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Block operator matrices, optical potentials, trace class perturbations and scattering

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Abstract

For an operator-valued block-matrix model, which is called in quantum physics a Feshbach decomposition, a scattering theory is considered. Under trace class perturbation the channel scattering matrices are calculated. Using Feshbach's optical potential it is shown that for a given spectral parameter the channel scattering matrices can be recovered either from a dissipative or from a Lax-Phillips scattering theory.

1 Introduction

Let L and L_0 be self-adjoint operators in a separable Hilbert space $\mathfrak{L}$ and denote by $P^{ac}(L_0)$ the orthogonal projection onto the absolutely continuous subspace $\mathfrak{L}^{ac}(L_0)$ of L_0. One says that the pair $\{L, L_0\}$ of self-adjoint operators performs a scattering system if the wave operators $W_\pm(L, L_0)$,

$$W_\pm(L, L_0) := s - \lim_{t\to\pm\infty} e^{itL}e^{-itL_0}P^{ac}(L_0), \tag{1.1}$$

exist, cf. [5]. If the wave operators exist, then they are isometries from the absolutely continuous subspace $\mathfrak{L}^{ac}(L_0)$ into the absolutely continuous subspace $\mathfrak{L}^{ac}(L)$, i.e. $\operatorname{ran}(W_\pm(L, L_0)) \subseteq \mathfrak{L}^{ac}(L)$. The scattering system $\{L, L_0\}$ is called complete if the ranges of the wave operators $W_\pm(L, L_0)$ coincide with $\mathfrak{L}^{ac}(L)$, cf. [5]. The operator

$$S(L, L_0) := W_+(L, L_0)^* W_-(L, L_0)$$

is called the scattering operator of the scattering system $\{L, L_0\}$. One easily verifies that the scattering operator acts from $\mathfrak{L}^{ac}(L_0)$ into $\mathfrak{L}^{ac}(L_0)$ and commutes with L_0. If the scattering system $\{L, L_0\}$ is complete, then $S(L, L_0)$ is an isometry from $\mathfrak{L}^{ac}(L_0)$ onto $\mathfrak{L}^{ac}(L_0)$. In physical applications L_0 is usually called the unperturbed or free Hamiltonian while L is called the perturbed or full Hamiltonian. Since $S(L, L_0)$ commutes with the free Hamiltonian H_0 the scattering operator is unitarily equivalent to a multiplication operator induced by a family $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of unitary operators in the spectral representation of H_0. This family is called the scattering matrix of the complete scattering system and is the most important quantity in the analysis of scattering processes.

In this paper we investigate the special case that the Hilbert space $\mathfrak{L}$ splits into two subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$,

$$\mathfrak{L} = \begin{array}{c}\mathfrak{H}_1\\ \oplus\\ \mathfrak{H}_2\end{array},$$

and the unperturbed Hamiltonian L_0 is of the form

$$L_0 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} : \begin{matrix} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{matrix} \longrightarrow \begin{matrix} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{matrix} . \tag{1.2}$$

In physics the subspaces $\mathfrak{H}_j$ and the self-adjoint operators H_j, $j = 1, 2$, are often called scattering channels and channel Hamiltonians, respectively. With respect to the decomposition (1.2) one introduces the channel wave operators

$$W_\pm(L, H_j) := s - \lim_{t\to\pm\infty} e^{itL} J_j e^{-itH_j} P^{ac}(H_j)$$

where $J_j : \mathfrak{H}_j \longrightarrow \mathfrak{L}$ is the natural embedding operator. Introducing the channel scattering operators

$$S_{ij} = W_+(L, H_i)^* W_-(L, H_j) : \mathfrak{H}_j \longrightarrow \mathfrak{H}_i, \quad i, j = 1, 2,$$

one obtains a channel decomposition of the scattering operator

$$S(L, L_0) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} . \tag{1.3}$$

In physics the decomposition (1.2) is often motivated either by the exclusive interest to scattering data in a certain channel or by the limited measuring process which allows to measure the scattering data only of a certain channel, say $\mathfrak{H}_1$. Thus, let us assume that only the channel scattering operator $S_{11} : \mathfrak{H}_1^{ac} \longrightarrow \mathfrak{H}_1^{ac}$ in the scattering channel $\mathfrak{H}_1$ is known. This rises the following problem: Is it possible to replace the full Hamiltonian L by an effective one H acting only in $\mathfrak{H}_1$ such that the scattering operator of the scattering system $\{H, H_1\}$ coincides with S_{11}? Since S_{11} is a contraction, in general, this implies that either the scattering system $\{H, H_1\}$ cannot be complete or H is not self-adjoint.

The problem has a solution within the scope of dissipative scattering systems developed in [18, 19, 20] for pairs $\{H, H_1\}$ of dissipative and self-adjoint operators in some separable Hilbert space. For such pairs the wave operators $W_\pm^D(H, H_1)$ are defined by

$$W_+^D(H, H_1) := s - \lim_{t\to+\infty} e^{itH^*} e^{-itH_1} P^{ac}(H_1)$$

and

$$W_-^D(H, H_1) := s - \lim_{t\to+\infty} e^{-itH} e^{itH_1} P^{ac}(H_1),$$

and the notion of completeness is generalized, cf. [18, 19, 20]. The scattering operator of a dissipative scattering system $\{H, H_1\}$ is defined by

$$S_D := W_+^D(H, H_1)^* W_-^D(H, H_1).$$

It turns out that S_D is a contraction acting on the absolutely continuous subspace $\mathfrak{H}_1^{ac}$ of H_1 which commutes with H_1. In [17, 18] it was shown that for any self-adjoint operator H_1 in $\mathfrak{H}_1$ and any contraction S_D acting on the absolutely continuous subspace $\mathfrak{H}_1^{ac}$ and commuting with H_1 there is a maximal dissipative operator H on $\mathfrak{H}_1$ such that $\{H, H_1\}$ performs a complete scattering system with scattering operator given by S_D. In particular, this holds for the self-adjoint operator H_1 and the channel scattering operator S_{11}. That means, there is a maximal dissipative operator H on $\mathfrak{H}_1$ such that the channel scattering operator S_{11} is the scattering operator of the complete dissipative scattering system $\{H, H_1\}$. Hence, roughly speaking, the scattering operator S_{11} can be always viewed as the scattering operator of a suitable chosen dissipative scattering system on $\mathfrak{H}_1$. The disadvantage of this fact is that H is not known explicitly.

Another approach to this problem was suggested by Feshbach in [10, 11], see also [6, 9]. He proposes a concrete dissipative perturbation V_1 of the channel Hamiltonian H_1, called "optical potential", such that the scattering operator S_1 of the dissipative scattering system $\{H_1+V_1, H_1\}$ approximates S_{11} with a certain accuracy. To explain this approach in more detail let us assume that the full Hamiltonian L is obtained from L_0 by an additive perturbation, $L = L_0 + V$, where V is given by

$$V = \begin{pmatrix} 0 & G \\ G^* & 0 \end{pmatrix} : \begin{matrix} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{matrix} \longrightarrow \begin{matrix} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{matrix} . \tag{1.4}$$

Introducing the "optical potential"

$$V_1(\lambda) := -G(H_2 - \lambda - i0)^{-1} G^*, \quad \lambda \in \mathbb{R}, \tag{1.5}$$

it was shown in [8, Theorem 4.4.4] that under strong assumptions indeed the scattering operator $S_1[\lambda]$ of the (in general dissipative) scattering system $\{H_1(\lambda), H_1\}$,

$$H_1(\lambda) := H_1 + V_1(\lambda), \quad \lambda \in \mathbb{R}, \tag{1.6}$$

coincides with the scattering operator S_{11} with an error of second order in the coupling constant.

We show that Feshbach's proposal can be made precise in another sense. Note first that the decomposition (1.2) leads not only to the decomposition (1.3) of the scattering operator S but also to a decomposition of the scattering matrix $\{S(\mu)\}_{\mu \in \mathbb{R}}$,

$$S(\mu) := \begin{pmatrix} S_{11}(\mu) & S_{12}(\mu) \\ S_{21}(\mu) & S_{22}(\mu) \end{pmatrix},$$

where $\{S_{ij}(\mu)\}_{\mu \in \mathbb{R}}$ are called the channel scattering matrices. Denoting the scattering matrix of the dissipative scattering system $\{H_1(\lambda), H_1\}$ by $\{S_1[\lambda](\mu)\}_{\mu \in \mathbb{R}}$ we prove that

$$S_{11}(\lambda) = S_1[\lambda](\lambda) \tag{1.7}$$

holds for a.e. $\lambda \in \mathbb{R}$. This shows, that Feshbach's proposal gives in fact a good approximation of the channel scattering matrix $\{S_{11}(\mu)\}_{\mu \in \mathbb{R}}$ in a neighborhood of the chosen spectral parameter λ of the optical potential $V_1(\lambda)$.

Moreover, Feshbach's proposal implies a second problem. Similarly to the optical potential $V_1(\lambda)$ in the first channel $\mathfrak{H}_1$ one can introduce an optical potential $V_2(\lambda)$ in the second channel,

$$V_2(\lambda) := -G^*(H_1 - \lambda - i0)^{-1}G, \quad \lambda \in \mathbb{R}, \tag{1.8}$$

and define a perturbed operator $H_2(\lambda)$,

$$H_2(\lambda) := H_2 + V_2(\lambda), \quad \lambda \in \mathbb{R}, \tag{1.9}$$

in $\mathfrak{H}_2$. We show below that the characteristic function $\Theta_2[\lambda](\xi)$, $\xi \in \mathbb{C}_-$, of the dissipative operator $H_2(\lambda)$ and the scattering matrix $\{S_{11}(\lambda)\}_{\lambda\in\mathbb{R}}$ are related by

$$S_{11}(\lambda) = \Theta_2[\lambda](\lambda)^* \tag{1.10}$$

for a.e. $\lambda \in \mathbb{R}$. By [1]-[4] the last relation also yields that the scattering matrix $\{S_{11}(\lambda)\}_{\lambda\in\mathbb{R}}$ can be regarded as the scattering matrix $S_{LP}[\lambda](\mu)$ of a Lax-Phillips scattering system at the point λ.

Below we restrict ourself to a complete scattering system $\{L, L_0\}$, $L = L_0 + V$, where the perturbation V is a self-adjoint trace class operator. The assumption that V is a trace class operator is made for simplicity. Indeed, it would be sufficient to assume that the resolvent difference $(L - z)^{-p} - (L_0 - z)^{-p}$ is nuclear for a certain $p \in \mathbb{N}$ or, more generally, that the conditions of the so-called "stationary" scattering theory are satisfied, cf. [5, Section 14]. However, we emphasize that in contrast to [8] the smallness of the perturbation V is not assumed. Following the lines of [5] we show in Section 2 how the scattering matrix of the scattering system $\{L, L_0\}$ can be calculated. Under the additional assumptions (1.2) and (1.4) we find in Section 3 the channel scattering matrices $\{S_{ij}(\lambda)\}_{\lambda\in\mathbb{R}}$. In Section 4 we prove relation (1.7). Section 5 is devoted to the proof of (1.10). Moreover, the Lax-Phillips scattering theory for which $\{\Theta_2[\lambda](\mu)^*\}_{\mu\in\mathbb{R}}$ is the scattering matrix is indicated.

2 Scattering matrix

In this section we briefly recall the notion of the scattering matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ of a scattering system $\{L, L_0\}$, where it is assumed that the unperturbed operator L_0 is self-adjoint in the separable Hilbert space $\mathfrak{L}$ and the perturbed operator L differs from L_0 by a self-adjoint trace class operator $V \in \mathcal{B}_1(\mathfrak{L})$,

$$L = L_0 + V, \qquad V = V^* \in \mathcal{B}_1(\mathfrak{L}). \tag{2.1}$$

Let $E_0(\cdot)$ be the spectral measure of L_0 and denote by $\mathfrak{B}(\mathbb{R})$ the set of all Borel subsets of the real axis $\mathbb{R}$. Without loss of generality we assume throughout the paper that the condition

$$\mathfrak{L} = \text{clospan}\{E_0(\Delta)\text{ran}\,(|V|) : \Delta \in \mathfrak{B}(\mathbb{R})\} \tag{2.2}$$

is satisfied, where $|V| := (V^*V)^{1/2}$. By Theorem X.4.4 of [13] the scattering system $\{L, L_0\}$ is complete, that is, the ranges of the wave operators $W_\pm(L, L_0)$ in (1.1) coincide with the absolutely continuous subspace $\mathfrak{L}^{ac}(L)$ of L. The operator V admits the representation

$$V = |V|^{1/2} C |V|^{1/2}, \quad |V| = (V^*V)^{1/2}, \quad C = \operatorname{sgn}(V), \tag{2.3}$$

where $|V|^{1/2}$ belongs to the Hilbert-Schmidt class $\mathcal{B}_2(\mathfrak{L})$ and $\operatorname{sgn}(\cdot)$ is the signum function. By Proposition 3.14 of [5] the limits

$$|V|^{1/2}(L - \lambda \pm i0)^{-1}|V|^{1/2} = \lim_{\epsilon \to +0} |V|^{1/2}(L - \lambda \pm i\epsilon)^{-1}|V|^{1/2} \tag{2.4}$$

exist in $\mathcal{B}_2(\mathfrak{L})$ for a.e. $\lambda \in \mathbb{R}$. The same holds for the limits

$$|V|^{1/2}(L_0 - \lambda \pm i0)^{-1}|V|^{1/2}.$$

Moreover by Proposition 3.13 of [5] the derivative

$$M_0(\lambda) := \frac{|V|^{1/2} E_0(d\lambda) |V|^{1/2}}{d\lambda} \geq 0 \tag{2.5}$$

exists in $\mathcal{B}_1(\mathfrak{L})$ for a.e. $\lambda \in \mathbb{R}$. We set

$$\mathfrak{Q}_\lambda := \operatorname{clo}\{\operatorname{ran}(M_0(\lambda))\} \subseteq \mathfrak{L}.$$

By $\{Q(\lambda)\}_{\lambda \in \mathbb{R}}$ we denote the family of orthogonal projections from $\mathfrak{L}$ onto $\mathfrak{Q}_\lambda$. One verifies that $\{Q(\lambda)\}_{\lambda \in \mathbb{R}}$ is measurable. Let us consider the standard Hilbert space $L^2(\mathbb{R}, d\lambda, \mathfrak{L})$. On $L^2(\mathbb{R}, d\lambda, \mathfrak{L})$ we introduce the projection Q

$$(Qf)(\lambda) := Q(\lambda) f(\lambda), \quad \lambda \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, d\lambda, \mathfrak{L}),$$

and set $\mathfrak{Q} = \operatorname{ran}(Q)$. Further, in $L^2(\mathbb{R}, d\lambda, \mathfrak{L})$ we define the multiplication operator $M_\mathfrak{L}$ by

$$\begin{aligned} (M_\mathfrak{L} f)(\lambda) &:= \lambda f(\lambda), \quad \lambda \in \mathbb{R}, \\ \operatorname{dom}(M_\mathfrak{L}) &:= \{f \in L^2(\mathbb{R}, d\lambda, \mathfrak{L}) : \lambda f(\lambda) \in L^2(\mathbb{R}, d\lambda, \mathfrak{L})\}. \end{aligned}$$

Obviously, the multiplication operator $M_\mathfrak{L}$ and the projection Q commute. We set

$$M_\mathfrak{Q} := M_\mathfrak{L} \restriction \operatorname{dom}(M_\mathfrak{L}) \cap \mathfrak{Q}.$$

From Section 4.5 of [5] one gets that the absolutely continuous part L^{ac} of the perturbed operator L and the operator $M_\mathfrak{Q}$ are unitarily equivalent. In the following we denote the subspace $\mathfrak{Q}$ by $L^2(\mathbb{R}, d\lambda, \mathfrak{Q}_\lambda)$ which can be regarded as the direct integral of the family of subspaces $\{\mathfrak{Q}_\lambda\}_{\lambda \in \mathbb{R}}$ with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}$, cf. [5].

Since the scattering operator $S = W_+(L, L_0)^* W_-(L, L_0)$ acts on L_0^{ac} and commutes with L_0^{ac} there is a measurable family $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ of operators

$$S(\lambda) : \mathfrak{Q}_\lambda \longrightarrow \mathfrak{Q}_\lambda$$

such that S is unitarily equivalent to the multiplication operator

$$\begin{array}{rcl} (M_{\mathfrak{L}}(S)f)(\lambda) & := & S(\lambda)f(\lambda), \\ \mathrm{dom}\,(M_{\mathfrak{L}}(S)) & := & L^2(\mathbb{R}, d\lambda, \mathfrak{L}_\lambda). \end{array}$$

The family $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ is called the *scattering matrix* of the scattering system $\{L, L_0\}$. Since the scattering system $\{L, L_0\}$ is complete the operator $S(\lambda)$ is unitary on $\mathfrak{L}_\lambda$ for a.e. $\lambda \in \mathbb{R}$.

The following representation theorem of the scattering matrix is a consequence of Corollary 18.9 of [5], see also [5, Section 18.2.2].

Theorem 2.1 *Let L, L_0 and V be self-adjoint operators in $\mathfrak{L}$ as in* (2.1). *Then $\{L, L_0\}$ is a complete scattering system and the corresponding scattering matrix matrix $\{S(\lambda)\}_{\lambda\in\mathbb{R}}$ admits the representation*

$$S(\lambda) = I_{\mathfrak{L}_\lambda} - 2\pi i\, M_0^{1/2}(\lambda)\left\{C - C|V|^{1/2}(L-\lambda-i0)^{-1}|V|^{1/2}C\right\}M_0^{1/2}(\lambda)$$

for a.e. $\lambda \in \mathbb{R}$.

3 Channel scattering matrices

Let us now assume that the Hilbert space $\mathfrak{L}$ is the orthogonal sum of two subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, $\mathfrak{L} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, that L_0 is a diagonal block operator matrix of the form

$$L_0 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} : \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array} \longrightarrow \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array}, \tag{3.1}$$

cf. (1.2), where H_1 and H_2 are self-adjoint operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and that $V \in \mathcal{B}_1(\mathfrak{L})$ is a self-adjoint trace class operator of the form

$$V = \begin{pmatrix} 0 & G \\ G^* & 0 \end{pmatrix} : \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array} \longrightarrow \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array}, \tag{3.2}$$

see (1.4). The operator $G : \mathfrak{H}_2 \longrightarrow \mathfrak{H}_1$ describes the interaction between the channels. Since V is a trace class operator we have

$$G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1).$$

The perturbed or full Hamiltonian L has the form

$$L := L_0 + V = \begin{pmatrix} H_1 & G \\ G^* & H_2 \end{pmatrix} : \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array} \longrightarrow \begin{array}{c} \mathfrak{H}_1 \\ \oplus \\ \mathfrak{H}_2 \end{array}. \tag{3.3}$$

The following lemma is known as the Feshbach decomposition in physics, cf. [10, 11]. We use the notation

$$H_1(z) = H_1 + V_1(z) \quad \text{and} \quad H_2(z) = H_2 + V_2(z), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.4}$$

where

$$V_1(z) = -G(H_2 - z)^{-1}G^* \quad \text{and} \quad V_2(z) = -G^*(H_1 - z)^{-1}G, \tag{3.5}$$

see (1.6), (1.9), (1.5) and (1.8).

Lemma 3.1 *Let L, $H_1(z)$ and $H_2(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, be given by* (3.3) *and* (3.4), *respectively. Then we have $z \in \operatorname{res}(H_i(z))$, $i = 1, 2$, for all $z \in \mathbb{C}\backslash\mathbb{R}$ and*

$$(L - z)^{-1} = \begin{pmatrix} (H_1(z) - z)^{-1} & -(H_1 - z)^{-1}G(H_2(z) - z)^{-1} \\ -(H_2(z) - z)^{-1}G^*(H_1 - z)^{-1} & (H_2(z) - z)^{-1} \end{pmatrix}. \tag{3.6}$$

Proof. From

$$\operatorname{Im}\big((H_1(z) - z)h_1, h_1\big) = \operatorname{Im}\overline{z}\|h_1\|^2 + \operatorname{Im}\overline{z}\|(H_2 - z)^{-1}G^*h_1\|^2,$$

$z \in \mathbb{C}\backslash\mathbb{R}$, $h_1 \in \mathfrak{H}_1$, we conclude that $(H_1(z) - z)^{-1}$ exists for $z \in \mathbb{C}\backslash\mathbb{R}$. Analogously one verifies that $(H_2(z) - z)^{-1}$ exists for all $z \in \mathbb{C}\backslash\mathbb{R}$. A straightforward computation shows

$$(L - z)^{-1} =$$
$$\begin{pmatrix} (H_1 - z)^{-1} + (H_1 - z)^{-1}G(H_2(z) - z)^{-1}G^*(H_1 - z)^{-1} & -(H_1 - z)^{-1}G(H_2(z) - z)^{-1} \\ -(H_2(z) - z)^{-1}G^*(H_1 - z)^{-1} & (H_2(z) - z)^{-1} \end{pmatrix}$$

for $z \in \mathbb{C}\backslash\mathbb{R}$. From the identity

$$\big(I - G^*(H_1 - z)^{-1}G(H_2 - z)^{-1}\big)^{-1}G^* = G^*\big(I - (H_1 - z)^{-1}G(H_2 - z)^{-1}G^*\big)^{-1}$$

we obtain

$$\begin{aligned} &(H_1 - z)^{-1} + (H_1 - z)^{-1}G(H_2(z) - z)^{-1}G^*(H_1 - z)^{-1} \\ &= (H_1 - z)^{-1}\Big\{I + G(H_2 - z)^{-1}\big(I - G^*(H_1 - z)^{-1}G(H_2 - z)^{-1}\big)^{-1}G^*(H_1 - z)^{-1}\Big\} \\ &= (H_1 - z)^{-1}\big\{I + G(H_2 - z)^{-1}G^*(H_1(z) - z)^{-1}\big\} = (H_1(z) - z)^{-1} \end{aligned}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$, which proves (3.6). $\square$

In the next lemma we calculate the limit $|V|^{1/2}(L - \lambda - i0)^{-1}|V|^{1/2}$, $\lambda \in \mathbb{R}$, cf. (2.4). Here and in the following it is convenient to use the functions

$$\begin{aligned} N_1(z) &:= |G^*|^{1/2}(H_1 - z)^{-1}|G^*|^{1/2}, \\ N_2(z) &:= |G|^{1/2}(H_2 - z)^{-1}|G|^{1/2}, \end{aligned} \qquad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.7}$$

and

$$\begin{aligned} F_1(z) &:= |G^*|^{1/2}(H_1(z) - z)^{-1}|G^*|^{1/2}, \\ F_2(z) &:= |G|^{1/2}(H_2(z) - z)^{-1}|G|^{1/2}, \end{aligned} \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.8}$$

Lemma 3.2 *Let $V \in \mathcal{B}_1(\mathfrak{L})$ be given by (3.2) with $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$. Then the limits*

$$N_i(\lambda) := \lim_{\epsilon \to +0} N_i(\lambda + i\epsilon) \quad \textit{and} \quad F_i(\lambda) := \lim_{\epsilon \to +0} F_i(\lambda + i\epsilon), \quad i = 1, 2, \tag{3.9}$$

exist in $\mathcal{B}_2(\mathfrak{H}_i)$ for a.e. $\lambda \in \mathbb{R}$ and the representation

$$|V|^{1/2}(L - \lambda - i0)^{-1}|V|^{1/2} = \begin{pmatrix} F_1(\lambda) & -N_1(\lambda)UF_2(\lambda) \\ -F_2(\lambda)U^*N_1(\lambda) & F_2(\lambda) \end{pmatrix} \tag{3.10}$$

holds for a.e. $\lambda \in \mathbb{R}$.

Proof. By $|G|^{1/2} \in \mathcal{B}_2(\mathfrak{H}_2)$ and $|G^*|^{1/2} \in \mathcal{B}_2(\mathfrak{H}_1)$ the existence of the limits $N_i(\lambda)$ in (3.9) for a.e. $\lambda \in \mathbb{R}$ follows from Proposition 3.13 of [5]. Using the representations $F_1(z) = |G^*|^{1/2}P_1(L - z)^{-1} \restriction_{\mathfrak{H}_1} |G^*|^{1/2}$ and $F_2(z) = |G|^{1/2}P_2(L - z)^{-1} \restriction_{\mathfrak{H}_2} |G|^{1/2}$, $z \in \mathbb{C}\backslash\mathbb{R}$, which follow from (3.6), and taking into account [5, Proposition 3.13] we again obtain the existence of $F_i(\lambda)$, $i = 1, 2$, for a.e. $\lambda \in \mathbb{R}$. It is easy to see that

$$|V|^{1/2} = (V^*V)^{1/4} = \begin{pmatrix} |G^*|^{1/2} & 0 \\ 0 & |G|^{1/2} \end{pmatrix} \tag{3.11}$$

holds. Let U be a partial isometry from $\mathfrak{H}_2$ into $\mathfrak{H}_1$ such that $G = U|G|$. Making use of the factorizations

$$G = |G^*|^{1/2}U|G|^{1/2} \quad \text{and} \quad G^* = |G|^{1/2}U^*|G^*|^{1/2}, \tag{3.12}$$

the block matrix representation of $(L - z)^{-1}$ in Lemma 3.1 and relation (3.11) one verifies (3.10). □

We note that if U is a partial isometry such that $G = U|G|$ holds and C is defined by

$$C := \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}, \tag{3.13}$$

then the operator V in (3.2) can be written in the form $|V|^{1/2}C|V|^{1/2}$, cf. (2.3). Let $E_1(\cdot)$ and $E_2(\cdot)$ be the spectral measures of H_1 and H_2, respectively. The operator function $M_0(\cdot)$ from (2.5) here admits the representation

$$M_0(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ 0 & M_2(\lambda) \end{pmatrix} \tag{3.14}$$

for a.e $\lambda \in \mathbb{R}$, where the derivatives

$$M_1(\lambda) = \frac{|G^*|^{1/2}E_1(d\lambda)|G^*|^{1/2}}{d\lambda} \quad \text{and} \quad M_2(\lambda) = \frac{|G|^{1/2}E_2(d\lambda)|G|^{1/2}}{d\lambda} \tag{3.15}$$

exist in $\mathcal{B}_1(\mathfrak{H}_1)$ and $\mathcal{B}_1(\mathfrak{H}_2)$ for a.e. $\lambda \in \mathbb{R}$, respectively. Setting

$$\mathfrak{Q}_{j,\lambda} := \operatorname{clo}\{\operatorname{ran}(M_j(\lambda))\}, \quad j = 1, 2,$$

and

$$\mathfrak{L}_\lambda := \mathfrak{L}_{1,\lambda} \oplus \mathfrak{L}_{2,\lambda} \tag{3.16}$$

for a.e. $\lambda \in \mathbb{R}$ we obtain the decomposition

$$L^2(\mathbb{R}, d\lambda, \mathfrak{L}_\lambda) = L^2(\mathbb{R}, d\lambda, \mathfrak{L}_{1,\lambda}) \oplus L^2(\mathbb{R}, d\lambda, \mathfrak{L}_{2,\lambda}),$$

cf. Section 2. From (2.2) the conditions

$$\begin{array}{rcl} \mathfrak{H}_1 & = & \mathrm{clospan}\{E_1(\Delta)\mathrm{ran}\,(|G^*|) : \Delta \in \mathfrak{B}(\mathbb{R})\}, \\ \mathfrak{H}_2 & = & \mathrm{clospan}\{E_2(\Delta)\mathrm{ran}\,(|G|) : \Delta \in \mathfrak{B}(\mathbb{R})\} \end{array} \tag{3.17}$$

follow. Moreover, the converse is also true, that is, condition (3.17) implies (2.2). Hence, without loss of generality we assume that condition (3.17) is satisfied. Therefore the reduced multiplication operators $M_{\mathfrak{L}_j}$,

$$M_{\mathfrak{L}_j} := M_{\mathfrak{H}_j} \restriction \mathrm{dom}\,(M_{\mathfrak{H}_j}) \cap L^2(\mathbb{R}, d\lambda, \mathfrak{L}_{j,\lambda}),$$

where

$$\begin{array}{rcl} (M_{\mathfrak{H}_j} f)(\lambda) & := & \lambda f(\lambda), \quad \lambda \in \mathbb{R}, \\ \mathrm{dom}\,(M_{\mathfrak{H}_j}) & := & \left\{ f \in L^2(\mathbb{R}, d\lambda, \mathfrak{H}_j) : \lambda f(\lambda) \in L^2(\mathbb{R}, d\lambda, \mathfrak{H}_j) \right\}. \end{array}$$

are unitary equivalent to the absolutely continuous parts H_j^{ac} of the operators H_j, $j = 1, 2$.

With respect to the decomposition (3.16) the scattering matrix $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ admits the decomposition

$$S(\lambda) = \begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix} : \begin{array}{c} \mathfrak{L}_{1,\lambda} \\ \oplus \\ \mathfrak{L}_{2,\lambda} \end{array} \longrightarrow \begin{array}{c} \mathfrak{L}_{1,\lambda} \\ \oplus \\ \mathfrak{L}_{2,\lambda} \end{array} \tag{3.18}$$

for a.e. $\lambda \in \mathbb{R}$. The entries $\{S_{ij}(\lambda)\}_{\lambda \in \mathbb{R}}$, $i, j = 1, 2$, are called *channel scattering matrices*. We note that the multiplication operators induced by the channel scattering matrices are unitary equivalent to the channel scattering operators $S_{ij} = P_i S P_j$, $i, j = 1, 2$, where P_i is the orthogonal projection in $\mathfrak{L}$ onto the subspace $\mathfrak{H}_j$ and S is the scattering operator of the complete scattering system $\{L, L_0\}$.

In the next proposition we give a more explicit description of the channel scattering matrices $S_{ij}(\lambda)$. The proof is an immediate consequence of Theorem 2.1, Lemma 3.2 and relations (3.14), (3.15) and (3.13).

Proposition 3.3 *Let L_0, V and L be given in accordance with* (3.1), (3.2) *and* (3.3), *respectively. Then the scattering matrix $\{S(\lambda)\}_{\lambda \in \mathbb{R}}$ of the complete scattering system $\{L, L_0\}$ admits the representation* (3.18) *with entries $S_{ij}(\lambda)$ given by*

$$\begin{array}{rcl} S_{11}(\lambda) & = & I_{\mathfrak{L}_{1,\lambda}} + 2\pi i M_1(\lambda)^{1/2} U F_2(\lambda) U^* M_1(\lambda)^{1/2}, \\ S_{12}(\lambda) & = & -2\pi i M_1(\lambda)^{1/2} \{U + U F_2(\lambda) U^* N_1(\lambda) U\} M_2(\lambda)^{1/2}, \\ S_{21}(\lambda) & = & -2\pi i M_2(\lambda)^{1/2} \{U^* + U^* N_1(\lambda) U F_1(\lambda) U^*\} M_1(\lambda)^{1/2}, \\ S_{22}(\lambda) & = & I_{\mathfrak{L}_{2,\lambda}} + 2\pi i M_2(\lambda)^{1/2} U^* F_1(\lambda) U M_2(\lambda)^{1/2}. \end{array} \tag{3.19}$$

for a.e. $\lambda \in \mathbb{R}$.

4 Dissipative channel scattering

In this section we consider the (dissipative) scattering system $\{H_1(\lambda), H_1\}$ for a.e. $\lambda \in \mathbb{R}$, where

$$H_1(\lambda) = H_1 + V_1(\lambda), \tag{4.1}$$

is defined for a.e. $\lambda \in \mathbb{R}$, and H_1 is the self-adjoint operator in $\mathfrak{H}_1$ from (3.1). The limit $V_1(\lambda) = \lim_{\epsilon\to+0} V_1(\lambda + i\epsilon)$ (see Lemma 4.1) is called the *optical potential* of the channel $\mathfrak{H}_1$. In Theorem 4.4 below we establish a connection between the scattering matrices corresponding to the scattering systems $\{H_1(\lambda), H_1\}$ and the channel scattering matrix $S_{11}(\lambda)$ from (3.18) and (3.19).

Lemma 4.1 *Let $V_1(z) = -G(H_2 - z)^{-1}G^*$, $z \in \mathbb{C}\backslash\mathbb{R}$, be defined by (3.5) with $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$. Then the limit $V_1(\lambda) = \lim_{\epsilon\to+0} V_1(\lambda + i\epsilon)$ exists in $\mathcal{B}_1(\mathfrak{H}_1)$ and $V_1(\lambda)$ is dissipative for a.e. $\lambda \in \mathbb{R}$.*

Proof. Using the factorizations (3.12) of G and G^* we find

$$V_1(z) = -|G^*|^{1/2} U N_2(z) U^* |G^*|^{1/2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{4.2}$$

where $N_2(z)$ is given by (3.7). According to Lemma 3.2 the limit $\lim_{\epsilon\to+0} N_2(\lambda + i\epsilon)$ exists in $\mathcal{B}_2(\mathfrak{H}_1)$ and since $|G^*|^{1/2} \in \mathcal{B}_2(\mathfrak{H}_1)$ we conclude that the limit

$$V_1(\lambda) = \lim_{\epsilon\to+0} V_1(\lambda + i\epsilon) = \lim_{\epsilon\to+0} -|G^*|^{1/2} U N_2(\lambda + i\epsilon) U^* |G^*|^{1/2}$$

exists in $\mathcal{B}_1(\mathfrak{H}_1)$ for a.e. $\lambda \in \mathbb{R}$. It is not difficult to see that $\operatorname{Im} V_1(z) \leq 0$ for $z \in \mathbb{C}^+$ and therefore also the limit $V_1(\lambda)$ is dissipative for a.e. $\lambda \in \mathbb{R}$. □

It follows from Lemma 4.1 that for a.e. $\lambda \in \mathbb{R}$ the operator $H_1(\lambda) = H_1 + V_1(\lambda)$ is maximal dissipative and therefore $\{H_1(\lambda), H_1\}$ is a *dissipative scattering system* in the sense of [19, 20]. By Theorem 4.3 of [20] the corresponding wave operators

$$W_+^D(H_1(\lambda), H_1) = s - \lim_{t\to+\infty} e^{itH_1(\lambda)^*} e^{-itH_1} P^{ac}(H_1)$$

and

$$W_-^D(H_1(\lambda), H_1) = s - \lim_{t\to+\infty} e^{-itH_1(\lambda)} e^{itH_1} P^{ac}(H_1)$$

exist and are complete which yields that $\{H_1(\lambda), H_1\}$ performs a complete dissipative scattering system for a.e. $\lambda \in \mathbb{R}$. The associated scattering operators are defined by

$$S_D[\lambda] := W_+^D(H_1(\lambda), H_1)^* W_-^D(H_1(\lambda), H_1)$$

and act on the absolutely continuous subspaces $\mathfrak{H}_1^{ac}(H_1)$. Since $S_D[\lambda]$ commutes with H_1 the scattering operator is unitary equivalent to a multiplication operator in the spectral representation $L^2(\mathbb{R}, d\lambda, \mathfrak{Q}_{1,\mu})$ of H_1 induced by a family of contractions $\{S_D[\lambda](\mu)\}_{\mu\in\mathbb{R}}$. The family $\{S_D[\lambda](\mu)\}_{\mu\in\mathbb{R}}$ is called the *scattering matrix* of the complete dissipative scattering system $\{H_1(\lambda), H_1\}$.

Using the fact that every maximal dissipative operator admits a self-adjoint dilation, i.e., there exists a self-adjoint operator in a (in general) larger Hilbert space such that its compressed resolvent coincides with the resolvents of the maximal dissipative operator for all $z \in \mathbb{C}^+$, cf. [7, Section 7], see also [12], one concludes from Proposition 3.14 of [5] that the limit

$$\mathfrak{F}_1[\lambda](\mu) = \lim_{\epsilon \to +0} \mathfrak{F}_1[\lambda](\mu + i\epsilon)$$

exist in $\mathcal{B}_1(\mathfrak{H}_1)$ for a.e. $\mu \in \mathbb{R}$, where

$$\mathfrak{F}_1[\lambda](z) := |G^*|^{1/2}(H_1(\lambda) - z)^{-1}|G^*|^{1/2}, \quad z \in \mathbb{C}_+,$$

is defined for a.e. $\lambda \in \mathbb{R}$. The next proposition is a direct consequence of Theorem 2.2 of [16], see also [15].

Proposition 4.2 *Let $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$ and $H_1(\lambda)$ be given by (4.1). Then for a.e. $\lambda \in \mathbb{R}$ the scattering matrix $\{S_D[\lambda](\mu)\}_{\mu \in \mathbb{R}}$ of the complete dissipative scattering system $\{H_1(\lambda), H_1\}$ admits the representation*

$$S_D[\lambda](\mu) = I_{\mathfrak{Q}_{1,\mu}} + 2\pi i M_1(\mu)^{1/2} U \Big\{ N_2(\lambda) + N_2(\lambda) U^* \mathfrak{F}_1[\lambda](\mu) U N_2(\lambda) \Big\} U^* M_1(\mu)^{1/2}$$

for a.e. $(\mu, \lambda) \in \mathbb{R}^2$ with respect to the Lebesgue measure in $\mathbb{R}^2$.

In the next lemma we show that the limit $\mathfrak{F}_1[\lambda](\lambda)$,

$$\mathfrak{F}_1[\lambda](\lambda) = \lim_{\epsilon \to +0} \mathfrak{F}_1[\lambda](\lambda + i\epsilon) = \lim_{\epsilon \to +0} |G^*|^{1/2}(H_1(\lambda) - \lambda - i\epsilon)^{-1}|G^*|^{1/2}, \tag{4.3}$$

exist in $\mathcal{B}_2(\mathfrak{H}_1)$ for a.e. $\lambda \in \mathbb{R}$.

Lemma 4.3 *Let L_0, V and L be given by (3.1), (3.2) and (3.3), respectively, with $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$. Further, let $F_1(\lambda)$ be as in Lemma 3.2 and let $H_1(\lambda)$ be defined by (4.1). Then the limit $\mathfrak{F}_1[\lambda](\lambda)$ in (4.3) exists in $\mathcal{B}_2(\mathfrak{H}_1)$ for a.e. $\lambda \in \mathbb{R}$ and the relation*

$$\mathfrak{F}_1[\lambda](\lambda) = F_1(\lambda) \tag{4.4}$$

holds for a.e. $\lambda \in \mathbb{R}$.

Proof. We have

$$\begin{aligned} F_1(z) - \mathfrak{F}_1[\lambda](z) &= |G^*|^{1/2}\big((H_1(z) - z)^{-1} - (H_1(\lambda) - z)^{-1}\big)|G^*|^{1/2} \\ &= |G^*|^{1/2}(H_1(z) - z)^{-1}(V_1(\lambda) - V_1(z))(H_1(\lambda) - z)^{-1}|G^*|^{1/2}. \end{aligned} \tag{4.5}$$

From (4.2) we obtain

$$V_1(\lambda) - V_1(z) = |G^*|^{1/2} U \big(N_2(z) - N_2(\lambda)\big) U^* |G^*|^{1/2}$$

and inserting this expression into (4.5) and using the definitions of $F_1(z)$ in (3.8) and $\mathfrak{F}_1[\lambda](z)$ yields

$$F_1(z) - \mathfrak{F}_1[\lambda](z) = F_1(z)U(N_2(z) - N_2(\lambda))U^*\mathfrak{F}_1[\lambda](z).$$

Hence

$$F_1(z) = \{I_{\mathfrak{H}_1} + F_1(z)U(N_2(z) - N_2(\lambda))U^*\}\,\mathfrak{F}_1[\lambda](z)$$

and for $z = \lambda + i\epsilon$, $\epsilon > 0$ sufficiently small, the operator

$$\{I_{\mathfrak{H}_1} + F_1(z)U(N_2(z) - N_2(\lambda))U^*\}$$

is invertible. Therefore we conclude

$$\{I_{\mathfrak{H}_1} + F_1(\lambda + i\epsilon)U(N_2(\lambda + i\epsilon) - N_2(\lambda))U^*\}^{-1}\,F_1(\lambda + i\epsilon) = \mathfrak{F}_1[\lambda](\lambda + i\epsilon).$$

From this representation we get the existence of $\mathfrak{F}_1[\lambda](\lambda)$ in $\mathcal{B}_2(\mathfrak{H}_1)$ and the equality (4.4) for a.e. $\lambda \in \mathbb{R}$. □

The next theorem is the main result of this section. We show how the channel scattering matrix $S_{11}(\lambda)$ of the scattering system $\{L, L_0\}$ is connected with the scattering matrices $S_D[\lambda](\mu)$ of the dissipative scattering systems $\{H_1(\lambda), H_1\}$.

Theorem 4.4 *Let $\{L, L_0\}$ be the scattering system from Section* 3*, where L_0, V and L are given by* (3.1)*,* (3.2) *and* (3.3)*, respectively, and $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$. Further, let $\{S_{ij}(\lambda)\}$, $i, j = 1, 2$, be the corresponding scattering matrix from* (3.18) *and let $S_D[\lambda](\mu)$ be the scattering matrices of the dissipative scattering systems $\{H_1(\lambda), H_1\}$. Then the scattering matrix $S_D[\lambda](\lambda)$ exists for a.e $\lambda \in \mathbb{R}$ and satisfies the relation*

$$S_D[\lambda](\lambda) = S_{11}(\lambda)$$

for a.e. $\lambda \in \mathbb{R}$.

Proof. From Proposition 4.2 and Lemma 4.3 we obtain that $S_D[\lambda](\lambda)$ exists for a.e $\lambda \in \mathbb{R}$ and has the form

$$S_D[\lambda](\mu) = I_{\mathfrak{L}_{1,\mu}} + 2\pi i M_1(\mu)^{1/2}U\Big\{N_2(\lambda) + N_2(\lambda)U^*F_1(\lambda)UN_2(\lambda)\Big\}U^*M_1(\mu)^{1/2} \quad (4.6)$$

A similar calculation as in the proof of Lemma 3.1 shows

$$F_2(z) = N_2(z) + N_2(z)U^*F_1(z)UN_2(z), \quad z \in \mathbb{C}_+.$$

If z tends to $\lambda \in \mathbb{R}$, then we get

$$F_2(\lambda) = N_2(\lambda) + N_2(\lambda)U^*F_1(\lambda)UN_2(\lambda) \qquad (4.7)$$

for a.e. $\lambda \in \mathbb{R}$. Inserting (4.7) into (4.6) we obtain

$$S_D[\lambda](\lambda) = I_{\mathfrak{L}_{1,\lambda}} + 2\pi i M_1(\lambda)^{1/2}UF_2(\lambda)U^*M_1(\lambda)^{1/2}$$

and by Proposition 3.3 this coincides with $S_{11}(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. □

5 Lax-Phillips channel scattering

Similarly to Lemma 4.1 one verifies that $V_2(\lambda) = \lim_{\epsilon\to+0} V_2(\lambda + i\epsilon)$ exists in $\mathcal{B}_1(\mathfrak{H}_2)$ for a.e. $\lambda \in \mathbb{R}$. The limit $V_2(\lambda)$, which is called the optical potential of the channel $\mathfrak{H}_2$, is dissipative for a.e. $\lambda \in \mathbb{R}$. The optical potential defines the maximal dissipative operator

$$H_2(\lambda) := H_2 + V_2(\lambda)$$

for a.e. $\lambda \in \mathbb{R}$. The operator $H_2(\lambda)$ decomposes for a.e. $\lambda \in \mathbb{R}$ into a self-adjoint part and a completely non-self-adjoint part. Let $\Theta_2[\lambda](\xi)$, $\xi \in \mathbb{C}_-$, be the characteristic function, cf. [12], of the completely non-self-adjoint part of $H_2(\lambda)$. We are going to verify $S_{11}(\lambda) = \Theta_2[\lambda](\lambda)^*$ for a.e. $\lambda \in \mathbb{R}$ which shows that for a.e. $\lambda \in \mathbb{R}$ the scattering matrix $S_{11}(\lambda)$ can be regarded as the result of a certain Lax-Phillips scattering theory, cf. [1, 2, 3, 4, 14].

There is an orthogonal decomposition

$$\mathfrak{H}_2 = \mathfrak{H}^{cns}_{2,\lambda} \oplus \mathfrak{H}^{self}_{2,\lambda}$$

for a.e. $\lambda \in \mathbb{R}$ such that $\mathfrak{H}^{cns}_{2,\lambda}$ and $\mathfrak{H}^{self}_{2,\lambda}$ reduce $H_2(\lambda)$ into a completely non-self-adjoint operator $H^{cns}_2(\lambda)$ and a self-adjoint operator $H^{self}_2(\lambda)$,

$$H_2(\lambda) = H^{cns}_2(\lambda) \oplus H^{self}_2(\lambda).$$

Taking into account Proposition 3.14 of [5] we get that

$$\Im\mathrm{m}\,(V_2(\lambda)) = -\pi|G|^{1/2}U^* M_1(\lambda) U|G|^{1/2}$$

for a.e. $\lambda \in \mathbb{R}$. Let us introduce the operator

$$\alpha(\lambda) := \sqrt{2\pi M_1(\lambda)}\, U|G|^{1/2}. \tag{5.1}$$

Notice that

$$\mathrm{clo}\{\mathrm{ran}\,(\alpha(\lambda))\} = \mathfrak{Q}_{1,\lambda}$$

for a.e. $\lambda \in \mathbb{R}$. With the completely non-self-adjoint part $H^{cns}(\lambda)$ one associates the characteristic function $\Theta_2[\lambda](\cdot) : \mathfrak{Q}_{1,\lambda} \longrightarrow \mathfrak{Q}_{1,\lambda}$ defined by

$$\Theta_2[\lambda](\xi) := I_{\mathfrak{Q}_{1,\lambda}} - i\alpha(\lambda)(H_2(\lambda)^* - \xi)^{-1}\alpha(\lambda)^*,$$

$\xi \in \mathbb{C}_-$. The characteristic function is a contraction-valued holomorphic function in $\mathbb{C}_-$. From [12, Section V.2] we get that the boundary values

$$\Theta_2[\lambda](\mu) := s - \lim_{\epsilon\to+0} \Theta_2[\lambda](\mu - i\epsilon)$$

exist for a.e. $\mu \in \mathbb{R}$.

Theorem 5.1 *Let L_0, V and L be given by* (3.1), (3.2) *and* (3.3). *If the condition $G \in \mathcal{B}_1(\mathfrak{H}_2, \mathfrak{H}_1)$ is satisfied, then the limit $\Theta_2[\lambda](\lambda)$,*

$$\Theta_2[\lambda](\lambda) := s - \lim_{\epsilon \to +0} \Theta_2[\lambda](\lambda - i\epsilon)$$

exists for a.e. $\lambda \in \mathbb{R}$ and the relation

$$S_{11}(\lambda) = \Theta_2[\lambda](\lambda)^*$$

holds for a.e. $\lambda \in \mathbb{R}$.

Proof. We set

$$\Theta_2^*[\lambda](\xi) := \Theta_2[\lambda](\overline{\xi})^* = I_{\mathfrak{Q}_{1,\lambda}} + i\alpha(\lambda)(H_2(\lambda) - \xi)^{-1}\alpha(\lambda)^*$$

$\xi \in \mathbb{C}_+$. Using (5.1) we get

$$\Theta_2^*[\lambda](\xi) = I_{\mathfrak{Q}_{1,\lambda}} + 2\pi i\sqrt{M_1(\lambda)}\, U\mathfrak{F}_2[\lambda](\xi)U^*\sqrt{M_1(\lambda)}$$

for a.e. $\lambda \in \mathbb{R}$, where

$$\mathfrak{F}_2[\lambda](\xi) := |G|^{1/2}(H_2(\lambda) - \xi)^{-1}|G|^{1/2}, \quad \xi \in \mathbb{C}_+.$$

Similar to the proof of Lemma 4.3 one verifies that the limit $\mathfrak{F}_2[\lambda](\lambda)$

$$\mathfrak{F}_2[\lambda](\lambda) = \lim_{\epsilon \to +0} \mathfrak{F}_2[\lambda](\lambda + i\epsilon)$$

exist in $\mathcal{B}_2(\mathfrak{H}_2)$ for a.e. $\lambda \in \mathbb{R}$ and satisfies the relation $\mathfrak{F}_2[\lambda](\lambda) = F_2(\lambda)$. Hence the limit $\Theta_2^*[\lambda](\lambda) = s - \lim_{\epsilon\to+0} \Theta_2[\lambda](\lambda - i\epsilon)^*$ exists for a.e. $\lambda \in \mathbb{R}$ and the relation

$$\Theta_2^*[\lambda](\lambda) = I_{\mathfrak{Q}_{1,\lambda}} + 2\pi i\sqrt{M_1(\lambda)}\, U\mathfrak{F}_2[\lambda](\lambda)U^*\sqrt{M_1(\lambda)}$$

holds for a.e $\lambda \in \mathbb{R}$. From (3.19) we obtain that $S_{11}(\lambda) = \Theta_2^*[\lambda](\lambda)$ for a.e. $\lambda \in \mathbb{R}$. Since the limit $\Theta_2^*[\lambda](\lambda)$ exists for a.e. $\lambda \in \mathbb{R}$ one concludes that

$$\Theta_2[\lambda](\lambda) := s - \lim_{\epsilon \to +0} \Theta_2[\lambda](\lambda - i\epsilon)$$

exists for a.e. $\lambda \in \mathbb{R}$ and $\Theta_2[\lambda](\lambda)^* = \Theta_2^*[\lambda](\lambda)$ is valid. This completes the proof Theorem 5.1. $\square$

The last theorem admits an interpretation of the scattering matrix $S_{11}(\lambda)$ as the result of a Lax-Phillips scattering. Indeed, let us introduce the minimal self-adjoint dilation $K_2(\lambda)$ of the maximal dissipative operator $H_2(\lambda)$. We set

$$\mathfrak{K}_{2,\lambda} = \mathfrak{D}_{-,\lambda} \oplus \mathfrak{H}_2 \oplus \mathfrak{D}_{+,\lambda},$$

where

$$\mathfrak{D}_{\pm,\lambda} := L^2(\mathbb{R}_\pm, dx, \mathfrak{Q}_{1,\lambda}).$$

Further, we define

$$K_2(\lambda)\begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} := \begin{pmatrix} -i\frac{d}{dx}f_- \\ \Re e\,(H_2(\lambda))f - \frac{1}{2}\alpha(\lambda)^*[f_+(0)+f_-(0)] \\ -i\frac{d}{dx}f_+ \end{pmatrix}$$

for elements of the domain

$$\mathrm{dom}\,(K_2(\lambda)) := \left\{ \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} : \begin{array}{c} f \in \mathrm{dom}\,(H_2(\lambda)) \\ f_\pm \in W^{1,2}(\mathbb{R}_\pm, dx, \mathfrak{Q}_{1,\lambda}) \\ f_+(0) - f_-(0) = -i\alpha(\lambda)f \end{array} \right\}.$$

The operator $K_2(\lambda)$ is self-adjoint and is a minimal self-adjoint dilation of the maximal dissipative operator $H_2(\lambda)$, that is,

$$(H_2(\lambda) - z)^{-1} = P^{\mathfrak{K}_{2,\lambda}}_{\mathfrak{H}_2}(K_2(\lambda) - z)^{-1} \upharpoonright \mathfrak{H}_2$$

for $z \in \mathbb{C}_+$ and

$$\mathfrak{K}_{2,\lambda} = \mathrm{clospan}\big\{E_{K_2(\lambda)}(\Delta)\mathfrak{H}_2 : \Delta \in \mathcal{B}(\mathbb{R})\big\},$$

where $E_{K_2(\lambda)}(\cdot)$ is the spectral measure of $K_2(\lambda)$. It turns out that $\mathfrak{D}_{\pm,\lambda}$ are incoming and outgoing subspaces with respect to $K_2(\lambda)$, i.e.,

$$e^{-itK_2(\lambda)}\mathfrak{D}_{+,\lambda} \subseteq \mathfrak{D}_{+,\lambda}, \quad t \geq 0,$$

and

$$e^{-itK_2(\lambda)}\mathfrak{D}_{-,\lambda} \subseteq \mathfrak{D}_{-,\lambda}, \quad t \leq 0.$$

However, we remark that the completeness condition

$$\mathfrak{K}_{2,\lambda} = \mathrm{clospan}\big\{e^{-itK_2(\lambda)}\mathfrak{D}_{\pm,\lambda} : t \in \mathbb{R}\big\} \tag{5.2}$$

is in general not satisfied. Condition (5.2) holds if and only if the maximal dissipative operator $H_2(\lambda)$ is completely non-selfadjoint and H_2 is singular, that means, the absolutely continuous part H_2^{ac} of H_2 is trivial.

On the subspace $\mathfrak{D}_\lambda$,

$$\mathfrak{D}_\lambda = \mathfrak{D}_{-,\lambda} \oplus \mathfrak{D}_{+,\lambda} = L^2(\mathbb{R}, dx, \mathfrak{Q}_{1,\lambda}) \subseteq \mathfrak{K}_{2,\lambda},$$

let us define the operator $K_0(\lambda)$,

$$(K_0(\lambda)g)(x) := -i\frac{d}{dx}g(x), \quad \mathrm{dom}\,(K_0(\lambda)) := W^{1,2}(\mathbb{R}, dx, \mathfrak{Q}_{1,\lambda}).$$

The self-adjoint operator $K_0(\lambda)$ generates the shift group, i.e,

$$(e^{-itK_0(\lambda)}g)(x) = g(x-t), \quad g \in \mathfrak{D}_\lambda.$$

Using the Fourier transform $\mathcal{F} : L^2(\mathbb{R}, dx, \mathfrak{Q}_{1,\lambda}) \longrightarrow L^2(\mathbb{R}, d\mu, \mathfrak{Q}_{1,\lambda})$,

$$(\mathcal{F}f)(\mu) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\; e^{-i\mu x}f(x),$$

the operator $K_0(\lambda)$ transforms into the multiplication operator on the Hilbert space $L^2(\mathbb{R}, d\mu, \mathfrak{Q}_{1,\lambda})$. Furthermore, one has

$$e^{-itK_2(\lambda)} \restriction \mathfrak{D}_{+,\lambda} = e^{-itK_0(\lambda)} \restriction \mathfrak{D}_{+,\lambda}, \quad t \geq 0,$$

and

$$e^{-itK_2(\lambda)} \restriction \mathfrak{D}_{-,\lambda} = e^{-itK_0(\lambda)} \restriction \mathfrak{D}_{-,\lambda}, \quad t \leq 0.$$

The last properties yield the existence of the Lax-Phillips wave operators

$$W_\pm^{LP}[\lambda] := s - \lim_{t\to\pm\infty} e^{itK_2(\lambda)} J_\pm(\lambda) e^{-itK_0(\lambda)},$$

cf. [5, 14] where $J_\pm(\lambda) : \mathfrak{D}_{\pm,\lambda} \longrightarrow \mathfrak{K}_{2,\lambda}$ is the natural embedding operator. The Lax-Phillips scattering operator $S_{LP}(\lambda)$ is defined by

$$S_{LP}[\lambda] := W_+^{LP}[\lambda]^* W_-^{LP}[\lambda],$$

cf. [5, 14]. With respect to the spectral representation $L^2(\mathbb{R}, d\mu, \mathfrak{Q}_{1,\lambda})$ the Lax-Phillips scattering matrix $\{S_{LP}[\lambda](\mu)\}_{\mu\in\mathbb{R}}$ coincides with $\{\Theta_2[\lambda](\mu)^*\}_{\mu\in\mathbb{R}}$, see [1, 2, 3, 4]. Hence the scattering matrix $\{S_{11}(\lambda)\}_{\lambda\in\mathbb{R}}$ can be regarded as the result of a Lax-Phillips scattering for a.e. $\lambda \in \mathbb{R}$.

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