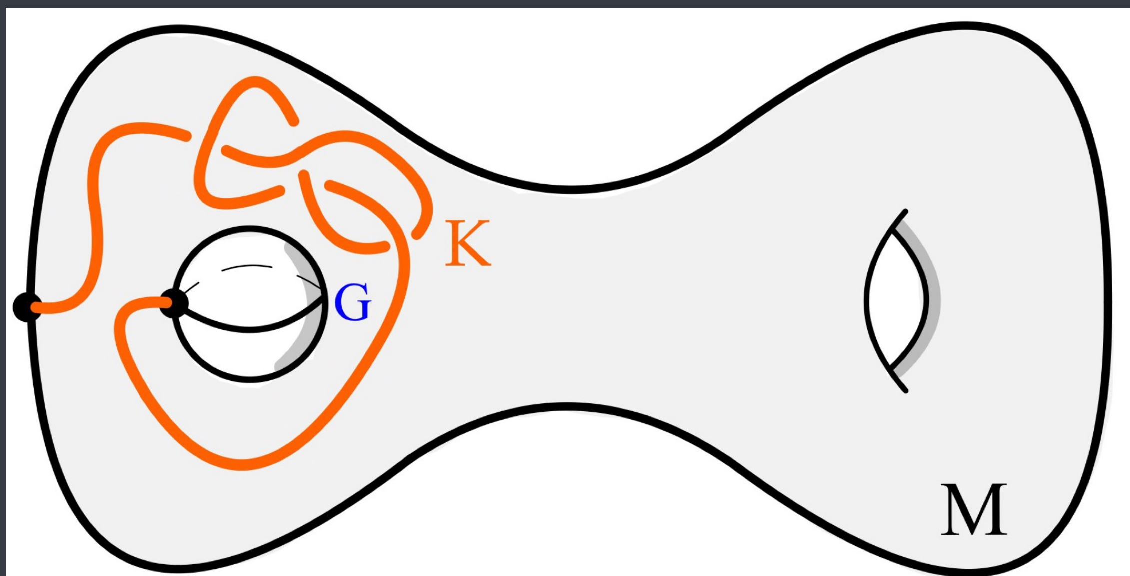
Isotopy classification of $\frac{1}{2}$-disks in 4-manifolds,

joint with Danica Kosanović.



Nov. 4, 2022

Banff, Canada

4.5-dim. topology

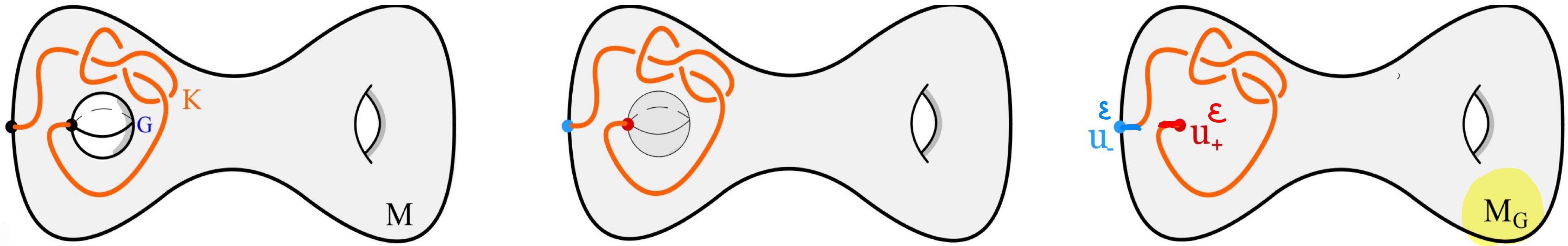
A 13 pictures talk based on 2 recent papers.

Classical LBT: $\mathrm{Emb}_\partial(D^1, M^3) \hookrightarrow C^\infty_\partial(D^1, M^3) \simeq \Omega M$

induces isom. on π_0, i.e. isotopy $\Longleftrightarrow$ homotopy

for knotted arcs K s.t. ∂K has a **dual** $G: S^2 \hookrightarrow \partial M$.

Space-level: $\forall d \geq 2$, $\mathrm{Emb}_\partial(D^1, M^d) \simeq \Omega_{u_-}^{u_+}\left(\mathbb{S}^{d-1}\left(M \cup_G D^d\right)\right)$



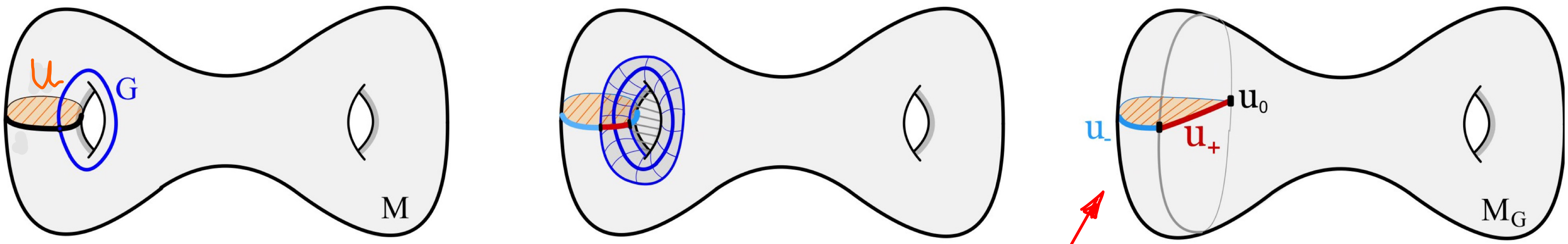
Thm: $\forall d \geq 2$, $1 \leq n \leq d$: $\mathrm{Emb}_\partial(D^n, M^d) \simeq \Omega_{u_-^\varepsilon}^{u_+^\varepsilon}\mathrm{Emb}^\varepsilon_\partial(D^{n-1}, M_G)$

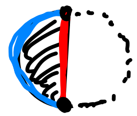
$M_G \cong M \cup_G h^{d-n+1}$.

[KT-high-dim, following Cerf $(n=d)$]

if ∂D^n has framed dual sphere $G: S^{d-n} \hookrightarrow \partial M$.

$n = 2,\ d = 3$:

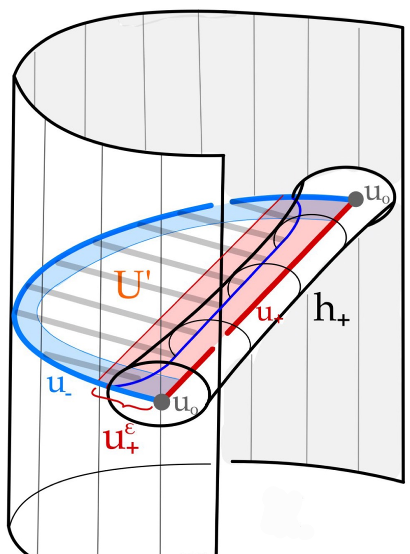


Proof in 2 Steps using $\frac{1}{2}$-disks $\text{ᗡ}^n :=$  :

$$\mathrm{Emb}_\partial(\mathbb{D}^n, M) \simeq \mathrm{Emb}_{\partial^\varepsilon}(\text{ᗡ}^n, M_G) \xrightarrow[\simeq]{\text{foliate}} \Omega\, \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^{n-1}, M_G)$$

$U' \longleftrightarrow U$

$\partial^\varepsilon = u_-^\varepsilon \cup u_+^\varepsilon$



$M \cong M_G \setminus \nu(u_+)$ uses only $\mathrm{Emb}_{u_-^\varepsilon}(\text{ᗡ}^n, M_G) \simeq *$

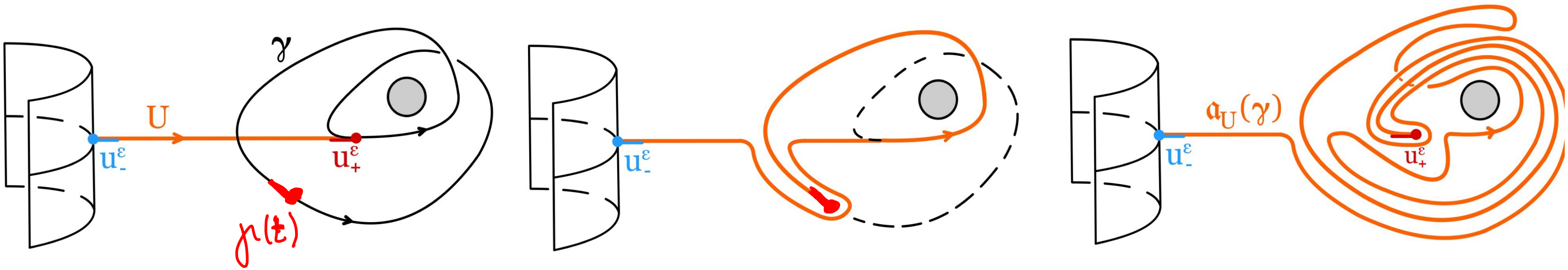
which is the cheap unknotting !

On each homotopy group π_i, an inverse of

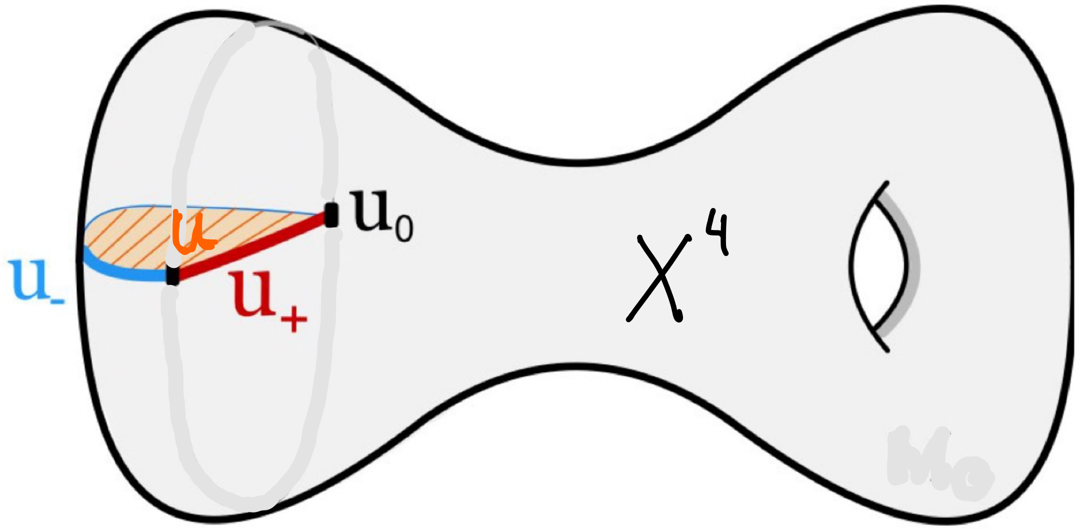
$$\mathrm{Emb}_{\partial^\varepsilon}(D^n, M_G) \xrightarrow[\simeq]{\text{foliate}} \Omega\, \mathrm{Emb}^{\varepsilon}_{\partial}(D^{n-1}, M_G)$$

is given by the i-parameter version of ambient isotopy theorem applied to $\mathcal{U}$,

e.g. $(n, d, i) = (1, 3, 0)$, $\gamma : [0,1] \xrightarrow[\text{isotopy}]{\text{one}} \mathrm{Emb}_{\partial}([0,\varepsilon), M_G)$



Focus on $(n,d) = (2,4)$ and on π_0, i.e. on isotopy classes.



isotopy classes of $\frac{1}{2}$-disks:

Cor. 1:

$$\mathbb{Z}[\pi \setminus 1]\big/ \mathsf{dax}(\pi_3 X) \underset{\mathbb{D}\mathsf{ax}}{\overset{U+\mathrm{fm}(\bullet)}{\rightleftarrows}} \mathcal{D}(X;k) \xrightarrow{-U\cup\bullet} \pi_2 X$$

[Dax, Dave, Danica]

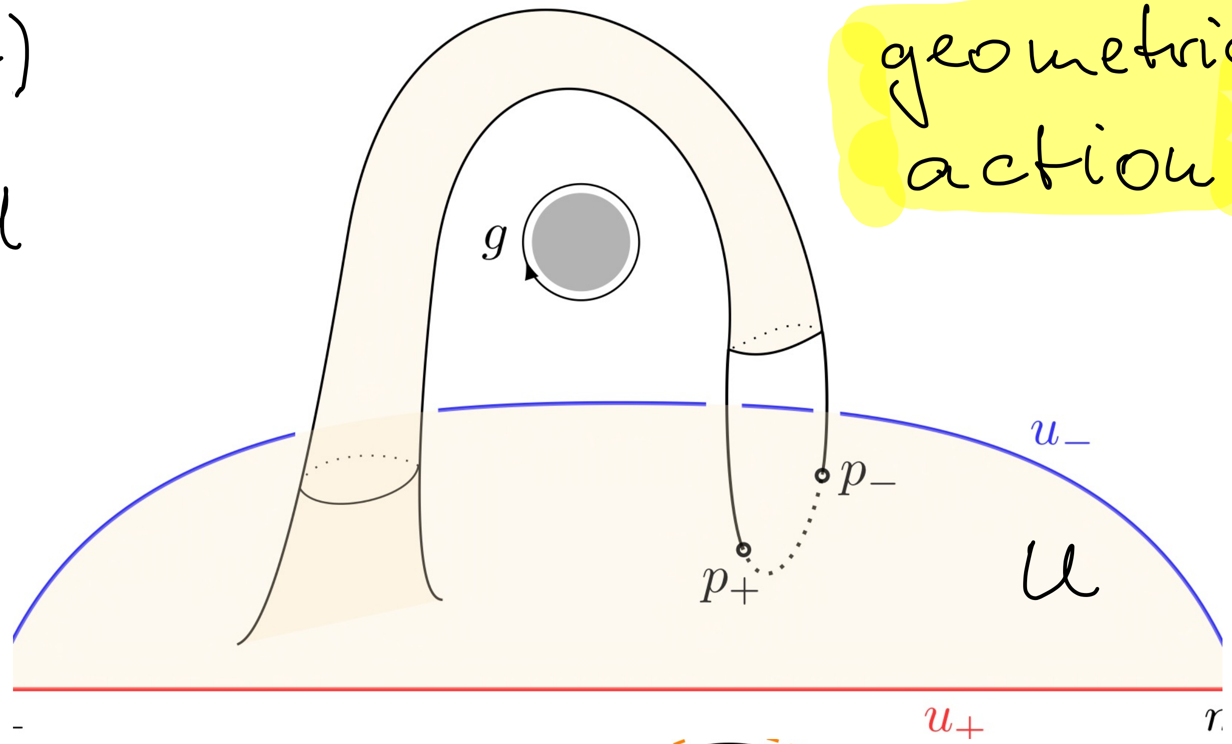
- $k = u_- \cup u_+$ is the $\frac{1}{2}$-boundary condition,
- $\pi = \pi_1 X$, X^4 oriented 4-mfd. with $\partial X \neq \emptyset$,
- $U =$ "un-$\frac{1}{2}$-disk" with boundary k,
- $\mathbb{D}\mathrm{ax} =$ Dax-invariant for homotopic $\frac{1}{2}$-disks.

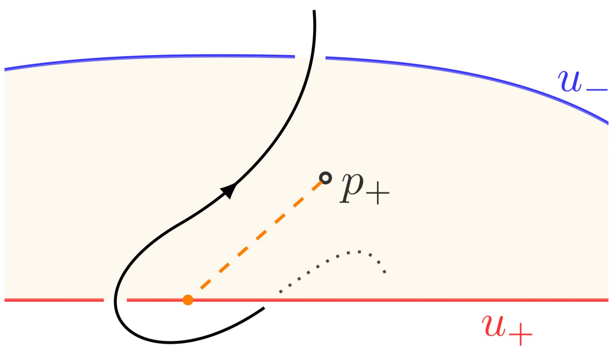
geometric action

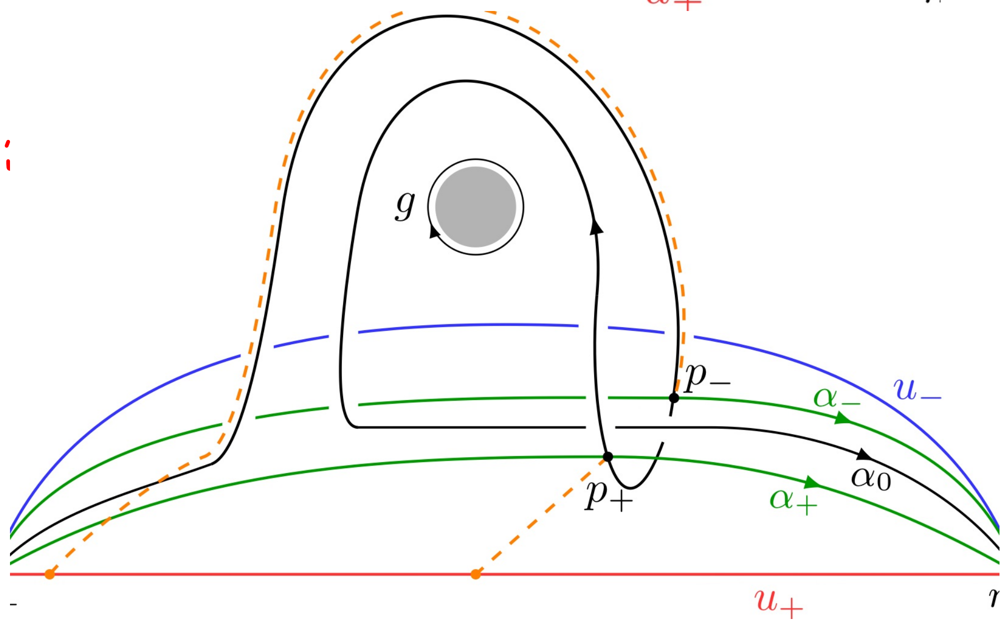
$$\mathcal{U} \xmapsto{\; g \in \pi \;} \mathcal{U} + fm(g)$$

1) finger move on $\mathcal{U}$ along $g \in \pi$,

2) push $p_{\pm}$ off free boundary $u_{\pm}$ along distinct sheets:







Back to neat disks $(D^2, \partial) \hookrightarrow (M^4, \partial)$ with ∂-condition & that has dual $G: S^2 \hookrightarrow \partial M$.

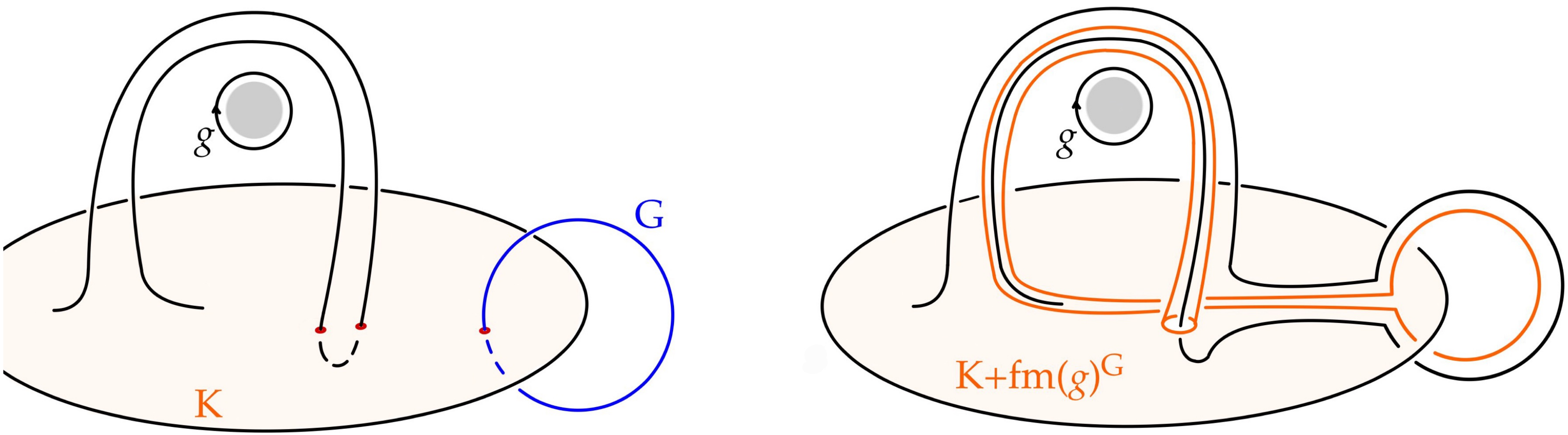
Cor. 2: There is a group structure on isotopy classes fitting into a central extension

$$\mathbb{Z}[\pi]/\mathsf{dax}(\pi_3 M) \underset{\mathbb{D}\mathsf{ax}\,\times\, e_{\mathrm{U}}/2}{\overset{\mathrm{U}+\mathrm{fm}(-)^G}{\rightleftarrows}} \underset{\ni\, \mathrm{U}}{\mathbb{D}(M;k)} \xrightarrow{p_{\mathrm{U}}} \pi_2 M/\mathbb{Z}[\pi]\cdot G$$

The group commutator of K_1, K_2 is

$$[K_1, K_2] = \mathrm{U} + \mathrm{fm}(\lambda(-\mathrm{U} \cup K_1, -\mathrm{U} \cup K_2))^G.$$

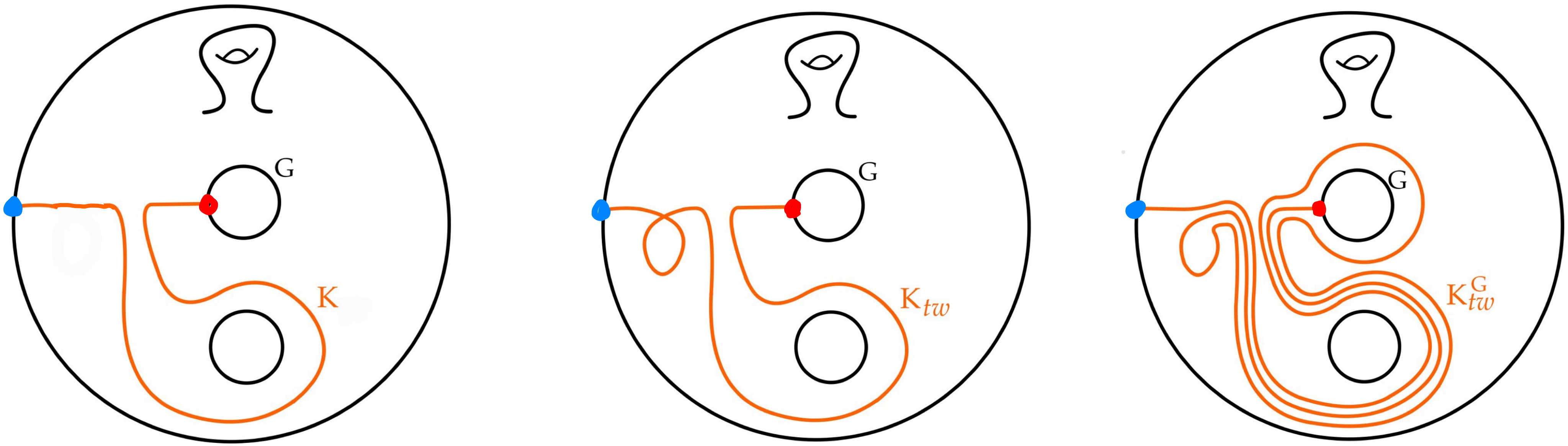
Here the group $\mathbb{Z}[\pi \setminus 1] / \mathrm{dax}(\pi_3 M)$ acts via



In particular, the group $\mathbb{D}(M;\varepsilon)$ is 2-step nilpotent but usually non-abelian, the extension does not split.

We use a subtle extension from $\mathbb{Z}[\pi \setminus 1]$- to $\mathbb{Z}[\pi]$-action by letting $1 \in \pi$ act via

$$K \longmapsto K_{tw} \longmapsto K_{tw}^{G} :$$



My favorite algebraic topology result in [KT-4-dim]

Theorem 3.15. *There is a commutative diagram of short exact sequences of abelian groups* for any connected 4-manifold X with $\partial X \neq \emptyset$

$$\begin{tikzcd}
\Gamma(\pi_2 X) \arrow[r, tail, "\Gamma(-\circ H)"] \arrow[d, "\Gamma(\mu_2)"] & \pi_3 X \arrow[r, two heads, "\mathrm{Hur}"] \arrow[d, "\mathsf{dax}"] & H_3 \widetilde{X} \arrow[d, "\mu_3"] \\
\mathbb{Z}[\pi \setminus 1] / \langle \overline{g} - g \rangle \arrow[r, tail, "g \mapsto g + \overline{g}"] & \mathbb{Z}[\pi \setminus 1]^\sigma \arrow[r, two heads] & \mathbb{Z}[\pi]^\sigma / \langle 1, g + \overline{g} \rangle \cong \mathbb{F}_2[T_\pi]
\end{tikzcd}$$

In particular, $\mathsf{dax}(a \circ H) = \mu_2(a) + \overline{\mu_2(a)} = \lambda(a, a)$ *for all* $a \in \pi_2 X$,

$$\mathsf{dax}\left([a_1, a_2]_{\mathsf{Wh}}\right) = \lambda(a_1, a_2) + \lambda(a_2, a_1).$$

and $\mathbb{Z}[\pi \setminus 1]^\sigma / \mathsf{dax}(\pi_3 X)$ is, up to an extension, determined by μ_2, μ_3 !