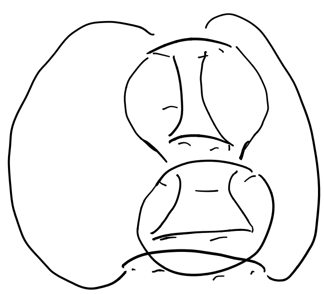
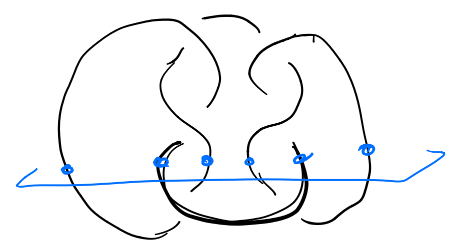
Uniqueness of Bridge Trisections
with Mark Hughes and Seungwon Kim

Intro: Describing surfaces in X^4 with bridge trisections

$S^2 \hookrightarrow S^4$ $\qquad\qquad$ $S^1 \hookrightarrow S^3$





Bridge position wrt Heegaard spl of S^3

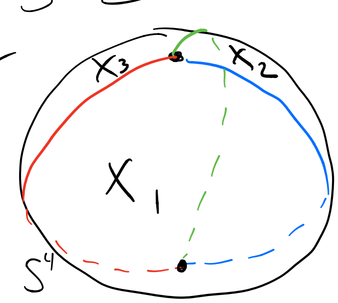
Def (Gay-Kirby 2012)

A (g,k)-trisection of X^4 is (X_1, X_2, X_3)

$$X^4 = X_1 \cup X_2 \cup X_3 \qquad \mathring{X}_i \cap \mathring{X}_j = \emptyset$$

$$X_i \cong \natural_k S^1 \times B^3, \quad X_i \cap X_j \cong \natural_g S^1 \times D^2$$

$$X_1 \cap X_2 \cap X_3 = \Sigma_g$$

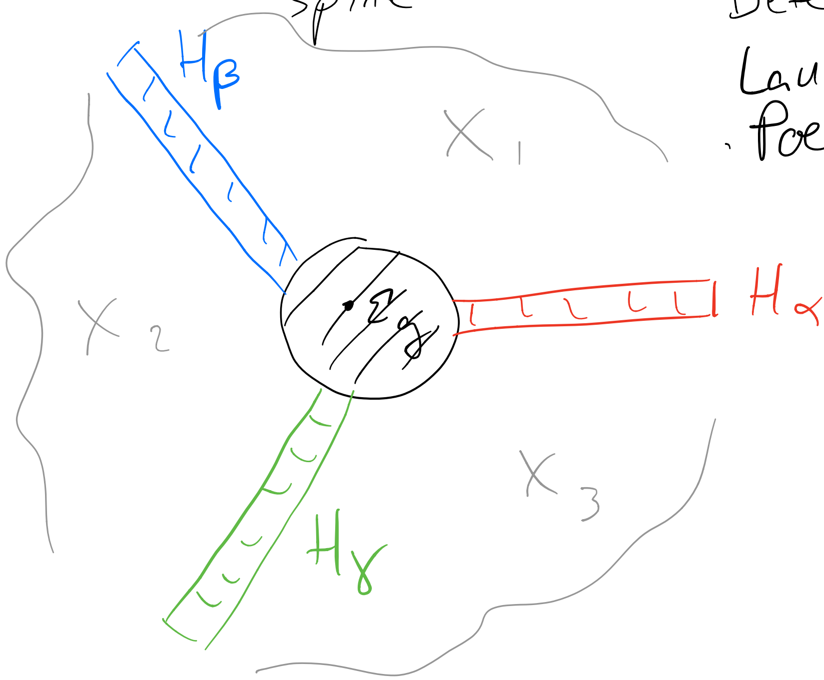


$(0,0)$-trisection of S^4

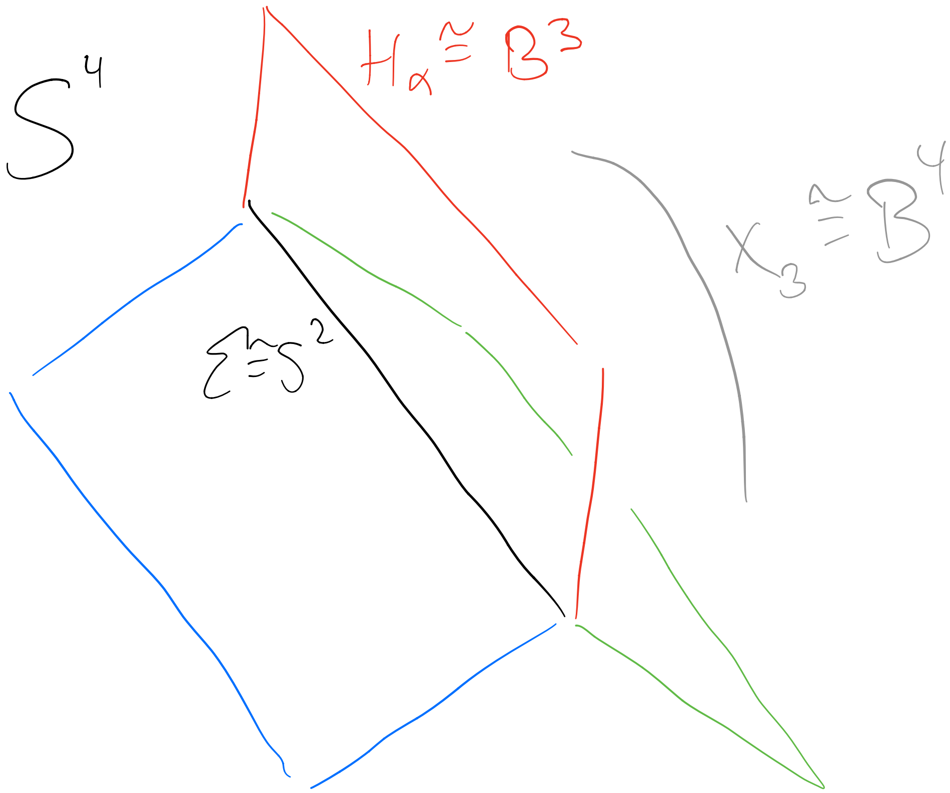
Diagram of (X_1, X_2, X_3) is $(\Sigma_g, \alpha, \beta, \gamma)$ where each α, β, γ are collections of g scc disjoint independent curves on Σ_g and

$$X = \underbrace{\Sigma_g \times D^2 \cup \underbrace{\overset{\partial \text{ in } \Sigma_g \times 0}{H_\alpha}}_{X_1 \cap X_2} \cup \underbrace{\overset{\Sigma_g \times e^{\frac{2\pi i}{3}}}{H_\beta}}_{X_2 \cap X_3} \cup \underbrace{\overset{\Sigma_g \times e^{\frac{4\pi i}{3}}}{H_\gamma}}_{X_3 \cap X_1}}_{\text{spine}} \cup \text{other stuff}$$

Determined by Laudenbach Poenaru



S^4



Gay-Kirby:

X^4 always admits a trisection which is unique up to a stabilization move

($\underset{\text{trisection}}{T} = T'$ ⇝ decompositions ambient isotopic in X^4)

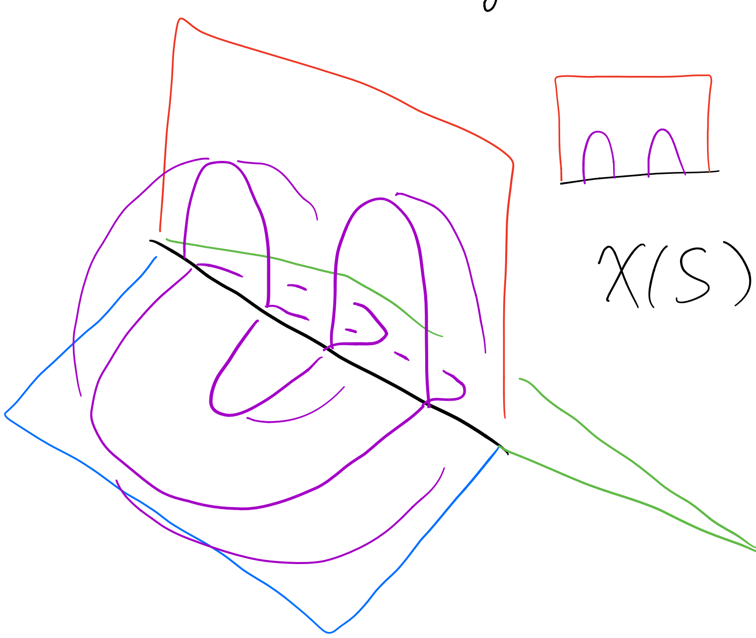
Bridge Trisections (Meier-Zupan
2015 S^4
2017 general)

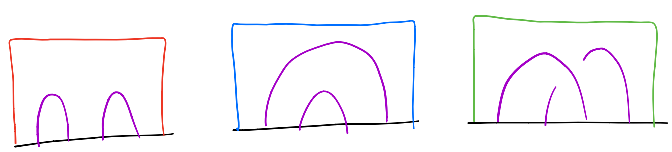
surface $\qquad\qquad$ trisection

$S \hookrightarrow (X^4, \mathcal{T}) \qquad \mathcal{T} = (X_1, X_2, X_3)$

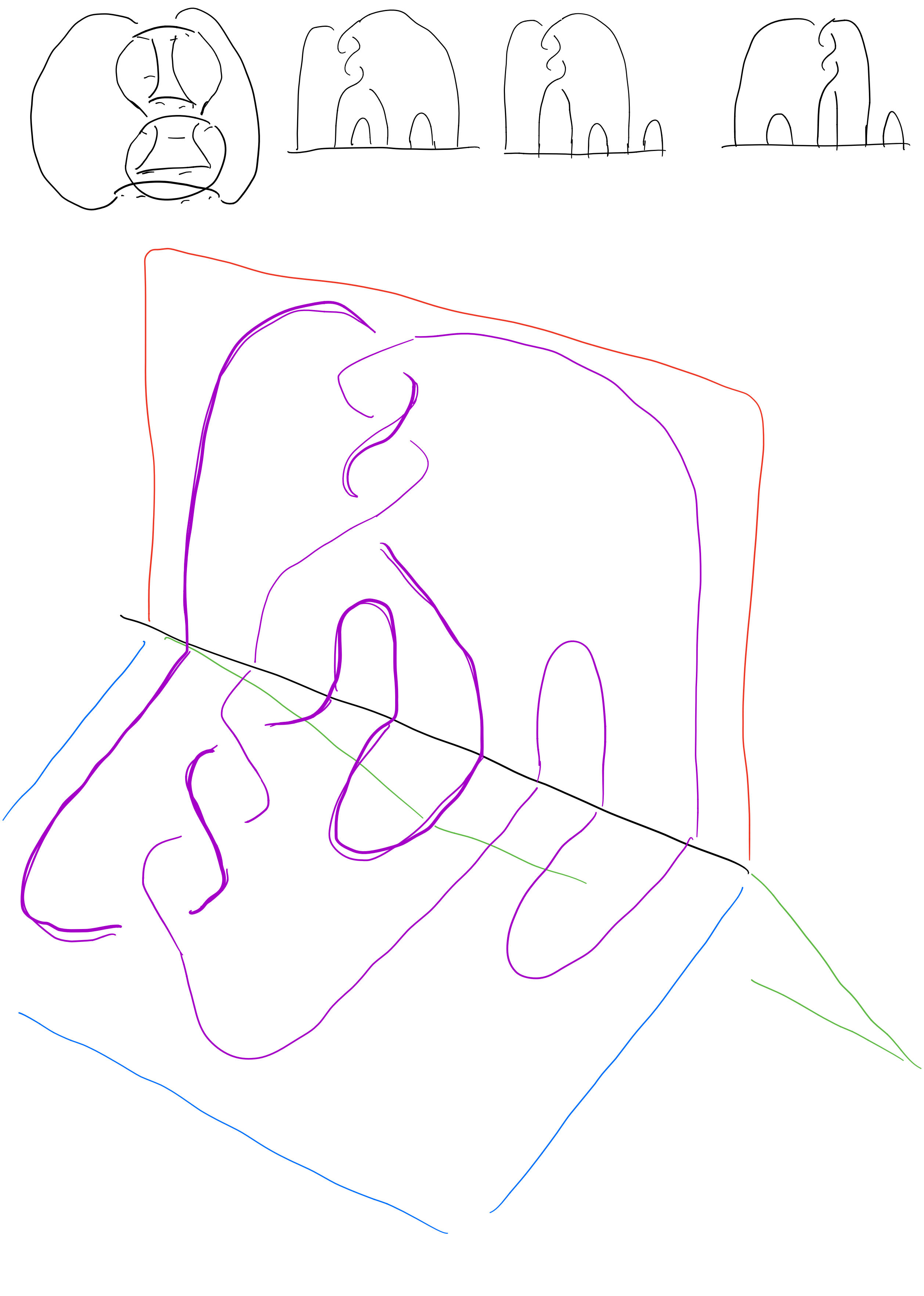
In <u>bridge position</u> wrt $\mathcal{T}$ if

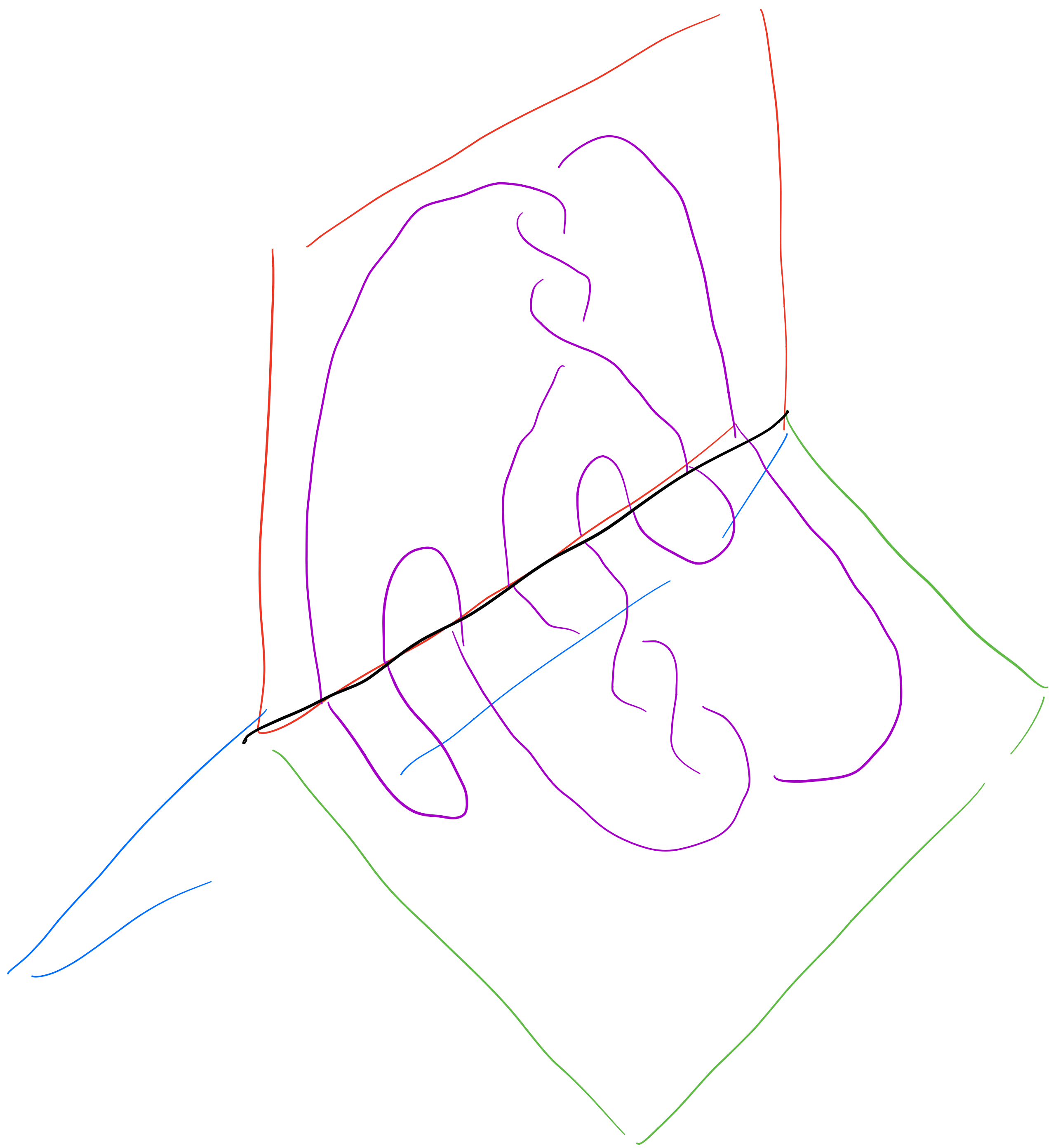
- $S \pitchfork \Sigma = X_1 \cap X_2 \cap X_3$
- $S \cap X_i = \partial$-parallel collection of disks
- $S \cap X_i \cap X_j = \partial$-parallel tangle

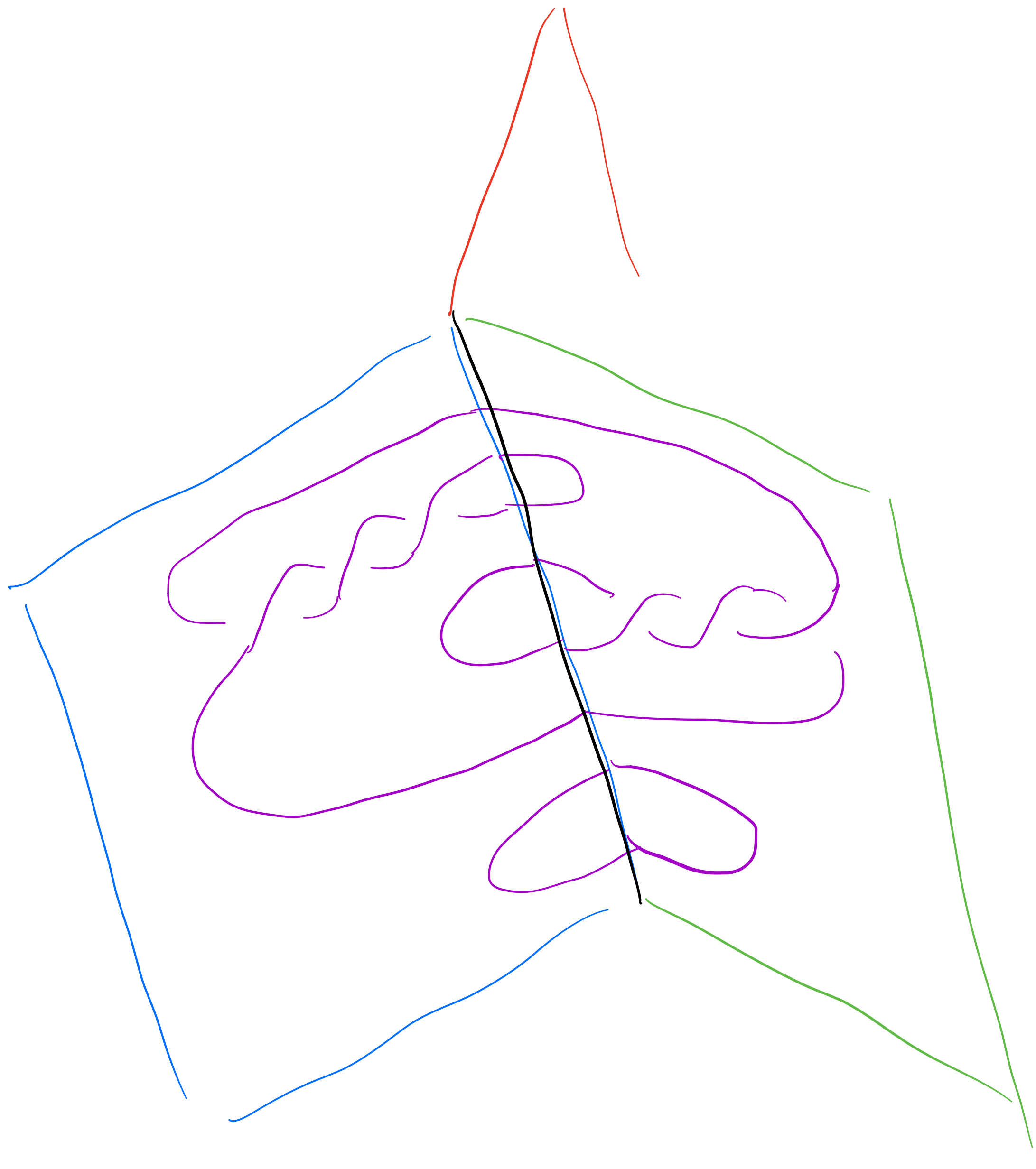




$\chi(S) = 3 - 6 + 4 = 1$





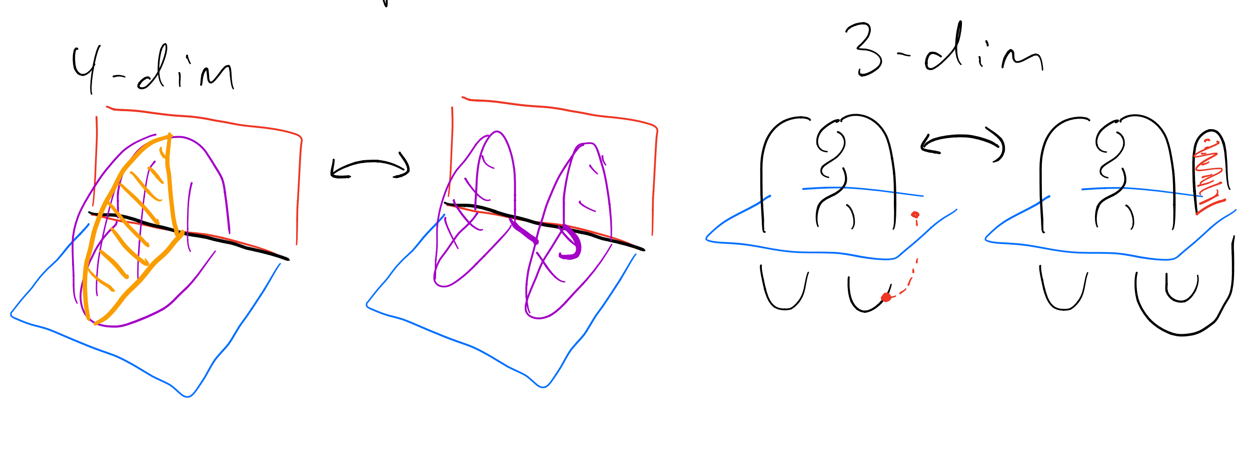


S determined by tangles $\{S \cap X_i \cap X_j\}$

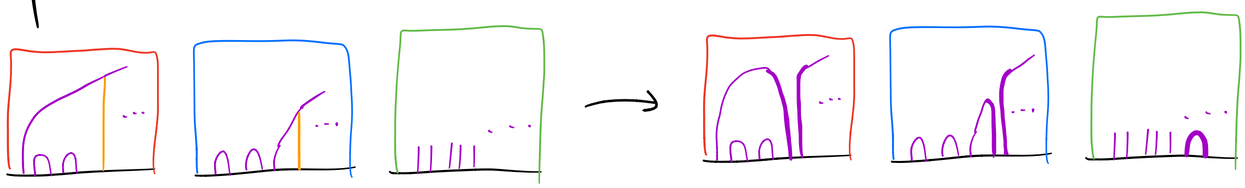
Meier - Zupan:

can always isotope S into bridge position. If $(X, T) = (S^4, (0,0)\text{-trisection})$ then bridge position unique up to

- isotopy in X_i } $\rightsquigarrow$ isotopy of tangles
- perturbation



perturbation



Analogue in Dim' 3 (easy)

Two bridge positions of $K \hookrightarrow (M, \Sigma) \leftarrow$ Heegaard split 3-mfd M

are related by perturbation and ambient isotopy respecting the Heegaard splitting

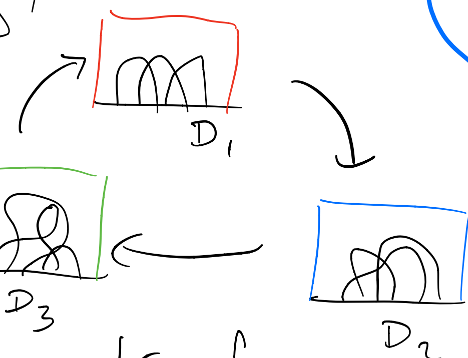
Thm (Hughes-Kim-M)

Bridge position of $S \subset (X^4, T)$ is unique up to perturbation (+ ambient isotopy preserving T)

Potential applications

Saltz: invariant of knotted surfaces in S^4

(Certain chain complex $CS(D_i \overline{D_{i+1}})$ associated to each pair $D_i \overline{D_{i+1}}$ = diagram of unlink



trivial homology: unique generator of highest grading $\Theta_{i,i+1}$

Triple mult $CS(D_i \overline{D_{i+1}}) \otimes CS(D_{i+1} \overline{D_{i+2}}) \otimes CS(D_{i+2} \overline{D_i}) \to CS(D_i \overline{D_i})$

$\Theta_{i,i+1} \otimes \Theta_{i+1,i+2} \otimes \Theta_{i+2,i} \to a_i \Theta_{ii} + \cdots$

$q(S) = a_1 + a_2 + a_3 \mod 2$

Could this be extended, at least for $k = 0$? (so $\partial X_i = S^3$)

Remark: status of this paper unclear; see ArXiV

Thm (Lambert-Cole – Meier; Spreer-Tillmann)
Every even, indefinite intersection form consistent with $11/8$ conjecture can be realized by smooth 4-mfd X admitting a $(g,0)$-trisection.

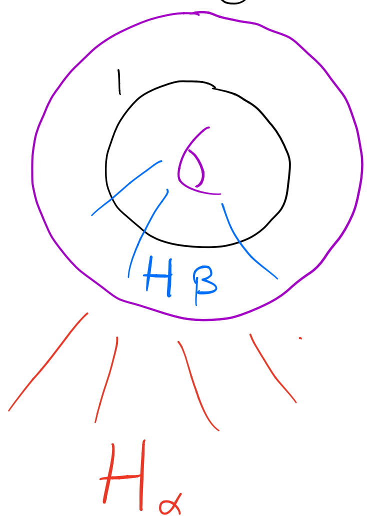
Lambert-Cole: symplectic surfaces in $\mathbb{CP}^2$ admit "nice" bridge trisections

$\rightsquigarrow$ any bridge trisection of symplectic surface can be made "nice" by perturbation.

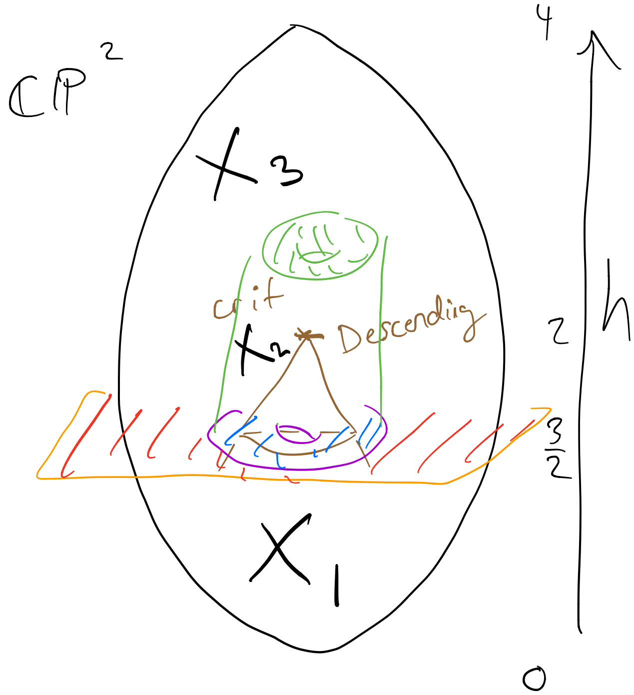
(1,0)-trisection

Pf

Kirby diagram (2-handle attaching circles $\subset \#S^1\times S^2$)

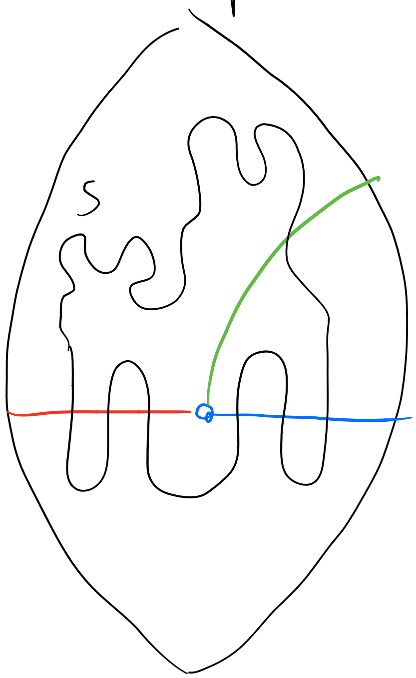


Σ = Heegaard surface for $\#_0 S^1\times S^2 \cong B^4 = H_\alpha \cup H_\beta$

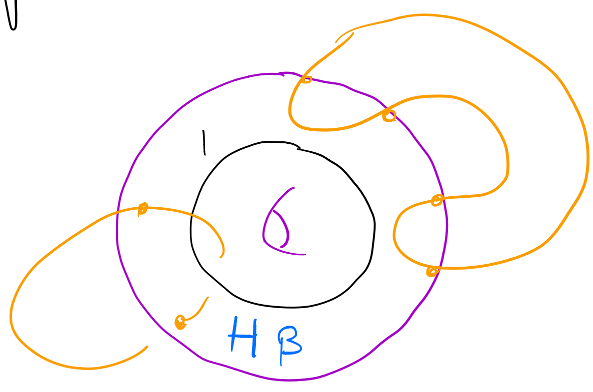


Isotope S so $S \cap \underbrace{h \leq \frac{3}{2}}_{X_1}$

= ∂-parallel disks



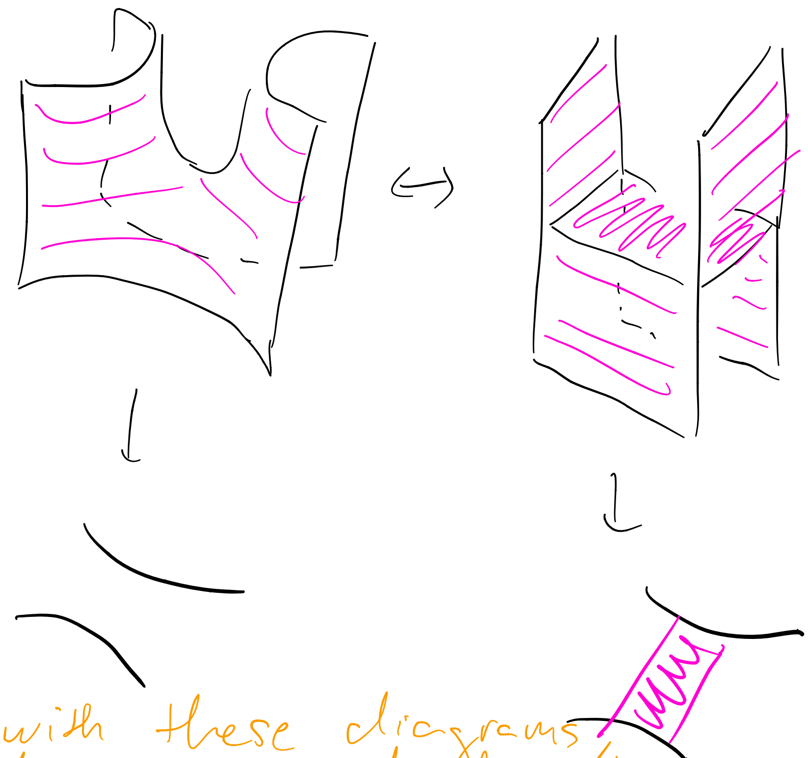
Near $h = \frac{3}{2}$, put $S \cap \{h = \frac{3}{2}\}$ into bridge position WRT Σ



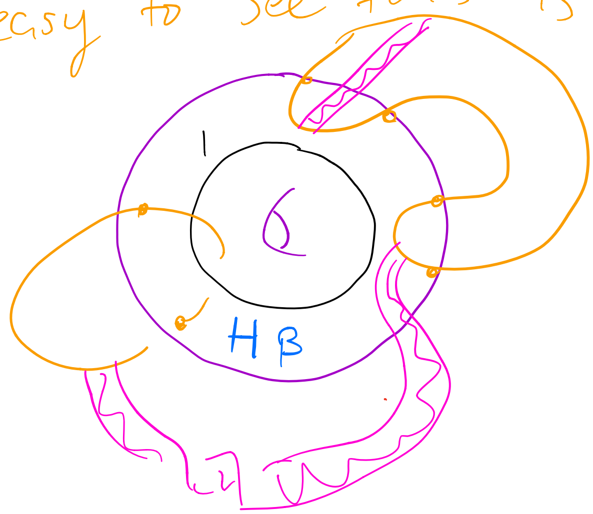
H_α

Isotope $S \cap \{h \geq \frac{3}{2}\}$ so index-1 crit's just above $h = \frac{3}{2}$

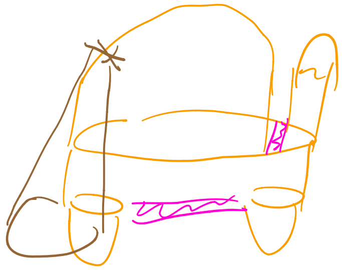
Flatten to bands and project to $h = \frac{3}{2}$



If we are comfortable with these diagrams easy to see this is standard $\mathbb{CP}^1$

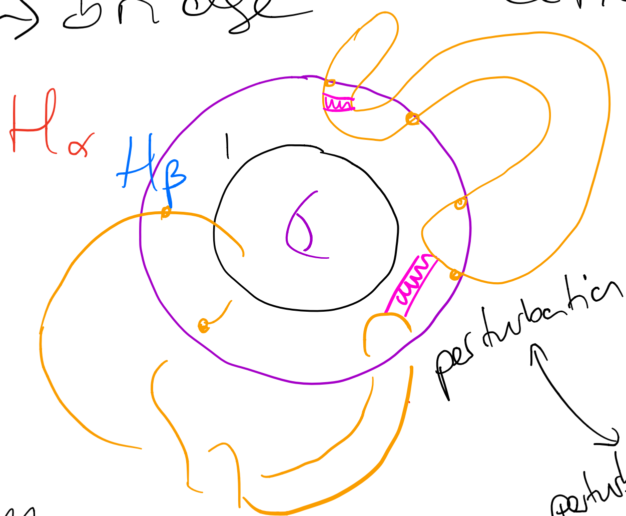


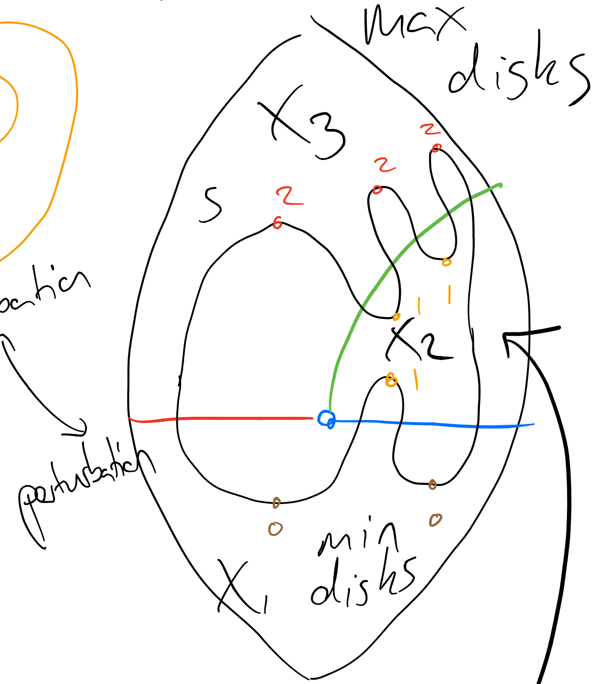
Isotope near $h = \frac{3}{2}$ so bands in H_β and parallel to Σ



2-sphere in homology class $\pm[\mathbb{CP}^1] = \pm 1$

⇝ bridge trisection

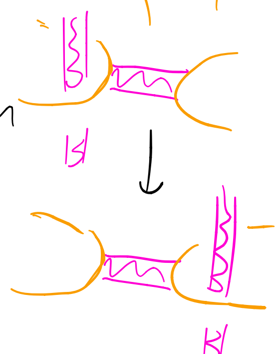




Thm
In Kirby diagram, isotopy of S looks like

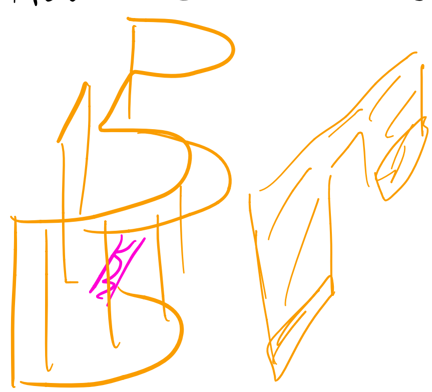
- Isotopy
- band slides

- band swim



- cup/cap

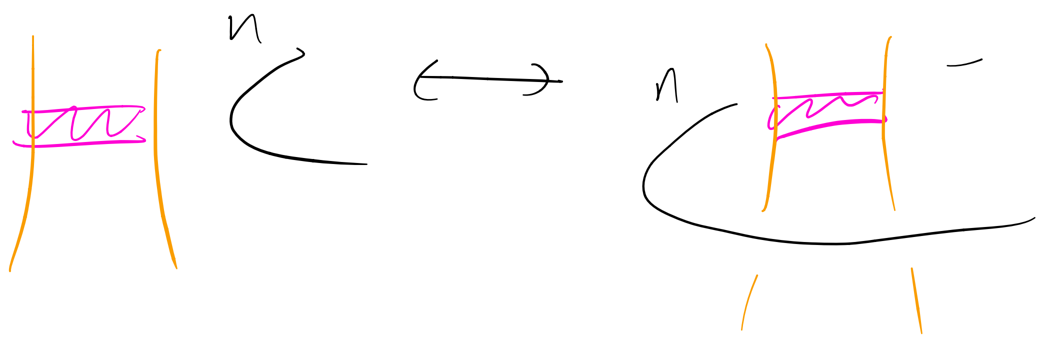
In here, surface with only index-1 crits perturb so far from each other



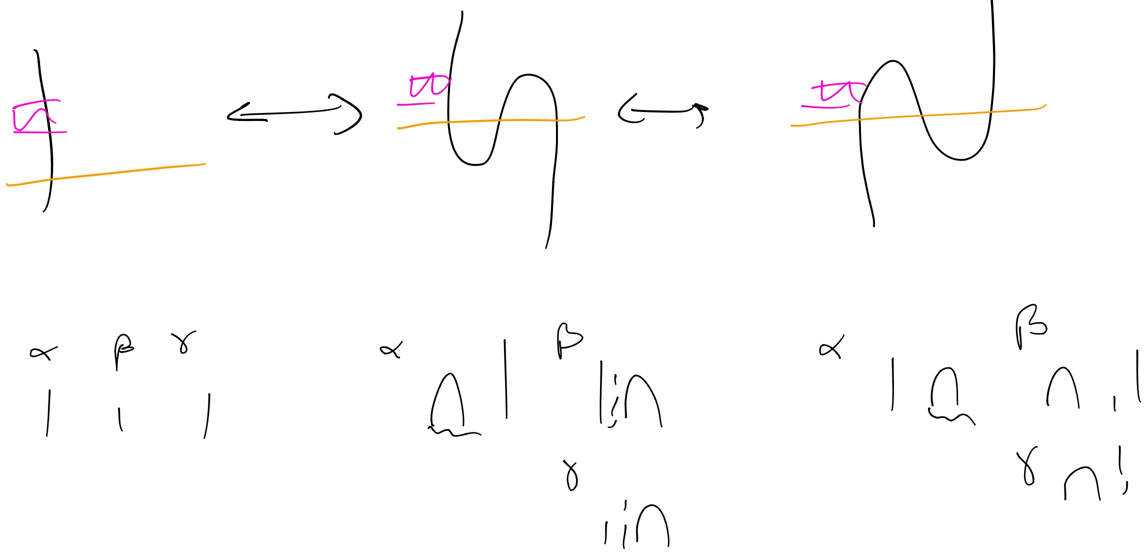
AND 2-handle slide/swim

- slide over 2-handle circle

- swim

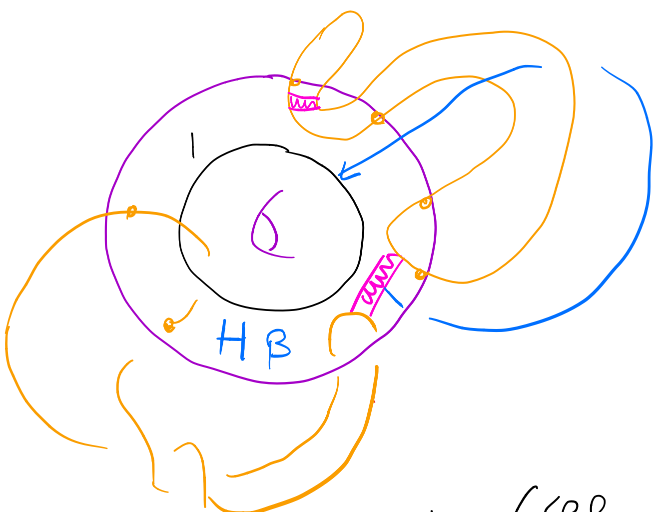


Use elementary perturbations

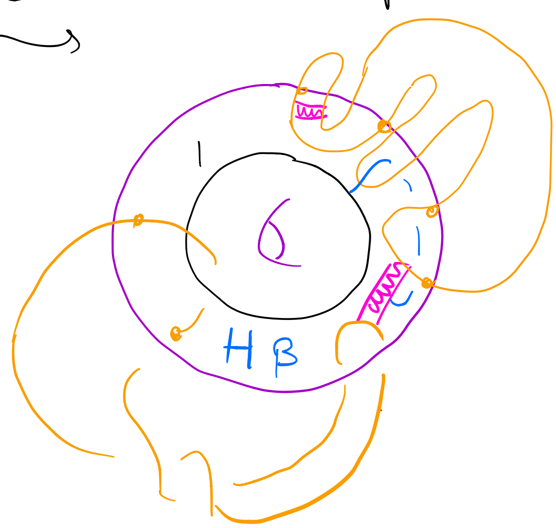


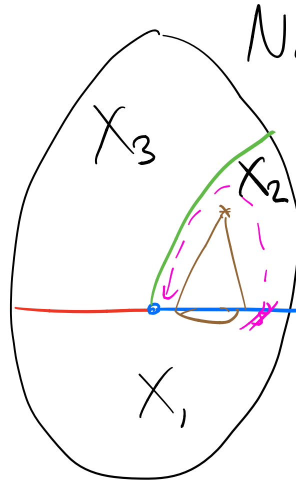
correspond to perturbation

Slide a band over 2-handle
by perturbation + Heegaard-respecting
isotopy

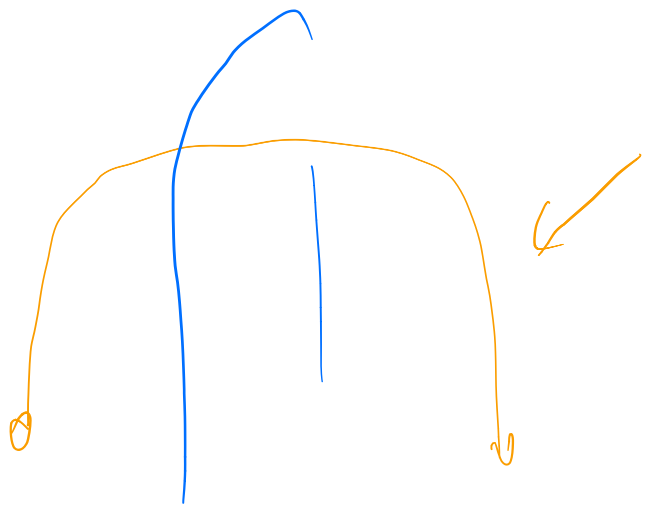


isotope arc into H_β (see next page)

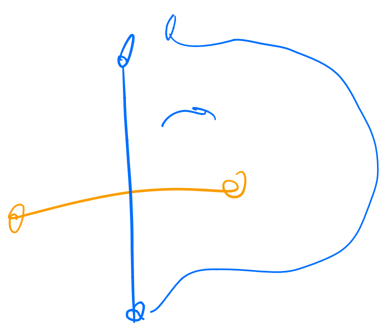




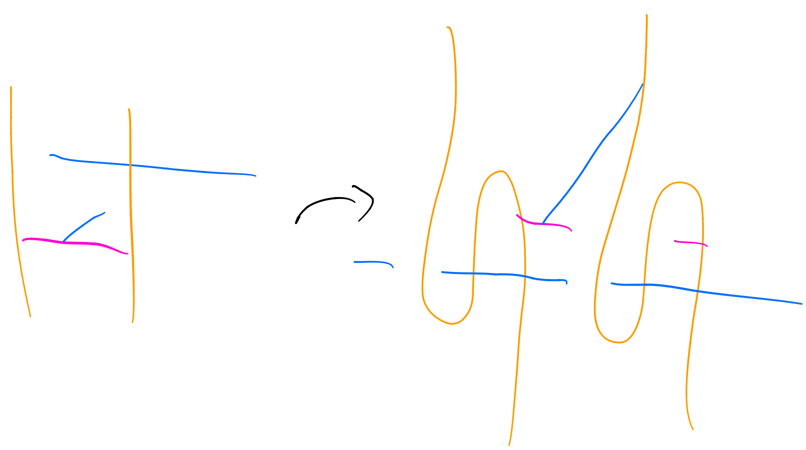
Now isotopy supported in X_2

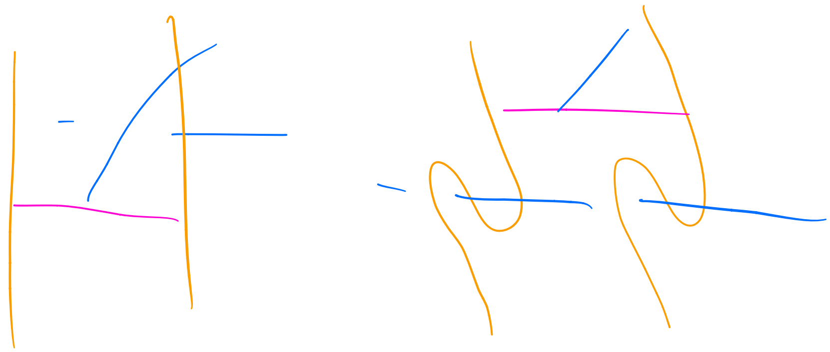


trivial tangle



self crossings





Framing