IRREDUCIBLE DYNAMICS OF QUANTUM SYSTEMS ASSOCIATED WITH LÉVY PROCESSES

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Abstract. Quantum systems described by the Schrödinger operators $H = \sum_{j=1}^{N} \Phi(p_j) + W(x_1, \ldots, x_N)$, $p_j = -\imath\nabla_j$, $x_j \in \mathbb{R}^\nu$ with Φ being continuous functions such that the pseudo-differential operators $\Phi(p_j)$ generate Lévy processes, are considered. It is proven that the linear span of the operators $\alpha_{t_1}(F_1)\cdots\alpha_{t_n}(F_n)$ is dense in the algebra of all observables in the σ-strong and hence in the σ-weak and strong topologies. Here $\alpha_t(F) = \exp(\imath tH)F\exp(-\imath tH)$ are time automorphisms and F's are taken from families of multiplication operators obeying conditions described in the paper. This result implies that a linear functional continuous in either of these topologies is fully determined by its values on such products. In the case of KMS states this yields a representation of such states in terms of path integrals.

1. Introduction

Gibbs (equilibrium) states of infinite-particle quantum systems described by unbounded Schrödinger operators cannot be constructed directly as Kubo-Martin-Schwinger (KMS) states. The only way here is to use path integrals to represent local Gibbs states (describing finite subsystems) in terms of probability measures. Then the Gibbs states of the whole system are constructed as probability measures with the help of the DLR technique known in classical statistical mechanics, see [5]. For models like quantum crystals, the corresponding approach is well elaborated, see e.g., [1]. In the present article we make first steps in developing a similar technique for a wider class of quantum systems by proving the basic statement for representing local Gibbs states in terms of probability measures.

We consider a system of interacting quantum particles with the kinetic energy of a particle being a pseudo-differential operator $\Phi(p)$, $p = -\imath\nabla$. Such operators appear if one 'quantizes' the relativistic expression $[p^2c^2 + m^2c^4]^{1/2} - mc^2$ and were used in particular in studying problems of stability of matter, see [12, 13, 14] and the references therein. At the same time, such Schrödinger operators generate stochastic processes, see [4, 8], which makes them interesting objects of their own. Recently operators of this kind have been thoroughly investigated [2, 10]. The results obtained open a possibility to study dynamics of the corresponding systems by passing to imaginary values of time and employing properties of the processes. It was realized in [7, 9], where a path integral representation of the semi-group $\exp(-tH)$, $t > 0$, H being a relativistic Schrödinger operator, was constructed. This representation allows one to describe 'imaginary-time' evolution of wave functions

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$\psi_t = \exp(-tH)\psi_0$ by mean of path integrals. A more important task here is to construct an integral representation of Gibbs states

$$A \mapsto \mathrm{trace}[A\exp(-\beta H)]/\mathrm{trace}[A\exp(-\beta H)], \quad \beta > 0. \tag{1.1}$$

By virtue of such representations, they are applicable to multiplication operators only and the question to which extent they determine the corresponding states on the algebras of all observables remains open. In this paper we study this question for a class of models, which includes the one with the relativistic kinetic energy.

In what follows, we consider the Schrödinger operators having the form

$$H = \sum_{j=1}^{N} \Phi(p_j) + W(x_1, \ldots, x_N), \quad x_j \in \mathbb{R}^\nu, \quad j = 1, \ldots, N, \tag{1.2}$$

where one of the possible choices of Φ is the above mentioned $\Phi(p_j) = [-\hbar^2 c^2 \Delta_j + m^2c^4]^{1/2} - mc^2$, which is a quantized version of the kinetic energy of a free relativistic particle of mass m and momentum $p_j \in \mathbb{R}^\nu$. Here c is the speed of light and 'the rest energy term' mc^2 is subtracted in order to shift the beginning of the spectrum of $\Phi(p_j)$ to the origin. For Φ obeying certain conditions (see the next section), the expression (1.2) defines a pseudo-differential operator in the physical Hilbert space $L^2(\mathbb{R}^{\nu N})$. Along with the mentioned example of Φ one considers also its zero-mass version $\Phi(p_j) = c|p_j|$, or generalizations like $\Phi(p_j) = |p_j|^\alpha$, $\alpha \in (0, 2]$. For a non-relativistic particle, one has $\Phi(p_j) = |p_j|^2/2m = -\hbar^2\Delta_j/2m$. Schrödinger operators of the type of (1.2) with such Φ generate stochastic processes, see [4]. In particular, the pseudo-differential operator with symbol Φ generates a Lévy process.

Assuming that Φ belongs to a certain class and that the operator (1.2) is essentially self-adjoint, we prove that the linear span of the products

$$\begin{aligned} &\alpha_{t_1}(F_1)\cdots\alpha_{t_n}(F_n), \quad n \in \mathbb{N}, \quad t_1, \ldots, t_n \in \mathbb{R}, \\ &\alpha_t(F) = \exp((\imath/\hbar)tH)F\exp(-(\imath/\hbar)tH), \end{aligned} \tag{1.3}$$

with all possible choices of bounded multiplication operators $F_1, \ldots, F_n$ from families satisfying certain conditions, is dense in the whole algebra in the σ-strong and hence in the σ-weak and strong topologies. Thereby, if a state ω is continuous in either of these topologies, its values on the whole algebra may be obtained from the values on the products (1.3). Suppose now that the state is such that these values as functions of the time variables in a unique way are continued to imaginary $t_1, \ldots, t_n$. Then the continuations can be written as path integrals, which thus uniquely determine the state. As is well known, the state (1.1) is continuous in the σ-weak topology and obeys the KMS condition, that yields a possibility of analytic continuation to imaginary values of the time variables and hence by our result to represent this state by means of path integrals.

We prove the above result with the help of the von Neumann density theorem by showing that the commutant of the set $\{\alpha_t(F) \mid t \in \mathbb{R},\ F - \text{multiplication operator}\}$ consists of multiplication operators by complex numbers only. Such a property is called *irreducibility* (see [3]), which is reflected in the title of the paper.

In Section 2, we introduce all necessities and give a precise formulation of the result in the form of Theorem 2.2. Section 3 is devoted to the study of irreducible C^*-dynamical systems, which in Section 4 is used to prove Theorem 2.2. The proof is performed in two steps. First we show that the free dynamics (i.e., generated by the kinetic energy only) is irreducible. Here we employ the properties of the functions Φ. Thereby, we extend this result to the dynamics defined by the whole Schrödinger operator by making use of the Trotter-Kato formula. Finally, in Section 5 we prove (Theorem 5.2) that Theorem 2.2 and the KMS property of the state (1.1) yield a representation of this state by means of path integrals.

2. Setup and Main Result

To simplify notations we use the units in which $c = \hbar = 1$. Given $\nu, N \in \mathbb{N}$, let $\mathfrak{C}$ be the C^*-algebra of all bounded linear operators on the complex Hilbert space $L^2(\mathbb{R}^{\nu N})$. Along with the norm topology on $\mathfrak{C}$, we will use also the σ-strong, σ-weak, and strong topologies, the definition of which is standard (see e.g., page 65 of [3] or page 19 of [16]). Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{C}$ consisting of multiplication operators by essentially bounded complex valued functions. We denote its elements and the corresponding elements of $L^\infty(\mathbb{R}^{\nu N})$ by the same letters. One observes that $\mathfrak{M}$ is a maximal commutative $*$-subalgebra of $\mathfrak{C}$. Similarly, the algebra of bounded continuous complex valued functions $C_{\rm b}(\mathbb{R}^{\nu N})$ can be associated with the subalgebra $\mathfrak{M}_{\rm cont} \subset \mathfrak{M}$.

We study the system of quantum particles described by the Schrödinger operator (1.2) with the function $\Phi : \mathbb{R}^\nu \to \mathbb{R}$ obeying the following conditions. The first one is

$$\int_{\mathbb{R}^\nu} \exp(-t\Phi(p)){\rm d}p < \infty, \quad \text{for all} \ \ t > 0. \tag{2.1}$$

Furthermore, we assume that Φ is a continuous negative definite function. The latter means that $\Phi(0) = 0$ and for any $t > 0$, the function $p \mapsto \exp(-t\Phi(p))$ is positive definite in the Bochner sense, i.e., for any $n \in \mathbb{N}$ and any choice of $p_1, \dots, p_n \in \mathbb{R}^\nu$, the matrix $A = (a_{ij})_{n\times n}$ with $a_{ij} = \exp(-t\Phi(p_i - p_j))$ is positive definite. These assumptions imply that Φ possesses the Lévy-Khinchine representation (see [4] and page 31 of [10])

$$\Phi(p) = \Psi(p) + \int_{\mathbb{R}^\nu} (1 - \cos(q,p)) \cdot \left[(1 + |q|^2)/|q|^2\right] \sigma({\rm d}q), \tag{2.2}$$

where Ψ is a symmetric non-negative definite quadratic form on $\mathbb{R}^\nu$, $(\cdot,\cdot)$ stands for the Euclidean scalar product in $\mathbb{R}^\nu$, and σ is a finite Borel measure such that $\sigma(\{0\}) = 0$. Each of the functions $\Phi(p) = \sqrt{|p|^2 + m^2} - m$, $\Phi(p) = |p|^\alpha$, $\alpha \in (0, 2]$ obeys the above assumptions. The pseudo-differential operator $\Phi(-\imath\nabla)$ with such a symbol generates a Lévy process with values in $\mathbb{R}^\nu$, see [10]. In the sequel, we suppose that both operators H and W are essentially self-adjoint on a joint domain, which contains the space of Schwartz test functions $\mathcal{S}(\mathbb{R}^{\nu N})$.

Definition 2.1. A family $\mathfrak{F} \subset \mathfrak{M}$ is called complete if it contains the identity operator I and the minimal subalgebra of $\mathfrak{C}$ which contains $\mathfrak{F}$ (i.e., which is generated by $\mathfrak{F}$) is strongly dense in $\mathfrak{M}$.

For $\mathfrak{F} \subset \mathfrak{C}$, by $\mathfrak{F}'$ we denote its commutant – the set of elements of $\mathfrak{C}$ which commute with all elements of $\mathfrak{F}$. By von Neumann's bicommutant and density theorems (see pages 72-74 of [3]), we have for a complete $\mathfrak{F} \subset \mathfrak{M}$, that $\mathfrak{F}' = \mathfrak{F}'' = \mathfrak{M}$.

The main result of the paper is contained in the following statement.

Theorem 2.2. *For the Schrödinger operator H (1.2) with Φ obeying the above conditions, let $\alpha_t(A) = \exp(\imath tH)A\exp(-\imath tH)$, $t \in \mathbb{R}$ be time automorphisms, which define the dynamics of the system. Let also $\mathfrak{F} \subset \mathfrak{M}$ be complete. Then the linear* *span of the operators* $\alpha_{t_1}(F_1)\cdots\alpha_{t_n}(F_n)$ *with all possible choices* $n\in\mathbb{N}$, $t_1,\ldots,t_n\in\mathbb{R}$, *and* $F_1,\ldots,F_n\in\mathfrak{F}$ *is* σ*-strongly and hence* σ*-weakly and strongly dense in* $\mathfrak{C}$.

Corollary 2.3. *Let* ω *be a linear functional on* $\mathfrak{C}$*, continuous in either of the topologies mentioned in Theorem 2.2. Then its values on the whole algebra* $\mathfrak{C}$ *are fully determined by the values on the products (1.3) with the operators* $F_1,\ldots,F_n$ *belonging to a complete family* $\mathfrak{F}$.

To obtain conditions for families $\mathfrak{F}\subset\mathfrak{M}$ to be complete we use the following obvious generalization of Theorem 20 of [15], page 28.

Proposition 2.4. *Let* $\mathbb{E}$ *be a nonempty set and* $\mathbb{V}$ *be the complex vector space of functions* $F:\mathbb{E}\to\mathbb{C}$*, including constant functions. Suppose that* $\mathbb{V}$ *is closed under both uniform convergence and point-wise convergence of monotone uniformly bounded sequences of nonnegative functions (such that* $F:\mathbb{E}\to[0,+\infty)$*; these functions define the order on* $\mathbb{V}$*). Let a subset* $\mathfrak{V}\subset\mathbb{V}$ *be self-adjoint and closed under multiplication; let also* $\mathcal{B}(\mathfrak{V})$ *be the* σ*-algebra of subsets of* $\mathbb{E}$ *generated by* $\mathfrak{V}$*. Then the space* $\mathbb{V}$ *contains all bounded* $\mathcal{B}(\mathfrak{V})$*-measurable functions.*

Two Borel subsets of $\mathbb{R}^{\nu N}$ are said to be equivalent if their symmetric difference is of zero Lebesgue measure. Let $\mathcal{B}$ be the corresponding factor σ-algebra. For a family $\mathfrak{F}\subset\mathfrak{M}$, we say that $\mathfrak{F}$ generates $\mathcal{B}$ if $\mathcal{B}$ is the smallest σ-algebra with respect to which the corresponding elements of $L^\infty(\mathbb{R}^{\nu N})$ are measurable. Since a monotone bounded net of positive elements of $\mathfrak{M}$ converges to its least upper bound in the σ-strong topology (see Lemma 2.4.19 in [3], page 76), the above statement has the following

Corollary 2.5. *If a subset of* $L^\infty(\mathbb{R}^{\nu N})$ *contains almost everywhere constant elements and generates* $\mathcal{B}$*, then the set* $\mathfrak{F}$ *of the corresponding multiplication operators is complete.*

Proof. By Proposition 2.4, the $*$-algebra $\mathfrak{M}(\mathfrak{F})$ generated by $\mathfrak{F}$ is σ-strongly closed in $\mathfrak{M}$. Then by the von Neumann density theorem, $\mathfrak{M}$ is the von Neumann algebra for $\mathfrak{M}(\mathfrak{F})$. Then $\mathfrak{F}''=\mathfrak{M}(\mathfrak{F})''=\mathfrak{M}$. Furthermore, $\mathfrak{F}'=\mathfrak{F}'''$, hence $\mathfrak{F}'=\mathfrak{M}'=\mathfrak{M}$. □

We say that a family $\mathfrak{F}\subset\mathfrak{M}_{\rm cont}$ separates points of $\mathbb{R}^{\nu N}$ if for any two distinct $x,x'\in\mathbb{R}^{\nu N}$, there exists $F\in\mathfrak{F}$ such that the corresponding continuous function has the property $F(x)\neq F(x')$. By Theorem 1.2, page 6 of [19], every $\mathfrak{F}\subset\mathfrak{M}_{\rm cont}$ separating points of $\mathbb{R}^{\nu N}$ generates $\mathcal{B}$. Thus, by Corollary 2.5, we have the following

Lemma 2.6. *Let* $\mathfrak{F}\subset\mathfrak{M}_{\rm cont}$ *separate points of* $\mathbb{R}^{\nu N}$ *and* $I\in\mathfrak{F}$*. Then* $\mathfrak{F}'=\mathfrak{M}$*, hence* $\mathfrak{F}$ *is complete.*

In view of von Neumann's bicommutant and density theorems, the proof of Theorem 2.2 may be done by showing that the commutant of $\{\alpha_t(F)\mid t\in\mathbb{R},\ F\in\mathfrak{F}\}$ consists of multiplication operators by complex numbers only. To prepare the use of such arguments we put the things into a more general setting. Let $\mathcal{A}=\{\alpha_t:\mathfrak{C}\to\mathfrak{C}\mid t\in\mathbb{R}\}$ be a one-parameter group of time automorphisms on $\mathfrak{C}$ acting as

$$\alpha_t(A)=\exp(\imath tH)A\exp(-\imath tH),\quad A\in\mathfrak{C},$$

with a certain essentially self-adjoint operator H. We say that the group $\mathcal{A}$ is defined by H. For $\mathfrak{B}\subset\mathfrak{C}$, we set

$$\mathfrak{D}(\mathfrak{B},\mathcal{A})=\{\alpha_t(B)\mid \alpha_t\in\mathcal{A},\ \ B\in\mathfrak{B}\},\tag{2.3}$$

and let $\mathfrak{C}(\mathfrak{B},\mathcal{A})$ be the $*$-subalgebra of $\mathfrak{C}$ generated by $\mathfrak{D}(\mathfrak{B},\mathcal{A})$.

Definition 2.7. The C^*-dynamical system $(\mathfrak{C}, \mathcal{A})$ is called irreducible if

$$\mathfrak{D}(\mathfrak{M}, \mathcal{A})'' = \mathfrak{C}. \tag{2.4}$$

Remark 2.8. Since the algebra $\mathfrak{M}$ contains the unit element $I \in \mathfrak{C}$, the algebra $\mathfrak{C}(\mathfrak{M}, \mathcal{A})$ is nondegenerate. Clearly, $\mathfrak{D}(\mathfrak{M}, \mathcal{A})'' = \mathfrak{C}(\mathfrak{M}, \mathcal{A})''$, then the von Neumann density theorem and (2.4) yield that $\mathfrak{C}(\mathfrak{M}, \mathcal{A})$ is dense in $\mathfrak{C}$ in the σ-strong and hence in σ-weak and strong topologies.

In what follows, Theorem 2.2 may be reformulated as follows.

Proposition 2.9. *Let a system of quantum particles be described by the Schrödinger operator H obeying the conditions of Theorem 2.2. Let also $\mathcal{A} = \{\alpha_t \mid t \in \mathbb{R}\}$ be the group of time automorphisms defined by H and $\mathfrak{F} \subset \mathfrak{M}$ be complete. Then*

$$\mathfrak{D}(\mathfrak{F}, \mathcal{A})'' = \mathfrak{C}, \tag{2.5}$$

and hence $(\mathfrak{C}, \mathcal{A})$ is irreducible.

3. Properties of Irreducible C^*-Dynamical Systems

In this section $\mathcal{A}$ stands for a group of time automorphisms on $\mathfrak{C}$, not necessarily defined by the operator H (1.2). By $\mathbb{C}I$ we denote the set of multiplication operators by complex numbers. Since $\mathfrak{M}$ is the maximal algebra containing all bounded multiplication operators, one has $\mathfrak{M}' = \mathfrak{M}'' = \mathfrak{M}$, which will be used below.

Lemma 3.1. *If $(\mathfrak{C}, \mathcal{A})$ is irreducible and for a given $B \in \mathfrak{C}$, one has $\alpha_t(B) \in \mathfrak{M}$ for all $t \in \mathbb{R}$, then $B \in \mathbb{C}I$.*

Proof. Let $\mathfrak{U}$ be the set of all unitary elements of $\mathfrak{M}$. Obviously, every $F \in \mathfrak{M}$ may be written in the form

$$F = a_1 U_1 + \cdots + a_4 U_4, \quad U_1, \ldots, U_4 \in \mathfrak{U}, \quad a_1, \ldots, a_4 \in \mathbb{C}. \tag{3.1}$$

Hence

$$\mathfrak{D}(\mathfrak{U}, \mathcal{A})' = \mathfrak{D}(\mathfrak{M}, \mathcal{A})'. \tag{3.2}$$

Let B be such that $\alpha_t(B) \in \mathfrak{M}$ for all $t \in \mathbb{R}$. Then for any $t, s \in \mathbb{R}$ and $U \in \mathfrak{U}$, we have

$$\begin{aligned} \alpha_t(B) &= \alpha_s\left(U \alpha_{t-s}(B) U^*\right) = \alpha_s(U) \alpha_s\left(\alpha_{t-s}(B)\right) \alpha_s(U^*) \\ &= \alpha_s(U) \alpha_t(B) \alpha_s(U)^*, \end{aligned}$$

which means that

$$\alpha_t(B) \in \mathfrak{D}(\mathfrak{U}, \mathcal{A})' = \mathfrak{D}(\mathfrak{M}, \mathcal{A})' = \mathfrak{D}(\mathfrak{M}, \mathcal{A})''' = \mathfrak{C}' = \mathbb{C}I.$$

□

Lemma 3.2. *Let $\mathfrak{F}$ be complete. Then*

$$\left[\mathfrak{D}(\mathfrak{M}, \mathcal{A})'' = \mathfrak{C}\right] \implies \left[\mathfrak{D}(\mathfrak{F}, \mathcal{A})'' = \mathfrak{C}\right]. \tag{3.3}$$

Proof. By Definition 2.1 and von Neumann's density theorem, $\mathfrak{F}' = \mathfrak{M}$. Take $B \in \mathfrak{D}(\mathfrak{F}, \mathcal{A})'$. Then for all $t \in \mathbb{R}$ and $F \in \mathfrak{F}$,

$$\alpha_t\left(B \alpha_{-t}(F)\right) = \alpha_t\left(\alpha_{-t}(F) B\right),$$

which means $\alpha_t(B) F = F \alpha_t(B)$, hence $\alpha_t(B) \in \mathfrak{F}' = \mathfrak{M}$ for all $t \in \mathbb{R}$. Then by Lemma 3.1, $B \in \mathbb{C}I$, yielding $\mathfrak{D}(\mathfrak{F}, \mathcal{A})' = \mathbb{C}I$, hence $\mathfrak{D}(\mathfrak{F}, \mathcal{A})'' = \mathfrak{C}$. □

Let the group $\mathcal{A} = \{\alpha_t \mid t \in \mathbb{R}\}$ be defined by the operator $H = H_0 + W$, such that H, H_0, and W are essentially self-adjoint on a joint domain. Let also the group $\mathcal{A}_0$ be defined by H_0 and W be a multiplication operator by a measurable function $W : \mathbb{R}^{\nu N} \to \mathbb{R}$.

Lemma 3.3. *If $(\mathfrak{C}, \mathcal{A}_0)$ is irreducible, then $(\mathfrak{C}, \mathcal{A})$ is irreducible as well.*

Proof. We prove this statement by developing an argument used by R. Høegh-Krohn in [6]. Its main ingredient is the Trotter-Kato product formula (see Theorem 1.1 page 4 of [18]), which in our context may be written

$$\begin{aligned} \exp(\imath t H) &= \exp[\imath t(H_0 + W)] \\ &= \lim_{n\to+\infty} \{\exp[\imath(t/n)H_0] \exp[\imath(t/n)W]\}^n , \end{aligned} \tag{3.4}$$

where the convergence holds in the strong topology. The proof will be done by showing that $\mathfrak{D}(\mathfrak{M}, \mathcal{A}_0)' = \mathbb{C}I$ implies

$$\mathfrak{D}(\mathfrak{M}, \mathcal{A})' = \mathbb{C}I. \tag{3.5}$$

Since each $B \in \mathfrak{D}(\mathfrak{M}, \mathcal{A})'$ commutes with every $\alpha_t(F)$, $t \in \mathbb{R}$, $F \in \mathfrak{M}$, one has

$$\exp(-\imath t H)\, B \exp(\imath t H)\, F = F \exp(-\imath t H)\, B \exp(\imath t H)\,, \tag{3.6}$$

for any $t \in \mathbb{R}$ and $F \in \mathfrak{M}$. Then by (3.6), $\alpha_t(B) \in \mathfrak{M}' = \mathfrak{M}$ for all $t \in \mathbb{R}$. By the assumptions of the lemma, the unitary operators $\exp(\imath s W)$, $s \in \mathbb{R}$ are multiplication operators, hence they commute with all $\alpha_t(B)$, $t \in \mathbb{R}$. Then for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\exp(-\imath(t/n)W)\alpha_{t/n}(B) \exp(\imath(t/n)W) = \alpha_{t/n}(B),$$

which yields

$$\begin{aligned} &\exp(-\imath(t/n)W) \exp[\imath(t/n)(H_0 + W)]B \\ &\quad\times \exp[-\imath(t/n)(H_0 + W)] \exp(\imath(t/n)W) = \alpha_{t/n}(B), \end{aligned}$$

or

$$\begin{aligned} &\alpha_{t/n} [\exp(-\imath(t/n)W) \exp[\imath(t/n)(H_0 + W)]B \\ &\quad\times \exp[-\imath(t/n)(H_0 + W)] \exp(\imath(t/n)W)] = \alpha_{2t/n}(B). \end{aligned}$$

By repetition,

$$\begin{aligned} &[\exp(-\imath(t/n)W) \exp(\imath(t/n)(H_0 + W))]^n\, B \\ &\quad\times [\exp(\imath(t/n)W) \exp(-\imath(t/n)(H_0 + W))]^n = \alpha_t(B). \end{aligned} \tag{3.7}$$

Since $\alpha_t(B) \in \mathfrak{M}$ for any $t \in \mathbb{R}$, so does the expression on the left-hand side of (3.7). Then the limit, as $n \to +\infty$, of such expressions is an element of $\mathfrak{M}$ since the latter algebra is strongly closed. By (3.4), this limit is $\exp(\imath t H_0)B \exp(-\imath t H_0)$ which yields that for all $t \in \mathbb{R}$,

$$\exp(\imath t H_0)B \exp(-\imath t H_0) = \alpha_t(B) \in \mathfrak{M}.$$

By the assumption of the lemma, $(\mathfrak{C}, \mathcal{A}_0)$ is irreducible, hence by Lemma 3.1, it follows that $B \in \mathbb{C}I$. □

For $\tau \in \mathbb{R}^{\nu N}$, we set

$$P(\tau) = \exp(\imath\tau \cdot p), \quad Q(\tau) = \exp(\imath\tau \cdot q). \tag{3.8}$$

Here $q = (q_j^{(\kappa)})$, $j = 1, \dots, N$, $\kappa = 1, \dots, \nu$ and $q_j^{(\kappa)}$ is a multiplication operator by $x_j^{(\kappa)}$ affiliated with the von Neumann algebra $\mathfrak{M}$. Furthermore, $p = (p_j^{(\kappa)})$, $j = 1, \dots, N$, $\kappa = 1, \dots, \nu$, $p_j^{(\kappa)} = -\imath \partial / \partial x_j^{(\kappa)}$, and

$$\tau \cdot p = \sum_{j=1}^{N} \sum_{\kappa=1}^{\nu} \tau_j^{(\kappa)} p_j^{(\kappa)}, \quad \tau \cdot q = \sum_{j=1}^{N} \sum_{\kappa=1}^{\nu} \tau_j^{(\kappa)} q_j^{(\kappa)}.$$

Both $P(\tau)$, $Q(\tau)$ are unitary and $P(\tau)^* = P(-\tau)$, $Q(\tau)^* = Q(-\tau)$ for all $\tau \in \mathbb{R}^{\nu N}$. Set

$$\mathfrak{P} = \{P(\tau) \mid \tau \in \mathbb{R}^{\nu N}\}, \quad \mathfrak{Q} = \{Q(\tau) \mid \tau \in \mathbb{R}^{\nu N}\}. \tag{3.9}$$

For $\psi \in \mathcal{S}(\mathbb{R}^{\nu N})$, we set

$$\hat{\psi}(p) = (V\psi)(p) = (2\pi)^{-\nu N/2} \int_{\mathbb{R}^{\nu N}} \exp(-\imath p \cdot x)\psi(x)\mathrm{d}x, \tag{3.10}$$

which one can extend to a unitary operator on the whole space $L^2(\mathbb{R}^{\nu N})$. Its inverse is

$$\psi(x) = (V^*\hat{\psi})(x) = (2\pi)^{-\nu N/2} \int_{\mathbb{R}^{\nu N}} \exp(\imath p \cdot x)\hat{\psi}(p)\mathrm{d}p. \tag{3.11}$$

By obvious reasons, one has the following facts

Proposition 3.4. *It follows that*

$$V\mathfrak{P}V^* = \mathfrak{Q}, \qquad V\mathfrak{Q}V^* = \mathfrak{P}; \qquad \mathfrak{P}' \bigcap \mathfrak{Q}' = \mathbb{C}I, \tag{3.12}$$

which means that the family $\mathfrak{P} \cup \mathfrak{Q}$ is irreducible.

4. Proof of Theorem 2.2

First we derive a property of continuous negative definite functions obeying (2.1).

Lemma 4.1. *Let $\Phi : \mathbb{R}^\nu \to \mathbb{R}$ obey the condition (2.1) and be continuous and negative definite. If for given $p_1, p_2 \in \mathbb{R}^\nu$, one has*

$$\Phi(p_1) - \Phi(p_1 - \vartheta) = \Phi(p_2) - \Phi(p_2 - \vartheta), \tag{4.1}$$

holding for all $\vartheta \in \mathbb{R}^\nu$, then $p_1 = p_2$.

Proof. Set

$$\Xi(\vartheta) = \Phi(p_1 - \vartheta) - \Phi(p_2 - \vartheta).$$

Then by (4.1), $\Xi(\vartheta) = \Xi(0)$ for all $\vartheta \in \mathbb{R}^\nu$. Therefore, by writing

$$\begin{aligned} \Xi(p_1) &= \Phi(0) - \Phi(p_2 - p_1) \\ \Xi(p_2) &= \Phi(p_1 - p_2) - \Phi(0), \end{aligned}$$

we get

$$2\Xi(0) = \Phi(p_1 - p_2) - \Phi(p_2 - p_1).$$

By (2.2), it follows that $\Phi(p) = \Phi(-p)$. Hence $\Xi(0) = 0$ and $\Phi(\vartheta) = \Phi(p_2 - p_1 + \vartheta)$ for all $\vartheta \in \mathbb{R}^\nu$, which contradicts the condition (2.1) if $p_1 \neq p_2$. $\square$

For $t \in \mathbb{R}$, $\tau \in \mathbb{R}^{\nu N}$, we set

$$G(p|t, \tau) = \exp\left(-\imath t \sum_{j=1}^{N} [\Phi(p_j) - \Phi(p_j - \tau_j)] \right). \tag{4.2}$$

Corollary 4.2. *Given $t \in \mathbb{R}$, let $\mathfrak{G}_t$ be the family of multiplication operators by the functions $G(\cdot|t,\tau)$, $\tau \in \mathbb{R}^{\nu N}$. Then $\mathfrak{G}_t' = \mathfrak{M}$.*

Proof. Since $\varPhi$ is continuous, the functions (4.2) are continuous; furthermore, $G(\cdot|t,0) = I$. By Lemma 4.1, the family $\mathfrak{G}_t$ separates points, thus by Lemma 2.6, one has the property stated. □

For a function $\varPhi$ obeying the conditions of Lemma 4.1, we set

$$H_0 = \sum_{j=1}^{N} \varPhi(p_j), \quad p_j = -\imath \nabla_j. \tag{4.3}$$

For $t \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R}^{\nu N})$, one has

$$\begin{aligned}(K_t\psi)(x) &\stackrel{\text{def}}{=} \left[\exp(\imath t H_0)\psi\right](x) \\ &= \frac{1}{(2\pi)^{\nu N/2}} \int_{\mathbb{R}^{\nu N}} \exp\left(\imath t \sum_{j=1}^{N} \varPhi(p_j) + \imath p \cdot x\right) \hat{\psi}(p) \mathrm{d}p,\end{aligned} \tag{4.4}$$

where $\hat{\psi}$ is defined by (3.10). It defines time automorphisms

$$\alpha_t^0(A) = K_t A K_{-t}, \quad A \in \mathfrak{C}. \tag{4.5}$$

Set $\mathcal{A}_0 = \{\alpha_t^0 \mid t \in \mathbb{R}\}$.

Lemma 4.3. *The C^*-dynamical system $(\mathfrak{C}, \mathcal{A}_0)$ is irreducible.*

Proof. Let us show that $\mathfrak{D}(\mathfrak{M}, \mathcal{A}_0)' = \mathbb{C}I$. Take $B \in \mathfrak{D}(\mathfrak{M}, \mathcal{A}_0)'$ and set

$$B(t,\tau) = Q(\tau)\alpha_t^0(B)Q(-\tau), \quad t \in \mathbb{R}, \quad \tau \in \mathbb{R}^{\nu N}, \tag{4.6}$$

where $Q(\tau)$ is the same as in (3.8). For any $t \in \mathbb{R}$ and $F \in \mathfrak{M}$, one has

$$BK_tFK_{-t} = K_tFK_{-t}B,$$

which immediately yields $\alpha_t^0(B) \in \mathfrak{M}$ for all $t \in \mathbb{R}$ since $\mathfrak{M}' = \mathfrak{M}$. Then

$$B(t,\tau) = \alpha_t^0(B), \tag{4.7}$$

for all $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^{\nu N}$. On the other hand, one may write

$$B(t,\tau) = R(t,\tau)\alpha_t^0(B)R(t,\tau)^*, \tag{4.8}$$

where

$$R(t,\tau) = Q(\tau)K_tQ(-\tau)K_{-t}, \tag{4.9}$$

is a unitary operator on $L^2(\mathbb{R}^{\nu N})$. Then by (4.7), all $\alpha_t^0(B)$, $t \in \mathbb{R}$ commute with all such $R(t,\tau)$. Given $t \in \mathbb{R}$, we set $\mathfrak{R}_t = \{R(t,\tau) \mid \tau \in \mathbb{R}^{\nu N}\}$. Thereby, for all $t \in \mathbb{R}$,

$$\alpha_t^0(B) \in \mathfrak{Q}' \bigcap \mathfrak{R}_t', \tag{4.10}$$

hence by (3.12)

$$V\alpha_t^0(B)V^* \in \mathfrak{P}' \bigcap \left(V\mathfrak{R}_tV^*\right)', \tag{4.11}$$

which holds for all $t \in \mathbb{R}$. But by (3.10), (3.11), (4.4), (4.9), it follows that $VR(t,\tau)V^*$ is a multiplication operator by the function $G(\cdot|t,\tau)$ given by (4.2), i.e., $V\mathfrak{R}_tV^* = \mathfrak{G}_t$, hence by Corollary 4.2,

$$\left(V\mathfrak{R}_tV^*\right)' = \mathfrak{M} = \mathfrak{Q}'. \tag{4.12}$$

Now by means of Proposition 3.4 and (4.11), (4.12) we get $V\alpha_t^0(B)V^* \in \mathbb{C}I$, hence $\alpha_t^0(B) \in \mathbb{C}I$, in particular $B \in \mathbb{C}I$. □

Proof of Theorem 2.2. Under the assumptions of the theorem, Lemma 3.3 may be applied to the model (1.2) yielding together with Lemma 4.3 that the system $(\mathfrak{C}, \mathcal{A})$ is irreducible. Then by Lemma 3.2 and Corollary 2.5, we prove Proposition 2.9 and hence Theorem 2.2. □

5. Application to KMS States

The Gibbs state of the quantum system described by the Schrödinger operator (1.2) is the functional (1.1), where β^{-1} is temperature, defined on the algebra $\mathfrak{C}$. It is a β-KMS normal state (see Definition 1.2 in [11]), which may be interpreted as a consistency between 'real time' and 'imaginary time' evolutions. In our case this consistency allows one to reconstruct the 'real time' dynamics of the whole algebra from the 'imaginary time' dynamics of a complete family of multiplication operators. The precise meaning of this is given in Theorem 5.2 below. Given $n \in \mathbb{N}$, we set

$$\mathcal{D}_n^\beta = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid 0 < \Im(z_1) < \cdots < \Im(z_n) < \beta\}, \tag{5.1}$$

$\bar{\mathcal{D}}_n^\beta$ to be the closure of $\mathcal{D}_n^\beta$, and

$$\mathcal{D}_n^\beta(0) = \{(z_1, \dots, z_n) \in \mathcal{D}_n^\beta \mid \Re(z_1) = \cdots = \Re(z_n) = 0\}. \tag{5.2}$$

The following statement is a direct corollary of Theorem 2.1 in [11].

Proposition 5.1. *Let ω be a β-KMS state on $\mathfrak{C}$ relative to the group of time automorphisms $\mathcal{A}$ defined by the Schrödinger operator (1.2) and $A_1, \dots, A_n$ be elements of $\mathfrak{C}$. Then the function*

$$G^\omega_{A_1,\dots,A_n}(t_1, \dots t_n) = \omega\left(\alpha_{t_1}(A_1) \cdots \alpha_{t_n}(A_n)\right), \tag{5.3}$$

is the restriction of a continuous function defined on the set $\bar{\mathcal{D}}_n^\beta$, which is analytic in $\mathcal{D}_n^\beta$.

We denote the above mentioned analytic functions also by $G^\omega_{A_1,\dots,A_n}$.

Theorem 5.2. *Let ω and ω' be β-KMS normal states on the algebra $\mathfrak{C}$ relative to the group of time automorphisms $\mathcal{A}$ defined by the Schrödinger operator (1.2). Let also $\mathfrak{F} \subset \mathfrak{M}$ be a complete family, such that for any $n \in \mathbb{N}$ and any choice of $F_1, \dots, F_n \in \mathfrak{F}$, the restrictions of the analytic functions $G^\omega_{F_1,\dots,F_n}$ and $G^{\omega'}_{F_1,\dots,F_n}$ to the set $\mathcal{D}_n^\beta(0)$ coincide. Then $\omega = \omega'$.*

Proof. The set $\mathcal{D}_n^\beta(0)$ is an inner set of uniqueness for functions analytic in $\mathcal{D}_n^\beta$ (see pages 101 and 352 of [17]). Thereby, $G^\omega_{F_1,\dots,F_n} = G^{\omega'}_{F_1,\dots,F_n}$, which by Theorem 2.2 implies the property stated since normal states are σ-weakly continuous. □

Remark 5.3. The point connecting the above theory with the path integral representation of the semi-group $\exp(-tH)$ is that the restrictions of $G^\omega_{F_1,\dots,F_n}$ to imaginary values of the time variables may be written as such path integrals. Thereby, the KMS states are uniquely determined by these integrals.

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