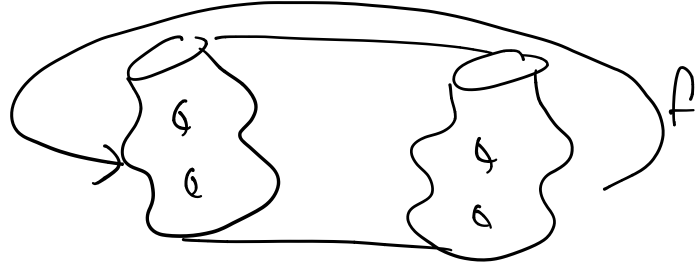
EVERYTHING SMOOTH

Background



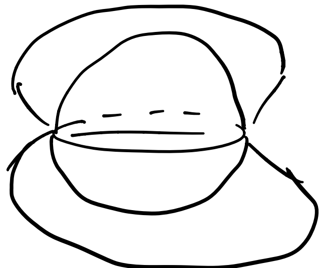
$K \subset S^3$ knot

K <u>fibered</u> if $S^3 \setminus \nu(K) = \underbrace{\Sigma}_{\text{Seifert surface}} \times I \,/\, (x,1) \sim (f(x), 0)$

$f : \Sigma \xrightarrow{\cong} \Sigma$

$= \Sigma \times_f S^1$

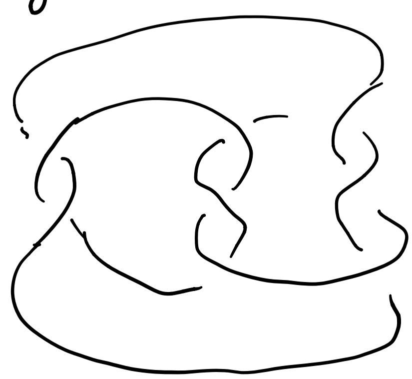
e.g. unknot U



$S^3 \setminus \nu(U)$

$= D^2 \times S^1 \quad (f = id)$

e.g. 8_{20} (put up special board)



e.g. torus knot $K_{p,q}$

$\{x^p + y^q = 0\} \cap S^3$ in $\mathbb{C}^2$

$g : S^3 \setminus \nu(K_{p,q}) \xrightarrow{\text{fibration}} S^1$

$(x,y) \longmapsto \dfrac{x^p + y^q}{|x^p + y^q|}$

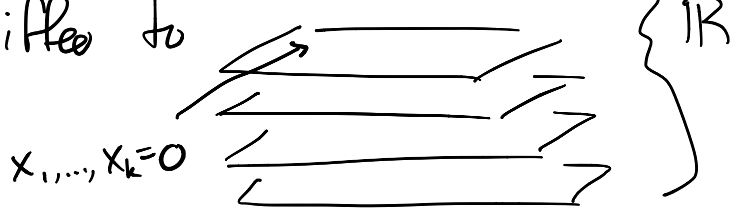
Σ

genus-2

Why natural to study:

- Study knots topologically from $S^3 \setminus \nu(K)$ .. simplest case
- Σ min genus surface for knot K $\underset{\text{Gabai 1980's}}{\Rightarrow}$ taut foliation on $S^3 \setminus \nu(K)$ w/ Σ = leaf

codim-k foliation = decomp of M^n into $(n-k)$-mfds ← leaves so of M^n locally diffeo to



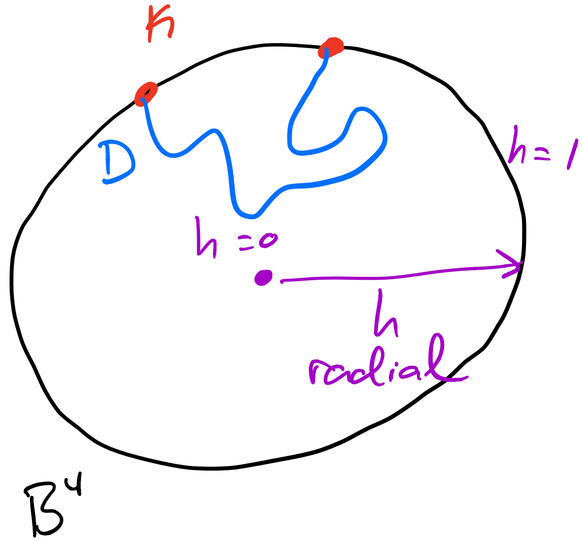
Thm (Classical but e.g. Novikov)

If codim-1 foliation has only compact leaves, then it's a <u>fibration</u>

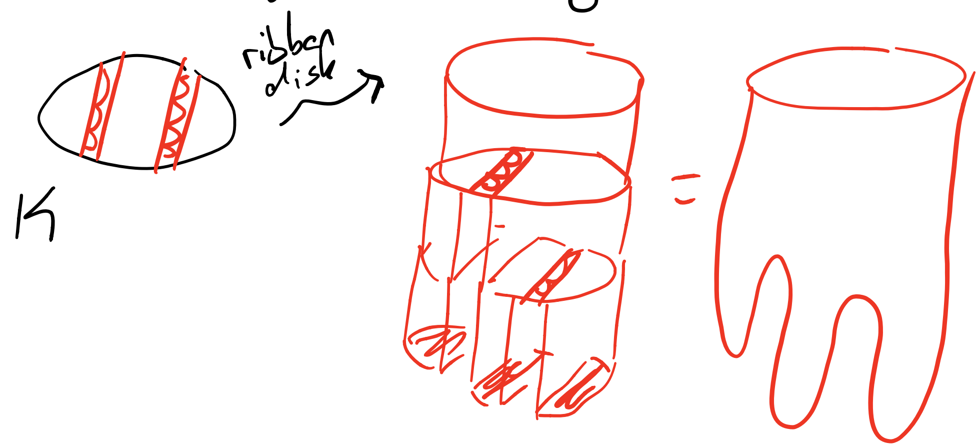
Note $S^3 = \partial B^4$

Say K is <u>slice</u> if $K = \partial D^2 \hookrightarrow B^4$

Say K is <u>ribbon</u> if $K = \partial D^2 \hookrightarrow B^4$ where $h|_{D^2}$ Morse and no maxima (index-2 pts)



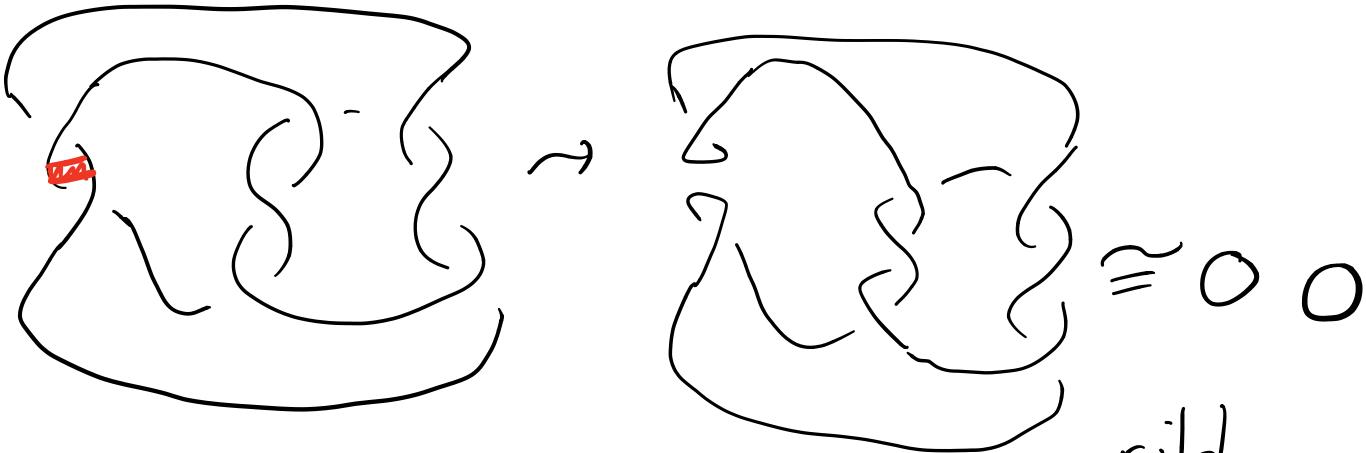
Described by K + bands $b_1, \dots, b_n$ so K surgered along $b_1, \dots, b_n \rightsquigarrow (n+1)$-comp unlink



bands ~ index-1

unlink ⇝ index-0

8_{20}



ribbon

Conj (Fox 1962)
Every slice knot is ribbon.

Thm (Casson–Gordon 1983)

K fibered + $\underbrace{\text{ribbon}}_{K=\partial D^2}$ $\Rightarrow \exists\,(E, V)$ s.t.

E = htpy 4-ball

handlebody 

$V \subset E$ = disk w/ $E \setminus \nu(V) = \underbrace{H \times S^1}_{f}$

and $\partial(V, E) = (S^3, K)$

E fibered by handlebodies

Q1 Is $V \cong B^4$? (If no, Poincaré fails)

Q2 If $V \cong B^4$, is $E = D$?

Obsv If K = fibered + ribbon and K bounds no disk fibered by handlebodies, then 4D Poincaré is false.

Thm If fibered ribbon $K = \partial D^2$ defined by bands $b_1, \dots, b_n$ where each b_i transverse to fibration on $S^3 \setminus \nu(K)$, then $B^4 \setminus \nu(D)$ fibered by handlebodies.

Rmk (Meier–Larson 2015)
If D fibered, fibers are handlebodies.

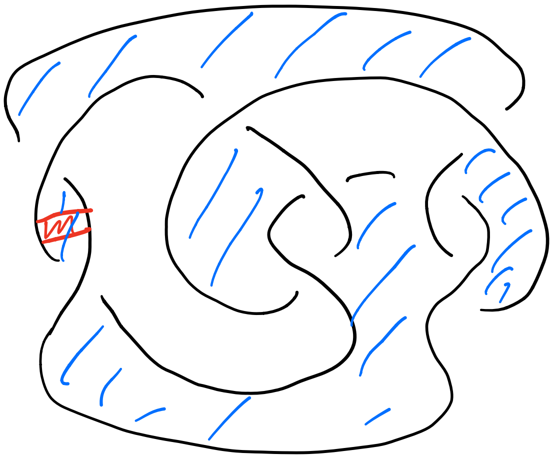
Cor K prime fibered + ribbon ≤ 12 crossings
$\Rightarrow K = \partial$ fibered ribbon disk

Cor If K fibered + ribbon $= \partial D^2$ with 2 min (one band) then $B^4 \setminus \nu(D)$ fibered (by handlebodies).

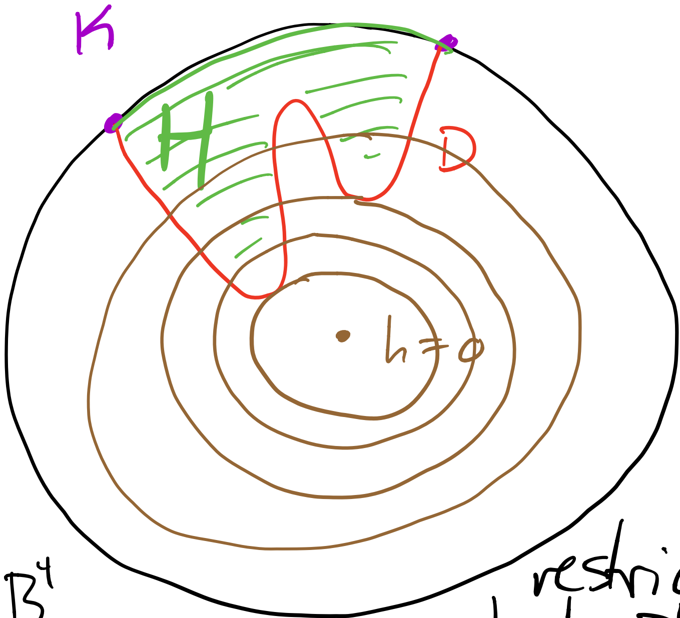
Pf of Cor
Scharlemann–Thompson: band b lies in min-genus surface (= leaf of fibration)

Pf of thm: constructive

Let's try example 8_{20}



Goal: build fibration (over S^1) (i.e. codim-1 foliation) of $B^4 \setminus \nu(D)$

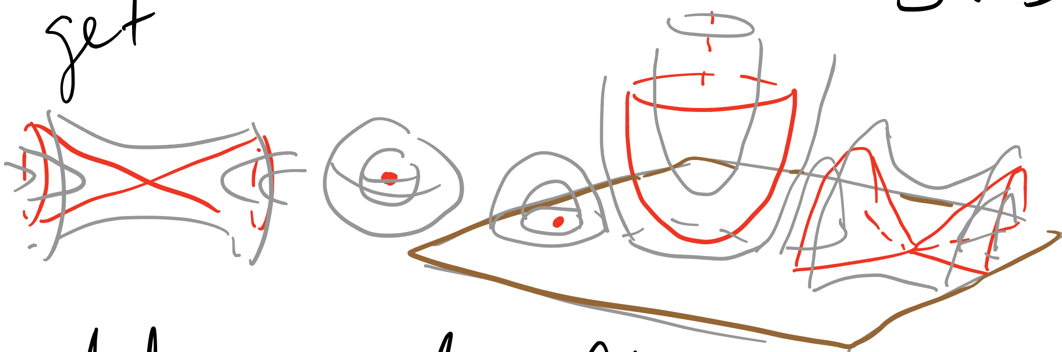


restrict h to $B^4 \setminus \nu(D)$

On each $h^{-1}(t) \cong S^3$, expect singular fibration

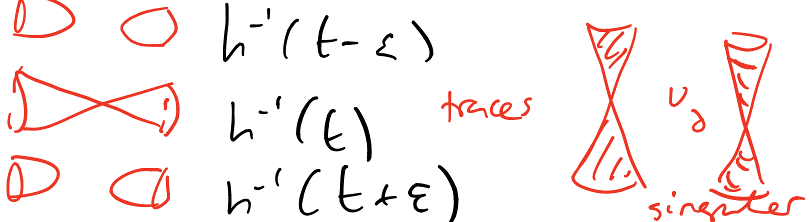
some H's are tangent to $h^{-1}(t)$, so get

$h^{-1}(t) \cap H \supset$



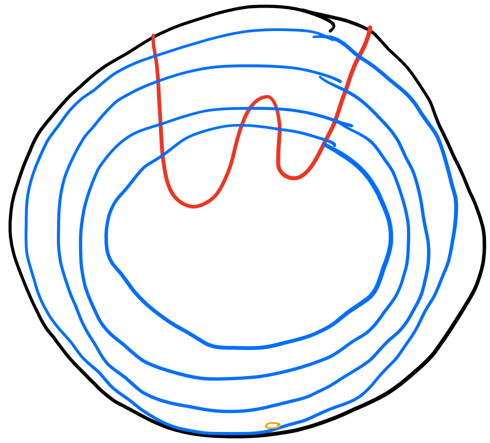
So idea: build singular fibration on each $h^{-1}(t)$ which vary smoothly (extra work to prove actual fibration)

e.g. don't want

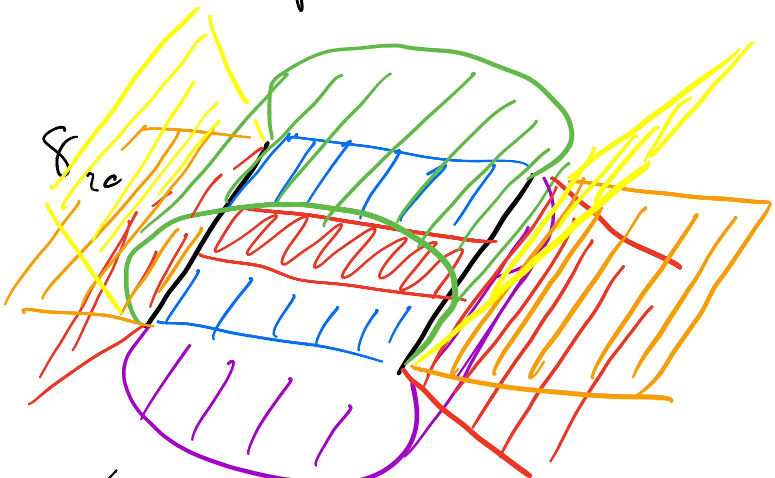


First step: sing. fibration on each $h^{-1}(t)$

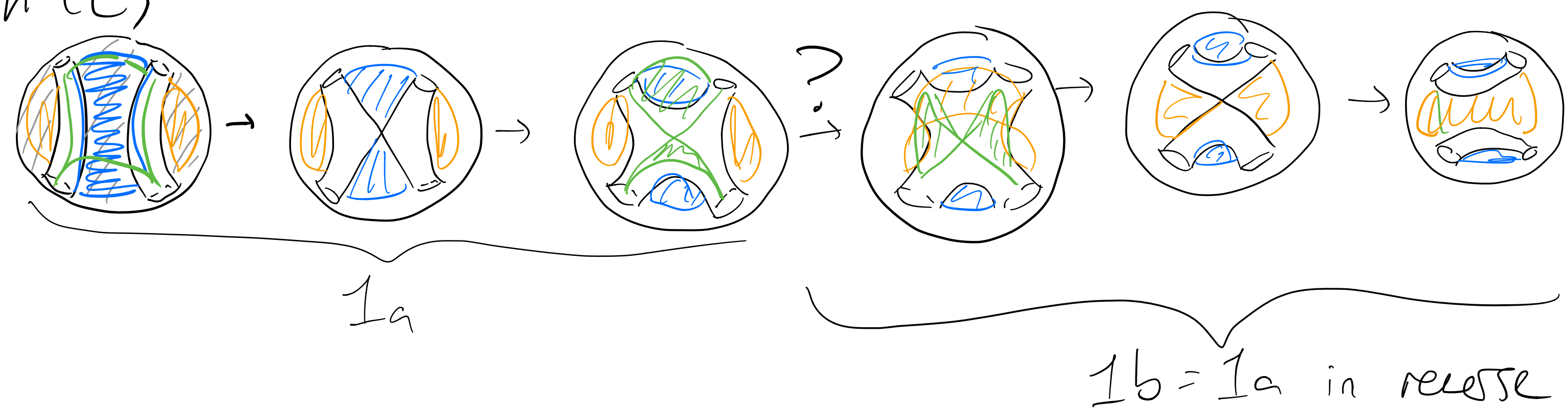
$t_b - \varepsilon \leq t \leq 1$ $t_b = h(\text{index-1 crit pt})$



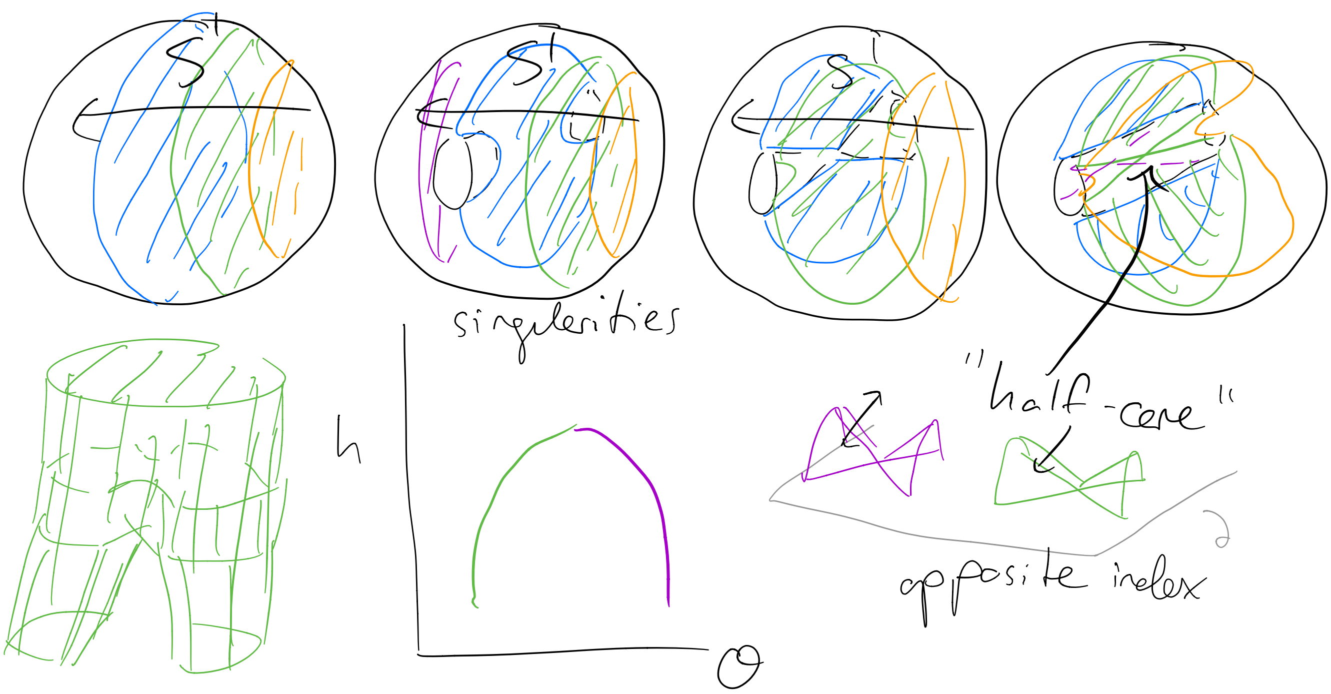
Close up of band



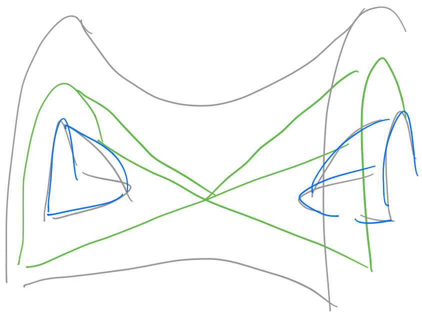
$h^{-1}(t)$

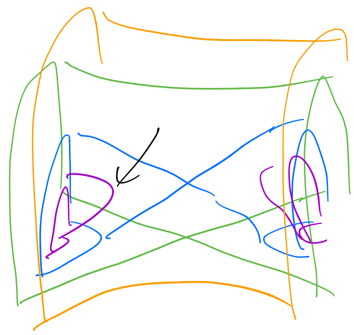


1a

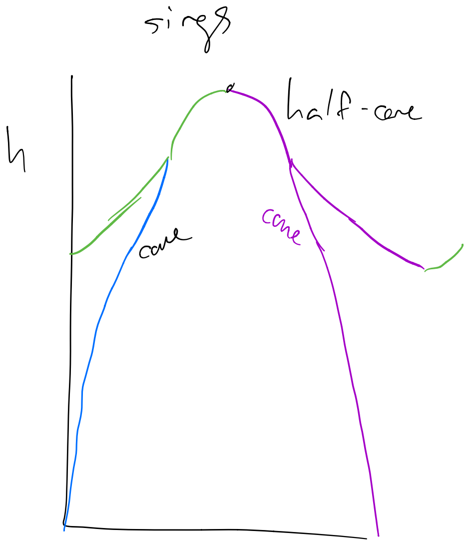


"rotate" the 2 half-cone singularities

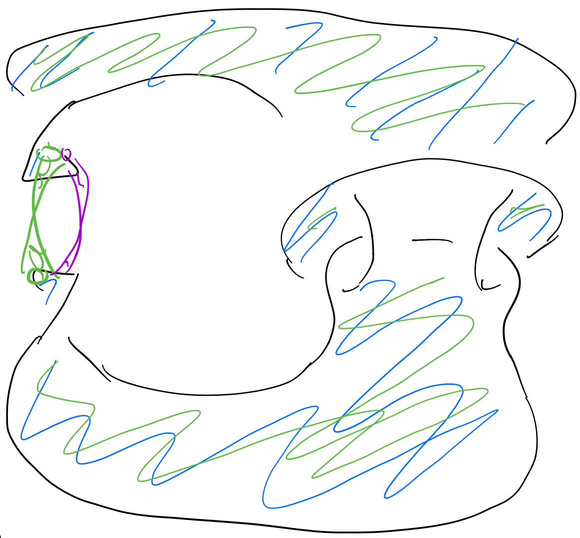




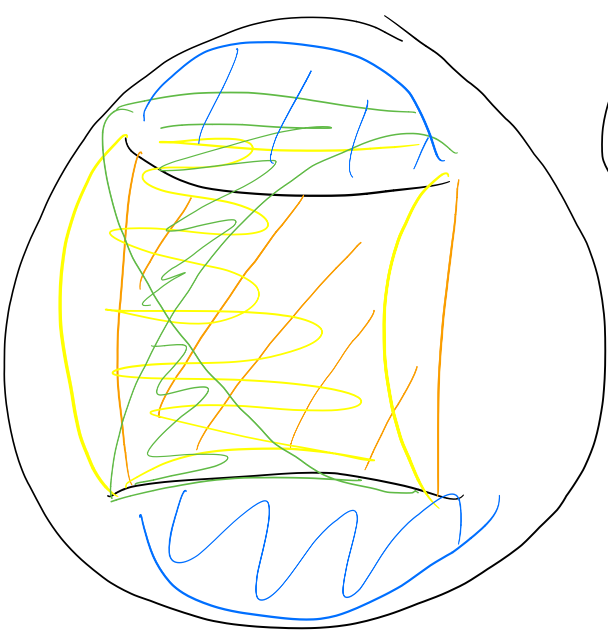
close up



new in $h^{-1}(t_b - \varepsilon)$ have sing fibration w/ 2 cones



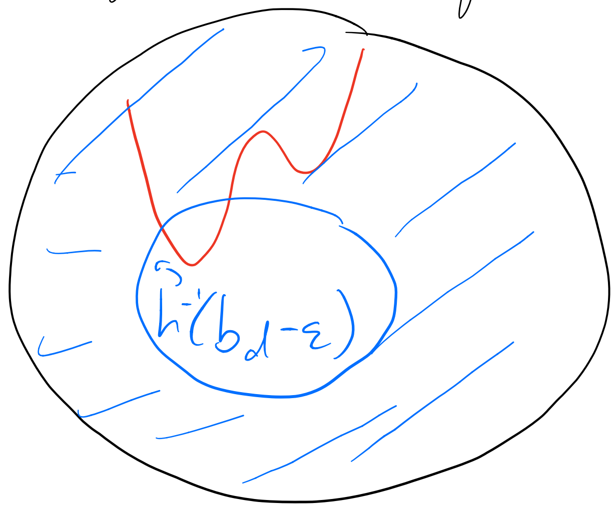
Q



now extend sing fibration to $h^{-1}(t)$

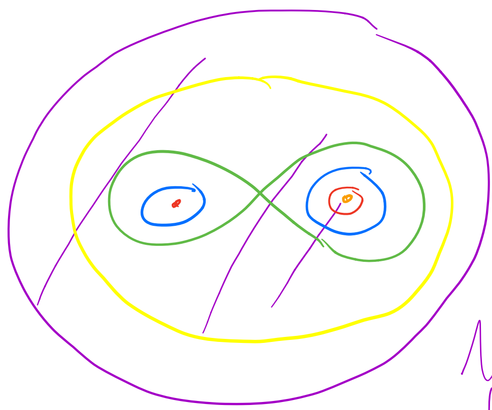
$b_d - \varepsilon \le t \le 1$ $\quad$ b_d = height of heighest min

$(h^{-1}(b_d-\varepsilon) \cong S^1 \times D^2)$



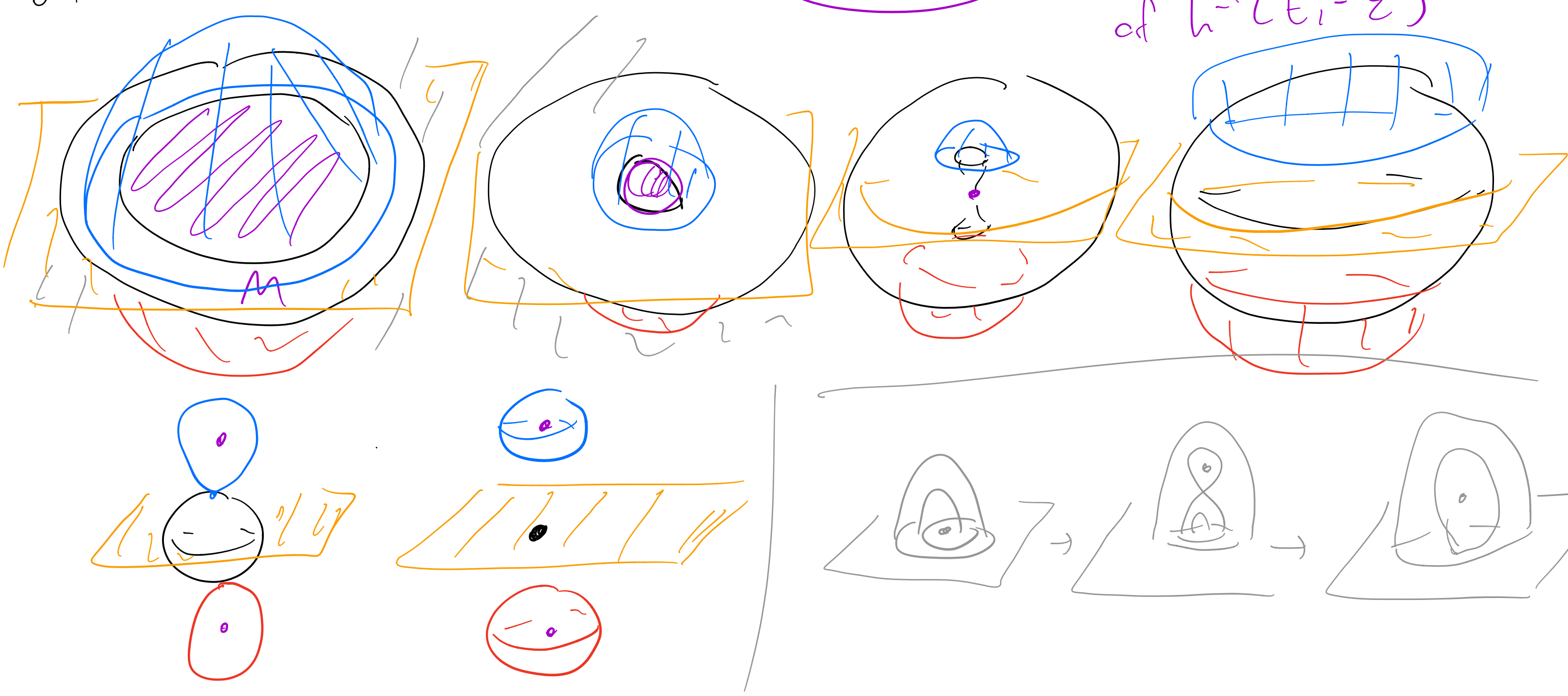
As h decreases, M compresses

minimum (perturbed to be degenerate disk M)



∂ torus

$M \cap$ fibration of $h^{-1}(t_1-\varepsilon)$



compress ints in M, inside to out over time

makes k dots k cones

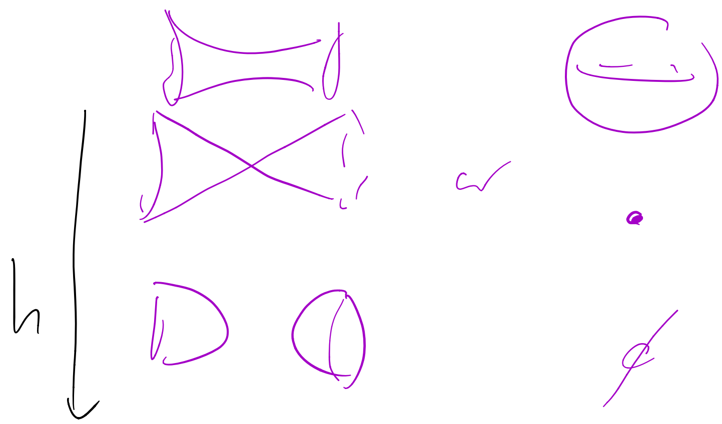
now M a leaf up to isotopy

<u>next</u> extend fibration to $h^{-1}(t_b - \varepsilon)$

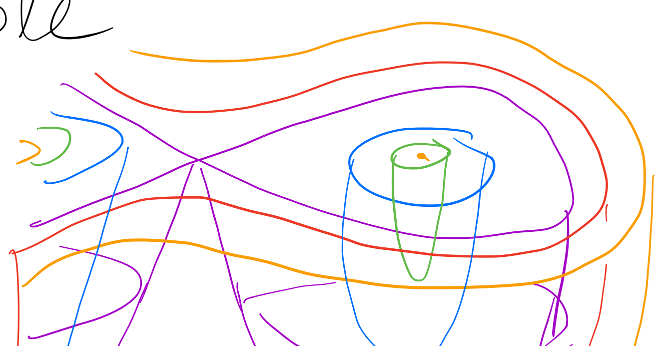
makes k cones $k+2$ dots

$\rightsquigarrow$ now in $h^{-1}(t_b - \varepsilon)$ have $k+2$ cones $k+2$ dots

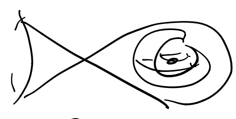
Moreover all resched

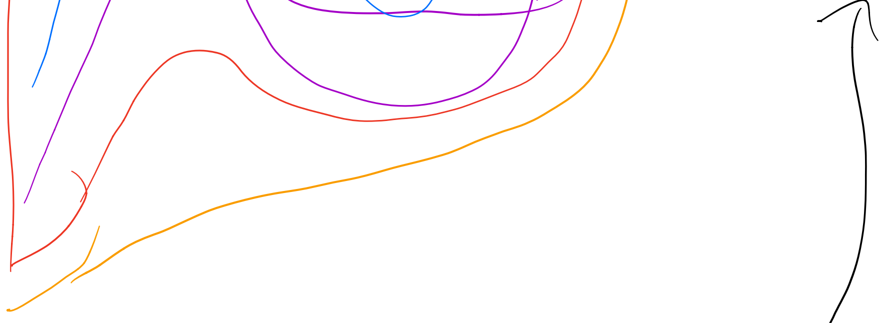


cancellable



if locally





They do all look like :

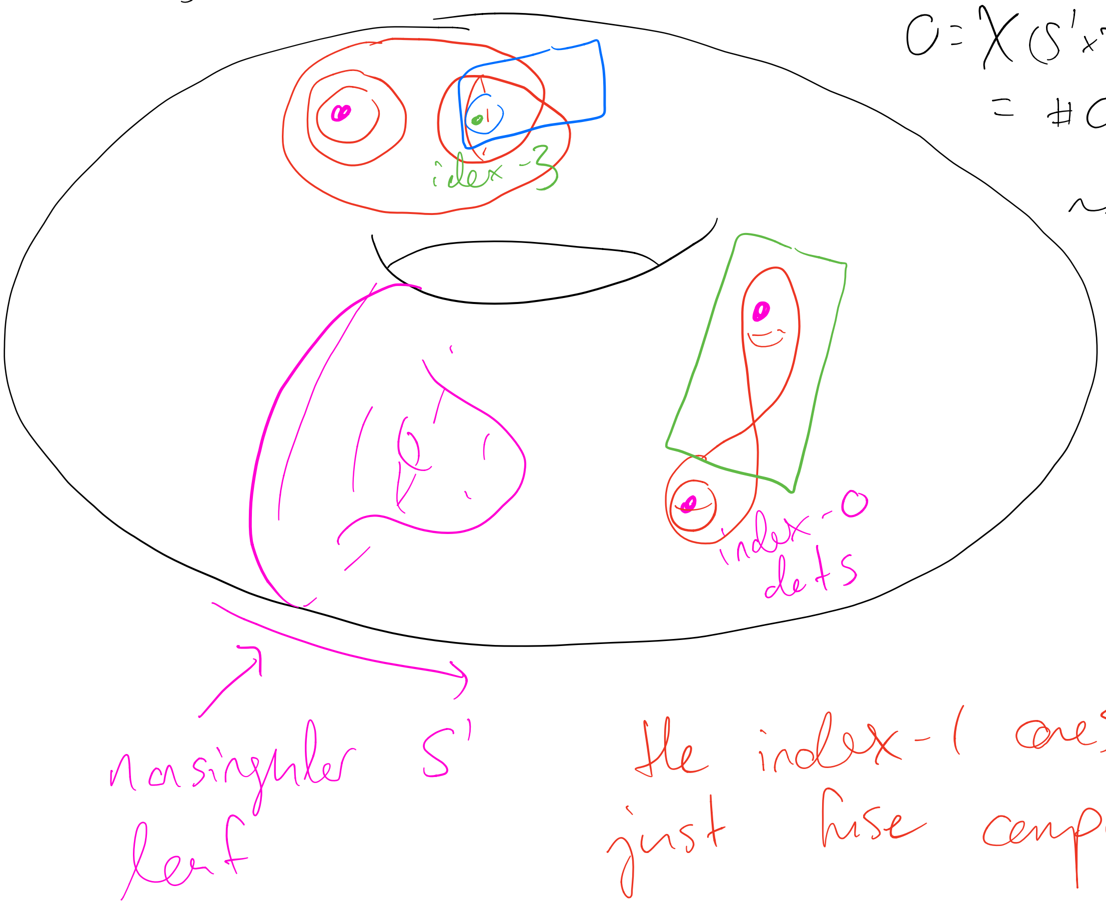
$h^{-1}(t_b-\varepsilon) \cong S^1\times D^2$

have # 1,2 sings = #0,3

$0 = \#1+\#2-\#0-\#3$

$0=\chi(S^1\times D^2)$
$= \#0-\#1+\#2-\#3$

$\rightsquigarrow \#0=\#1$
$(\#2=\#3)$

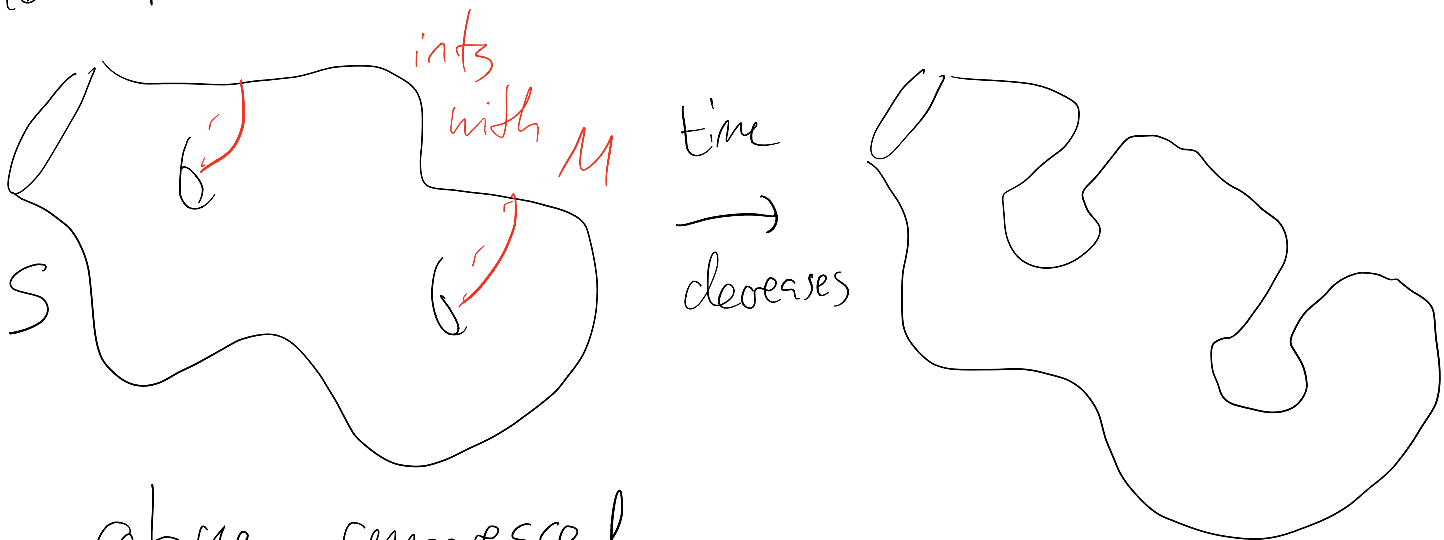


the index-1 ones just fuse components

$\rightsquigarrow$ find a cone and dot to cancel, repeat inductively to get nonsing fibration

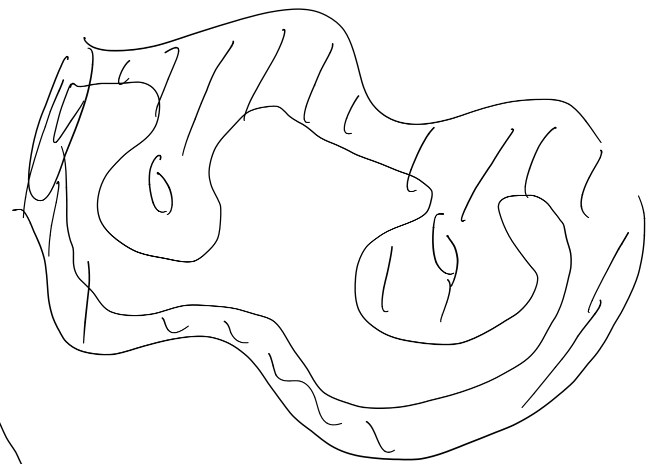
cap off with $B^4\setminus\nu(D)\cap h^{-1}[0,t_b-2\varepsilon]$
$\cong S^1\times B^3$ (fibered)

Understand fibers:

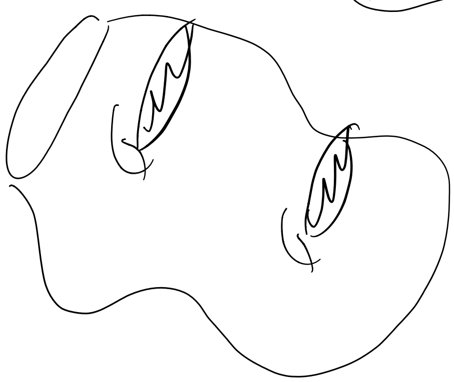


above compressed min M

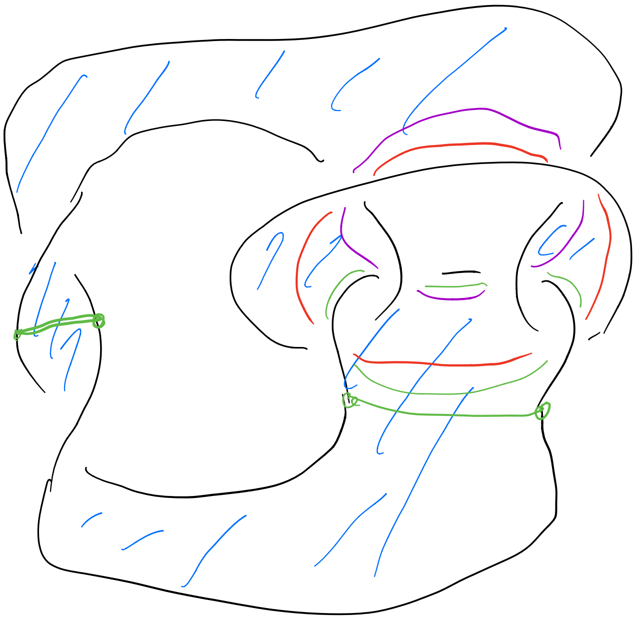
traces out S + 2-handles



$\simeq$



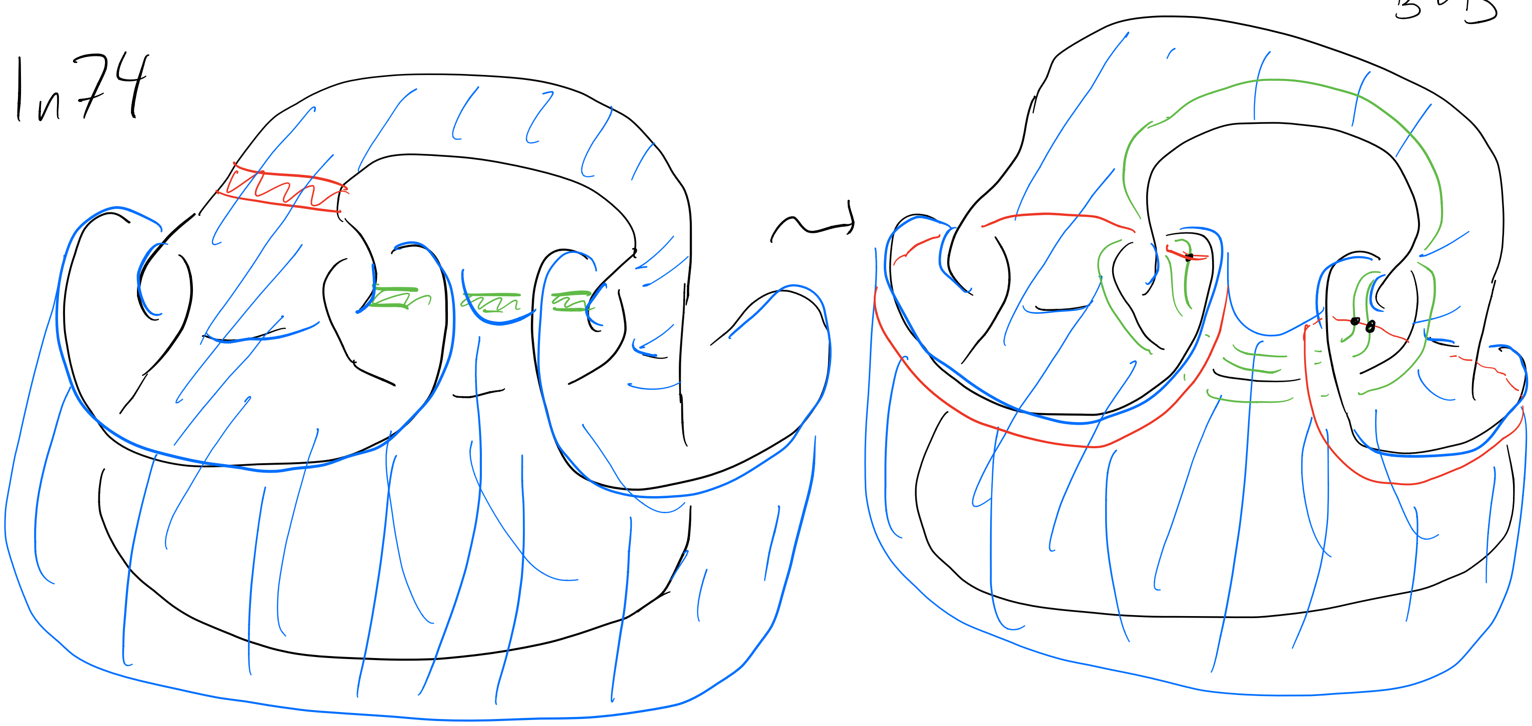
$\therefore$ 2-handles of fiber $H = \Sigma \cup$ 2-handles are attached along $\Sigma \cap$ min disks

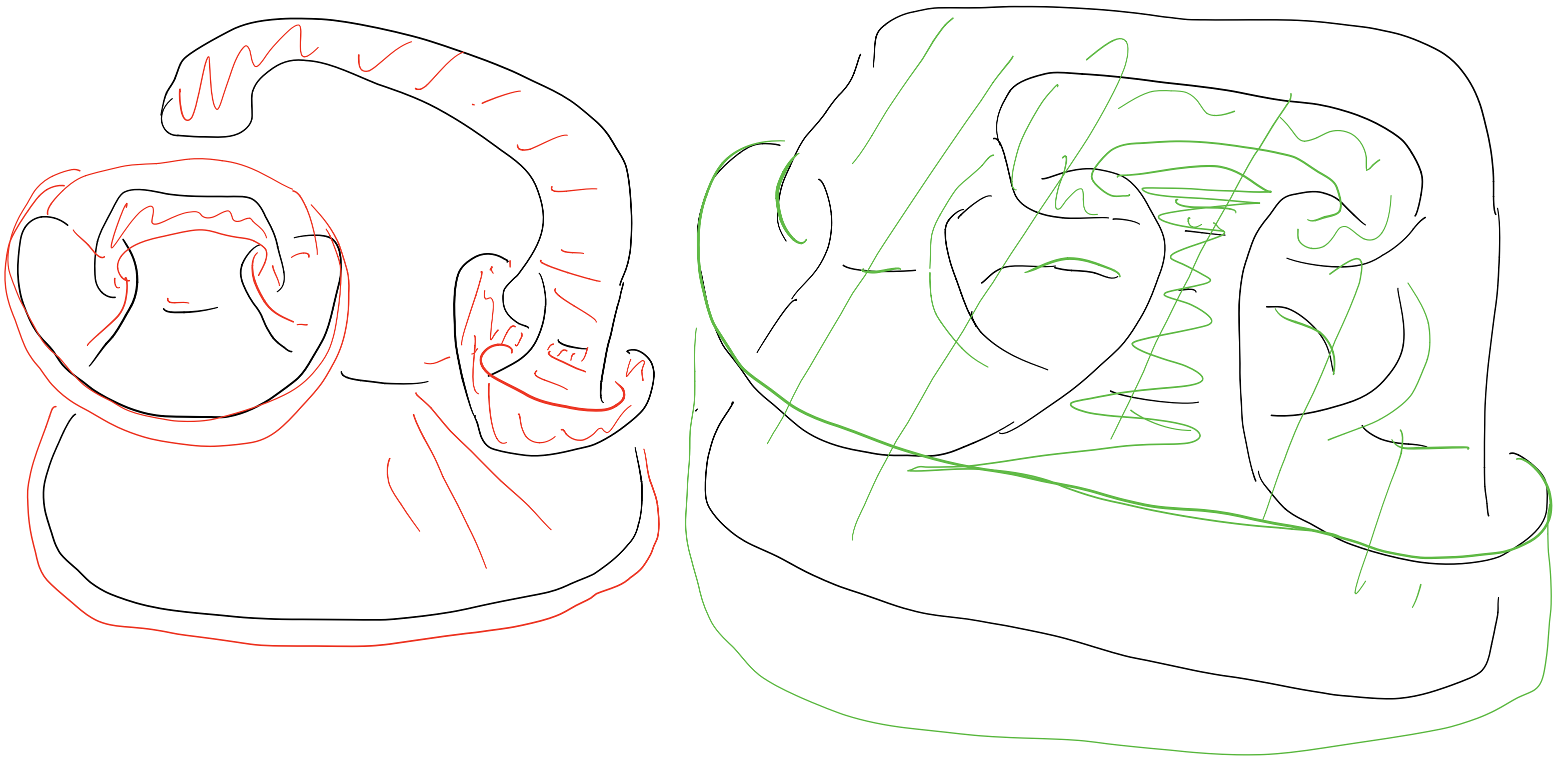


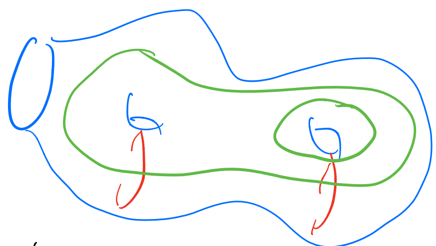
Σ

$\rightsquigarrow$ Given two disks D_1, D_2 with same ∂, can give Heegaard splitting of fiber for fibered 2-knot $D_1 \cup \bar{D}_2 \subset S^4$
$= B^4 \cup \bar{B}^4$

11n74





Heegaard spl $\cong$  of B^3

$\therefore\ D_1 \cup \overline{D}_2$ = unknot $\Rightarrow$ 11n74 doubly slice