BERNOULLI JETS AND THE ZERO MEAN CURVATURE EQUATION

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Abstract. We consider an elliptic PDE problem related with fluid mechanics. We show that level sets of rescaled solutions satisfy the zero mean curvature equation in a suitable weak viscosity sense. In particular, such level sets cannot be touched by below (above) by a convex (concave) paraboloid in a suitably small neighborhood.

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Contents

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1. Introduction

Given $p \in (1, +\infty)$, we consider here a problem driven by the p-Laplacian operator[1] and related with fluid dynamics. Some mean curvature estimates on the level sets of the solutions will be obtained, following a technique recently developed in [13] for smooth phase transition models. The setting in which these mean curvature estimates arise is a weak, quantitative, viscosity sense. Roughly speaking, we will consider a homogeneous ε rescaling of the solution and prove that the level sets of such rescaled solutions cannot be touched by a curved paraboloid in a neighborhood of order $\sqrt{\varepsilon}$. More formally, the main result dealt with in this paper is the following:

Theorem 1.1. *Let $\varpi > 0$ and $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be so that:*[2]

$$u(0) = 0\,, \tag{1.1}$$

$$|u| \le 1\,, \tag{1.2}$$

$$\Delta_p u = 0 \text{ for any } x \in \{|u| < 1\}, \tag{1.3}$$

Let us suppose that the following two free boundary growth conditions hold:

- *if there is an open ball $B_+ \subseteq \{|u| < 1\}$ touching $\partial\{u < 1\}$ at x_+, then*

$$u(x_+ + t_j \nu_0) \ge 1 - \varpi t_j - \mathfrak{f}_+(t_j)\,, \tag{1.4}$$

for an infinitesimal positive sequence $t_j \searrow 0^+$, where ν_0 is the interior normal of B_+ at x_+ and $\mathfrak{f}_+ : (0,1) \to \mathbb{R}$ is so that

$$\lim_{t \to 0^+} \frac{\mathfrak{f}_+(t)}{t} = 0\,; \tag{1.5}$$

- *if there is an open ball $B_- \subseteq \{u = -1\}$ touching $\partial\{u < -1\}$ at x_-, then*

$$u(x_- - t_j \nu_0) \ge -1 + \varpi t_j - \mathfrak{f}_-(t_j)\,, \tag{1.6}$$

for an infinitesimal positive sequence $t_j \searrow 0^+$, where ν_0 is the interior normal of B_- at x_- and $\mathfrak{f}_- : (0,1) \to \mathbb{R}$ is so that

$$\lim_{t \to 0^+} \frac{\mathfrak{f}_-(t)}{t} = 0\,. \tag{1.7}$$

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[1] As usual, we denote here the p-Laplacian by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, in the distributional sense.

[2] We will often denote a point $x \in \mathbb{R}^N$ by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Quantities depending only on N, p and ϖ will be referred to as (universal) constants.

Assume also that u satisfies the following decay property: there exists a universal $L > 0$ such that:

$$\begin{aligned} &\text{if } \{u = 0\} \cap \{|x'| \le l\} \subseteq \{x_N \ge -l/100\} \\ &\text{then } u(x) = -1 \text{ for any } x \text{ so that } |x'| \le l/2 \text{ and } x_N \le -l/10, \end{aligned} \tag{1.8}$$

provided $l \ge L$. Let $\mathfrak{d} \in (0,1)$ and $M \in \mathrm{Mat}((N-1) \times (N-1))$ with

$$\operatorname{tr} M > \mathfrak{d}\|M\| \qquad \text{and} \qquad \|M\| \le \mathfrak{d}^{-1}.$$

Let

$$\Gamma := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \;\; s.t. \;\; x_N = \frac{1}{2} x' \cdot M x' \right\}.$$

Let also $u_\varepsilon(x) := u(x/\varepsilon)$. Then, there exist a universal $\mathfrak{d}^\star > 0$ and a function $\sigma_0 : (0,1) \longrightarrow (0,1)$ such that if $\varepsilon \in (0, \sigma_0(\mathfrak{d}))$ and $\mathfrak{d} \in (0, \mathfrak{d}^\star)$, then Γ cannot touch $\{u_\varepsilon = 0\}$ by below in $B_{\mathfrak{d}\sqrt{\varepsilon}/\sqrt{\operatorname{tr} M}}$: more explicitly,

$$\{u_\varepsilon = 0\} \cap \left\{ x_N < \frac{1}{2} x' \cdot M x' \right\} \cap \left\{ |x| < \frac{\mathfrak{d}\sqrt{\varepsilon}}{\sqrt{\operatorname{tr} M}} \right\} \ne \emptyset.$$

Using an informal language, one may describe the content of Theorem 1.1 here above by stating that solutions of the problem considered enjoy a *weak viscosity zero mean curvature property*, in the sense that the level set $\{u_\varepsilon = 0\}$ cannot be touched from below by a convex[3] paraboloid in a neighborhood of the origin (which gets small with ε). It is known (see [4]) that level sets of rescaled minimizers approach a minimal surface in the Γ-convergence setting. Thus, in a way, we may think that Theorem 1.1 says that the level set $\{u_\varepsilon = 0\}$ attains a zero mean curvature property (though in a weak, quantitative, viscosity sense) even "before" converging to a limit surface. The fact that level sets inherit further properties from the minimal surface limit case may be related with the flat regularity of low dimensional level sets first conjectured by De Giorgi (see [9]). Also, Theorem 1.1 here holds for more general solutions than minimizers, differently from the Γ-convergence results in [4], and it provides a geometric and quantitative connection between the problem discussed here and the minimal surfaces.

Let us now briefly discuss some physical motivation behind the model considered here. The problem dealt with in Theorem 1.1 is inspired by ideal fluid jets. For instance, if $N = p = 2$, then u may be seen as the stream function[4] of a fluid, that is, the particles of the fluid move along

[3]Of course, an analogous statement holds for concave paraboloids touching from above.

[4]In such setting, we remark that the level sets of u, which we study here, have some physical relevance, since the particles of the fluid move along them.

the level sets $\{u = \theta\}$, for $\theta \in (-1, 1)$. In this sense, (1.3) is just the continuity equation. On the free boundary $\partial\{|u| < 1\}$, Bernoulli's law states that the speed of the fluid (which agrees with $|\nabla u|$), must be balanced by the exterior pressure. This is the physical meaning of (1.4) and (1.6), in the sense that these assumptions are just weak versions of the Bernoulli condition "$|\nabla u| = \varpi$ on the free boundary".

More precisely, the solutions dealt with in Theorem 1.1 are related with the minimizers of a functional widely studied in the literature both from the pure and applied mathematics point of view. Namely, for $\lambda > 0$, let us define[5] the following functional on $W^{1,p}(\Omega)$:

$$\mathcal{F}_\Omega(u) = \int_\Omega |\nabla u(x)|^p + \lambda \chi_{(-1,1)}\big(u(x)\big)\, dx\,.$$

The functional $\mathcal{F}$ is[6] a model for ideal fluid jets and cavitation problems (see, e.g., [1], [2], [3] and [12]); roughly speaking, the "kinetic" part $|\nabla u|^p$ leads to the PDE equation satisfied by the stream function of the ideal fluid, while the free boundary imposed by the discontinuity of the characteristic function yields the balance with the exterior pressure, according to Bernoulli's law. We refer to the above cited papers for further discussions upon these facts. Also, similar functionals provide models for flame propagation, combustion and electro-chemical processes (see, e.g., [7], [8] and references therein).

In this setting, we derive from Theorem 1.1 the following result:

Theorem 1.2. *Let $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$, $|u| \le 1$, be a Class A minimizer of $\mathcal{F}$, i.e., assume that*

$$\mathcal{F}_\Omega(u) \,\le\, \mathcal{F}_\Omega(u + \phi)\,,$$

for any $\phi \in C^\infty_0(\Omega)$, for any bounded domain Ω. Suppose that $u(0) = 0$. Let $u_\varepsilon(x) := u(x/\varepsilon)$ and

$$\Gamma := \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ \ s.t.\ \ x_N = \frac{1}{2} x' \cdot M x' \right\}\,.$$

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[5]As a matter of fact, the quantities λ and ϖ will be related by (5.2).

[6]Here above and in the sequel, we use the standard notation for the characteristic function of a set E, namely

$$\chi_E(\xi) \;=\; \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if } \xi \notin E. \end{cases}$$

Assume also that $\{|u|<1\}$ *possesses the inner ball property, i.e., suppose that*

$$\begin{aligned} &\textit{for any } \bar{x} \in \partial\{|u|<1\}, \textit{ there exists} \\ &\textit{an open ball } B \subseteq \{|u|<1\} \\ &\textit{which touches } \partial\{|u|<1\} \textit{ at } \bar{x}. \end{aligned} \tag{1.9}$$

Then, there exist a universal $\mathfrak{d}^\star > 0$ *and a function* $\sigma_0 : (0,1) \longrightarrow (0,1)$ *such that if* $\varepsilon \in (0, \sigma_0(\mathfrak{d}))$ *and* $\mathfrak{d} \in (0, \mathfrak{d}^\star)$*, then* Γ *cannot touch* $\{u_\varepsilon = 0\}$ *by below in* $B_{\mathfrak{d}\sqrt{\varepsilon}/\sqrt{\operatorname{tr} M}}$.

Theorems 1.1 and 1.2 have also the following consequence: if $\{u_\varepsilon = 0\}$ converges uniformly to a hypersurface, then this surface satisfies the zero mean curvature equation in the (standard) viscosity sense. We omit here the details on this, referring the interested reader to Theorem 2.1 of [14] (see also [12] for conditions under which the uniform convergence of level sets holds).

The proof of the results of this paper is deeply inspired by the geometric construction developed in the masterpiece [13] (see also [14] and [15] for related results). Also, Theorems 1.1 and 1.2 will be bridged together via some results in [7] and [12].

We organize this paper in the following way. In § 2, we will construct suitable barriers, in order to estimate the curvatures of the level sets. Barriers are like ships and solutions are like the land where the ships are going to dock. Inspired by the case $N = 1$, one suspects that such lands look like flat hills of slope ϖ. Thus, a first ship will be constructed with a protruding zero level set (as a little rostrum), in such a way the dock will occur on it. The second ship is a modification of the flat distance function: on the one hand, the distance function is expected to play a rôle, since it encodes curvature information; on the other hand, we need to bend the distance function a bit to get apart from the free boundary. In § 3, the ships will sail the sea to touch the land: barriers will be slided towards the solution to check the curvatures of its zero level set: a Comparison Principle of [6] will be employed for this. The proof of Theorem 1.1 will then follow at once, by a scaling argument presented in § 4. The proof of Theorem 1.2 is contained in § 5.

2. Useful barriers

Following the ideas in [13] and [14], we now construct some barriers in order to trap our solution. Roughly, the crucial idea, which goes back to De Giorgi, is that one-dimensional solutions are the ones which encode much information of the system. We will therefore modify the one-dimensional broken line to get suitable (one-dimensional or rotation) supersolutions. For other heuristic justification of a similar construction, see also § 5 in [14].

The first barrier we introduce is smooth but on the levels 0 and ± 1. This will confine touching points on these levels. Actually, a free boundary analysis will avoid touching points to occur at ± 1-levels, and this will localize the touching points on the zero level set of the solution.

Lemma 2.1. *Let κ, $\kappa^\star > 0$ be suitably small and define*

$$\varpi_\pm \,:=\, \sqrt{(\varpi \pm \kappa^\star)^2 \pm 4\kappa}$$

and

$$\mathfrak{a}_\pm \,:=\, \frac{\varpi_\pm - \sqrt{\varpi_\pm^2 \mp 4\kappa}}{2\kappa}\,.$$

Let

$$g(s) \,:=\, \begin{cases} \varpi_+ s - \kappa s^2 & \text{if } s \in [0,\, \mathfrak{a}_+], \\ \varpi_- s - \kappa s^2 & \text{if } s \in [\mathfrak{a}_-,\, 0), \\ \quad -1 & \text{if } s \in (-\infty,\, \mathfrak{a}_-). \end{cases}$$

Let[7] *also $y \in \mathbb{R}^N$, $l > 1$ and*

$$\Psi^{y,l}(x) \,:=\, g(|x-y| - l)\,,$$

for any $x \in B_{l+\mathfrak{a}_+}(y)$. Then, there exists a uniform constant $\bar c \in (0,1)$, so that, if $l \geq 1/(\bar c\kappa)$ and κ, $\kappa^\star \in (0, \bar c)$,

$$\Delta_p \Psi^{y,l}(x) \,\leq\, -\bar c \kappa \,<\, 0\,,$$

for any x so that $\Psi^{y,l}(x) \neq 0, \pm 1$.

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[7]Note that g is defined and continuous in $(-\infty,\, \mathfrak{a}_+]$, that

$$g(a_\pm) = \pm 1$$

and that g is smooth (with $g' > 0$) except on the level sets 0 and -1. Furthermore,

$$\begin{aligned} g'(\mathfrak{a}_-) &= \varpi - \kappa^\star < \\ &< \varpi < \\ &< \varpi + \kappa^\star = \\ &= g'(\mathfrak{a}_+)\,. \end{aligned}$$

Also,

$$\mathfrak{a}_\pm \,\sim\, \pm\frac{1}{\varpi}$$

for small positive κ and $\kappa^\star$. We will freely use these elementary observations in the sequel. Notice also that the rôle of κ and $\kappa^\star$ with respect to l is quite different: while $\kappa^\star$ is l-independent, κ will be taken of the order of $1/l$.

Proof. We take x in the interior of the domain of $\Psi^{y,l}$ and such that $\Psi^{y,l}(x) \neq 0, \pm 1$ and we use the short hand notation $t := |x-y| - l$. Since $0 < |g(t)| < 1$, we have that

$$\frac{\varpi}{2} \leq g'(t) \leq 2\varpi\,,$$

if κ and $\kappa^\star$ are small enough. Then, a direct computation of the p-Laplacian shows that

$$\Delta_p \Psi^{y,l}(x) = (g'(t))^{p-2}\Big((p-1)\, g''(t) + (N-1)\, g'(t)/|x-y|\Big)\,.$$

Also, if $\Psi^{y,l}(x) > -1$, then $|x-y| \geq l/2$ by construction. Thus,

$$\Delta_p \Psi^{y,l}(x) \leq (g'(t))^{p-2}\left(-2\kappa\,(p-1) + \frac{4\varpi\,(N-1)}{l}\right),$$

from which the desired result follows. □

Lemma 2.2. *Let Ω be an open domain, let $u \in W^{1,p}(\Omega) \cap C(\mathbb{R}^N)$ satisfy (1.2), (1.3), (1.4) and (1.6) and let $\Psi^{y,l}$ as in Lemma 2.1. Let us assume that $\Psi^{y,l} \geq u$ in their common domain of definition. If $x^\star \in \Omega$ is so that $\Psi^{y,l}(x^\star) = u(x^\star)$, then either $x^\star$ is in the interior of $\{u = -1\}$ or $u(x^\star) = 0$.*

Proof. Assume that $u(x^\star) \neq 0$. If also $|u(x^\star)| < 1$, then $\Delta_p \Psi^{y,l} < 0 = \Delta_p u$ and $\nabla \Psi^{y,l} \neq 0$ in a neighborhood of $x^\star$ (by Lemma 2.1), and a contradiction then follows from the Strong Comparison Principle (see, e.g., Theorem 1.4 in [6] or Theorem 3.2 in [14]). Thus, $|u(x^\star)| = 1$. If $x^\star$ is in the interior of $\{u = 1\}$, then $u \equiv 1$ in a neighborhood U of $x^\star$; thus, since $\{\Psi^{y,l} = 1\}$ is an $(N-1)$-dimensional sphere, it would exist $\hat{x} \in U$ so that $1 = u(\hat{x}) > \Psi^{y,l}(\hat{x})$, contradicting our hypothesis. Therefore, $x^\star$ is either in the interior of $\{u = -1\}$, as claimed, or $x^\star$ lies on the free boundary $\partial\{u = \pm 1\}$. We exclude this possibility by arguing as follows. First, we exclude that $x^\star \in \partial\{u = 1\}$. For this, we argue by contradiction and we assume that $x^\star \in \partial\{u = 1\}$. Then, from the free boundary growth (1.4), if $\nu_0 \in \mathrm{S}^{N-1}$ is the interior normal of $\{\Psi^{y,l} = 1\}$ at $x^\star$ (note that ν_0 points towards $\{u < 1\}$, since $u \leq \Psi^{y,l}$), we have that

$$u(x^\star + t_j \nu_0) \geq 1 - \varpi t_j - \mathfrak{f}_+(t_j)\,,$$

for an infinitesimal positive sequence $t_j \searrow 0^+$. On the other hand, recalling the footnote on page 6, the fact that $\Psi^{y,l}(x^\star) = 1$ gives that

$$\frac{\partial \Psi^{y,l}}{\partial \nu_0}(x^\star) = -(\varpi + \kappa^\star)\,,$$

thence

$$\Psi^{y,l}(x^\star + t_j\nu_0) \le 1 - \left(\varpi + \frac{\kappa^\star}{2}\right) t_j \,.$$

The fact that $\Psi^{y,l} \ge u$ and the above estimates thus imply that

$$1 - \varpi t_j - \mathfrak{f}_+(t_j) \le 1 - \left(\varpi + \frac{\kappa^\star}{2}\right) t_j \,,$$

which easily yields a contradiction with (1.5). This shows that $x^\star$ does not lie on the free boundary $\partial\{u = 1\}$.

Thus, to complete the proof of the desired claim, we need to exclude that $x^\star \in \partial\{u = -1\}$. For this, assume, by contradiction, that $x^\star \in \partial\{u = -1\}$. Then, the fact that $\Psi^{y,l} \ge u$ implies that also $x^\star \in \partial\{\Psi^{y,l} = -1\}$. Thus, let ν_0 be the inner normal of $\partial\{\Psi^{y,l} = -1\}$ at $x^\star$. Since $\Psi^{y,l} \ge u$, it follows that ν_0 points towards $\{u = -1\}$. Thus, the free boundary growth (1.6) yields that

$$u(x^\star - t_j\nu_0) \ge -1 + \varpi t_j - \mathfrak{f}_-(t_j)$$

for $t_j \searrow 0^+$. Also, recalling the footnote on page 6 agin, we infer from the fact that $x^\star \in \partial\{\Psi^{y,l} = -1\}$ that

$$\Psi^{y,l}(x^\star - t_j\nu_0) \le -1 + \left(\varpi - \frac{\kappa^\star}{2}\right) t_j \,.$$

Finally, the fact that $\Psi^{y,l} \ge u$, together with the above estimates, gives that

$$-1 + \varpi t_j - \mathfrak{f}_-(t_j) \le -1 + \left(\varpi - \frac{\kappa^\star}{2}\right) t_j \,,$$

contradicting (1.7) and thus ending the proof of the desired result. □

We now introduce another barrier, in order to deal with the distance function and control the curvature of the level sets of the solution.

Lemma 2.3. *Let $0 < \varepsilon \le \sigma \le \delta < 1$, $\xi \in \mathbb{R}^{N-1}$, $M \in \mathrm{Mat}((N-1)\times(N-1))$. Let Γ be the hypersurface defined as*

$$\Gamma := \{\, x_N = \frac{\varepsilon}{2} x' \cdot M x' + \sigma\xi \cdot x' \,\} \cap \{\, |x'| < \frac{\sigma}{\varepsilon} \,\}$$

and assume that

$$\mathrm{tr}\, M \ge \delta \,, \qquad \|M\| \le \frac{3}{\delta} \,, \qquad |\xi| \le \frac{3}{\delta} \,.$$

We define $d_\Gamma(x)$ as the signed distance from x to Γ, with the assumption that d_Γ is positive[8] *above Γ. Let also $c_1 > 0$ be a suitably small constant.*

[8]Of course, here "above" is intended with respect to the e_N-direction. For some properties of the distance function, see [11]. As usual, $\{e_1, \ldots, e_N\}$ denotes here the standard base of $\mathbb{R}^N$.

Let us define

$$\mathfrak{b}_\pm := \frac{-\varpi + \sqrt{\varpi^2 \pm 4c_1\varepsilon\delta}}{2c_1\varepsilon\delta}$$

and

$$\breve{g}(s) := \varpi s + c_1\varepsilon\delta s^2 , \qquad \textit{for any } s \in [\mathfrak{b}_-,\, \mathfrak{b}_+].$$

Let us also define $\breve{g}(s) = -1$ *for any* $s < \mathfrak{b}_-$. *Then*[9], *there exists a functions* $\sigma_0 : (0,+\infty) \longrightarrow (0,1)$ *such that, if* $\varepsilon \le \sigma \le \sigma_0(\delta)$, *then*

$$\Delta_p\Big(\breve{g}(d_\Gamma(x))\Big) < -\tilde{c}c_1\varepsilon\delta < 0$$

at any point x *for which* $d_\Gamma(x) \in (\mathfrak{b}_-,\, \mathfrak{b}_+)$, *for a suitable small positive constant* $\tilde{c}$.

Proof. If $d_\Gamma(x) \in (\mathfrak{b}_-,\, \mathfrak{b}_+)$,

$$|d_\Gamma(x)| \le \frac{2}{\varpi} . \tag{2.1}$$

Also, if $s := d_\Gamma(x)$,

$$2\varpi \ge \breve{g}'(s) \ge \frac{\varpi}{2} , \tag{2.2}$$

for small c_1, ε and δ.

In an appropriate system of coordinates we have that

$$D^2 d_\Gamma = \operatorname{diag}\left(\frac{-k_1}{1-d_\Gamma k_1}, \dots, \frac{-k_{N-1}}{1-d_\Gamma k_{N-1}}, 0\right) \in \operatorname{Mat}(N\times N) , \tag{2.3}$$

where the k_i's are the principal curvatures of Γ at the point where the distance is realized (see § 14.6 in [11] for further details). By construction,

$$|k_i| \le C_1(\delta)\varepsilon ,$$

for a suitable $C_1(\delta)$.

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[9] Note that $\breve{g}$ is continuous in $(-\infty, \mathfrak{b}_+]$ and smooth (with $\breve{g}' > 0$) in $(\mathfrak{b}_-,\, \mathfrak{b}_+)$. Also, $\mathfrak{b}_\pm \sim \pm 1/\varpi$ for small positive c_1, ε and δ. What is more, $\breve{g}(\mathfrak{b}_\pm) = \pm 1$ and $\breve{g}'(\mathfrak{b}_\pm) = \sqrt{\varpi^2 \pm 4c_1\varepsilon\delta}$. In particular,

$$\breve{g}'(\mathfrak{b}_+) > \varpi > \breve{g}'(\mathfrak{b}_-) .$$

A second order Taylor expansion shows that

$$\mathfrak{b}_+ \le \frac{1}{\varpi} - \frac{c_1\varepsilon\delta}{\varpi^3} + \text{const}\,\frac{(c_1\varepsilon\delta)^2}{\varpi^5} ,$$

thus

$$\mathfrak{b}_+ < \frac{1}{\varpi}$$

for small positive c_1, δ and ε. These elementary properties of $\breve{g}$ will be freely used in what follows.

We denote by P the paraboloid describing Γ, i.e.,

$$P(x') := \frac{\varepsilon}{2} x' \cdot M x' + \sigma \xi \cdot x' .$$

Note that

$$|\nabla P| \,\leq\, \mathrm{const}\, \frac{\sigma}{\delta} \tag{2.4}$$

is a small quantity. Therefore, by the mean curvature equation (see, for instance, equation (14.103) of [11]), it follows that

$$\begin{aligned} \sum_{i=1}^{N-1} k_i &= \sum_{i=1}^{N-1} \partial_i \left(\frac{\partial_i P}{\sqrt{1+|\nabla P|^2}} \right) \\ &= \frac{\Delta P}{\sqrt{1+|\nabla P|^2}} - \frac{(D^2 P\, \nabla P)\cdot \nabla P}{(1+|\nabla P|^2)^{3/2}} \\ &\geq \frac{1}{2} \Delta P - \mathrm{const}\, |\nabla P|^2 \| D^2 P\| \,. \end{aligned}$$

Thus, by using also (2.1), (2.3) and (2.4), we infer that

$$\begin{aligned} \Delta d_\Gamma &= \sum_{i=1}^{N-1} \frac{-k_i}{1 - d_\Gamma k_i} = \\ &= -\sum_{i=1}^{N-1} k_i - \sum_{i=1}^{N-1} \frac{d_\Gamma k_i^2}{1-d_\Gamma k_i} \leq \\ &\leq -\frac{1}{2} \Delta P + C_2(\delta) \Big(|\nabla P|^2 \|D^2 P\| + \varepsilon^2 \Big) \leq \\ &\leq -\frac{\varepsilon\delta}{2} + C_3(\delta)\,(\varepsilon\sigma^2 + \varepsilon^2)\,, \end{aligned}$$

for suitable $C_i(\delta)$. In particular,

$$\Delta d_\Gamma \,\leq\, -\frac{\varepsilon\delta}{4} \,.$$

Then, a direct computation on the p-Laplacian and (2.2) give that

$$\begin{aligned} & \Delta_p \Big(\breve{g}(d_\Gamma(x)) \Big) = \\ = \;& \Big(\breve{g}'(d_\Gamma(x)) \Big)^{p-2} \Big[(p-1)\, \breve{g}''(d_\Gamma(x)) + \breve{g}'(d_\Gamma(x)) \Delta d_\Gamma(x) \Big] \leq \\ \leq \;& \Big(\frac{\varpi}{2} \Big)^{p-2} \Big[2c_1 \varepsilon\delta\, (p-1) - \frac{\varpi\varepsilon\delta}{8} \Big] \,, \end{aligned}$$

from which the desired result follows by taking c_1 conveniently small.[10]

□

In analogy with Lemma 2.2 we point out the following result for the barrier $\breve{g} \circ d_\Gamma$ constructed here above. Though the proof is similar in spirit to the one of Lemma 2.2, we provide full details of it for the reader's convenience.

Lemma 2.4. *Let Ω be an open domain, let $u \in W^{1,p}(\Omega) \cap C(\mathbb{R}^N)$ satisfy (1.2), (1.3), (1.4) and (1.6) and let Γ and $\breve{g}$ be as in Lemma 2.3. Let us assume that $\breve{g} \circ d_\Gamma \geq u$ in their common domain of definition (which is supposed to be nonempty). Then, if $x^\star \in \Omega$ is so that $\breve{g}(d_\Gamma(x^\star)) = u(x^\star)$, then $x^\star$ lies in the interior of $\{u = -1\}$.*

Proof. Let us first observe that $x^\star$ cannot lie on the free boundary $\partial\{u = \pm 1\}$. First, we show that $x^\star \notin \partial\{u = -1\}$. We argue by contradiction, assuming that $x^\star \in \partial\{u = -1\}$. Then, since $\breve{g} \circ d_\Gamma \geq u$, we have that also $x^\star \in \partial\{\breve{g} \circ d_\Gamma = -1\}$. Thus, let ν_0 be the normal of $\partial\{\breve{g} \circ d_\Gamma = -1\}$ at $x^\star$ pointing towards $\{\breve{g} \circ d_\Gamma = -1\}$. The fact that $\breve{g} \circ d_\Gamma \geq u$ also implies that ν_0 points towards $\{u = -1\}$. Thus, the fact that $x^\star \in \partial\{u = -1\}$ and the free boundary growth (1.6) yield that

$$u(x_- - t_j \nu_0) \geq -1 + \varpi t_j - \mathfrak{f}_-(t_j)\,,$$

with $t_j \searrow 0^+$. Also, recalling the footnote on page 9, the fact that $x^\star \in \partial\{\breve{g} \circ d_\Gamma = -1\}$ gives that

$$\breve{g}\big(d_\Gamma(x^\star - t_j \nu_0)\big) \leq -1 + \sqrt{\varpi^2 - 2c_1\varepsilon\delta}\; t_j\,.$$

Using again that $\breve{g} \circ d_\Gamma \geq u$, one thus gets that

$$-1 + \varpi t_j - \mathfrak{f}_-(t_j) \leq -1 + \sqrt{\varpi^2 - 2c_1\varepsilon\delta}\; t_j\,.$$

The latter estimate and (1.7) lead to a contradiction, thus $x^\star \notin \partial\{u = -1\}$.

We now show that $x^\star \notin \partial\{u = 1\}$. To see this, let us argue by contradiction and let us assume that $x^\star \in \partial\{u = 1\}$. By construction, $x^\star$ also belongs to the $(N-1)$-dimensional surface

$$\Sigma = \left\{ x \in \mathbb{R}^N \;\middle|\; d_\Gamma(x) = \mathfrak{b}_+ \right\}.$$

Let $\nu_0 \in \mathrm{S}^{N-1}$ be the interior normal of Σ at $x^\star$. Note that ν_0 points towards $\{|u| < 1\}$, since $u \leq \breve{g} \circ d_\Gamma$. Then, by the free boundary

[10] The last passage in the proof also gives a good hint on how such barrier has been constructed: namely, the curvature of Γ, which is of order $-\varepsilon\delta$, is going to compensate the one of $\breve{g}$, which is of order $c_1\varepsilon\delta$.

growth (1.4),

$$u(x^\star + t_j\nu_0) \geq 1 - \varpi t_j - \mathfrak{f}_+(t_j)\,,$$

for an infinitesimal positive sequence t_j. Also, by construction, $u \leq \breve{g} \circ d_\Gamma$ and thus

$$\breve{g}\big(d_\Gamma(x^\star + t_j\nu_0)\big) \geq 1 - \varpi t_j - \mathfrak{f}_+(t_j)\,.$$

By the elementary properties described in the footnote on page 9, we also have that

$$\frac{\partial(\breve{g} \circ d_\Gamma)}{\partial\nu_0}(x^\star) \;=\; -\sqrt{\varpi^2 + 4c_1\varepsilon\delta}$$

and therefore

$$\breve{g}\big(d_\Gamma(x^\star + t_j\nu_0)\big) \leq 1 - \sqrt{\varpi^2 + 2c_1\varepsilon\delta}\; t_j$$

if j is large enough. These estimates give that

$$1 - \varpi t_j - \mathfrak{f}_+(t_j) \;\leq\; 1 - \sqrt{\varpi^2 + 2c_1\varepsilon\delta}\; t_j$$

which easily provides a contradiction with (1.5) for j large.

We thus have that $x^\star$ does not lie on the free boundary $\partial\{u = \pm 1\}$. Furthermore, $x^\star$ cannot lie in the interior of $\{u = 1\}$. To see this, let us assume, by contradiction, that $x^\star \in U \subset \{u = 1\}$, for a suitable open set U. Then, since Σ, as defined here above, is an $(N-1)$-dimensional surface, there would exist $x_0 \in U$ such that $\breve{g}(d_\Gamma(x_0)) < 1$. Therefore,

$$\breve{g}(d_\Gamma(x_0)) < 1 = u(x_0)\,,$$

in contradiction with our assumptions.

We have thus proved that either $x^\star$ is in the interior of $\{u = -1\}$ or $u(x^\star) \in (-1, 1)$. The latter possibility, however, cannot hold, due to the Strong Comparison Principle (see, e.g., Theorem 1.4 in [6] or Theorem 3.2 in [14]). $\square$

3. Sliding methods

We now use the barriers introduced here above in § 2 to deduce an estimate on the curvature of the paraboloids which may touch our solution. In particular, we will show that a zero mean curvature property is attained by the level sets of our solution, though in a weak viscosity sense. Next result, which, in our opinion, is interesting in itself, will play a crucial rôle in the proof of Theorem 1.1, which indeed will follow via a natural rescaling.

Theorem 3.1. *Let*

$$\Omega := \Big\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \;\Big|\; |x'| < l\,,\; |x_N| < l\Big\} \subset \mathbb{R}^N\,.$$

Let $l, \theta, \delta > 0$ *and* $M_1 \in \mathrm{Mat}((N-1)\times(N-1))$*. Let* u *be as in Theorem 1.1. Assume that* $u(x) < 0$ *for any* $x = (x', x_N) \in \Omega$ *so that*

$$x_N < \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l}\xi \cdot x'\,.$$

Then, there exist a universal constant $\delta_0 > 0$ *and a function* $\sigma : (0,1) \longrightarrow (0,1)$*, so that, if*

$$\delta \in (0, \delta_0]\,, \quad \delta \le \theta\,, \quad \frac{\theta}{l} \in \Big(0, \sigma(\delta)\Big]\,, \quad \|M_1\| \le \frac{1}{\delta} \quad \textit{and} \quad |\xi| \le \frac{1}{\delta}\,,$$

then

$$\mathrm{tr} M_1 \le \delta\,.$$

Proof. The proof is similar to the one of Lemma 3.2.2 in [13] and Lemma 6.6 in [14]. However, due to the technicalities[11] involved, we provide full details for the reader's convenience. We will apply Lemmas 2.1 and 2.3 by choosing

$$\begin{aligned} \kappa &:= \frac{1}{\bar c\, l} \\ \text{and } \varepsilon &:= \frac{\theta}{2l^2}\,. \end{aligned} \tag{3.1}$$

Note that, by our assumptions, $l \ge \delta/\sigma(\delta)$ may and will be assumed to be a large quantity. In particular, we may and do assume κ and $\kappa^\star$ to be suitably small with respect to δ. Let also

$$l_+ := l/4 + \mathfrak{a}_+\,, \tag{3.2}$$

so that, by Lemma 2.1, the barrier $\Psi^{y,l/4}$ is defined in $B_{l_+}(y)$. Define also

$$\begin{aligned} \Gamma_1 \;:=\; \Big\{ & x = (x', x_N) \in \mathbb{R}^N \text{ s.t. } |x'| < l \\ & \text{and } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l}\xi \cdot x' \Big\}\,. \end{aligned}$$

Let us make some elementary observations upon the above paraboloid. First of all, by construction, u is negative below Γ_1 in $[-l, l]^N$. What is more, the principal curvatures of Γ_1 are bounded by $\mathrm{const}\,\sigma(\delta)/(l\delta)$: thus, if $\sigma(\delta)/\delta$ is sufficiently small, then, given any $\breve{x} \in \Gamma_1$, there exists

[11] In particular, several quantitative estimates here differ from similar ones in [13] and [14]. The main reason for such difference is that the smooth transitions considered in [13] and [14] lead to exponentially deacying barriers, while the barriers constructed here have "something like" a linear decay.

a ball of radius $l/4$ which touches Γ_1 from below at $\breve{x}$. Given $\breve{x} \in \Gamma_1$, let $\nu_{\breve{x}}$ be the normal direction of Γ_1 at $\breve{x}$ pointing downwards. Let

$$\mathcal{K} := \{|x'| \le l/4\} \cap \left\{ d_{\Gamma_1}(x) \in \left[-\frac{l}{8}, \mathfrak{a}_+ \right] \right\} . \tag{3.3}$$

We now claim that

$$u(x) \le g(d_{\Gamma_1}(x)) \tag{3.4}$$

for any $x \in \mathcal{K}$. To prove (3.4), first notice that, by construction, the zero level set of u is above Γ_1, hence above the hyperplane $\{x_N = -l/100\}$; thus, from (1.8),

$$u(x) < \Psi^{-(l/2)e_N, l/4}(x)$$

for any x in their common domain of definition, provided

$$\Psi^{-(l/2)e_N, l/4}(x) \neq -1 .$$

Then, for a given $\breve{x} \in \Gamma_1$ we define

$$x_0 = x_0(\breve{x}) := \breve{x} + (l/4)\nu_{\breve{x}} , \tag{3.5}$$

where we have denoted by $\nu_{\breve{x}}$ the normal direction of Γ_1 at $\breve{x}$ pointing downwards. In particular, from the above observation, $B_{l/4}(x_0)$ touches Γ_1 from below at $\breve{x}$.

We now slide the surface $\Psi^{-(l/2)e_N, l/4}$ in the direction of the vector

$$v = v(\breve{x}) := x_0 + (l/2)e_N ,$$

that is, we will consider the surface $\Psi^t := \Psi^{-(l/2)e_N + tv, l/4}$ for $t > 0$. We will show that

$$\Psi^t(\mathfrak{x}) > u(\mathfrak{x}) \tag{3.6}$$

for any $t \in [0, 1)$ and any $\mathfrak{x}$ in their common domain of definition, provided $\Psi^t(\mathfrak{x}) \neq -1$. Indeed, as a consequence of Lemma 2.1, we have that Ψ^t and u cannot touch each other on the free boundary $\partial\{u = -1\}$. In the light of this observation, we may take $t \in [0, 1)$ as the first time (if any) on which Ψ^t touches u at a point in $\{\Psi^t \neq -1\}$. Note that, since $t < 1$, we have that the domain of Ψ^t lies below Γ_1 and thence the domain of Ψ^t lies in $\{u < 0\}$. Therefore, u cannot be equal to Ψ^t and touching points between u and Ψ^t cannot occur on $\{\Psi^t = 0\}$. On the other hand, Lemma 2.2 says that touching points cannot occur anywhere else. This proves (3.6).

We are now in the position to complete the proof of (3.4), by arguing as follows.

We deduce from (3.6) that $\Psi^1(\mathfrak{x}) \geq u(\mathfrak{x})$ for any $\mathfrak{x}$ in the common domain of definition of Ψ^1 and u, that is, for any $\mathfrak{x} \in B_{l_+}(x_0)$. Take now any $x \in \mathcal{K}$ and let $\breve{x}$ realize $d_{\Gamma_1}(x)$. Let also x_0 be as in (3.5), so that $x \in B_{l_+}(x_0)$: then,

$$\begin{aligned} g\Big(d_{\Gamma_1}(x)\Big) &= g(|x-x_0|-l/4) \\ &= \Psi^{x_0,l/4}(x) \\ &= \Psi^1(x) \\ &\geq u(x)\,. \end{aligned}$$

This proves (3.4).

We now complete the proof of the desired result arguing by contradiction and supposing that $\mathrm{tr} M_1 > \delta$. We define

$$\begin{aligned} \Gamma_2 \;:=\; \Big\{ & x=(x',x_N)\in[-l,l]^N \;\text{ s.t.} \\ & x_N = \frac{\theta}{2l^2} x'\cdot M_1 x' + \frac{\theta}{l}\xi\cdot x' - \frac{\varepsilon\delta}{2(N-1)}|x'|^2 \Big\}, \end{aligned}$$

where ε has been introduced in (3.1). By Lemma 2.3, we get that $\breve{g}\circ d_{\Gamma_2}$ is strictly p-superharmonic.[12]

Furthermore, by the definitions of Γ_1 and Γ_2, if $|x'|=l/4$, then

$$d_{\Gamma_2}(x) \geq d_{\Gamma_1}(x) + c(\delta) \tag{3.7}$$

for a suitable $c(\delta)\in(0,1)$. We now take κ and $\kappa^\star$ in Lemma 2.1 to be positive and suitably[13] small (possibly in dependence of δ) in such a way that

$$|\varpi_\pm - \varpi| \leq \frac{c(\delta)\,\varpi^2}{4}\,. \tag{3.8}$$

For further use, recalling the footnote on page 6, we will also assume that

$$\mathfrak{a}_+ \geq \frac{1}{\varpi} - \frac{c(\delta)}{2}\,.$$

We now recall the footnote on page 9 and note that the latter assumption and (3.7) thus imply the following estimate: if $\hat{x}$ is so that $d_{\Gamma_1}(\hat{x}) \geq \mathfrak{a}_+$ and $|\hat{x}'| = l/4$, then

$$d_{\Gamma_2}(\hat{x}) \geq \mathfrak{a}_+ + c(\delta) \geq \frac{1}{\varpi} + \frac{c(\delta)}{2} > \mathfrak{b}_+\,, \tag{3.9}$$

[12] The quantities M, σ and ξ in Lemma 2.3 correspond here to $2M_1 - \delta/(N-1)$, $\theta/(2l)$ and 2ξ, respectively.

[13] Note that we required $l \sim 1/\kappa$, thus the fact that κ is small in dependence of δ is warranted by the fact that $1/l$ is small in dependence of δ: recall (3.1).

which will be of later use.

The choice in (3.8) implies that, if s is in the domain of g,

$$\begin{aligned} g(s) \;&\le\; \varpi s + \frac{c(\delta)\,\varpi^2\,|s|}{4} - \kappa s^2 \le \\ &\le\; \varpi s + \frac{c(\delta)\,\varpi}{2}\,. \end{aligned}$$

Thus, in the light of (3.7), we have that

$$\begin{aligned} g\big(d_{\Gamma_1}(x)\big) \le& \varpi d_{\Gamma_1}(x) + \frac{c(\delta)\,\varpi}{2} \le \\ \le& \varpi d_{\Gamma_2}(x) - \frac{c(\delta)\,\varpi}{2} \le \\ \le& \breve g\big(d_{\Gamma_2}(x)\big) - \frac{c(\delta)\,\varpi}{2} < \\ <& \breve g\big(d_{\Gamma_2}(x)\big)\,. \end{aligned} \tag{3.10}$$

for any x for which the above functions are defined, so that $|x'| = l/4$.

We point out that if $x \in \mathcal{K}$, then, by (3.3), we have that

$$d_{\Gamma_1}(x) \le \mathfrak{a}_+$$

and so $d_{\Gamma_1}(x)$ is in the domain of g. ¿From this observation, (3.4) and (3.10), we gather that

$$u(x) \,<\, \breve g(d_{\Gamma_2}(x))\,, \tag{3.11}$$

for any $x \in \mathcal{K}$ so that $|x'| = l/4$ and $d_{\Gamma_2}(x)$ is in the domain of $\breve g$.

With these estimates, we are now ready to deduce the contradiction that will finish the proof of the desired result. To this end, we define[14]

$$\mathcal{K}^\star \,:=\, \{|x'| \le l/4\} \cap \left\{ d_{\Gamma_1}(x) \ge -\frac{l}{8} \right\}$$

and (recalling Lemma 2.4) we slide $\breve g \circ d_{\Gamma_2}$ from $-\infty$ in the e_N-direction until we touch[15] u in the closed domain $\mathcal{K}^\star$ at some point, say $x^\star$, with $u(x^\star) > -1$. For this, we consider, for $t \in \mathbb{R}$,

$$g^t(x) := \breve g\Big(d_{\Gamma_2}(x - te_N)\Big)$$

and we increase t from $-\infty$. By Lemma 2.4, we have that the first touching points between g^t and u at a level greater than -1 must

[14]Note that, by (3.3), $\mathcal{K}^\star \supset \mathcal{K}$.

[15]This touching must indeed occur sooner or later, since $u(0) = 0$, due to (1.1).

occur on $\partial\mathcal{K}^\star$. Note that, by definition, $\partial\mathcal{K}^\star$ is composed by two parts: the "side", given by the cylinder $|x'| = l/4$ and the "bottom", given by

$$\left\{ d_{\Gamma_1}(x) = -l/8 \right\}.$$

We now show that no first touching points between g^t and u at a level greater than -1 may occur on $\partial\mathcal{K}^\star$ and this will give the desired contradiction.

By the definition of Γ_1, one sees that the bottom of $\mathcal{K}^\star$ lies in $\{x_N \leq -l/9\}$, thus, by (1.8), $u = -1$ on the bottom of $\mathcal{K}^\star$. This excludes that first touching points between g^t and u at a level greater than -1 may occur on the bottom of $\partial\mathcal{K}^\star$.

But these touching points cannot occur on the side of $\partial\mathcal{K}^\star$ either, thanks to the following argument. We assume, by contradiction that there exists a first touching point $x^\star$ between g^t and u lying on the side of $\partial\mathcal{K}^\star$ (that is, $|(x^\star)'| = l/4$). There are two cases: either $d_{\Gamma_1}(x^\star) \leq \mathfrak{a}_+$ or the converse. Let us first assume that $d_{\Gamma_1}(x^\star) \leq \mathfrak{a}_+$. Then, by (3.3), $x^\star \in \mathcal{K}$. Observe also that the fact that $u(0) = 0$ implies $t \leq 0$; thus, an elementary observation gives that

$$d_{\Gamma_2}(x^\star - te_N) \geq d_{\Gamma_2}(x^\star).$$

But then, since $\breve{g}$ is non-decreasing, we deduce from the fact that $x^\star \in \mathcal{K}$ and (3.11) that

$$\breve{g}(d_{\Gamma_2}(x^\star - te_N)) = g^t(x^\star) = u(x^\star) < \breve{g}(d_{\Gamma_2}(x^\star)) \leq \breve{g}(d_{\Gamma_2}(x^\star - te_N)).$$

This contradiction shows that only the second case may occur, that is, $d_{\Gamma_1}(x^\star) > \mathfrak{a}_+$. But even this last case cannot hold, since, from (3.9), we would get that $x^\star$ is strictly above the domain of $\breve{g} \circ d_{\Gamma_2}$ and thus, since $t \leq 0$, strictly above the domain of g^t.

This contradiction concludes the proof of Theorem 3.1. □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 can be now completed by arguing as follows.

We will apply Theorem 3.1 by making use of the following choice of parameters:

$$l := \frac{\mathfrak{d}}{\sqrt{\varepsilon \operatorname{tr} M}}, \quad \delta := \theta := \mathfrak{d}^2, \quad M_1 := \frac{1}{\operatorname{tr} M} M, \quad \xi := 0.$$

If, by contradiction, the claim of Theorem 1.1 were false, by scaling back and using the above parameters, we would have that Γ_1 touches the zero level set of u by below , where

$$\Gamma_1 = \left\{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \text{ s.t. } x_N = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\} .$$

By Theorem 3.1, we gather that $1 > \delta \geq \operatorname{tr} M_1 = 1$, which is the contradiction that proves Theorem 1.1.

5. Proof of Theorem 1.2

The proof of Theorem 1.2 will be accomplished once we show that either the function u in Theorem 1.2 or the function $\tilde{u} := -u$ satisfies the assumptions[16] of Theorem 1.1. Namely, we have to prove the continuity of u, (1.3), (1.4), (1.6) and (1.8) for either u or $\tilde{u}$.

First of all, u is uniformly continuous by the celebrated result in [10]. More precisely, it is uniformly Lipschitz continuous: see Theorem 2.1 in [12].

The proof of (1.3) is standard and we omit it.

In order to proof (1.4) we argue as follows. First observe that the continuity of u implies that the free boundary $\partial\{u = -1\}$ is uniformly separated from the free boundary $\partial\{u = 1\}$. Therefore, by elementary observations (see, e.g., Lemma 3.3 in [12]), there exists a universal $\rho > 0$ so that, if

$$x_\pm \in \partial\{u = \pm 1\} = \partial\{\tilde{u} = \mp 1\} ,$$

then $1 \mp u = 1 \pm \tilde{u}$ is an absolute minimizer for the functional

$$\widetilde{\mathcal{F}}(w) = \int_{B_\rho(x_\pm)} |\nabla w(x)|^p + \lambda \chi_{(0,+\infty)}\big(w(x)\big)\, dx .$$

Then, by Theorem 7.1 in [7], if $x \in \mathbb{R}^N$ and $r > 0$ are so that $B_r(x) \subset B_\rho(x_\pm)$ and $B_r(x) \cap \partial\{\pm u < 1\} \neq \emptyset$, we have that

$$\sup_{B_r(x)} |\nabla u| \leq \varpi + C r^\alpha , \tag{5.1}$$

for suitable universal positive C and α and

$$\varpi := \frac{\lambda}{(p-1)^{1/p}} . \tag{5.2}$$

In particular, if there is a ball $B \subseteq \{|u| < 1\}$ touching $\partial\{\pm u = 1\}$ at $x_\pm$, we define ν_0 as the interior normal of B at $x^\mp$, we fix small positive

[16]Note that $\tilde{u}$ also is a Class A minimizer of $\mathcal{F}$.

$s < t$ and η and take $x := x_\pm + t\nu_0$, $y := x_\pm + s\nu_0$ and $r := t + \eta$. With this choice, both y and $x_\pm$ are in $B_r(x)$, so (5.1) yields that

$$|u(x) - u(y)| \le (\varpi + Cr^\alpha)\,|x - y| = (\varpi + C(t+\eta)^\alpha)\,(t - s)\,.$$

Sending s and η to zero, the continuity of u implies that

$$1 \mp u(x_\pm + t\nu_0) \le |u(x_\pm + t\nu_0) \mp 1| \le \varpi t + Ct^{1+\alpha}\,,$$

thus

$$\begin{aligned} & u(x_+ + t\nu_0) \ge 1 - \varpi t - Ct^{1+\alpha} \\ \text{and} \qquad & \tilde{u}(x_- + t\nu_0) \ge 1 - \varpi t - Ct^{1+\alpha}\,. \end{aligned}$$

This proves (1.4) for both u and $\tilde{u}$.

We now prove (1.6) by arguing in the following way. Let us assume that a ball lying in $\{\pm u = -1\}$ touches $\partial\{\pm u = -1\}$ at $x^\mp$. Then, by the inner ball property in (1.9), another ball touches $\partial\{\pm u = -1\}$ from the other side, that is, the free boundary $\partial\{\pm u = -1\}$ is trapped between two balls which are tangent at $x^\mp$. Then, by Theorem 9.1 of [7], $\partial\{\pm u = -1\}$ is smooth (and, in fact, analytic) near $x^\mp$ (see also Theorems 8.1, 8.2 and 8.4 in [1] for the case $p = 2$). Then, (1.6) follows from Theorems 5.5-5.6 in [7] (see also Theorem 4.8 for the case $p = 2$).

The proof of (1.8) is by contradiction. Assume that $\{u = 0\} \cap \{|x'| \le l\}$ is above the hyperplane $\{x_N = -l/100\}$ and that there exists $x \in \{|u| < 1\}$ so that $|x'| \le l/2$ and $x_N \le -l/10$. Then, by the Density Estimate in Theorem 2.2 of [12] (see also [5] for analogous results in the case $p = 2$),

$$\mathfrak{L}^N\Big(B_C(x) \cap \{u = 1\}\Big) \;>\; 0$$

and

$$\mathfrak{L}^N\Big(B_C(x) \cap \{u = -1\}\Big) \;>\; 0\,,$$

where $\mathfrak{L}^N$ is the Lebesgue measure and C is a suitable positive constant. In particular, $B_C(x)$ contains points $x_\pm$ so that $u(x_\pm) = \pm 1$. Thus, by the continuity of u, there must be a point x_* on the segment joining x_+ and x_- so that $u(x_*) = 0$. By construction, $x_* \in B_C(x)$, thus

$$|x'_*| \le |x'| + C \le \frac{l}{2} + C < l$$

and

$$x_{*,N} \le x_N + C \le -\frac{l}{10} + C < -\frac{l}{100}\,,$$

if l is large enough. This contradicts the assumption that $\{u = 0\} \cap \{|x'| \le l\}$ is above the hyperplane $\{x_N = -l/100\}$. Thus $|u(x)| = 1$

for any x so that $|x'| \leq l/2$ and $x_N \leq -l/8$. This proves that (1.8) is fulfilled by either u or $\tilde{u}$.

Then, Theorem 1.2 follows by applying Theorem 1.1 to either u or $\tilde{u}$ and by noticing that $\{\tilde{u}_\varepsilon = 0\} = \{u_\varepsilon = 0\}$.

References

[1] H. W. Alt, L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. 325 (1981), 105–144.
[2] H. W. Alt, L. A. Caffarelli, A. Friedman, *A free boundary problem for quasilinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), no. 1, 1–44.
[3] H. W. Alt, L. A. Caffarelli, A. Friedman, *Jets with two fluids. I. One free boundary*, Indiana Univ. Math. J. 33 (1984), no. 2, 213–247.
[4] G. Bouchitté, *Singular perturbations of variational problems arising from a two-phase transition model*, Appl. Math. Optim. 21 (1990), no. 3, 289–314.
[5] L. A. Caffarelli, A. Córdoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. 48 (1995), no. 1, 1–12.
[6] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré. Analyse non linéaire, 15(4), 1998, pp. 493 – 516.
[7] D. Danielli, A. Petrosyan, *A minimum problem with free boundary for a degenerate quasilinear operator*, to appear in Calc. Var. Partial Differential Equations.
[8] D. Danielli, A. Petrosyan, H. Shahgholian, *A singular perturbation problem for the p-Laplace operator*, Indiana Univ. Math. J. 52 (2003), no. 2, 457-476.
[9] E. De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 131–188, Pitagora, Bologna, 1979.
[10] M. Giaquinta, E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. 148 (1982), 31–46.
[11] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001.
[12] A. Petrosyan, E. Valdinoci, *Geometric properties of Bernoulli-type minimizers*, to appear in Interfaces Free Bound.
[13] V. O. Savin, *Phase Transitions: Regularity of Flat Level Sets*, Ph.D. thesis at the University of Texas at Austin, 2003.
[14] B. Sciunzi, E. Valdinoci, *Mean curvature properties for p-Laplace phase transitions*, to appear in J. Eur. Math. Soc. (JEMS).
[15] E. Valdinoci, B. Sciunzi, V. O. Savin, *Flat level set regularity of p-Laplace phase transitions*, preprint.

Publications [7] and [12] are available on-line at
`http://www.math.purdue.edu/~arshak/publications.html`

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