Centre de Physique Théorique[*] - CNRS - Luminy, Case 907
F-13288 Marseille Cedex 9 - France

ONE REFINEMENT OF THE GUROV-RESHETNYAK INEQUALITY

Anatoli KORENOVSKII[1,2]

Abstract

An estimate of the rearrangement of a function over its relative mean oscillation is obtained. From this estimate follows, in particular, the Gurov-Reshetnyk theorem and some another known results.

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anonymous ftp or gopher: cpt.univ-mrs.fr

[*]Unité Propre de Recherche 7061
[1]and PHYMAT, Université de Toulon et du Var, F-83957 La Garde cedex, France
[2]On leave of absence from Odessa State University, Petra Velikogo, 2, 270000, Odessa, Ukraine

1. Introduction and main results. Let $f \in L(Q_0)$ be a nonnegative function, where $Q_0 \subset \mathrm{R}^{\mathrm{N}}$ be a cube with the sides parallel to the axes (only such cubes will be considered below). For the measurable set $E \subset Q_0$ we denote $f_E \equiv \mid E \mid^{-1} \int_E f(x)dx$ where $\mid E \mid$ is the Lebesgue measure of E. We suppose that $f_{Q_0} > 0$ and write

$$J\nu(f;\sigma) = \sup_{|Q|\leq\sigma} \frac{\frac{1}{|Q|}\int_Q \mid f(x) - f_Q \mid dx}{\frac{1}{|Q|}\int_Q f(x)dx}, \qquad 0 < \sigma \leq \mid Q_0 \mid,$$

where the supremum is taked over cubes $Q \subset Q_0$ with measure not exceeding σ and such that $f_Q > 0$. It is obvious that $J\nu(f;\sigma) \leq 2,\ 0 < \sigma \leq \mid Q_0 \mid$, for any f.

In the Gurov and Reshetnyak paper [1] it has be shown that if $J\nu(f;\sigma)$ is bounded by some small constant then the function f^p is integrable for some $p > 1$. More precisely, the following was obtained:

Theorem of Gurov-Reshetnyak. *If for some $\varepsilon,\ 0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(N) > 0$, the inequality*

$$J\nu(f;\sigma) \leq \varepsilon, \quad 0 < \sigma \leq \mid Q_0 \mid,$$

holds, then $f \in L^p(Q_0)$ for every $p < p_0$, where $1 < p_0 = p_0(\varepsilon, N) \to \infty$ when $\varepsilon \to 0+$.

Different proofs and generalisations of this Theorem were given by number of authors (cf. [2-8]). In particular, in [7,8] are considered the properties of the function f, which satisfies the condition

$$J\nu(f;\sigma) \to 0, \quad \sigma \to 0+.$$

In this paper we study some properties of functions f expressed in terms of $\nu(f,\sigma)$. They formulate in terms of rearrangement

$$f^*(t) \equiv \sup_{|e|\geq t} \ \mathrm{ess} \inf_{x\in e} f(x), \qquad 0 < t \leq \mid Q_0 \mid.$$

The function $f^*(t)$ is nonincreasing on $(0, \mid Q_0 \mid]$ and equimeasurable with f.

The rearrangements have been used to study the similar problems by various authors. Their application is effective because they have the more simple form and in some sense keep the properties of original function. In Theorem 1 we obtain one of such properties of rearrangement. We will denote

$$f^{**}(t) \equiv \frac{1}{t}\int_0^t f^*(u)du, \qquad 0 < t \leq \mid Q_0 \mid.$$

Theorem 1. *Let $f \in L(Q_0)$. Then the inequality*

$$\frac{1}{t}\int_0^t \mid f^*(u) - f^{**}(t) \mid du \leq 3 \cdot 2^N \nu(f, 2^N t) f^{**}(t), \qquad 0 < t \leq 2^{-N} \mid Q_0 \mid, \qquad (1)$$

holds.

The proof of this Theorem is given in Section 2. Theorem 1 leads to the following main result:

J**Theorem 2.** *There are constants c_1 and c_2 depending only on the dimension N such that for $f \in L(Q_0)$ we have the inequality:*

$$f^{**}(t) \leq c_1 f_{Q_0} exp(c_2 \int_t^{|Q_0|} \nu(f,\sigma)\frac{d\sigma}{\sigma}), \quad 0 < t \leq \mid Q_0 \mid . \tag{2}$$

We will prove the Theorem 2 also in Section 2.

¿From inequality (2) follows obviously the Gurov-Reshetnyak Theorem. In addition, if $\nu(f,\sigma) \to 0+,\ \sigma \to 0+$, then we have, as a direct consequence of (2), the following estimate: $f^{**}(t) = O(t^{-1/p}),\ t \to 0+$, for every $p < \infty$. Hence is proved:

Corollary 1. *If $f \in L(Q_0)$ such that $\nu(f, 0+) = 0$, then $fJ \in L^p(Q_0)$ for every $p < \infty$.*

This assertion is contained in the paper of Franciosi [7]. In this paper it has been proved also that:*if $\nu(f,\sigma^N) = o(\sigma^\beta),\ \sigma \to 0+$, for some $\beta > 0$, then f is equivalent to an continuous function g, and the modulus of continuity:*

$$\omega(g;\sigma) \equiv \sup_{|x-y|\leq\sigma\ x,y\in Q_0} \mid g(x) - g(y) \mid, \quad 0 < \sigma \leq l(Q_0),$$

is $O(\nu(f,\sigma^N))$ when $\sigma \to 0+$. Here $l(Q)$ denotes the edgelength of cube Q. The proof of the last property is based on the application of the following result given in [9] (cf. also [10]):*if for some $\beta \in (0,1]$ $\quad \mu(f,\sigma^N) = O(\sigma^\beta),\ \sigma \to 0+$, then $\omega(f,\sigma) = O(\sigma^\beta),\ \sigma \to 0+$, where*

$$\mu(f,\sigma) \equiv \frac{1}{f_{Q_0}} \sup_{|Q|\leq\sigma,\ Q\subset Q_0} \frac{1}{\mid Q \mid}\int_Q \mid f(x) - f_Q \mid dx, \quad 0 < \sigma \leq \mid Q_0 \mid .$$

With the help of Theorem 2 the estimate of the modulus of continuity of function f over $\nu(f,\sigma)$ can be easily extended in the case when the condition

$$\int_0^{l(Q_0)} \nu(f,\sigma^N)\frac{d\sigma}{\sigma} < \infty \tag{3}$$

holds. In fact, from Theorem 2 it follows

Corollary 2. *If $f \in L(Q_0)$ satisfies (3), then f is essentially bounded on Q_0 and*

$$\mid\mid f \mid\mid_\infty \leq c_3 f_{Q_0}. \tag{4}$$

Further, from (4) we obtain the inequality

$$\nu(f,\sigma) \geq c_4 \mu(f,\sigma), \quad 0 < \sigma \leq \mid Q_0 \mid, \tag{5}$$

where $c_4 > 0$. Now, recall the following:

Theorem of Spanne ([11]).*Let $f \in L(Q_0)$ be a function such that:*

$$\int_0^{l(Q_0)} \mu(f,\sigma^N)\frac{d\sigma}{\sigma} < \infty.$$

Then f is equivalent to an continuous function g and

$$\omega(g;\sigma) \leq c_5 f_{Q_0} \int_0^{\sigma} \mu(f,\delta^N)\frac{d\delta}{\delta}, \quad 0 < \sigma \leq l(Q_0),$$

where the constant c_5 is independent of σ.

The last Theorem and inequality (5) lead directly to:

Theorem 3. *Let $f \in L(Q_0)$ be a function which satisfies (3). Then f is equivalent to an continuous function g and*

$$\omega(g;\sigma) \leq c_6 f_{Q_0} \int_0^{\sigma} \nu(f,\delta^N)\frac{d\delta}{\delta}, \quad 0 < \sigma \leq l(Q_0),$$

where the constant c_6 is independent of σ.

J2. Proof of Theorems 1 and 2. The proof of Theorem 1 is based on the following Lemma.

Lemma of Calderon-Zygmund ([12]).*Let f be a nonnegative function in $L(Q_0)$ and α be a number not less than f_{Q_0}. Tnen there exist two families of cubes Q_s and $R_s \supset Q_s$ such that the interiors of cubes Q_s are pointwise intersecting, $\mid R_s \mid = 2^N \mid Q_s \mid$,*

$$f_{R_s} \leq \alpha < f_{Q_s} \leq 2^N\alpha, \quad s = 1,2,...,$$

and $f(x) \leq \alpha$ almost everywhere on $Q_0 \setminus E$, where $E \equiv \bigcup_s Q_s$.

Proof of Theorem 1. Fix $t, 0 < t \leq 2^{-N} \mid Q_0 \mid$. Applying the Lemma of Calderon-Zygmund with $\alpha = f^{**}(t)$, we construct the cubes Q_s and R_s with the indicated abowe properties. Since

$$\frac{1}{\mid E \mid}\int_E f(x)dx = \frac{1}{\mid EJ}\sum_s \int_{Q_s} f(x)dx \geq \alpha,$$

then from the inequality

$$\frac{1}{t}\int_0^t f^*(u)du = f^{**}(t) = \alpha \leq \frac{1}{\mid E \mid}\int_E f(x)dx \leq \frac{1}{\mid EJ}\int_0^{|E|} f^*(u)du$$

and from the monotonicity of f^* it follows that $\mid E \mid\leq t$. Therefore $\mid Q_s \mid\leq t, \mid R_s \mid\leq 2^N t,\ s=1,2,...$ If we take into account the properties of rearrangement, of the mean value of function, of the cubes Q_s and R_s, of set E, and next the monotonicity of the function $\nu(f,\sigma)$, we obtain

$$\int_0^t \mid f^*(u)-f^{**}(t)\mid du = 2\int_{\{u:\ f^*(u)>\alpha\}}(f^*(u)-\alpha)du = 2\int_{\{x\in Q_0:\ f(x)>\alpha\}}(f(x)-\alpha)dx =$$

$$= 2\int_{\{x\in Q_0:\ f(x)>\alpha\}\bigcap E}(f(x)-\alpha)dx = 2\sum_s\int_{\{x\in Q_0:\ f(x)>\alpha\}\bigcap Q_s}(f(x)-\alpha)dx =$$

$$= 2\sum_s\int_{\{x\in Q_s:\ f(x)>\alpha\}}(f(x)-\alpha)dx =$$

$$= 2\sum_s\int_{\{x\in Q_s:\ f(x)>\alpha\}}(f(x)-f_{Q_s})dx + 2\sum_s(f_{Q_s}-\alpha)\mid \{Jx\in Q_s:\ f(x)>\alpha\}\mid =$$

$$= 2\sum_s\int_{\{x\in Q_s:\ f(x)>f_{Q_s}\}}(f(x)-f_{Q_s})dx +$$

$$+2\sum_s\int_{\{x\in Q_s:\ \alpha<f(x)\leq f_{Q_s}\}}(f(x)-f_{Q_s})dx + 2\sum_s(f_{Q_s}-\alpha)\mid \{x\in Q_s:\ f(x)>\alpha\}\mid\leq$$

$$\leq \sum_s\int_{Q_s}\mid f(x)-f_{Q_s}\mid dx + 2\sum_s \mid Q_s\mid (f_{Q_s}-f_{R_s}) \leq$$

$$\leq \sum_s\int_{Q_s}\mid f(x)-f_{Q_s}\mid dx + 2\sum_s\int_{Q_s}\mid f(x)-f_{R_s}\mid dx \leq$$

$$\leq \sum_s \int_{Q_s} \mid f(x) - f_{Q_s} \mid dx + 2\sum_s \int_{R_s} \mid f(x) - f_{R_s} \mid dx \leq$$

$$\leq \nu(f,t) \sum_s \int_{Q_s} f(x)dx + 2\nu(f, 2^N t) \sum_s \int_{R_s} f(x)dx \leq$$

$$\leq 3\nu(f, 2^N t) \sum_s \int_{R_s} f(x)dx \leq 3\alpha \cdot \nu(f, 2^N t) \sum_s \mid R_s \mid \leq$$

$$\leq 3 \cdot 2^N \alpha \cdot \nu(f, 2^N t) \sum_s \mid Q_s \mid \leq 3 \cdot 2^N t \cdot f^{**}(t) \nu(f, 2^N t).$$

¿From the abowe follows (1). This completes the proof.

We will needed below the following Lemma in the proof of Theorem 2.

Lemma ([6]).*Let $f \in L(Q_0)$. Then for any $a > 1$ the inequality*

$$f^{**}(\frac{t}{a}) - f^{**}(t) \leq \frac{a}{2} \cdot \frac{1}{t} \int_0^t \mid f^*(u) - f^{**}(t) \mid du, \quad 0 < t \leq \mid Q_0 \mid,$$

holds.

Proof of Theorem 2. By the last lemma and by the Theorem 1 we have

$$f^{**}(\frac{t}{2}) - f^{**}(t) \leq \frac{1}{t} \int_0^t \mid f^*(u) - f^{**}(t) \mid du \leq 3 \cdot 2^N \nu(f, 2^N t) f^{**}(t),$$

that is

$$f^{**}(\frac{t}{2}) \leq (1 + 3 \cdot 2^N \nu(f, 2^N t)) f^{**}(t), \quad 0 < t \leq 2^{-N} \mid Q_0 \mid .$$

The recurrent application of this inequality gives

$$f^{**}(\frac{\mid Q_0 \mid}{2^{N+s}}) \leq \prod_{i=0}^{s-1} (1 + 3 \cdot 2^N \nu(f, \frac{\mid Q_0 \mid}{2^i})) f^{**}(\frac{\mid Q_0 \mid}{2^N}), \quad s = 1, 2, ...$$

Therefore, using the inequality $\nu(f, \sigma) \leq 2$ with $0 < \sigma \leq \mid Q_0 \mid$, and monotonicity of function $J\nu(f, \sigma)$, for $s = 1, 2, ...,$ we have

$$f^{**}(\frac{\mid Q_0 \mid}{2^{N+s}}) \leq exp(ln\ f^{**}(\frac{\mid Q_0 \mid}{2^N}) + \sum_{i=0}^{s-1} ln(1 + 3 \cdot 2^N \nu(f, \frac{\mid Q_0 \mid}{2^i}))) \leq$$

$$\leq f^{**}(\frac{\mid Q_0 \mid}{2^N}) exp(3 \cdot 2^N \sum_{i=0}^{s-1} \nu(f, \frac{\mid Q_0 \mid}{2^i})) \leq$$

$$\leq f^{**}(\frac{\mid Q_0 \mid}{2^N}) exp(3 \cdot 2^N \nu(f, \mid Q_0 \mid) + 3 \cdot 2^{N+1} \sum_{i=0}^{s-1} \frac{\nu(f, \frac{|Q_0|}{2^i})}{\frac{|Q_0|}{2^{i-1}}} \frac{\mid Q_0 \mid}{2^i}) \leq$$

$$\leq exp(3 \cdot 2^{N+1}) \frac{2^N}{\mid Q_0 \mid} \int_0^{|Q_0| 2^{-N}} f^*(u) du \cdot exp(3 \cdot 2^{N+1} \sum_{i=1}^{s-1} \int_{|Q_0| 2^{-i}}^{|Q_0| 2^{-i+1}} \nu(f, \sigma) \frac{d\sigma}{\sigma}) \leq$$

$$\leq 2^N exp(3 \cdot 2^{N+1}) \frac{1}{\mid Q_0 \mid} \int_0^{|Q_0|} f^*(u) du \cdot exp(3 \cdot 2^{N+1} \int_{|Q_0| 2^{-s+1}}^{|Q_0|} \nu(f, \sigma) \frac{d\sigma}{\sigma}).$$

Let fix $t,\ 0 < t \leq \mid Q_0 \mid$, and find s such that $\mid Q_0 \mid 2^{-s} < t \leq \mid Q_0 \mid 2^{-s+1}$. Then from the last inequality, by monotonicity of f^{**}, follows

$$f^{**}(t) \leq f^{**}(\frac{\mid Q_0 \mid}{2^{N+s}}) \leq 2^N exp(3 \cdot 2^{N+1}) f_{Q_0}\ exp(3 \cdot 2^{N+1} \int_{|Q_0| 2^{-s+1}}^{|Q_0|} \nu(f, \sigma) \frac{d\sigma}{\sigma}) \leq$$

$$\leq 2^N exp(3 \cdot 2^{N+1}) f_{Q_0}\ exp(3 \cdot 2^{N+1} \int_t^{|Q_0|} \nu(f, \sigma) \frac{d\sigma}{\sigma}).$$

This completes the proof.

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