})} + \|U\|_{W^{0,\infty}_{l,2}(\mathbb{R}^{3K})} \|f\|_{H^2_{l,0}(\mathbb{R}^{3K})} \right) \\
&\leq c\, t \, \|U\|_{W^{2,\infty}_{l,2}(\mathbb{R}^{3K})} \|f\|_{H^2_{l,0}(\mathbb{R}^{3K})}
\end{aligned}
\tag{5.25}
$$

For the fourth term in (5.22) we have

$$
\begin{aligned}
&\left\| \int_0^t d\sigma e^{-i(t-\sigma)X_0} X_{0,l} U e^{-i\sigma X} R_l^2 f \right\|_{L^2(\mathbb{R}^{3K})} \\
&\leq \int_0^t d\sigma \Bigg[\|(X_{0,l} U) e^{-i\sigma X} R_l^2 f\|_{L^2(\mathbb{R}^{3K})} + 2 \sum_{s=1}^3 \|(D_{l,s} U) D_{l,s} e^{-i\sigma X} R_l^2 f\|_{L^2(\mathbb{R}^{3K})} \\
&+ \|U X_{0,l} e^{-i\sigma X} R_l^2 f\|_{L^2(\mathbb{R}^{3K})} \Bigg] \\
&\leq t \, \|X_{0,l} U\|_{L^\infty(\mathbb{R}^{3K})} \|R_l^2 f\|_{L^2(\mathbb{R}^{3K})} + 2 \sum_{s=1}^3 \|D_{l,s} U\|_{L^\infty(\mathbb{R}^{3K})} \int_0^t d\sigma \, \|D_{l,s} e^{-i\sigma X} R_l^2 f\|_{L^2(\mathbb{R}^{3K})} \\
&+ \|U\|_{L^\infty(\mathbb{R}^{3K})} \int_0^t d\sigma \, \|X_{0,l} e^{-i\sigma X} R_l^2 f\|_{L^2(\mathbb{R}^{3K})} \\
&\leq c\, t \, \left(\|U_l\|_{W^{2,\infty}} \|f\|_{H^{0,2}_{l,2}(\mathbb{R}^{3K})} + \|U_l\|_{W^{1,\infty}} \|f\|_{H^1_{l,2}(\mathbb{R}^{3K})} + \|U\|_{L^\infty(\mathbb{R}^{3K})} \|f\|_{H^2_{l,2}(\mathbb{R}^{3K})} \right) \\
&\leq c\, t \|U\|_{W^{2,\infty}_{l,0}(\mathbb{R}^{3K})} \|f\|_{H^2_{l,2}(\mathbb{R}^{3K})}
\end{aligned}
\tag{5.26}
$$

Therefore, by (5.22), (5.23), (5.24), (5.25), (5.26) and proposition 5.3 we finally obtain

$$
\|(X_{0,l} + I)[R_l^2, e^{-itX}] f\|_{L^2(\mathbb{R}^{3K})} \;\; \leq \;\; t\, \bar{C} \|f\|_{H^{4,2}_{l,2}(\mathbb{R}^{3K})} \tag{5.27}
$$

where

$$\bar{C} = \bar{C}_1 + c \max_l \left[(1+T)\left(1 + T\,\|U\|_{W^{4,\infty}_{l,2}(\mathbb{R}^{3K})}\right) + \|U\|_{W^{2,\infty}_{l,2}(\mathbb{R}^{3K})} + \|U\|_{W^{2,\infty}_{l,0}(\mathbb{R}^{3K})} \right] \tag{5.28}$$

□

6. Application to decoherence

Some of the most peculiar aspects of Quantum Mechanics are direct consequences of the superposition principle, i.e. the fact that the normalized superposition of two quantum states is a possible state for a quantum system. Interference effects between the two states and their consequences on the statistics of the expected results of a measurement performed on the system do not have any explanation within the realm of classical probability theory.
On the other hand this highly non-classical behaviour is extremely sensitive to the interaction with the environment. The mechanism of irreversible diffusion of quantum correlations in the environment is generally referred to as decoherence. The analysis of this phenomenon within the frame of Quantum Theory is of great interest and, at the same time, of great difficulty inasmuch as results about the dynamics of large quantum systems are required in order to build up non-trivial models of environment.
In this section we consider the mechanism of decoherence on a heavy particles (the system) scattered by N light particles (the environment). For this purpose we follow closely the line of reasoning of Joos and Zeh ([JZ]) and we exploit formula (1.7) for the asymptotic wave function in the simpler case $U = 0$.
(For other rigorous analysis of the mechanism of decoherence see e.g. [D], [DS], [CCF]).
All the information concerning the dynamical behaviour of observables associated with the heavy particle is contained in the reduced density matrix, which in our case is the positive, trace class operator $\rho^\varepsilon(t)$ in $L^2(\mathbb{R}^3)$ with $\mathrm{Tr}\,\rho^\varepsilon(t) = 1$ and integral kernel given by

$$\rho^\varepsilon(t; R, R') = \int_{\mathbb{R}^{3N}} dr\, \Psi^\varepsilon(t; R, r)\overline{\Psi^\varepsilon}(t; R', r) \tag{6.1}$$

An immediate consequence of theorems 1, 1′ is that for $\varepsilon \to 0$ the operator $\rho^\varepsilon(t)$ converges in the trace class norm to the asymptotic reduced density matrix

$$\rho^a(t) = e^{-itX_0}\rho^a_0 e^{itX_0} \tag{6.2}$$

where ρ^a_0 is a density matrix whose integral kernel is

$$\rho_0^a(R,R') = \phi(R)\overline{\phi}(R')\mathcal{I}(R,R') \tag{6.3}$$

$$\mathcal{I}(R,R') = \prod_{j=1}^{N} \left(\Omega_+(R')^{-1}\chi_j, \Omega_+(R)^{-1}\chi_j\right)_{L^2} \tag{6.4}$$

and $(\cdot,\cdot)_{L^2}$ denotes the scalar product in $L^2(\mathbb{R}^3)$.
Notice that the asymptotic dynamics of the heavy particle described by $\rho^a(t)$ is generated by X_0, i.e. the Hamiltonian of the heavy particle when the light particles are absent. The effect of the interaction with the light particles is expressed in the change of the initial state from $\phi(R)\overline{\phi}(R')$ to $\phi(R)\overline{\phi}(R')\mathcal{I}(R,R')$. Significantly the new initial state is not in product form, meaning that entanglement between the system and the environment has taken place. Yet, at this level of approximation, entanglement is instantaneous and no result about the dynamics of the decoherence process can be extracted from the approximate reduced density matrix.
Moreover notice that $\mathcal{I}(R,R) = 1$, $\mathcal{I}(R,R') = \overline{\mathcal{I}}(R',R)$ and $|\mathcal{I}(R,R')| \leq 1$. For N large $\mathcal{I}(R,R')$ tends to be exponentially close to zero for $R \neq R'$.
In ([AFFT]) a concrete example was considered in the case $N = 1$. The initial condition for the heavy particle were chosen as a superposition of two identical wave packets heading one against the other. The wave packet of an isolated heavy particle would have shown interference fringes typical of a two slit experiment. The decrease in the interference pattern, induced by the interaction with a light particle, was computed and taken as a measure of the decoherence effect.
We want to give here a brief summary of the same analysis for any number of light particles where the enhancement of the decoherence effect due to multiple scattering is easily verified.
Let the initial state be the coherent superposition of two wave packets in the following form

$$\phi(R) = b^{-1}\left(f_\sigma^+(R) + f_\sigma^-(R)\right), \quad b \equiv \|f_\sigma^+ + f_\sigma^-\|_{L^2} \tag{6.5}$$

$$f_\sigma^\pm(R) = \frac{1}{\sigma^{3/2}} f\left(\frac{R \pm R_0}{\sigma}\right) e^{\pm i P_0 \cdot R}, \qquad R_0, P_0 \in \mathbb{R}^3 \tag{6.6}$$

where f is a real valued function in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with $\|f\|_{L^2} = 1$, $R_0 = (0,0,|R_0|)$, $P_0 = (0,0,-|P_0|)$.
It is clear that under the free evolution the two wave packets (6.5) exhibit a significant overlap and the typical interference effect is observed.
On the othe hand, if we take into account the interaction with the light particles and introduce the further assumption $\sigma\alpha\|\nabla V\|_{L^2} \ll 1$, it can be easily seen that $\rho^a(t)$ is approximated by

$$\rho^e(t) \quad = \quad e^{-itX_0}\rho_0^e\, e^{itX_0} \tag{6.7}$$

where ρ_0^e has integral kernel

$$\rho_0^e(R,R') = \frac{1}{b^2}\left(\left|f_\sigma^+(R)\right|^2 + \left|f_\sigma^-(R)\right|^2 + \Lambda f_\sigma^+(R)\overline{f}_\sigma^-(R') + \overline{\Lambda} f_\sigma^-(R)\overline{f}_\sigma^+(R')\right) \tag{6.8}$$

$$\Lambda \equiv \prod_{j=1}^{N}\left(\Omega_+(R_0)^{-1}\chi_j, \Omega_+(-R_0)^{-1}\chi_j\right)_{L^2} \tag{6.9}$$

The proof is easily obtained adapting the proof given in ([AFFT]) for the case $N = 1$.
It is clear from (6.9) that, if the interaction is absent, then $\Lambda = 1$ and (6.8) describes the pure state corresponding to the coherent superposition of f_σ^+ and f_σ^- evolving according to the free Hamiltonian.
If the interaction with the light particles is present then $\Omega_+(R_0)^{-1} \neq I$ and $|\Lambda| \ll 1$ for N large. For specific model interaction the factor Λ can also be explicitely computed (see e.g. the one dimensional case treated in [DFT]).
This means that the only effect of the interaction on the heavy particle is to reduce the non diagonal terms in (6.8) by the factor Λ and this means that the interference effects for the heavy particle are correspondingly reduced.
In this sense we can say that a (partial) decoherence effect on the heavy particle has been induced and, moreover, the effect is completely characterized by the parameter Λ.

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