A Riemannian Off-Diagonal Heat Kernel Bound for Uniformly Elliptic Operators

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Abstract

We find a Gaussian off-diagonal heat kernel estimate for uniformly elliptic operators with measurable coefficients acting on regions $\Omega \subseteq \mathbb{R}^N$, where the order $2m$ of the operator satisfies $N < 2m$. The estimate is expressed using certain Riemannian-type metrics, and a geometrical result is established allowing conversion of the estimate into terms of the usual Riemannian metric on Ω. Work of Barbatis [1] is applied to find the best constant in this expression.

1. Introduction

Let H be a differential operator with quadratic form

$$Q(f) = \int_\Omega \sum_{\substack{|i|\le m \\ |j|\le m}} a_{i,j}(x) D^i f(x) \overline{D^j f(x)} \mathrm{d}^N x, \tag{1}$$

where $a_{i,j}(x) = \overline{a_{j,i}(x)}$ are complex-valued bounded measurable functions on a region $\Omega \subseteq \mathbb{R}^N$. Dirichlet boundary conditions are imposed upon H by restricting the domain of the quadratic form to be the Sobolev space $W_0^{m,2}(\Omega)$, defined as the Hilbert space of all functions $f \in L^2(\Omega)$ whose weak derivatives $D^\alpha f$ lie in $L^2(\Omega)$ for all multi-indices α such that $|\alpha| \le m$. It is equipped with the inner product

$$\langle f, g\rangle_{m,2} := \sum_{|\alpha|\le m} \langle D^\alpha f, D^\alpha g\rangle_2. \tag{2}$$

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In the special case where the coefficients $a_{i,j}$ are constant and chosen so that

$$\sum_{|i|=|j|=m} a_{i,j}\xi^{i+j} = |\xi|^{2m}$$

and $a_{i,j} = 0$ whenever $|i| + |j| \leq 2m - 1$, the associated operator $H =: H_{\Omega,m}$ is the polyharmonic operator $(-\Delta)^m|_{\mathrm{DIR}}$ of the region Ω. We denote this particular quadratic form by Q_m.

Throughout this paper we shall assume that the coefficients $a_{i,j}(x)$ are chosen in such a way that Q satisfies the Gårding inequality

$$\lambda Q_m(f) - c\|f\|_2^2 \leq Q(f) \leq \mu Q_m(f) + d\|f\|_2^2, \tag{3}$$

where $0 < \lambda \leq \mu$ and c, d are non-negative constants. We then say that H is uniformly elliptic. Quadratic forms satisfying the Gårding inequality are closed on the domain $W^{m,2}(\Omega)$. For a more detailed account of uniformly elliptic operators see [4].

Note 1.1: If the sum in equation (1) is only taken over non-negative multi-indices i, j with $|i| = |j| = m$ then the operator is said to be homogeneous of order $2m$. If this is the case then we may set $c = d = 0$ in the Gårding inequality (3).

Lemma 1.2: *For $N < 2m$ the operator H has a heat kernel $K(t, x, y)$ which satisfies*

$$|K(t, x, y)| \leq ct^{-N/2m}e^t \tag{4}$$

for all $x, y \in \Omega$ and all $t > 0$.

Proof: See [4, Corollary 15 and Lemma 17]. If the operator H is homogeneous then the bound is valid even without the term e^t. □

In [4, Lemma 19], Davies obtains the pointwise heat kernel bound

$$|K(t, x, y)| \leq c_1 t^{-N/2m} \exp[-c_2|y - x|^{2m/(2m-1)}t^{-1/(2m-1)} + kt], \tag{5}$$

for all $t > 0$ and $x, y \in \mathbb{R}^N$, where c_1, c_2, k are positive constants. If the operator is homogeneous of order $2m$ then the constant k may be set to zero. In [2], Barbatis and Davies find the sharp constant c_2 for this expression in the following sense: If H is uniformly elliptic, homogeneous of order $2m$, and satisfies the Gårding inequality

$$Q_m(f) \leq Q(f) \leq \mu Q_m(f) \tag{6}$$

then

$$|K(t, x, y)| \leq c_\epsilon t^{-N/2m} \exp[-(\sigma_m - O(\mu - 1) - \epsilon)|y - x|^{2m/(2m-1)}t^{-1/(2m-1)}], \tag{7}$$

where $\epsilon > 0$ and

$$\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin[\pi/(4m-2)]. \tag{8}$$

This is shown to be sharp by considering the case $H = H_{\mathbb{R}^N,m}$.

The Euclidean metric $d_0(x,y) := |y-x|$ is a relatively weak and unnatural way of expressing the heat kernel bound for non-convex regions. As an example, for a horse-shoe shaped region whose extremities are touching, the Euclidean distance $d_0(x,y)$ can be made arbitrarily small for internally distant points $x, y \in \Omega$, rendering the heat kernel bound useless at these points.

The aim of this paper is to find heat kernel bounds which are given in terms of the Riemannian metric d_g (see definition (12)), instead of the Euclidean metric d_0. This would improve the original bound; for a horse-shoe shaped region which touches itself or even overlaps with itself one may choose x and y to make $d_0(x,y)$ arbitrarily small whilst $d_g(x,y)$ remains large. In order to find bounds involving the Riemannian metric, we first find bounds involving Riemannian-type metrics.

Definition 1.3: *For $\beta > 0$ define the Riemannian-type metrics $d_{m,\beta} : \Omega^2 \to \mathbb{R}_+$ by*

$$d_{m,\beta}(x,y) = \sup\{\phi(y) - \phi(x) : \phi \in \mathscr{E}_{m,\beta}\}, \tag{9}$$

where $\mathscr{E}_{m,\beta}$ denotes the set of all bounded real valued smooth functions ϕ on a region Ω such that

$$\|\nabla\phi\|_\infty \leq 1 \quad \textit{and} \quad \|D^i\phi\|_\infty \leq \beta^{|i|-1} \tag{10}$$

for all non-negative multi-indices i such that $2 \leq |i| \leq m$.

We show, for an arbitrary uniformly elliptic operator whose coefficients need only be measurable, that

$$|K(t,x,y)| \leq c_1 t^{-N/2m} \exp[-c_2 d_{m,\beta}(x,y)^{2m/(2m-1)} t^{-1/(2m-1)} + k(1+\beta^{2m})t] \tag{11}$$

where $N < 2m$. Note that since $d_{m,0}$ is the Euclidean metric d_0, setting $\beta = 0$ in this equation retrieves equation (5).

If Ω is a region with C^2 boundary and radii of curvature uniformly bounded below by r then there exists a constant K, dependent only upon m and N, such that for $\beta \geq 4K/r$,

$$\left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g \leq d_{m,\beta} \leq d_g,$$

where d_g is the standard Riemannian distance

$$d_g(x,y) := \inf\{l(\gamma) : \gamma(0) = x, \gamma(1) = y, \gamma \subseteq \overline{\Omega}, \gamma \text{ is cts and piecewise } C^1\}, \tag{12}$$

and where $l(\gamma)$ is the length of the path γ. See Theorem 4.18. This feature allows us to convert the off-diagonal bound (11) into the bound

$$|K(t,x,y)| \le c_1 t^{-N/2m} \exp[-c_2 d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)} + kt] \tag{13}$$

involving the Riemannian metric.

Note 1.4: As a special case of [3, Theorem 3.2.7 and Corollary 3.2.8] Davies has the bound

$$0 \le K(t,x,y) \le c_\delta t^{-N/2} \exp\{-d_g(x,y)^2/4(1+\delta)t\}, \tag{14}$$

valid for all N, on the heat kernel $K(t,x,y)$ of the Dirichlet Laplacian on a region $\Omega \subseteq \mathbb{R}^N$. The regularity of the boundary of Ω, required for higher order operators is a genuine feature of the order. □

A more natural class of metrics for the determination of heat kernel bounds is, in a certain sense, the class of Finsler-type metrics $d_{a,M}$ induced by the operator itself. Here a denotes the principal symbol

$$a(x,\xi) = \sum_{\substack{|i|=m \\ |j|=m}} a_{i,j}(x)\xi^{i+j} \tag{15}$$

of the operator, and M is a positive constant.

Definition 1.5: *The Finsler-type metrics are defined by*

$$d_{a,M}(x,y) = \sup\{\phi(y) - \phi(x) : \phi \in \mathscr{F}_{a,M}\}, \tag{16}$$

where $\mathscr{F}_{a,M}$ denotes the set of all bounded real-valued smooth functions ϕ on Ω such that

$$a(x, \nabla\phi(x)) \le 1 \quad \text{and} \quad \|D^i\phi\|_\infty \le M \tag{17}$$

for all non-negative multi-indices i such that $2 \le |i| \le m$.

Note that the Riemannian-type metrics $d_{m,\beta}$ are similar to the Finsler-type metrics induced by the polyharmonic operator $(-\Delta)^m|_{\mathrm{DIR}}$, whose symbol is

$$a(x,\xi) = |\xi|^{2m}.$$

Under the following assumptions, Barbatis [1] uses Finsler-type metrics to express a sharp heat kernel bound.

Assumptions 1.6:

(i) *H is uniformly elliptic and homogeneous of order $2m > N$;*

(ii) *The coefficients $a_{i,j}$ lie in the Sobolev space $W^{m,\infty}(\Omega)$;*

(iii) *The symbol of H is strongly convex (see Definition 1.7 below).*

Definition 1.7: *For $|k| = 2m$ define*

$$\alpha_k(x) = \frac{k!}{(2m)!} \sum_{\substack{|i|=|j|=m\\ i+j=k}} a_{i,j}(x).$$

We may rewrite the principal symbol of H as

$$a(x,\xi) = \sum_{|k|=2m} \frac{(2m)!}{k!} a_k(x)\xi^k$$

We say that the symbol $a(x,\xi)$ is strongly convex if the quadratic form

$$\Gamma(x,\zeta) = \sum_{\substack{|p|=m\\ |q|=m}} a_{p+q}(x)\zeta_p\zeta_q$$

is non-negative for each $x \in \Omega$, where $\zeta = (\zeta_p)_{|p|=m} \in \mathbb{R}^\nu$ and

$$\nu = \binom{n+m-1}{n-1}$$

is the number of distinct multi-indices p with $|p| = m$.

Barbatis proves that, under these assumptions, we have the heat kernel bound

$$|K(t,x,y)| \le c_\delta t^{-N/2m} \exp[-(\sigma_m - \delta) d_{a,M}(x,y)^{2m/(2m-1)} t^{-1/(2m-1)}] \tag{18}$$

for M large and for $t/d_{a,M} \le T_{\delta,M}$. We shall apply this result under the same assumptions, to find the sharp constant c_2 in the bound (13).

2. The Heat Kernel Bounds

We shall use the technique found in [4] of twisting the operator H to define $H_{\alpha\phi} = e^{\alpha\phi} H e^{-\alpha\phi}$, which has quadratic form

$$Q_{\alpha\phi}(f) = \int_\Omega \sum_{\substack{|i|\le m\\ |j|\le m}} a_{i,j}(x)\{e^{\alpha\phi} D^i e^{-\alpha\phi} f(x)\}\{\overline{e^{-\alpha\phi} D^j e^{\alpha\phi} f(x)}\} \mathrm{d}^N x. \tag{19}$$

Here $\alpha > 0$ and $\phi \in \mathscr{E}_{m,\beta}$ for some $\beta > 0$. The following proposition is a generalisation of [4, Lemmas 1 and 2], treating derivatives of the function ϕ more delicately.

Proposition 2.1: *The twisted quadratic form satisfies the inequality*

$$|Q_{\alpha\phi}(f) - Q(f)| \leq \epsilon Q(f) + c_\epsilon(1 + \alpha^{2m} + \beta^{2m})\|f\|_2^2, \tag{20}$$

where ϵ may be taken arbitrarily small.

Proof: Each term in $Q_{\alpha\phi}(f)$ may be expanded using formulae of the type

$$e^{\alpha\phi} D^i e^{-\alpha\phi} f = D^i f + \sum c_k \Big(\prod_{r=1}^{p} \alpha D^{k_r}\phi \Big) D^{k_0} f, \tag{21}$$

where the sum is taken over all integers p and non-negative multi-indices $k_0, \ldots, k_p$ such that

$$\sum_{r=0}^{p} k_r = i, \quad k_0 \neq i, \quad \text{and} \quad k_1, \ldots k_p \neq 0. \tag{22}$$

Combining these terms,

$$Q_{\alpha\phi}(f) = Q(f) + \int_\Omega {\sum}' c_{k,l}(x) \alpha^p \Big(\prod_{r=1}^{p} D^{k_r}\phi \Big) D^{k_0} f . \alpha^q \Big(\prod_{s=1}^{q} D^{l_r}\phi \Big) \overline{D^{l_0} f} \mathrm{d}^N x, \tag{23}$$

where the sum $\sum'$ is taken over all integers p, q and non-negative multi-indices $k_0, \ldots, k_p, l_0, \ldots, l_q$ such that

$$\sum_{r=0}^{p} |k_r| \leq m, \quad \sum_{s=0}^{q} |l_s| \leq m, \quad \text{and} \quad |k_0| + |l_0| \leq 2m - 1. \tag{24}$$

Hence

$$|Q_{\alpha\phi}(f) - Q(f)| \leq {\sum}' \|c_{k,l}\|_\infty \|\alpha^p \Big(\prod_{r=1}^{p} D^{k_r}\phi \Big) D^{k_0} f\|_2 \|\alpha^q \Big(\prod_{s=1}^{q} D^{l_s}\phi \Big) D^{l_0} f\|_2. \tag{25}$$

Now

$$\begin{aligned} \|\alpha^p \Big(\prod_{r=1}^{p} D^{k_r}\phi \Big) D^{k_0} f\|_2 &\leq \Big(\prod_{r=1}^{p} \|D^{k_r}\phi\|_\infty \Big) \|\alpha^p D^{k_0} f\|_2 \\ &\leq \Big(\prod_{r=1}^{p} \beta^{|k_r|-1} \Big) \|\alpha^p D^{k_0} f\|_2 \\ &= \|\alpha^p \beta^{s-p-|k_0|} D^{k_0} f\|_2, \end{aligned}$$

where $s := \sum_{r=0}^{p} k_r \leq m$, and so

$$\begin{aligned}\|\alpha^p\Big(\prod_{r=1}^{p} D^{k_r}\phi\Big)D^{k_0}f\|_2^2 &\leq \int_{\mathbb{R}^N} \alpha^{2p}\beta^{2s-2p-2|k_0|}(i\xi)^{2k_0}|\hat{f}(\xi)|^2 \mathrm{d}^N\xi \\ &\leq \int_{\mathbb{R}^N} \alpha^{2p}\beta^{2s-2p-2|k_0|}|\xi|^{2|k_0|}|\hat{f}(\xi)|^2 \mathrm{d}^N\xi \\ &\leq \int_{\mathbb{R}^N}[\epsilon|\xi|^{2s} + c_\epsilon(\alpha^{2s}+\beta^{2s})]|\hat{f}(\xi)|^2 \mathrm{d}^N\xi \qquad (26) \\ &\leq \int_{\mathbb{R}^N}[\epsilon|\xi|^{2m} + c_\epsilon(\alpha^{2m}+\beta^{2m}+1)]|\hat{f}(\xi)|^2 \mathrm{d}^N\xi \qquad (27)\end{aligned}$$

where ϵ may be arbitrarily small in (26) provided $|k_0| \leq s-1$ and arbitrarily small in (27) provided $|k_0| \leq m-1$. Each term in $\sum'$ has at least one of $|k_0| \leq m-1$ or $|l_0| \leq m-1$ and so is dominated by

$$\epsilon Q_m(f) + c_\epsilon(1+\alpha^{2m}+\beta^{2m})\|f\|_2^2.$$

Using the Gårding inequality (3) this in turn is dominated by

$$\epsilon Q(f) + c_\epsilon(1+\alpha^{2m}+\beta^{2m})\|f\|_2^2.$$ □

Note 2.2: If H is homogeneous (see Definition 1.1), then $2s = 2m$ and the above proof finishes with equation (26) instead of (27). This yields the twisted form inequality

$$|Q_{\alpha\phi}(f) - Q(f)| \leq \epsilon Q(f) + c_\epsilon(\alpha^{2m}+\beta^{2m})\|f\|_2^2, \qquad (28)$$

instead of equation (20). This will induce a corresponding change in inequalities (29), (30) and (31). □

Lemma 2.3: *There exist positive constants $c, k > 0$ such that*

$$\|e^{-H_{\alpha\phi}t}\| \leq \exp[k(1+\alpha^4+\beta^4)t] \qquad (29)$$

and

$$\|H_{\alpha\phi}e^{-H_{\alpha\phi}t}\| \leq ct^{-1}\exp[k(1+\alpha^4+\beta^4)t]. \qquad (30)$$

for all $t > 0$, $\alpha, \beta > 0$, and $\phi \in \mathscr{E}_{m,\beta}$.

Proof: See [4, Lemmas 6 and 7]. □

Lemma 2.4: *Let $d_{m,\beta}$ be the Riemannian-type metrics of Definition* 1.3. *There exist positive constants c_1, c_2, k such that*

$$|K(t,x,y)| \leq c_1 t^{-N/2m}\exp[-c_2 d_{m,\beta}(x,y)^{2m/(2m-1)}t^{-1/(2m-1)} + k(1+\beta^{2m})t] \qquad (31)$$

for all $\beta, t > 0$ and all $x, y \in \Omega$.

Proof: Put $f_t = \exp[-H_{\alpha\phi}t]f$. Then

$$\begin{aligned}
\|f_t\|_\infty &\le c\|(-\Delta)^{m/2}f_t\|_2^{N/2m}\|f_t\|_2^{1-N/2m} \\
&\le cQ(f_t)^{N/4m}\|f_t\|_2^{1-N/2m} \\
&\le c\{\operatorname{Re} Q_{\alpha\phi}(f_t) + (1+\alpha^{2m}+\beta^{2m})\|f_t\|_2^2\}^{N/4m}\|f_t\|_2^{1-N/2m} \\
&\le c\{\|H_{\alpha\phi}f_t\|_2\|f_t\|_2 + (1+\alpha^{2m}+\beta^{2m})\|f_t\|_2^2\}^{N/4m}\|f_t\|_2^{1-N/2m} \\
&\le c\{t^{-1} + (1+\alpha^{2m}+\beta^{2m})\}^{N/4m}\exp[k(1+\alpha^{2m}+\beta^{2m})t]\|f\|_2 \\
&\le ct^{-N/4m}\exp[k(1+\alpha^{2m}+\beta^{2m})t]\|f\|_2.
\end{aligned}$$

Therefore

$$\|\exp[-H_{\alpha\phi}t]\|_{\infty,2} \le ct^{-N/4m}\exp[k(1+\alpha^{2m}+\beta^{2m})t]. \tag{32}$$

By duality,

$$\|\exp[-H_{\alpha\phi}t]\|_{\infty,1} \le ct^{-N/2m}\exp[k(1+\alpha^{2m}+\beta^{2m})t]. \tag{33}$$

But $\exp[-H_{\alpha\phi}t]$ has kernel

$$K_{\alpha\phi(x)}(t,x,y) = e^{\alpha\phi(x)}K(t,x,y)e^{-\alpha\phi(x)} \tag{34}$$

for all $t > 0$ and $x, y \in \Omega$. Equivalently,

$$|K(t,x,y)| \le ct^{-N/2m}\exp[\alpha(\phi(y)-\phi(x)) + k(1+\alpha^{2m}+\beta^{2m})t]. \tag{35}$$

Taking the infimum over all $\phi \in \mathscr{E}_{m,\beta}$ in this bound, we see that

$$|K(t,x,y)| \le ct^{-N/2m}\exp[-\alpha d_{m,\beta}(x,y) + k(1+\alpha^{2m}+\beta^{2m})t].$$

Optimising with respect to α gives

$$|K(t,x,y)| \le ct^{-N/2m}\exp[-k' d_{m,\beta}(x,y)^{2m/(2m-1)}t^{-1/(2m-1)} + k(1+\beta^{2m})t]. \quad \square$$

Theorem 2.5: *Let $\Omega \subseteq \mathbb{R}^N$ be a region whose boundary is C^2, with radii of curvature uniformly bounded below, and suppose that $N < 2m$. There exist positive constants c_1, c_2, k such that*

$$|K(t,x,y)| \le c_1 t^{N/2m}\exp[-c_2 d_g(x,y)^{2m/(2m-1)}t^{-1/(2m-1)} + kt] \tag{36}$$

Proof: This follows by setting $\beta = 4K/r$ and applying the main result, Theorem 4.18, of Section 4. $\square$

Note 2.6: The statement (36) is equivalent to the existence of positive constants c_1, c_2, T such that for $t \leq T d_g(x,y)$

$$|K(t,x,y)| \leq c_1 t^{-N/2m} \exp[-c_2 d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)}] \tag{37}$$

See Lemma 3.1. $\square$

3. Sharp Constants For The Heat Kernel Bound

Lemma 3.1: *Let H be a uniformly elliptic operator acting in $L^2(\Omega)$ where $\Omega \subseteq \mathbb{R}^N$ and $N < 2m$. Let $K(t,x,y)$ be the heat kernel of H, and let c_2 be fixed. The following conditions are equivalent:*

(i) *For all $\epsilon > 0$ there exist positive constants c_1, T such that*

$$|K(t,x,y)| \leq c_1 t^{-N/2m} \exp[-(c_2 - \epsilon) d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)}] \tag{38}$$

for $t/d_g(x,y) < T$.

(ii) *For all $\epsilon > 0$ there exist positive constants c_1, k such that*

$$|K(t,x,y)| \leq c_1 t^{-N/2m} \exp[-(c_2 - \epsilon) d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)} + kt] \tag{39}$$

for $t > 0$ and $x, y \in \Omega$.

Proof: Suppose that (i) holds. Let $k = (c_2 - \epsilon) T^{-2m/(2m-1)} + 1$. Then for $t/d_g \geq T$

$$-(c_2 - \epsilon) d_g^{2m/(2m-1)} t^{-1/(2m-1)} \geq -kt + t$$

and hence using Lemma 1.2,

$$\begin{aligned} |K(t,x,y)| &\leq c t^{-N/2m} \exp[t] \\ &\leq c t^{-N/2m} \exp[-(c_2 - \epsilon) d_g^{2m/(2m-1)} t^{-1/(2m-1)} + kt]. \end{aligned}$$

Conversely, suppose that (ii) holds. Let $T = (\epsilon/k)^{(2m-1)/2m}$. For $t/d_g(x,y) \leq T$ we have $kt \leq \epsilon d_g^{2m/(2m-1)} t^{-1/(2m-1)}$ and so

$$|K(t,x,y)| \leq c_1 t^{-N/2m} \exp[-c_2 d_g^{2m/(2m-1)} t^{-1/(2m-1)} + \epsilon d_g^{2m/(2m-1)} t^{-1/(2m-1)}]$$

$$= c_1 t^{-N/2m} \exp[-(c_2-\epsilon) d_g^{2m/(2m-1)} t^{-1/(2m-1)}]. \qquad \square$$

Theorem 3.2: *Suppose that Ω is a region with C^2 boundary and whose radii of curvature are bounded below. Suppose also that H satisfies the assumptions* 1.6. *For $\epsilon > 0$ there exist positive constants c_ϵ, k_ϵ such that*

$$|K(t,x,y)| \leq c_\epsilon t^{-N/2m} \exp[-(\sigma_m - \epsilon)\mu^{-1/(2m-1)} d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)} + k_\epsilon t]. \tag{40}$$

Equivalently, for $\epsilon > 0$ there exist positive constants c_ϵ, T_ϵ such that

$$|K(t,x,y)| \leq c_\epsilon t^{-N/2m} \exp[-(\sigma_m - \epsilon)\mu^{-1/(2m-1)} d_g(x,y)^{2m/(2m-1)} t^{-1/(2m-1)}] \tag{41}$$

for $t/d_g(x,y) < T_\epsilon$.

Proof: For $\epsilon > 0$ fixed let $\beta \geq 4K/r$ be large enough such that the result (18) of Barbatis [1] is valid for $M := \mu^{-1/2m}\beta^{m-1}$, and such that

$$(\sigma_m - \epsilon) \leq (\sigma_m - \delta)\left(1 - \sqrt{\frac{K}{\beta r}}\right).$$

The Gårding inequality implies that the symbol $a(x,\xi)$ defined in equation (15) satisfies

$$\lambda|\xi|^{2m} \leq a(x,\xi) \leq \mu|\xi|^{2m}$$

for all $\xi \in \mathbb{R}^N$. Let $\phi \in \mathscr{E}_{m,\beta}$ and define $\psi = \mu^{-1/2m}\phi$. Then

$$a(x, \nabla\psi(x)) = \mu^{-1} a(x, \nabla\phi(x)) \leq |\nabla\phi(x))|^{2m} \leq 1$$

and

$$\|D^i\psi\|_\infty = \mu^{-1/2m}\|D^i\phi\|_\infty \leq \mu^{-1/2m}\beta^{|i|-1} \leq M,$$

so $\psi \in \mathscr{F}_{a,M}$. Hence

$$\begin{aligned} \mu^{-1/2m} d_{m,\beta}(x,y) &= \sup\{\mu^{-1/2m}\phi(y) - \mu^{-1/2m}\phi(x) : \phi \in \mathscr{E}_{m,\beta}\} \\ &\leq \sup\{\psi(y) - \psi(x) : \psi \in \mathscr{F}_{a,M}\} \\ &= d_{a,M}(x,y). \end{aligned} \tag{42}$$

For $t/d_g(x,y) \leq (1 - \sqrt{K/\beta r})\mu^{-1/2m} T_{\delta,M} =: T_\epsilon$ we see, using equation (42) and Theorem 4.18, that

$$t/d_{a,M}(x,y) \leq \mu^{1/2m} t/d_{m,\beta}(x,y)$$

$$\leq \mu^{1/2m}(1 - \sqrt{K/\beta r})^{-1} t/d_g(x, y)$$
$$\leq T_{\delta,M},$$

and so using (18),

$$\begin{aligned}
&|K(t, x, y)| \\
&\leq c_\delta t^{N/2m} \exp[-(\sigma_m - \delta) d_{a,M}(x, y)^{2m/(2m-1)} t^{-1/(2m-1)}] \\
&\leq c_\delta t^{N/2m} \exp[-(\sigma_m - \delta) \mu^{-1/(2m-1)} d_{m,\beta}(x, y)^{2m/(2m-1)} t^{-1/(2m-1)}] \\
&\leq c_\delta t^{N/2m} \exp[-(\sigma_m - \delta) \mu^{-1/(2m-1)} \left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g(x, y)^{2m/(2m-1)} t^{-1/(2m-1)}] \\
&\leq c_\epsilon t^{N/2m} \exp[-(\sigma_m - \epsilon) \mu^{-1/(2m-1)} d_g(x, y)^{2m/(2m-1)} t^{-1/(2m-1)}].
\end{aligned}$$

Using Lemma 3.1 this is equivalent to the bound (40). □

4. A Comparison Of Riemannian-Type Metrics With The Standard Riemannian Metric For A Highly Non-Convex Region

The purpose of this section is to prove a geometrical result concerning metrics, which can be applied in Section 2. The Riemannian-type metrics of Definition 1.3 are used in Section 2 to express certain Gaussian heat kernel estimates. It is beneficial to compare these metrics for different values of β in order to convert the bound into terms of the standard Riemannian metric on a region. The comparison is particularly interesting when one notes that $d_{m,0} = d_0$ is the standard Euclidean metric

$$d_0(x, y) = |y - x|, \tag{43}$$

and $d_{m,\infty}$ is the standard Riemannian metric d_g, defined in equation (12). If Ω is convex then all the above metrics are identical. For non-convex regions however this is not the case, and a useful comparison is non-trivial. Clearly $d_{m,\beta}$ is an increasing function of β.

Let Ω be a region in $\mathbb{R}^N$, whose boundary $\partial\Omega$ is C^2, with radii of curvature uniformly bounded below. We shall prove that for $\beta \geq 4K/r$,

$$\left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g \leq d_{m,\beta} \leq d_g \tag{44}$$

where $r > 0$ is the greatest lower bound of the radii of curvature of the boundary of Ω, and K depends only on m and N. This result is also valid for locally Euclidean Riemannian manifolds. See Note 4.19.

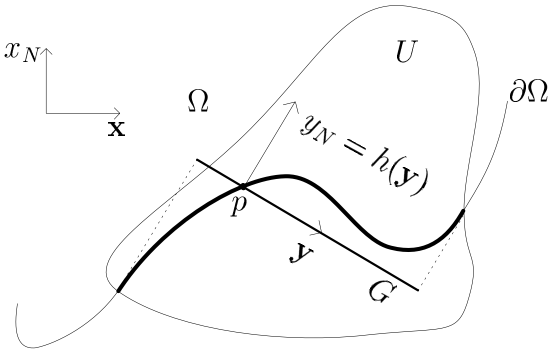

Figure 1: Representation of the boundary

We now develop tools for local representation of the surface $\partial\Omega$ of Ω.

Definition 4.1: *We say that a region $\Omega \subseteq \mathbb{R}^N$ has C^2 boundary $\partial\Omega$ if $\partial\Omega = \partial\overline{\Omega}$ and if for each point $p \in \partial\Omega$, there exist a set $U = U(p)$ open in $\mathbb{R}^N$ and containing p, a local coordinate system $\mathbf{y} = (y_1, \dots, y_{N-1})$ and y_N, with $(\mathbf{y}, y_N) = (\mathbf{0}, 0)$ at p, and a function $h = h(., p)$ such that $\partial\Omega \cap U$ has a representation*

$$y_N = h(\mathbf{y}) \qquad \mathbf{y} \in G, \quad h \in C^2(\overline{G}), \tag{45}$$

where $G = G(p)$ is open in $\mathbb{R}^{N-1}$ and convex.

The surface $\partial\Omega$ within U may equivalently be represented by the non-degenerate, bijective C^2 map $\sigma : G \to \partial\Omega \cap U$ defined in local coordinates by

$$\underline{\sigma}(\mathbf{y}) = (\mathbf{y}, h(\mathbf{y})). \tag{46}$$

Notation 4.2:

(i) *The matrix derivative of a function $f : \mathbb{R}^{N-1} \to \mathbb{R}^M$ is defined by*

$$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{y}} := \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_2}{\partial y_1} & \cdots & \frac{\partial f_M}{\partial y_1} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_1}{\partial y_{N-1}} & \cdots & \cdots & \frac{\partial f_M}{\partial y_{N-1}} \end{pmatrix} \tag{47}$$

where $f_i := \pi_i \circ f$ $(i = 1, \dots, M)$ are the coordinate functions of f;

(ii) *The second derivative of a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ is the matrix defined by*

$$\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2} := \frac{\mathrm{d}}{\mathrm{d}\mathbf{y}} \left(\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} \right) = \begin{pmatrix} \frac{\partial^2 h}{\partial y_1^2} & \cdots & \frac{\partial^2 h}{\partial y_1 \partial y_{N-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 h}{\partial y_{N-1} \partial y_1} & \cdots & \frac{\partial^2 h}{\partial y_{N-1}^2} \end{pmatrix}. \tag{48}$$

Let the y_N-axis point into Ω. The unit normal $n = n(p)$ to the surface at $p = \sigma(\mathbf{y})$ is defined in local coordinates by

$$\underline{n}(\mathbf{y}) = \left(\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}, -1\right)\left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-\frac{1}{2}}. \tag{49}$$

The C^1 map $\tau : G \times \mathbb{R} \to \mathbb{R}^N$ defined by

$$\tau(\mathbf{y}, u) = \sigma(\mathbf{y}) + un(\mathbf{y}) \tag{50}$$

is non-degenerate at $(\mathbf{0}, 0)$.

Define $\pi : \partial\Omega \times \mathbb{R} \to \partial\Omega \times \mathbb{R}$ by

$$\pi(p, u) = (p, 0) \qquad p \in \partial\Omega, \quad u \in \mathbb{R}, \tag{51}$$

and $\rho : \partial\Omega \times \mathbb{R} \to \mathbb{R}^N$ by

$$\rho(p, u) = p + un(p) \qquad p \in \partial\Omega, \quad u \in \mathbb{R}. \tag{52}$$

Definition 4.3: *For $\delta > 0$ we define the δ-neighbourhoods of $\partial\Omega$ and Ω by*

$$(\partial\Omega)_\delta := \{z \in \mathbb{R}^N : d(z, \partial\Omega) < \delta\} \tag{53}$$

and

$$\Omega_\delta := \{z \in \mathbb{R}^N : d(z, \Omega) < \delta\}. \tag{54}$$

Proposition 4.4: *The δ-neighbourhood $(\partial\Omega)_\delta$ of the boundary is the image of $\partial\Omega \times (-\delta, \delta)$ under the map ρ. Similarly $\Omega_\delta = \Omega \cup \rho(\partial\Omega \times [0, \delta))$.*

Proof: Suppose that $z = \rho(p, u) = p + un(p)$ for some $p \in \partial\Omega$ and some $-\delta < u < \delta$. Then $|z - p| = |u| < \delta$ so $d(z, \partial\Omega) < \delta$.

Conversely, suppose that $0 \leq d := d(z, \partial\Omega) < \delta$. Let $p \in \partial\Omega$ be such that $|z - p| = d$. Using the representation of the surface $\underline{\sigma}(\mathbf{y}) = (\mathbf{y}, h(\mathbf{y}))$ in the local coordinate system based at p we see that the function $\mathbf{y} \mapsto |\underline{z} - (\mathbf{y}, h(\mathbf{y}))|^2$ is minimized at $\mathbf{y} = \mathbf{0}$. Thus for $i = 1, \ldots, N-1$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial y_i}|\underline{z} - (\mathbf{y}, h(\mathbf{y}))|^2\bigg|_{\mathbf{y}=0} \\ &= 2\left\langle \underline{z} - (\mathbf{y}, h(\mathbf{y})), -\left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial h}{\partial y_i}\right)\right\rangle\bigg|_{\mathbf{y}=0} \end{aligned}$$

$$= -2\left\langle \underline{z} - (\mathbf{0}, 0), \left(0, \ldots, 0, 1, 0, \ldots, 0, \left.\frac{\partial h}{\partial y_i}\right|_{\mathbf{y}=0}\right)\right\rangle$$

so $z - p$ is normal to the surface. The vector $(z - p)/d$ has unit modulus so $(z - p)/d = \pm n(p)$, the sign being dependent on whether $(z - p)/d$ is inward or outward pointing. Thus $z = p \pm dn(p) \in \rho(\partial\Omega \times (-\delta, \delta))$.

The proof that $\Omega_\delta = \Omega \cup \rho(\partial\Omega \times [0, \delta))$ is similar. $\square$

Condition 4.5: *Let Ω be a region in $\mathbb{R}^N$ with C^2 boundary such that there exists an $r > 0$ whereby*

$$B(p \pm rn(p); r) \cap \partial\Omega = \emptyset \tag{55}$$

for all $p \in \partial\Omega$.

This condition is slightly stronger than requiring that the radii of curvature at points of the boundary are bounded below by r. This is done to exclude certain regions for which the results of this section still hold, but which require a more technical treatment. See Note 4.19.

Lemma 4.6: *Equation* (55) *in Condition* 4.5 *is equivalent to bijectivity of the restriction $\rho : \partial\Omega \times (-r, r) \to (\partial\Omega)_r$.*

Proof: Suppose that for some $p \in \partial\Omega$, $B(p + rn(p); r) \cap \partial\Omega \neq \emptyset$. By writing $B(p + rn(p); r) = \bigcup_{0<d<r} B_d$ where $B_d := B(p + dn(p); d)$ we see that $B_d \cap \partial\Omega \neq \emptyset$ for some $d < r$. Let $x \in B_d \cap \partial\Omega$. Then $d(x, p + dn(p)) < d$ so, by Proposition 4.4, $p + dn(p) \in (\partial\Omega)_d$. Hence

$$\rho(p, d) \in \rho(\partial\Omega \times (-d, d)),$$

and ρ is not injective. The same conclusion is drawn if $B(p - rn(p); r) \cap \partial\Omega \neq \emptyset$ for some $p \in \partial\Omega$.

Conversely suppose that the restriction of ρ is not injective. Then

$$p + un(p) = q + vn(q) \tag{56}$$

for some

$$(p, u) \neq (q, v). \tag{57}$$

Without loss of generality we may assume that $|v| \leq |u|$. Assume also that $0 \leq u < r$. Then

$$|p + rn(p) - q| = |vn(q) + (r - u)n(p)|$$

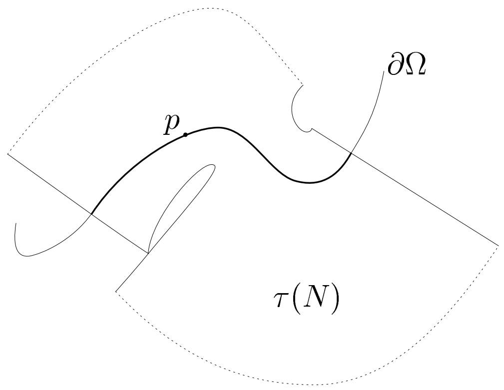

Figure 2: $\tau(N)$ for a typical region Ω.

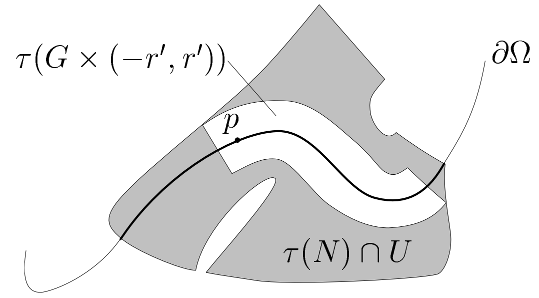

Figure 3: Construction of $G \times (-r', r')$.

$$\begin{aligned} &\leq |v| + r - u \\ &\leq r. \end{aligned} \tag{58}$$

Moreover, using inequality (56) we see that (57) implies that $un(p) \neq vn(q)$ and hence the inequality is strict. Thus

$$B(p + rn(p); r) \cap \partial\Omega \neq \emptyset. \tag{59}$$

If we assume that $-r < u \leq 0$ then similarly we obtain

$$B(p - rn(p); r) \cap \partial\Omega \neq \emptyset. \qquad \square$$

Proposition 4.7: *If Ω is bounded then it satisfies Condition* 4.5.

Proof: Since τ is non-degenerate at $(\mathbf{0}, 0)$, an application of the inverse function theorem shows that τ it is injective and non-degenerate in an open neighbourhood N of $(\mathbf{0}, 0)$. The set $\tau(N)$ is open because τ is non-degenerate, and since τ is continuous, $\tau^{-1}(\tau(N) \cap U)$ is open. See Figure 2. If G and $r' > 0$ are small enough,

$$G \times (-r', r') \subseteq \tau^{-1}(\tau(N) \cap U) \subseteq N.$$

Again, since τ is non-degenerate in N, the coordinate neighbourhood $V := \tau(G \times (-r', r'))$ is open and so there exists $r''_p > 0$ such that $B(p; \frac{1}{3}r''_p) \subseteq V$. See Figure 3. The collection $\{B(p; \frac{1}{3}r''_p) : p \in \partial\Omega\}$ forms an open covering of $\partial\Omega$ and has a finite subcovering $\{B(p_i; \frac{1}{3}r''_i)\}_{i=1}^m$ by compactness. Let $r = \frac{1}{3}\min_{i=1,\ldots,m} r''_i$.

This construction has been chosen so that if $p \in \partial\Omega$ then $p \in B(p_i; \frac{1}{3}r''_i)$ for some i and then $B(p; 2r) \subseteq B(p_i; r''_i) \subseteq V_i$.

We shall now prove that ρ is injective. For suppose otherwise, then $\rho(p,u) = \rho(q,v)$ where $p,q \in \partial\Omega$, $-r \le u,v < r$. By the construction above, $B(p;2r) \subseteq V_i$ for some i. Since $p + un(p) = q + vn(q)$ we see that $|p-q| \le |u| + |v| < 2r$ so $q \in V_i$. Thus $p,q \in V_i \cap \partial\Omega$ and so

$$\begin{aligned} p &= \tau_i(\mathbf{x},0) = \sigma_i(\mathbf{x}) \\ q &= \tau_i(\mathbf{y},0) = \sigma_i(\mathbf{y}) \end{aligned}$$

for some $\mathbf{x},\mathbf{y} \in G_i$. Now $\tau_i(\mathbf{x},u) = \rho(p,u) = \rho(q,v) = \tau_i(\mathbf{y},v)$, and since τ_i is injective we see that $\mathbf{x} = \mathbf{y}$ and $u = v$. Moreover, $p = \sigma_i(\mathbf{x}) = \sigma_i(\mathbf{y}) = q$. $\square$

From this point onwards we shall assume that all regions satisfy Condition 4.5.

Definition 4.8: *We say that a real symmetric $(N-1)\times(N-1)$ matrix A is non-negative, and write $A \ge 0$ if $\mathbf{a}A\mathbf{a}^{\mathrm{T}} \ge 0$ for all $\mathbf{a} \in \mathbb{R}^{N-1}$.*

Proposition 4.9: *Let $p \in \partial\Omega$ and let $h : G \to \mathbb{R}$ be the representation of the surface $\partial\Omega$ in the local coordinate system $(\mathbf{y}, y_N)$ based about p. Then*

$$\text{(i)} \quad I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} \ge r n_N \frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}; \tag{60}$$

$$\text{(ii)} \quad I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} \ge -r n_N \frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}, \tag{61}$$

where

$$n_N := -\left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1/2} \tag{62}$$

is the N-th entry of the unit normal $\underline{n}(\mathbf{y})$ in local coordinates.

Proof: Let $\underline{\tilde{\sigma}}(\mathbf{y}) = (\mathbf{y}, g(\mathbf{y}))$ represent, in local coordinates, the surface of the sphere, radius r touching the surface $(\mathbf{y}, h(\mathbf{y}))$ at the point $\underline{q} = (\mathbf{y_0}, h(\mathbf{y_0}))$ where the unit normal is $\underline{n} = (\mathbf{n}, n_N)$. Then

$$(\underline{\tilde{\sigma}}(\mathbf{y}) - \underline{q})(\underline{\tilde{\sigma}}(\mathbf{y}) - 2r\underline{n} - \underline{q})^{\mathrm{T}} = 0. \tag{63}$$

Differentiating with respect to $\mathbf{y} = (y_1, \ldots, y_{N-1})$ we see that

$$\begin{aligned} \mathbf{0}^{\mathrm{T}} &= \left(I_{N-1} \middle| \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}\right)(\underline{\tilde{\sigma}}(\mathbf{y}) - 2r\underline{n} - \underline{q})^{\mathrm{T}} + \left(I_{N-1} \middle| \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}\right)(\underline{\tilde{\sigma}}(\mathbf{y}) - \underline{q})^{\mathrm{T}} \\ &= 2\left(I_{N-1} \middle| \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}\right)(\underline{\tilde{\sigma}}(\mathbf{y}) - r\underline{n} - \underline{q})^{\mathrm{T}} \\ &= 2(\mathbf{y}^{\mathrm{T}} - r\mathbf{n}^{\mathrm{T}} - \mathbf{y_0}^{\mathrm{T}}) + 2(g - rn_N - h(\mathbf{y_0}))\frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}. \end{aligned} \tag{64}$$

Differentiating the transpose,

$$\mathbf{0} = I_{N-1} + \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} + (g - rn_N - h(\mathbf{y_0}))\frac{\mathrm{d}^2 g}{\mathrm{d}\mathbf{y}^2}. \tag{65}$$

Since the sphere touches the surface at $\mathbf{y} = \mathbf{y_0}$, we see that

$$g(\mathbf{y_0}) = h(\mathbf{y_0}), \quad \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}(\mathbf{y_0}) = \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}(\mathbf{y_0}), \quad \text{and} \quad \frac{\mathrm{d}^2 g}{\mathrm{d}\mathbf{y}^2}(\mathbf{y_0}) \le \frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}(\mathbf{y_0}). \tag{66}$$

Thus, at $\mathbf{y} = \mathbf{y_0}$,

$$I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} = I_{N-1} + \frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}g}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} = rn_N\frac{\mathrm{d}^2 g}{\mathrm{d}\mathbf{y}^2} \ge rn_N\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}. \tag{67}$$

This result holds for all $\mathbf{y_0} \in G$. The proof of part (ii) uses the fact that a ball of radius r fits inside the region. $\square$

For $\mathbf{a} \in \mathbb{R}^{N-1}$ define

$$\alpha = \frac{\left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1/2}\mathbf{a}\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}\mathbf{a}^{\mathrm{T}}}{\mathbf{a}\left(I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}}, \qquad \beta = \frac{\mathbf{a}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}}{\mathbf{a}\left(I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}}, \tag{68}$$

and let A be the real symmetric matrix

$$A = I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}. \tag{69}$$

Lemma 4.10:

(i) $\displaystyle \frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} = \left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}$;

(ii) $\alpha^2, \beta \in \left[0, 1/r^2\right]$;

(iii) $1 + 2\alpha r + \beta r^2 \ge 0$ *and* $1 - 2\alpha r + \beta r^2 \ge 0$;

(iv) $\displaystyle \left(1 - \frac{|u|}{r}\right)^2 \le 1 + 2\alpha u + \beta u^2 \le \left(1 + \frac{|u|}{r}\right)^2$ *for all* $u \in [-r, r]$.

Proof: (i) By differentiating the expression for the normal, we see that

$$\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}} = \left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-3/2}\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}\left[\left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)\left(I_{N-1}|\mathbf{0}^{\mathrm{T}}\right) - \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\left(\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}, -1\right)\right]$$

$$= \left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\left[\left(I_{N-1}|\mathbf{0}^{\mathrm{T}}\right)-\left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\left(\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}},-1\right)\right].$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} &= \left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\left[I_{N-1}-2\left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right.\\
&\quad\left.+\left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-2}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right]\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\\
&= \left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\left[I_{N-1}-\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\right]\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\\
&= \left(1+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}.
\end{aligned}$$

(ii) Proposition 4.9 immediately implies that

$$-\frac{1}{r}\le\alpha\le\frac{1}{r}.$$

Since A is a positive definite matrix, we may pre- and post-multiply the results of Proposition 4.9 by $A^{-1/2}$ to get

$$I_{N-1}+rn_NA^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1/2}\ge 0; \tag{70}$$

$$I_{N-1}-rn_NA^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1/2}\ge 0. \tag{71}$$

The matrices in inequalities (70) and (71) commute, so their product is also positive. Hence

$$I_{N-1}-r^2n_N^2A^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1/2}\ge 0. \tag{72}$$

By pre- and post-multiplying by $A^{1/2}$, we see that

$$\left(I_{N-1}+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)-r^2n_N^2\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\ge 0, \tag{73}$$

so using part (i),

$$r^2\mathbf{a}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}=r^2n_N^2\mathbf{a}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\mathbf{a}^{\mathrm{T}}$$

$$\leq \mathbf{a}\left(I_{N-1}+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}.$$

Thus $\beta \leq 1/r^2$.

(iii) Squaring the left hand side matrix of inequality (70) will yield a positive matrix

$$I_{N-1}+2rn_N A^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1/2}+r^2n_N^2A^{-1/2}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1/2}.$$

Pre- and post-multiplying by $A^{1/2}$, we see that

$$\mathbf{a}\left(I_{N-1}+\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}+2rn_N\mathbf{a}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\mathbf{a}^{\mathrm{T}}+r^2n_N^2\mathbf{a}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}A^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\mathbf{y}^2}\mathbf{a}^{\mathrm{T}}\geq 0$$

for all $\mathbf{a}\in\mathbb{R}^{N-1}$. Thus $1+2\alpha r+\beta r^2\geq 0$. Squaring the left hand side of inequality (71), we see that $1-2\alpha r+\beta r^2\geq 0$.

(iv) By part (iii), the polynomial $1+2\alpha u+\beta u^2$ dominates

$$1-2\frac{u}{r}+\frac{u^2}{r^2}$$

at $u=0,r$. Moreover, since $\beta\leq 1/r^2$, this holds true for all $u\in[0,r]$. Thus

$$1+2\alpha u+\beta u^2\geq\left(1-\frac{u}{r}\right)^2$$

for all $u\in[0,r]$. Similarly,

$$1+2\alpha u+\beta u^2\geq\left(1+\frac{u}{r}\right)^2$$

for all $u\in[-r,0]$. Also, by part (ii),

$$1+2\alpha u+\beta u^2\leq 1+2\frac{|u|}{r}+\frac{|u|^2}{r^2}=\left(1+\frac{|u|}{r}\right)^2.$$

□

Let $P:G\times(-r,r)\to G\times(-r,r)$ be the projection defined by $P(\mathbf{y},u)=(\mathbf{y},0)$. Let

$$\tau'(.,.):G\times(-r,r)\to M_N(\mathbb{R}) \tag{74}$$

denote the Jacobian matrix

$$\frac{\mathrm{d}\tau}{\mathrm{d}(\mathbf{y},u)}$$

of τ.

Proposition 4.11: *Let $(\mathbf{y}, u) \in G \times (-r, r)$ and let $v = (\mathbf{a}, a_N) \in \mathbb{R}^N$. Then*

$$\left(1 - \frac{|u|}{r}\right) |P(v)(\tau' \circ P)(\mathbf{y}, u)| \leq |v\tau'(\mathbf{y}, u)|. \tag{75}$$

An alternative formulation of the above in local coordinates is

$$\mathbf{a}\left(I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}\left(1 - \frac{|u|}{r}\right)^2 \leq |v\underline{\tau}'(\mathbf{y}, u)|^2. \tag{76}$$

Proof:

$$\begin{aligned}
|v\,\underline{\tau}'(\mathbf{y}, u)|^2 &= \left| v \left[\left(\begin{array}{c|c} I_{N-1} & \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}} \\ \hline \multicolumn{2}{c}{\underline{n}(\mathbf{y})} \end{array} \right) + u \left(\begin{array}{c} \frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}} \\ \hline \mathbf{0} \end{array} \right) \right] \right|^2 \\
&= v \left(\begin{array}{c|c} I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \hline \mathbf{0} & 1 \end{array} \right) v^{\mathrm{T}} \\
&\qquad + 2uv \left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1/2} \left(\begin{array}{c|c} \frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2} & \mathbf{0}^{\mathrm{T}} \\ \hline \mathbf{0} & 0 \end{array} \right) v^{\mathrm{T}} \\
&\qquad + u^2 v \left(\begin{array}{c|c} \frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \hline \mathbf{0} & 0 \end{array} \right) v^{\mathrm{T}} \\
&= \mathbf{a}\left(I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}} + a_N^2 \\
&\qquad + 2u\left(1 + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\right)^{-1/2} \mathbf{a}\frac{\mathrm{d}^2 h}{\mathrm{d}\mathbf{y}^2}\mathbf{a}^{\mathrm{T}} + u^2 \mathbf{a}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\underline{n}}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}} \\
&= \mathbf{a}\left(I_{N-1} + \frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}h}{\mathrm{d}\mathbf{y}}^{\mathrm{T}}\right)\mathbf{a}^{\mathrm{T}}[1 + 2\alpha u + \beta u^2] + a_N^2
\end{aligned}$$

The proposition follows by applying Lemma 4.10. □

We shall now remove references to local parametrisations of the boundary. If a point $z \in \mathbb{R}^N$ is within a distance r of $\partial\Omega$ then there is a unique nearest point of $\partial\Omega$ to z. We shall define maps ν_1, ν_2 which formalize the notion of a nearest point to $\partial\Omega$ and $\overline{\Omega}$ respectively. Define $\nu_1 : (\partial\Omega)_r \to \partial\Omega$ by

$$\nu_1 := \rho\pi\rho^{-1}. \tag{77}$$

This is well defined due to Lemma 4.6. Define $\nu_2 : \Omega_r \to \overline{\Omega}$ by

$$\nu_2(\omega) = \begin{cases} \omega, & \omega \in \Omega \\ \nu_1(\omega), & \omega \in \rho(\partial\Omega \times [0, r)). \end{cases} \tag{78}$$

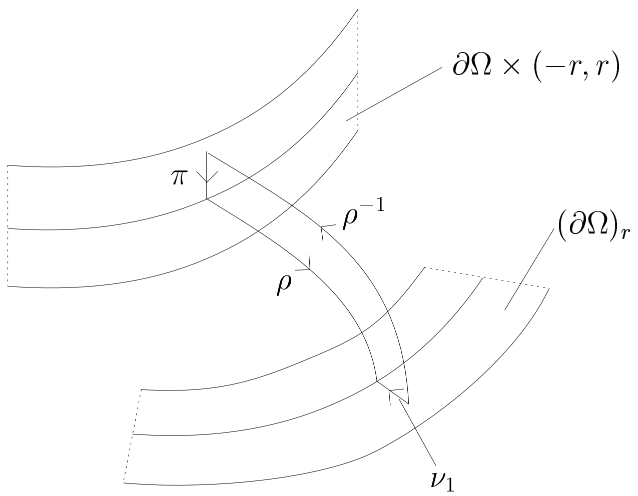

Figure 4: Construction Of ν_1

Note that $\rho(\partial\Omega \times [0, r)) \cap \Omega = \emptyset$ because of the construction of ρ.

Let $x \in \Omega$ and define $d_x : \Omega_r \to \mathbb{R}_+$ by

$$d_x(z) := d_g(x, \nu_2 z) \tag{79}$$

where we recall that the standard Riemannian metric $d_g : \overline{\Omega}^2 \to \mathbb{R}_+$ is defined by

$$d_g(x, y) := \inf\{l(\gamma) : \gamma(0) = x, \gamma(1) = y, \gamma \subseteq \overline{\Omega}, \gamma \text{ is cts and piecewise } C^1\}. \tag{80}$$

Construction 4.12: *Let $0 \le \delta \le r$ be fixed. Let γ_1 satisfy either of the following conditions, and define γ_2 accordingly.*

(i) *Let $\gamma_1 : [0, 1] \to (\partial\Omega)_\delta$ be a C^1 curve such that $\gamma_1 \nsubseteq \Omega$. Let*

$$T_0 = \inf\{t \in [0, 1] : \gamma_1(t) \notin \Omega\} \tag{81}$$

and let

$$T_1 = \sup\{t \in [0, 1] : \gamma_1(t) \notin \Omega\}. \tag{82}$$

Define $\gamma_2 : [0, 1] \to \overline{\Omega}$ by

$$\gamma_2(t) = \begin{cases} \nu_1\gamma_1(t), & t \in (T_0, T_1) \\ \gamma_1(t), & \textit{otherwise}; \end{cases} \tag{83}$$

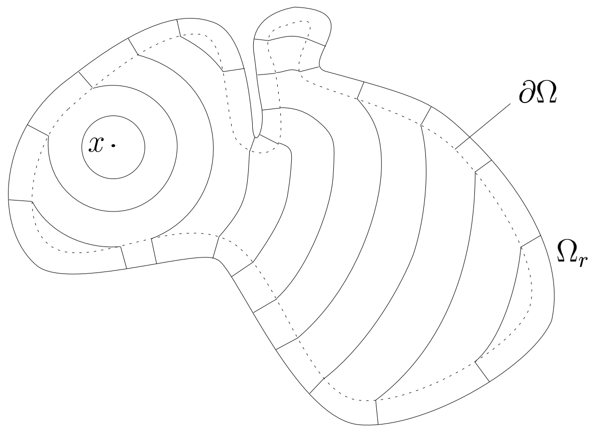

Figure 5: Contour sketch of d_x for a typical region Ω

(ii) *Let $\gamma_1 : [0,1] \to \Omega$ be a C^1 curve. Then let $\gamma_2 = \gamma_1$.*

Lemma 4.13: *In both cases of Construction 4.12, γ_2 is a piecewise C^1 curve such that $\gamma_2 \subseteq \overline{\Omega}$, $\gamma_2(0) = \nu_2[\gamma_1(0)]$, $\gamma_2(1) = \nu_2[\gamma_1(1)]$ and*

$$\left(1 - \frac{\delta}{r}\right) l(\gamma_2) \leq l(\gamma_1). \tag{84}$$

Proof: We shall only prove the lemma for case (i), as case (ii) is trivial. For $t \in (T_0, T_1)$ let $p \in \partial\Omega$ be the nearest point of $\partial\Omega$ to $\gamma_1(t)$ and let $(\mathbf{y}, h(\mathbf{y}))$ represent the surface $\partial\Omega$ in the local coordinate system based at p. Since γ_1 is continuous, there exists an open interval $I \subseteq (T_0, T_1)$, containing t such that $\gamma_1(I) \subseteq \tau(G \times (-r, r))$. Let γ_3, γ_4 be paths defined, for $s \in I$, by

$$\gamma_3 := \tau^{-1}\gamma_1 \qquad \gamma_4 := P\gamma_3. \tag{85}$$

Note that $\gamma_2|_I = \tau\gamma_4|_I$ and that $\gamma_2, \gamma_3, \gamma_4$ are C^1 on I because τ is C^1. By writing $\gamma_3(s) = (\mathbf{a}(s), a_N(s))$ we see that $|a_N(t)| = d(\gamma_1(t), \partial\Omega) < \delta$. Thus by Proposition 4.11,

$$\begin{aligned}\left(1 - \frac{\delta}{r}\right) |\gamma_2'(t)| &< \left(1 - \frac{|a_N(t)|}{r}\right) |\gamma_4'(t)\tau'(\gamma_4(t))| \\ &= \left(1 - \frac{|a_N(t)|}{r}\right) |P\gamma_3'(t)(\tau' \circ P)(\gamma_3(t))| \\ &\leq |\gamma_3'(t)\tau'(\gamma_3(t))|\end{aligned}$$

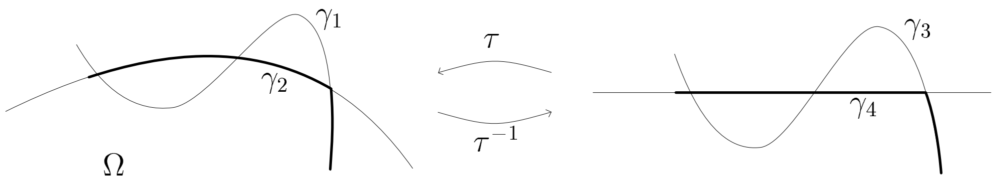

Figure 6: Construction of the paths γ_1, γ_2, γ_3, andγ_4

$$= |\gamma_1'(t)|.$$

Integration yields the result. □

Corollary 4.14:

(i) *Let $z_1 \in \Omega_\delta$, where $0 < \delta \leq r$. Then*

$$\limsup_{z_2 \to z_1} \frac{|d_x(z_2) - d_x(z_1)|}{|z_2 - z_1|} \left(1 - \frac{\delta}{r}\right) \leq 1; \tag{86}$$

(ii) *Let $x \in \Omega$ and let $z \in B(x; \delta)$, where $0 < \delta \leq r$. Then*

$$d_g(x, \nu_2 z) \leq \left(\frac{1}{\delta} - \frac{1}{r}\right)^{-1}; \tag{87}$$

(iii) *Let $x, y \in \Omega$ be such that $d_g(x, y) < 2r$. Then*

$$|y - x| \geq \frac{2r d_g(x, y)}{2r + d_g(x, y)}. \tag{88}$$

Proof: (i) Let $z_2 \in B(z_1; \delta - d(z_1, \overline{\Omega}))$ and let γ_1 be the straight line joining z_1 and z_2. Then γ_1 satisfies one of the two conditions in Construction 4.12. Accordingly,

$$\begin{aligned}
|d_x(z_2) - d_x(z_1)| &= |d_g(x, \nu_2 z_2) - d_g(x, \nu_2 z_1)| \\
&\leq d_g(\nu_2 z_1, \nu_2 z_2) \\
&\leq l(\gamma_2) \\
&\leq \left(1 - \frac{\delta}{r}\right)^{-1} l(\gamma_1) \\
&= \left(1 - \frac{\delta}{r}\right)^{-1} |z_2 - z_1|.
\end{aligned}$$

(ii) Let γ_1 be the straight line joining x and z. Then γ_1 satisfies one of the conditions in Construction 4.12. Hence, by Lemma 4.13

$$d_g(x, \nu_2 z)\left(1 - \frac{\delta}{r}\right) \le \left(1 - \frac{\delta}{r}\right) l(\gamma_2) \le l(\gamma_1) = \delta.$$

(iii) Let γ_1 be the straight line joining x and y. Then setting $\delta = \frac{1}{2}|y - x| < r$, we see that γ_1 satisfies one of the conditions in Construction 4.12. Hence, by Lemma 4.13

$$\left(1 - \frac{\frac{1}{2}|y - x|}{r}\right) d_g(x, y) \le |y - x|.$$

Rearranging this gives the desired inequality. $\square$

Let k_δ be approximate identities, defined as follows:

Definition 4.15: *Let $B(0;1)$ denote the unit ball in $\mathbb{R}^N$, and let $k : B(0;1) \to \mathbb{R}$ be smooth, non-negative and have unit integral. For $\delta > 0$ define $k_\delta : B(0;\delta) \to \mathbb{R}$ by*

$$k_\delta(z) = \delta^{-N} k\left(\frac{z}{\delta}\right). \tag{89}$$

Let $x \in \Omega$ and let $\beta > K/r$ where

$$K = K_{m,N,k} := \sup_{1 \le |j| \le m-1} \left(\int_{B(0;1)} |D^j k|\right)^{1/|j|}. \tag{90}$$

Define $f_{m,\beta,x} : \Omega \to \mathbb{R}$ by

$$\begin{aligned} f_{m,\beta,x}(y) &:= \left(1 - \frac{K}{\beta r}\right) \int_{B(y;K/\beta)} d_x(z) k_{K/\beta}(z - y) \mathrm{d}^N z \\ &= \left(1 - \frac{K}{\beta r}\right) \int_{B(0;K/\beta)} d_x(z + y) k_{K/\beta}(z) \mathrm{d}^N z. \end{aligned} \tag{91}$$

Lemma 4.16: *The functions $f_{m,\beta,x}$ belong to the class $\mathscr{E}_{m,\beta}$ (see Definition 1.3).*

Proof: Suppose that $y_1, y_2 \in \Omega$ and that $|j|$ is a non-negative multi-index such that $1 \le |j| \le m - 1$. Then by Corollary 4.14 (i),

$$\limsup_{y_2 \to y_1} \frac{|D^j f_{m,\beta,x}(y_2) - D^j f_{x,\beta}(y_1)|}{|y_2 - y_1|}$$

$$
\begin{aligned}
&= \limsup_{y_2 \to y_1} \frac{\left| \left(1 - \frac{K}{\beta r}\right) \int_{B(0;K/\beta)} \{d_x(z+y_2) - d_x(z+y_1)\} D^j k_{K/\beta}(z) \mathrm{d}^N z \right|}{|y_2 - y_1|} \\
&\leq \limsup_{y_2 \to y_1} \int_{B(0;K/\beta)} \frac{|d_x(z+y_2) - d_x(z+y_1)|}{|(z+y_2) - (z+y_1)|} \left(1 - \frac{K}{\beta r}\right) |D^j k_{K/\beta}(z)| \mathrm{d}^N z \\
&= \int_{B(0;K/\beta)} \left[\limsup_{y_2 \to y_1} \frac{|d_x(z+y_2) - d_x(z+y_1)|}{|(z+y_2) - (z+y_1)|} \left(1 - \frac{K}{\beta r}\right) \right] |D^j k_{K/\beta}(z)| \mathrm{d}^N z \\
&\leq \int_{B(0;K/\beta)} |D^j k_{K/\beta}(z)| \mathrm{d}^N z = K^{-|j|} \beta^{|j|} \int_{B(0;1)} |D^j k(z)| \mathrm{d}^N z \leq \beta^{|j|}.
\end{aligned}
$$

Since $f_{m,\beta,x}$ is C^∞ and satisfies the above inequality we see that $|\nabla f_{m,\beta,x}(y).\mathbf{u}^{\mathrm{T}}| \leq 1$ for all $\mathbf{u} \in \mathbb{R}^N$ with $|\mathbf{u}| = 1$, by taking $j = 0$, and hence that $|\nabla f_{m,\beta,x}(y)| \leq 1$. Also, letting $j \leq i$ be any non-negative multi-index such that $|j| = |i| - 1$,

$$
|D^i f_{m,\beta,x}(y)| \leq \beta^{|i|-1}. \qquad \square
$$

Recall the definition of the Riemannian-type metrics $d_{m,\beta} : \Omega^2 \to \mathbb{R}_+$

$$
d_{m,\beta}(x,y) := \sup\{\phi(y) - \phi(x) : \phi \in \mathscr{E}_{m,\beta}\} \tag{92}
$$

Lemma 4.17: *Let Ω satisfy Condition 4.5. Let $x, y \in \Omega$ and let $\beta > K/r$. Then*

$$
d_{m,\beta}(x,y) \geq d_g(x,y) \left(1 - \frac{K}{\beta r}\right) - \frac{2K}{\beta}. \tag{93}
$$

Proof: If $z \in B(x; K/\beta)$ then by Corollary 4.14 (ii),

$$
d_x(z) = d_g(x, \nu_{\overline{\Omega}} z) \leq \left(\frac{\beta}{K} - \frac{1}{r}\right)^{-1},
$$

so

$$
f_{m,\beta,x}(x) = \int_{B(x;K/\beta)} d_x(z) \left(1 - \frac{K}{\beta r}\right) k_{K/\beta}(z - x) \mathrm{d}^N z \leq \frac{K}{\beta}.
$$

Similarly, if $z \in B(y; K/\beta)$ then

$$
d_x(z) = d_g(x, \nu_{\overline{\Omega}} z) \geq d_g(x,y) - d_g(y, \nu_{\overline{\Omega}} z) \geq d_g(x,y) - \left(\frac{\beta}{K} - \frac{1}{r}\right)^{-1},
$$

so

$$
\begin{aligned}
f_{m,\beta,x}(y) &= \int_{B(y;K/\beta)} d_x(z)\left(1-\frac{K}{\beta r}\right) k_{K/\beta}(z-y)\mathrm{d}^N z\\
&\geq d_g(x,y)\left(1-\frac{K}{\beta r}\right)-\frac{K}{\beta}.
\end{aligned}
$$

Thus using Lemma 4.16,

$$
d_{m,\beta}(x,y) \geq f_{m,\beta,x}(y)-f_{m,\beta,x}(x) \geq d_g(x,y)\left(1-\frac{K}{\beta r}\right)-\frac{2K}{\beta}. \qquad \square
$$

The dependence of K upon k can be removed by taking the infimum of all values of $K_{m,N,k}$, where k is smooth, non-negative, supported in the unit ball and has unit integral. For example, $K_{2,N}=N^2$.

Theorem 4.18: *Let Ω satisfy Condition 4.5. For $\beta \geq 4K/r$ we have*

$$
\left(1-\sqrt{\frac{K}{\beta r}}\right) d_g \leq d_{m,\beta} \leq d_g.
$$

Proof: Let $\gamma \subseteq \overline{\Omega}$, $\gamma(0)=x$, $\gamma(1)=y$, γ is piecewise C^1 and let ϕ be such that $|\nabla\phi| \leq 1$. Then

$$
\begin{aligned}
|\phi(y)-\phi(x)| &= |\phi(\gamma(1))-\phi(\gamma(0))|\\
&= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t))\mathrm{d}t\right|\\
&= \left|\int_0^1 \nabla\phi.\gamma'(t)\mathrm{d}t\right|\\
&\leq \int_0^1 |\nabla\phi||\gamma'(t)|\mathrm{d}t\\
&\leq l(\gamma).
\end{aligned}
$$

Taking the supremum over all ϕ in $\mathscr{E}_{m,\beta}$, we see that

$$
d_{m,\beta}(x,y) \leq d_g(x,y).
$$

Let $\beta > 4K/r$ and let $\epsilon := \sqrt{K/\beta r} < 1/2$. For large distances $d_g(x,y)$, Lemma 4.17 is a useful result. We therefore consider two cases:

(i) Suppose $d_g(x,y) \le 2\epsilon r/(1-\epsilon)$. Then we may use Corollary 4.14 (iii) to obtain

$$\begin{aligned} d_{m,\beta}(x,y) &\ge |y-x| \\ &\ge \frac{2r d_g(x,y)}{2r + d_g(x,y)} \\ &\ge \frac{2r d_g(x,y)}{2r/(1-\epsilon)} \\ &= \left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g(x,y). \end{aligned}$$

(ii) For $d_g(x,y) \ge 2\epsilon r/(1-\epsilon)$ we see that

$$\begin{aligned} d_g(x,y)\left(\epsilon - \frac{K}{r\beta}\right) &\ge \frac{2\epsilon r}{1-\epsilon}(\epsilon - \epsilon^2) \\ &= 2\epsilon^2 r \\ &= \frac{2K}{\beta}. \end{aligned}$$

Hence

$$d_g(x,y)\left(1 - \frac{K}{r\beta}\right) - \frac{2K}{\beta} \ge (1-\epsilon) d_g(x,y)$$

Using Lemma 4.17,

$$d_{m,\beta}(x,y) \ge \left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g(x,y).$$

In both cases, we see that

$$\left(1 - \sqrt{\frac{K}{\beta r}}\right) d_g(x,y) \le d_{m,\beta}(x,y) \le d_g(x,y).$$

□

Note 4.19: Suppose that Ω is an N-dimensional locally Euclidean Riemannian manifold which posesses a locally injective isometric mapping into $\mathbb{R}^N$ (i.e. Ω is a covering space of some non-simply connected open subset of $\mathbb{R}^N$). The notion of a Euclidean metric in such manifolds degenerates, and so heat kernel bounds involving such metrics are not useful. Heat kernel bounds involving the Riemannian metric can be found by adapting the result of this section and using the methods of Section 2. In fact, by using a covering space, it can be seen that

Theorem 4.18 is valid where r is the greatest lower bound of the radii of curvature at all points of the boundary. $\square$

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