Dax's Work on Embedding Spaces

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TransCanada PipeLines Pavilion. Photo courtesy of The Banff Centre

Topology in Dimension 4.5

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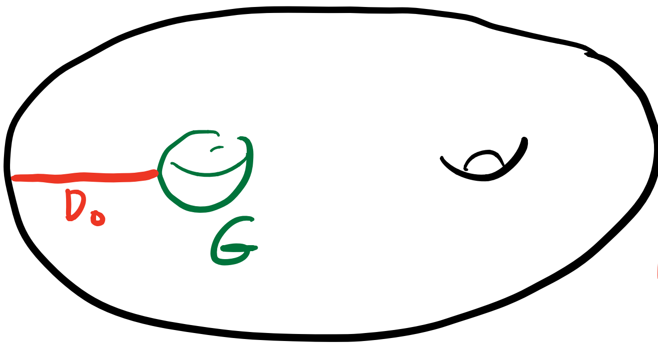
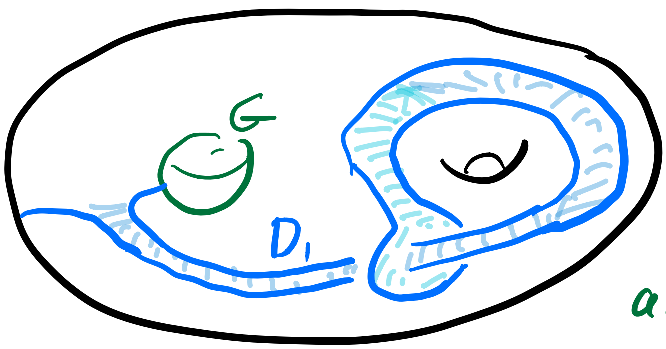
Motivating example

$M^4 =$ 

$S^2 \times D^2 \natural S^1 \times B^3$

$\times [-1,1]$

D_0 = std vertical disc



D_1 = self-referential disc

= D_0 tubed to an unknotted S^2

Theorem D_1 not isotopic to D_0 rel ∂

Facts 1) (LBT) If $N = M \cup$ 2-handle along ∂D_i, D_i's extended to S_i^2's then S_1 isotopic to S_0 fixing G ptwse.

2) $\exists\, f : M \longrightarrow M$ $f(D_0) = D_1$, $f \simeq id$ fixing ∂M ptwse
(Based on H. Schwartz for spheres using Cerf Palais)

3) FQ $(D_0, D_1) = 0$ $\pi_1(M)$ is torsion free
(see Schneiderman - Teichner)

4) STongs Km $(D_0, D_1) = 0$ D_0, D_1 have dual spheres
(see Klug - Miller)

<u>Theorem</u> D_0 properly embedded 2-disc $\subset M^4$ compact, with dual sphere $G \subset \partial M$. $\mathcal{D} =$ isotopy classes of embedded discs homotopic to D_0 rel ∂

Then there is a homomorphism

$$\Phi_{D_0} : \mathcal{D} \longrightarrow \pi_1^D(\mathcal{E}mb(I, M), I_0)$$

$$= \mathbb{Z}(\pi_1(M) \setminus 1)/D(I_0)$$

It maps onto subgroup generated by elements $g + g^{-1}$ and $\hat{\lambda}$ where $\hat{\lambda}^2 = 1$

$\mathcal{D}$ is an abelian group with $[D_0]$ the zero element.

<u>Example</u> when $M = S^2 \times D^2 \natural S^1 \times B^3$, $D(I_0) = 1$

$\mathcal{D}$ is $\cong$ to subgroup $\{[z^n + z^{-n}] \mid n \in \mathbb{N}\}$

z generator of π_1

Kosanovic - Teichner

1) Always an $\cong$

2) Understand the space of embeddings $D^2 \hookrightarrow M^4$ with boundary dual sphere

3) general group structure not necessarily abelian

Hannah Schwartz

LBT for discs with dual sphere G such that $\pi_1(M-G) \xrightarrow{\cong} \pi_1(M)$.

Question (Schwartz) Is there a LBT when $\pi_1(M-G) \to \pi_1(M)$ not $\cong$

Théorème A. — *Soient* V^n *et* M^m *deux variétés différentielles de classe* C^∞, *la variété* V^n *étant compacte sans bord.*

Soit $f : V^n \to M^m$ *une application continue. Si* $2\,m - 3\,n - 3 \geq 0$, f *est homotope à un plongement si et seulement si* $\alpha_0(f)$ *est l'élément neutre du groupe* $\Omega_{2n-m}(\mathcal{C}_f, \partial W; \theta_f)$.

Soient k *un entier* ≥ 1 *et* $f_0 : V^n \to M^m$ *un plongement. L'homomorphisme (application pointée si* $k = 1$*) :* n=1 m=4 k=2

$$\underset{2}{\alpha_{\cancel{k}}} : \quad \underset{2}{\pi_{\cancel{k}}}(\mathrm{Hom}\,(V^n, M^m),\ \mathrm{Pl},\ f_0) \to \Omega_{\underbrace{2n-m+k}_{0}}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

est un isomorphisme (bijection si $k = 1$*) pour* $k \leq 2\,m - 3\,n - 3$, *un épimorphisme (surjection si* $k = 1$*) pour* $k = 2\,m - 3\,n - 2$.

Jean-Pierre Dax 1972

Étude homotopique des espace de plongements

Dax Isomorphism Theorem:

$$\alpha_{\underset{2}{\cancel{k}}} : \pi_{\underset{2}{\cancel{k}}}(\mathrm{Hom}(V^{\overset{1}{\cancel{n}}}, M^{\overset{4}{\cancel{m}}}), \mathrm{Pl}, f_0) \xrightarrow{\simeq} \Omega_{\underset{0}{\cancel{2n-m+k}}}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

Let I_0 be a properly embedded closed interval in the oriented M^4.

i) $\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is generated by $\{\mathcal{Z}_g \mid g \neq 1, g \in \pi_1(M)\}$ and is canonically $\cong$ to

$$\mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

ii) There is a homomorphism

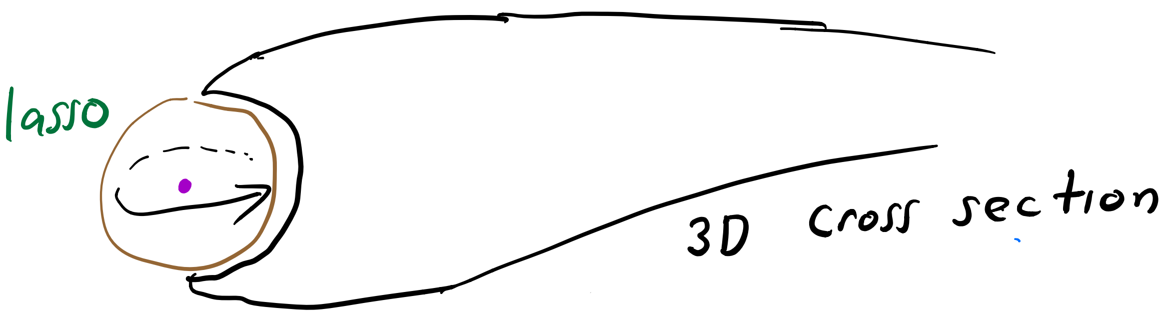
$$d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$$

with image $D(I_0)$ - <u>the Dax kernal</u>

$\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is the subgroup of loops that are $\simeq *$ in the space of maps - <u>The Dax group</u>

<u>Reference</u> "Self-Referential discs and the light bulb lemma"

Spinning (Definition of $\mathcal{J}_g$)



- conventions
  1) base of band below lasso on I_0
  2) an orientation rule determines sign

Fact: Spinnings commute

Dax's key idea:

Let $\alpha_t : I \longrightarrow M$

with $\alpha_0 = \alpha_1 = 1_{I_0}$ (id map to I_0)

$\alpha_t \in \pi_1^D(\mathrm{Emb}(I, M), I_0) \Longrightarrow$

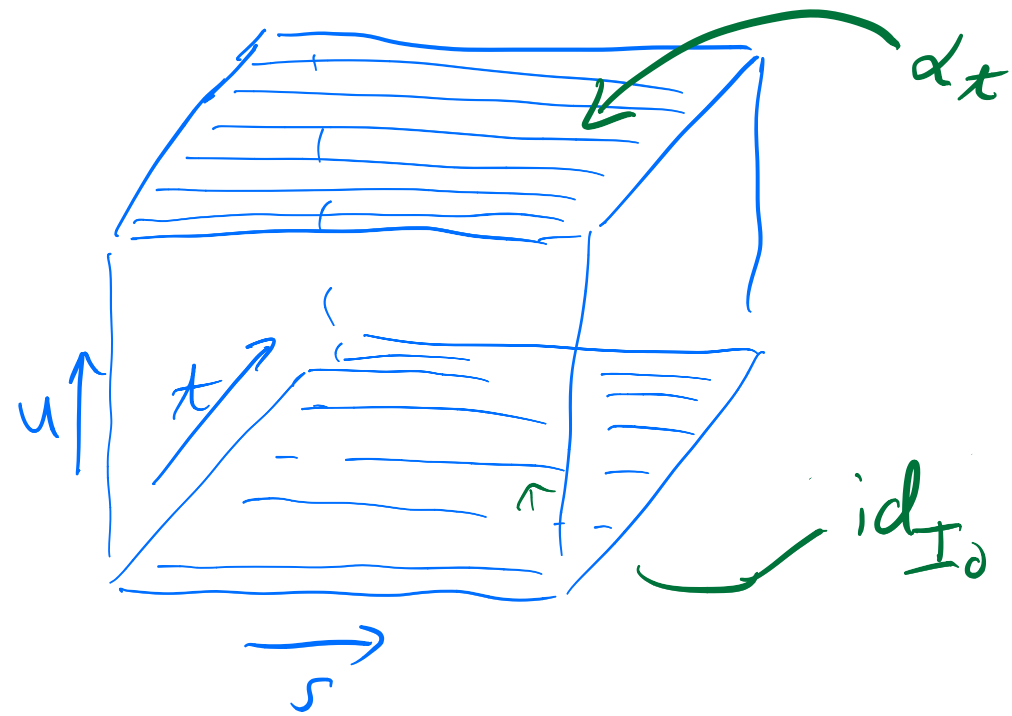
$\exists\, \alpha_{t,u} \in \mathrm{Maps}(I, M; I_0)$ with

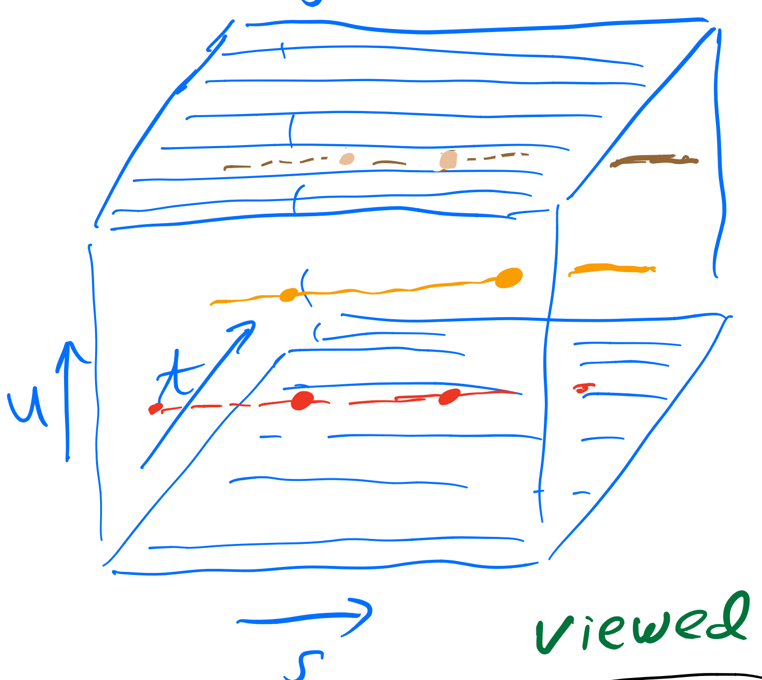
$\alpha_{t,u} = 1_{I_0}$ u near 0, $t \in I$; $= \alpha_t$ u near 1

Define $F_0 : I \times I^2 \longrightarrow M \times I^2$

$F_0(s, t, u) = \left(\alpha_{t,u}(s), t, u\right)$

we can assume F is an immersion with finitely many double points, no triple points. self ⋔ at double pts





Double Points of F_0

viewed in M

How to Compute g_x

sign $= \sigma_x$ oriented self intersection #

The orientation of the interval informs which tangent space comes first.

$$F_0 \rightsquigarrow d(\alpha_{t,u}) = \sum_{i=1}^{n} \sigma_{x_i} g_{x_i} \in \mathbb{Z}[\pi_1(M) \setminus 1]$$

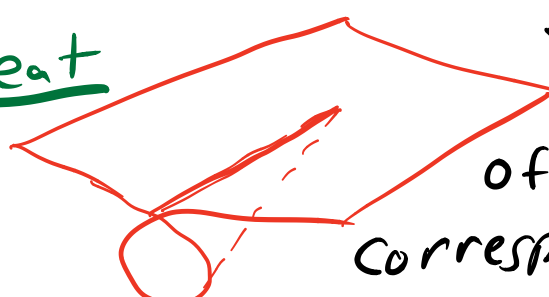
summed over double points with

$g_{x_i} \neq 1$

This is well defined

If $\alpha^0_{t,u}$, $\alpha^1_{t,u}$ two homotopies in $Maps(I, M; I_0)$ and $\alpha^0_{t,u} \simeq \alpha^1_{t,u}$ fix ∂ then the usual intersection theory argument – considering double curves of interpolating homotopy – shows $d(\alpha^0_{t,u}) = d(\alpha^1_{t,u})$

Caveat



Some double curves cone off – but this corresponds to loops $g_x = 1$.

If $\alpha^0_{t,u} \not\simeq \alpha^1_{t,u}$ then they differ by an element of π_3.

Define $d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$ where $a \in \pi_3$ is represented by $\alpha_{t,u}(s)$ where $\alpha_{t,0}, \alpha_{t,1} = I_{I_0}$

Define $D(I_0) = d_3(\pi_3(M, x_0))$

Then

$$d : \pi_1^D(\mathrm{Emb}(I, M; I_0)) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

is a homomorphism.

Looking closely at

$$F_0 \rightsquigarrow \sum_{i=1}^{n} \sigma_{x_i} g_{x_i} \in \mathbb{Z}[\pi_1(M)]$$

where the sum is without cancellation and g_{x_i} possibly $\underline{1}$ then one sees that α_t is a concatenation of the spin maps $\sigma_{x_i} g_{x_i}$.

I.e. α_t differs from 1_{I_0} by this concatenation of spinnings.

Since spin maps $\in \pi_1^D(\mathrm{Emb}\, I, M; I_0)$

It follows that

$$d : \pi_1^D(\mathrm{Emb}(I, M; I_0)) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1] / D(I_0)$$

is surjective. Since spinnings commute and spinning around $\underline{1}$ is $\simeq *$, d is injective.

Technical pt This avoids a double point s-cobordism like elimination argument of Dax

Example

$$\pi_1^D(\mathrm{Emb}(I, S^1 \times B^3; I_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0]$$

Proof $\pi_3 = 0$

$$\pi_1^D(\mathrm{Emb}(I, S^2 \times D^2 \natural S^1 \times B^3; I_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0]$$

Proof Same generators - less space to kill them.

Dax Isomorphism Theorem:

$$\alpha_{\cancel{k}\,2} : \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^{\cancel{n}\,1}, M^{\cancel{m}\,4}), \mathrm{Pl}, f_0) \xrightarrow{\cong} \Omega_{\cancel{2n-m+k}\,0}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

Let I_0 be a properly embedded closed interval in the oriented M^4.

i) $\pi_1^D(\mathrm{Emb}(I, M; I_0))$ is generated by $\{\mathfrak{Z}_g \mid g \neq 1, g \in \pi_1(M)\}$ and is canonically $\cong$ to

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ii) There is a homomorphism

$$d_3 : \pi_3(M, x_0) \longrightarrow \mathbb{Z}[\pi_1(M) \setminus 1]$$

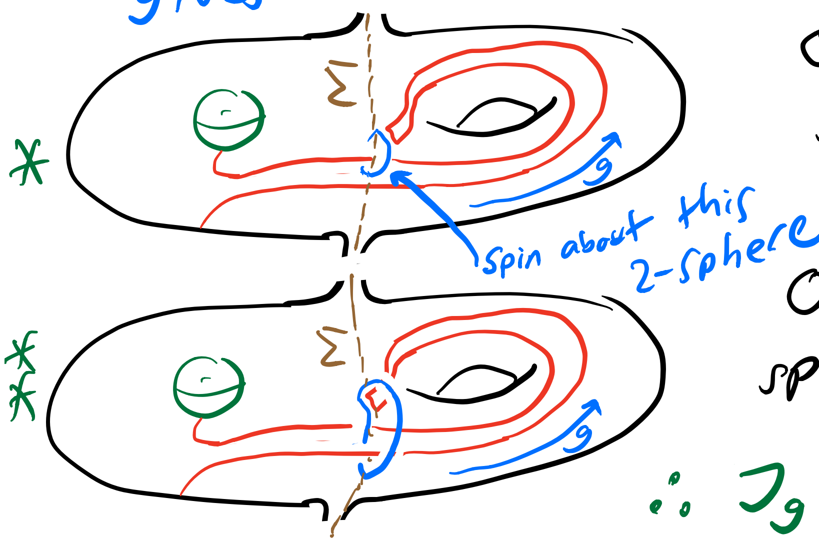
with image $D(I_0)$ - <u>the</u> <u>Dax</u> <u>Kernal</u>

Differences between two Dax $\cong$ Theorems

1) working in different spaces
2) Part ii) is not part of his theory
3) we identify the generators geometrically

$$\pi_1^D(\mathrm{Emb}(I, S^2 \times D^2 \# S^1 \times B^3; I_0)) = \mathbb{Z}[\mathbb{N}]$$

<u>Idea of Proof</u> The separating $S^3 := \Sigma$ gives relations. A 2-sphere $\subset \Sigma$ bounds B^3's



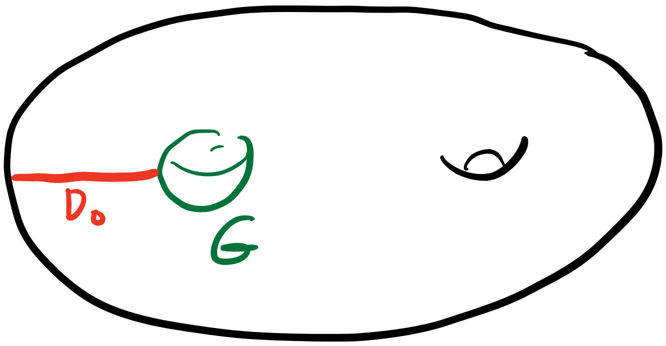
One gives spinning J_g

Other gives spinning J_{g-1}

$\therefore J_g \sim J_{g-1}$ up to sign

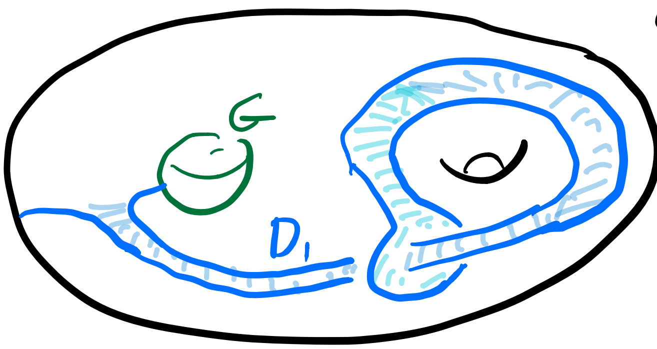
$$\alpha_{\cancel{k}\,2} : \pi_{\cancel{k}\,2}(\mathrm{Hom}(V^{\cancel{n}\,1}, M^{\cancel{m}\,4}), \mathrm{Pl}, f_0) \to \Omega_{\cancel{2n-m+k}\,0}(\mathcal{C}_{f_0}, \partial W; \theta_{f_0})$$

These "homotopies" *, ** represent different elements of the source of α_2, but are homotopic in π_1^D.



View $D_1 \in \pi_1^D(\mathrm{Emb}(I,M); I_0)$

$d(D_1) = g + g^{-1}$



(using correct choice of sweeping across D_1)

Proof

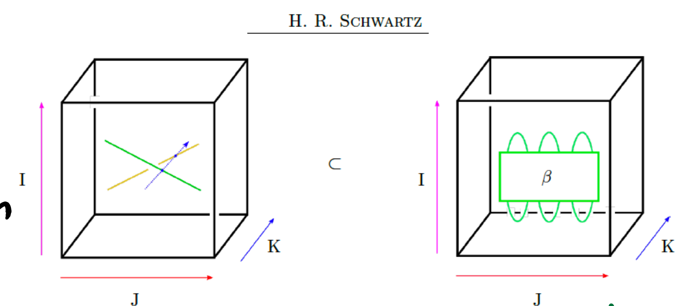
D_1 is the concatenation of J_g, $J_{g^{-1}}$ $\square$

How to compute $d(D_1)$ up to $I(D_0)$
(after H. Schwartz)

Reference (Schwartz) "A LBT for discs"

Step 1 Consider a regular homotopy $I \times D^2 \to I \times M^4$
of D_0 to D_1 $\rightsquigarrow$

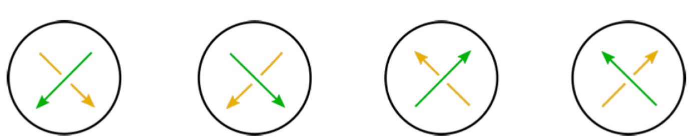
Step 2
Consider the self intersection locus viewed as a plat.



(Here K is the "interval direction")

Step 3 Add up crossings (projection into I, J plane) corresponding to identified arcs.

- each crossing comes with a sign and group element.



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Ignore crossings with $g_x = 1$