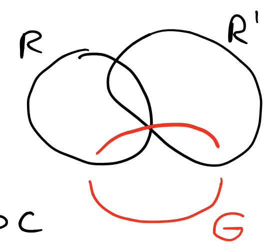
Light bulb concordance (20 min)

Thm (4D light bulb Gabai 2017)

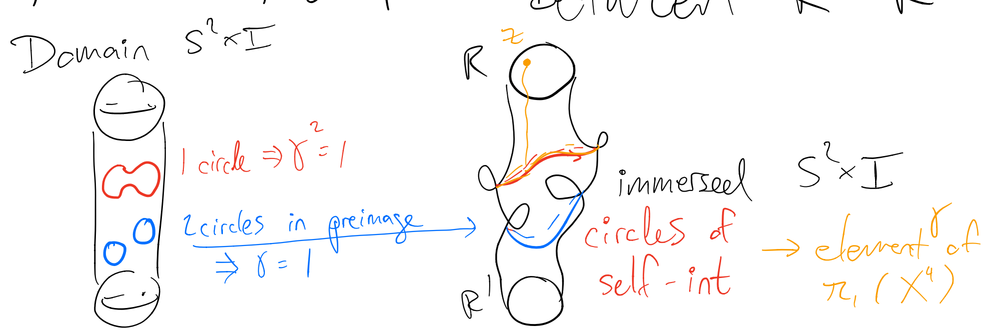


$R, R' \hookrightarrow X^4$ htpc 2-spheres which have common **dual** sphere

Def. Given $R \subset X^4$, say G is a **dual** sphere for R if G = 2-sphere w/ triv. normal bundle and $R \cap G = pt$

and $\pi_1(X^4)$ has no 2-torsion, then R and R' are ambiently isotopic.

$\exists$ modified statement when $\pi_1(X^4)$ has 2-torsion : Study regular homotopy between R R'



Gabai's condition: Every element of $T_2 - \{1\}$ appears as a self-intersection of track of homotopy an *even* # of times.

2-torsion subgp of $\pi_1 X$

$\Rightarrow R, R'$ isotopic

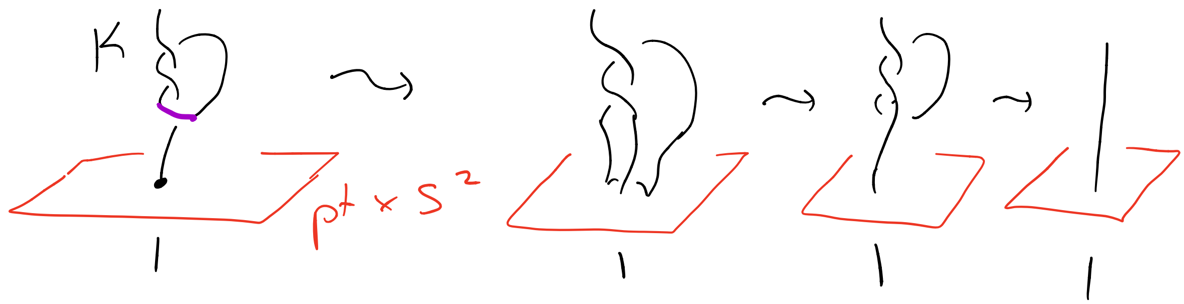
Thm (Schwartz)

This condition is necessary.

3D analogue

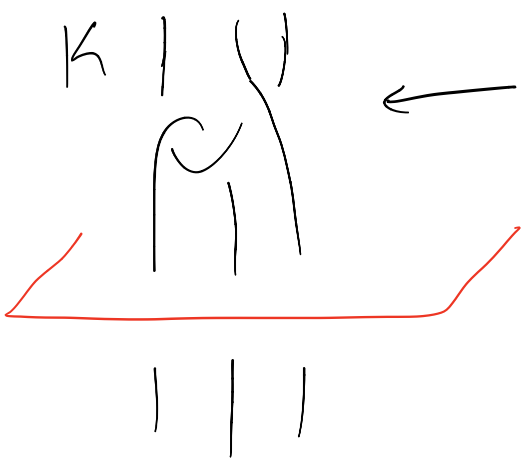
Let $K \subset S^1 \times S^2$ be a circle intersecting $pt \times S^2$ once (transversely). Then K isotopic to $S^1 \times pt$.

(Analogue: $K = R'$ $S^1 \times pt = R$, $pt \times S^2 = G$)



What if $|K \cap pt \times S^2| > 1$?

Say $[K] = [S^1 \times pt]$ in $H_1(S^1 \times S^2)$



K <u>not</u> isotopic to $S^1 \times pt$

$(\pi_1(S^1 \times S^2 \setminus K) \not\cong \mathbb{Z})$

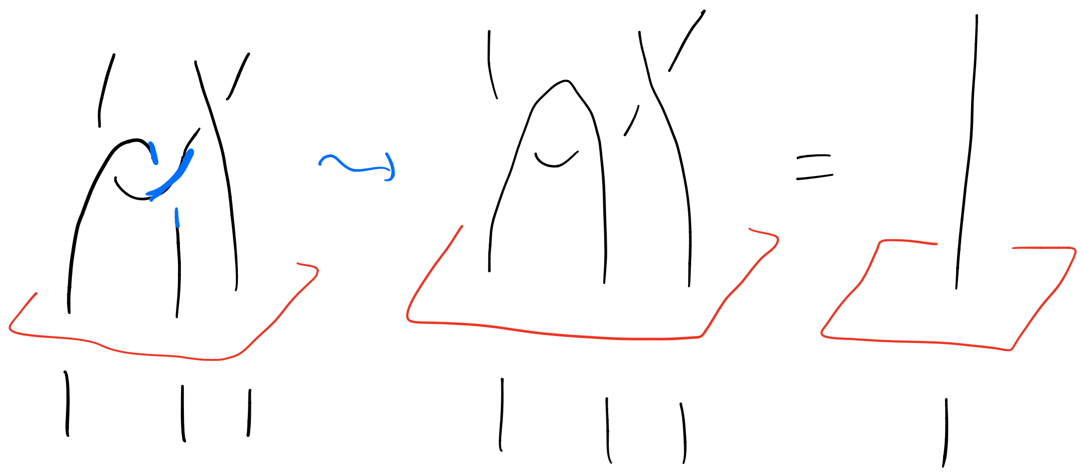
<u>But</u>

Thm (Yildiz; Davis-Nagel-Park-Ray)

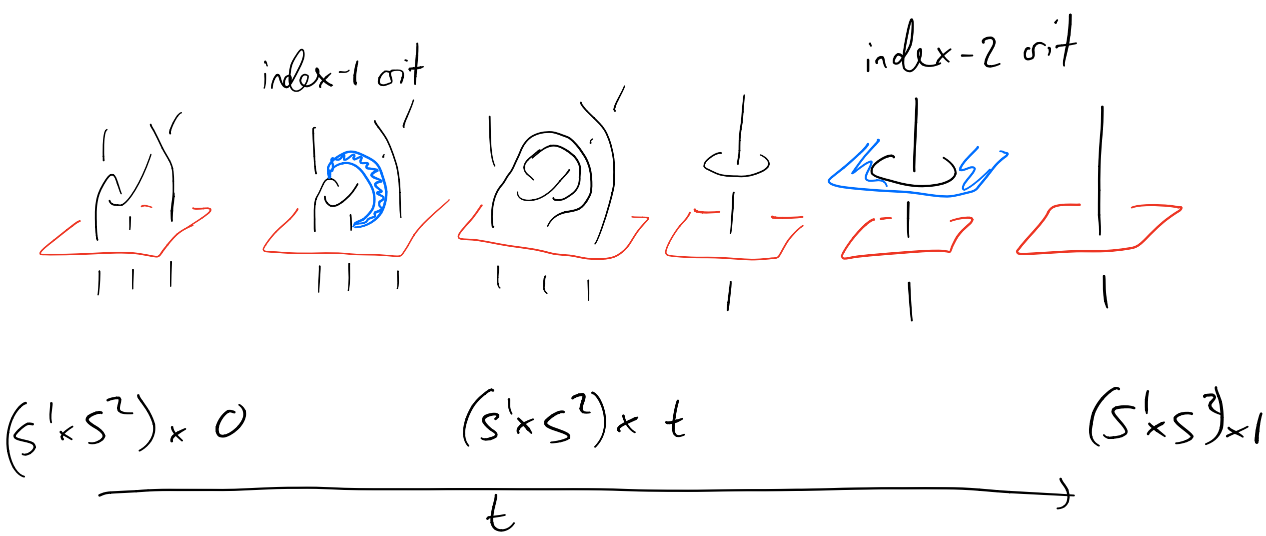
If $[K] = [S^1 \times pt]$ in $H_1(S^1 \times S^2)$ then K <u>concordant</u> to $S^1 \times pt$.

Pf that K concordant to $S^1 \times pt$

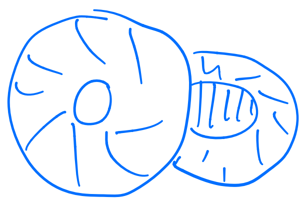
- K homotopic to $S^1 \times pt$ through crossing changes



Build concordance from crossing changes



This is a picture of surface



Annulus
$\rightsquigarrow$ concordence
from K to $S^1 \times pt$.

Back to dim'n 4

Thm

If $R, R' \hookrightarrow X^4$ htpc 2-spheres
$\underset{S^1\times pt}{\underbrace{R}}$ has transverse sphere G
~~and $\pi_1(X^4)$ has no 2-torsion~~

same condition on homotopy

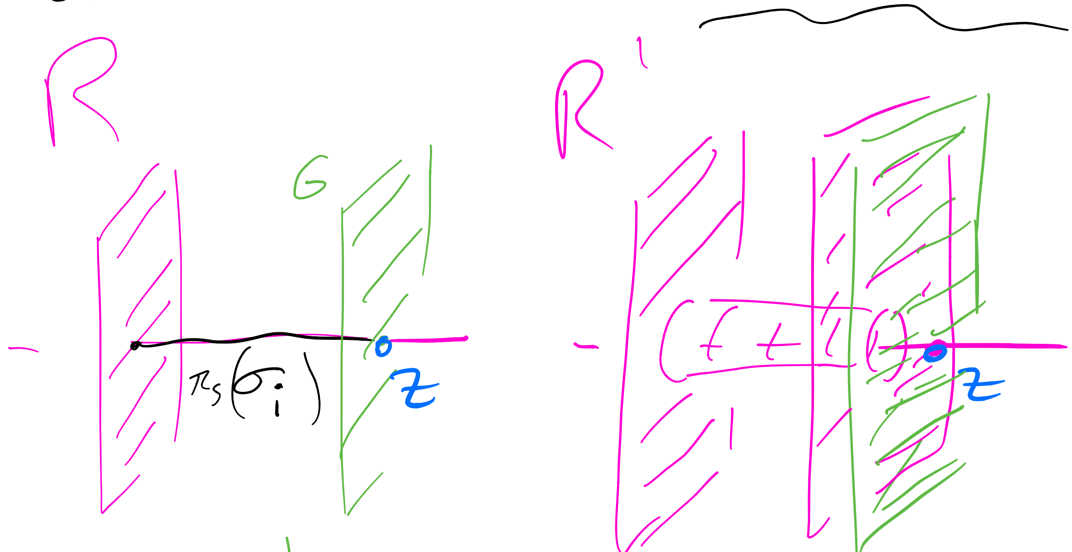
then R and R' are concordent.

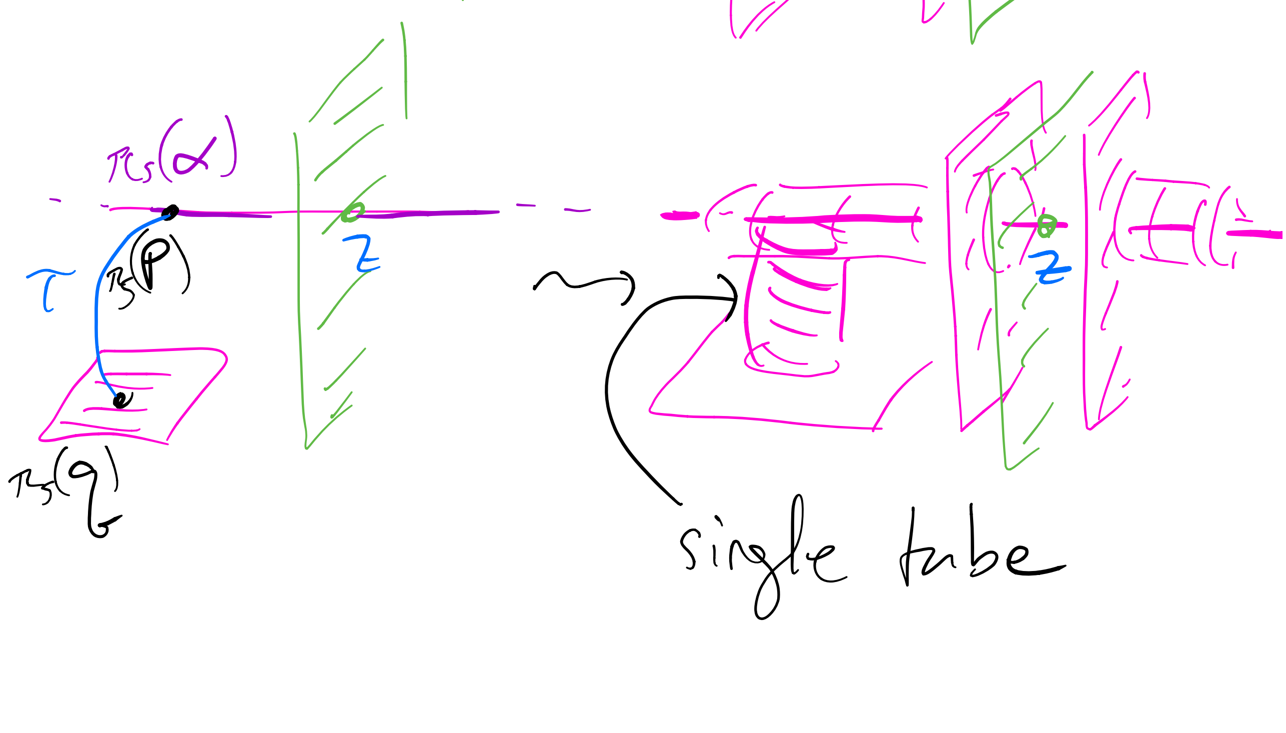
Zeeman:
5-twist spun trefoil^K has group $A_5 \times \underset{\langle \mu \rangle}{\mathbb{Z}}$
surger μ to get $S^2 \times S^2$, K htpc to $S^2 \times pt$
$\pi_1(S^2 \times S^2 \setminus K) \cong A_5$ so K not isotopic
to $S^2 \times pt$ (Due to Sato)

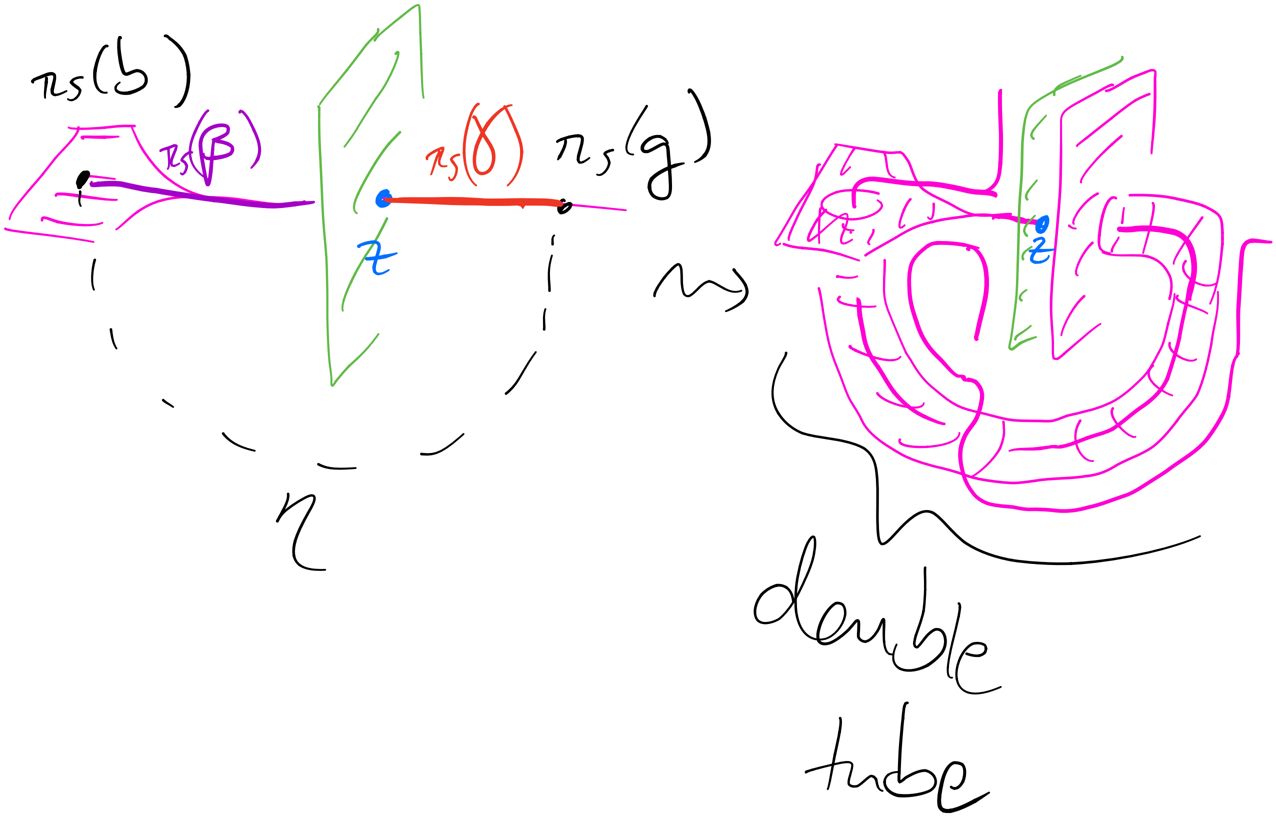
4DLBT Proof idea

Step 1

R' can be put into standard form: tubed surface





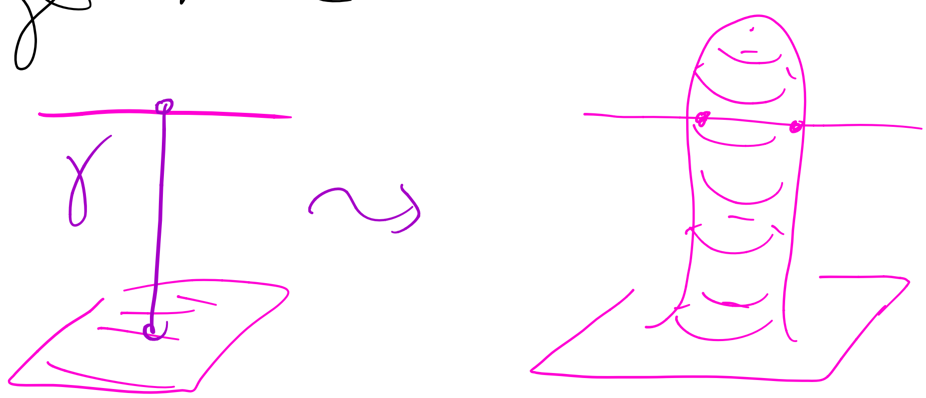


$\pi_S(b)$
$\pi_S(\beta)$
$\pi_S(\gamma)$
$\pi_S(g)$
z
η
$\rightsquigarrow$
z
double tube

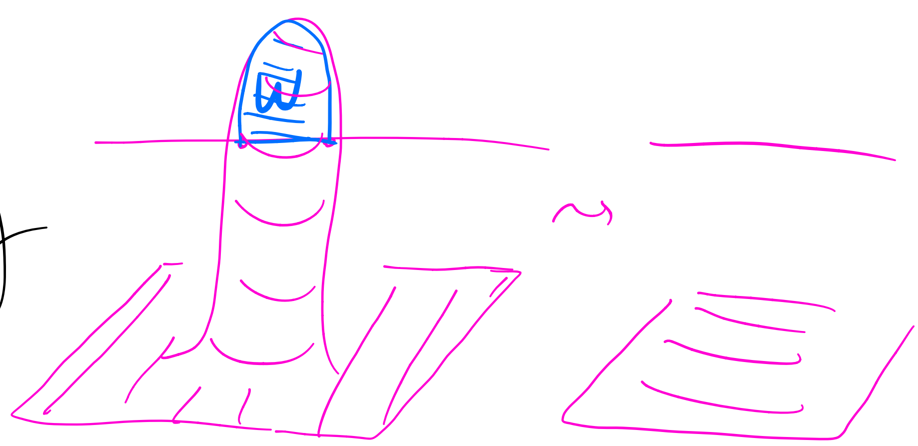
Regular Htpy of surfaces

(see e.g. Freedman-Quinn)

Finger move



Whitney move



Thm (Smale)

If R, R' embedded htpc surfaces in X^4 then R and R' are reg htpc

Thm (Quinn?)

Then $\exists$ a sequence of finger moves $f_1, \dots, f_n$ followed by Whitney moves $w_1, \dots, w_n$ (w/ intermediate isotopies) so

$$R' \to f_1, f_2, \dots, f_n, w_1, \dots, w_n \to R$$

Have R' isotopic to tubed surface R'' on R
realization

Gabai

$\rightsquigarrow$ List

$\{[\eta] \in \pi_1(X^4, z)$

η a double tube of $R''\}$

If every 2-torsion element of $\pi_1(X^4)$ appears an even # of times, then R'' isotopic to R.

Now prove concordance theorem

<u>Setup</u> R, R' (neg) htpc 2-spheres in X^4

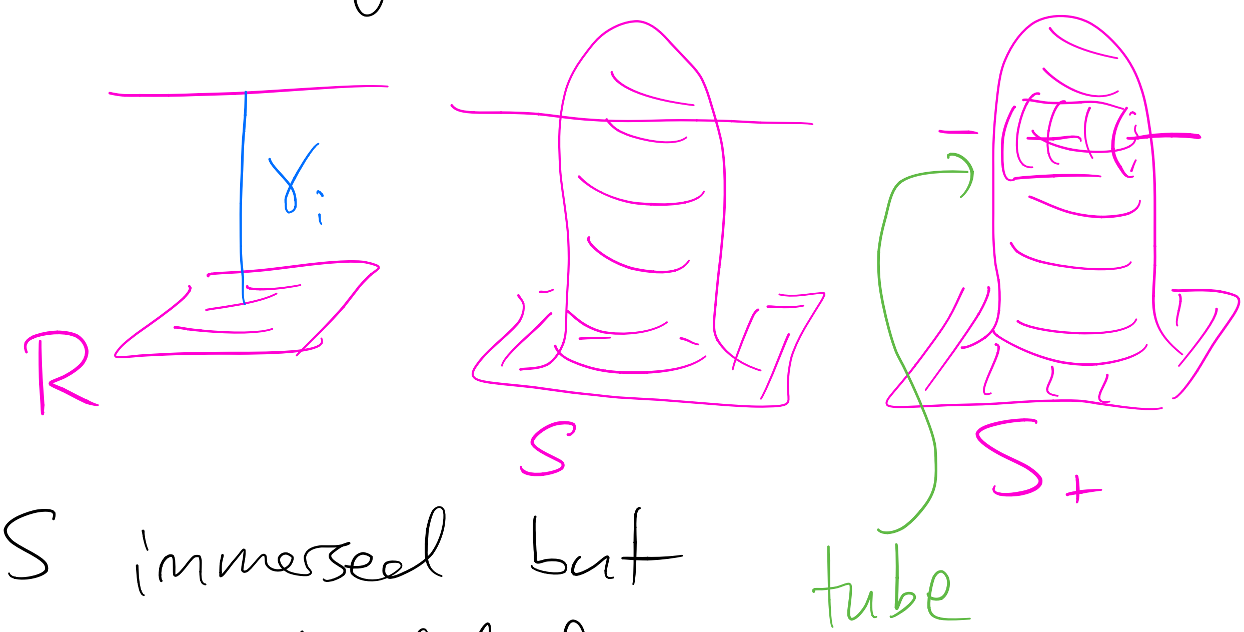
G a transverse sphere for R

finger moves $f_1, \dots, f_n$ then Whitney moves $w_1, \dots, w_n$ takes R' to R.

Step 1

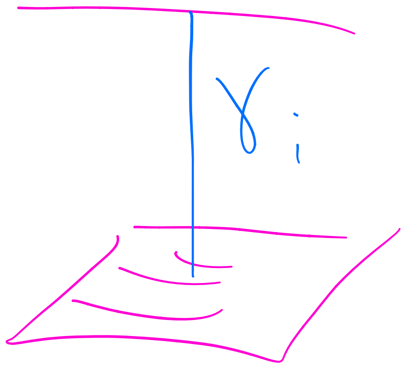
$S = R'$ after $f_1, \ldots, f_n$

$S_+ =$ genus-n surface



S immersed but S_+ embedded.

Can also obtain S_+ by attaching tubes to R.

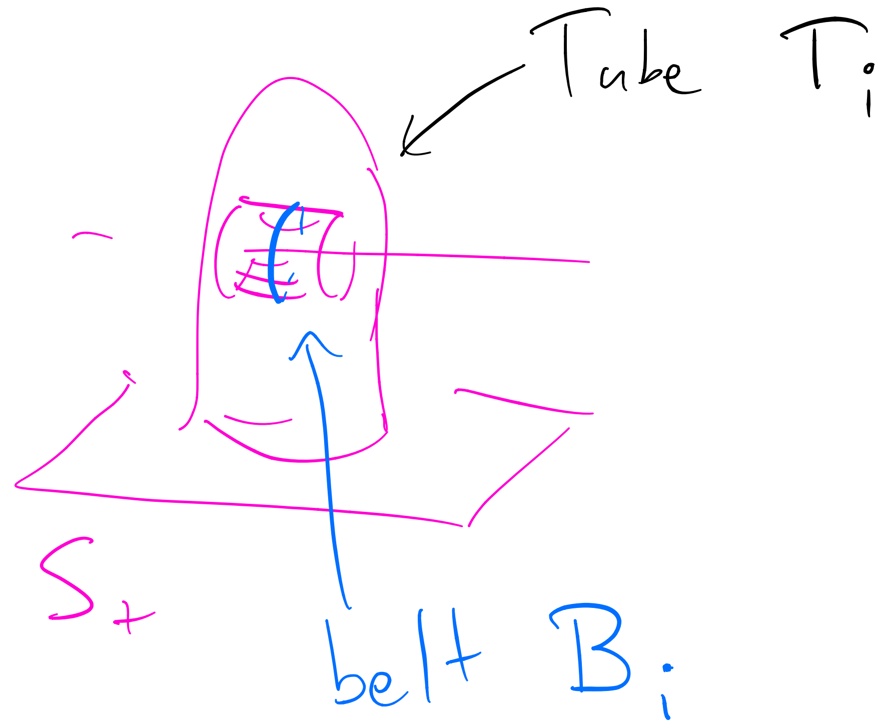


R

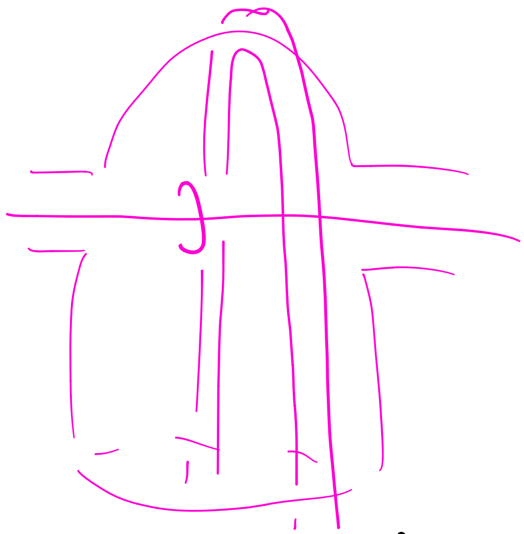
S_+

This gives a cobordism M, from R to S_+ in $X^4 \times I$ where $M = R \times I \cup n$ 1-handles

To geometrically cancel 1-handles, we want to attach 2-handles along these circles:

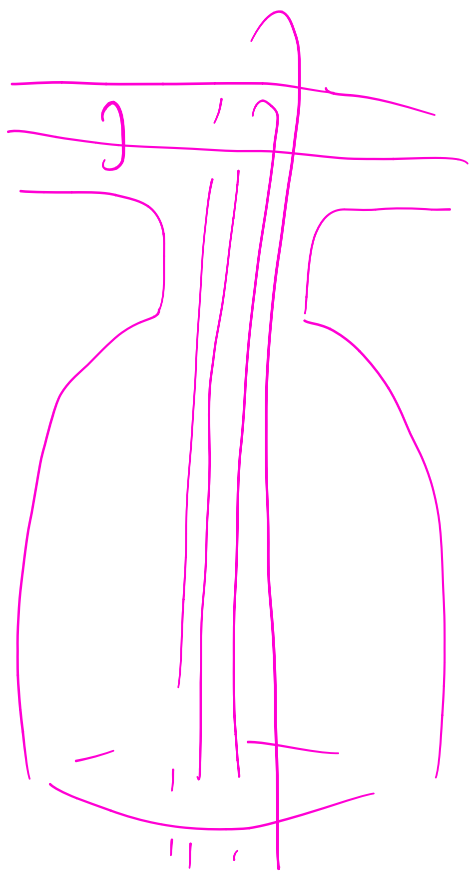


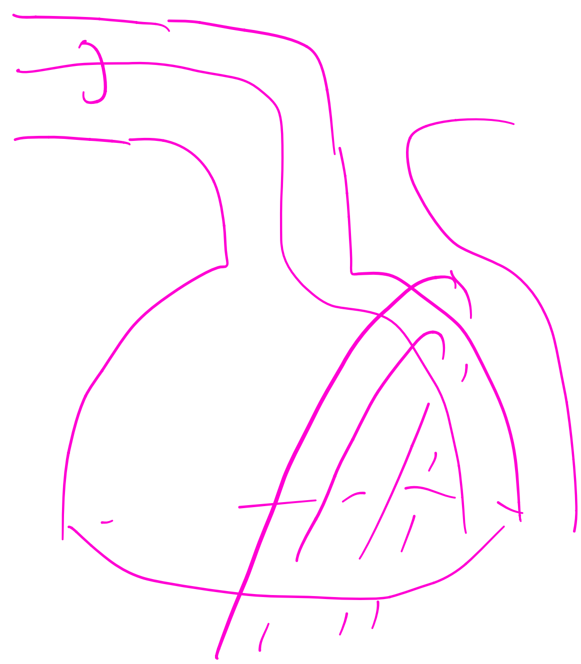
Isotope S_+ near Whitney moves (in order, tubes disjoint from interior of disk by dimensionality)



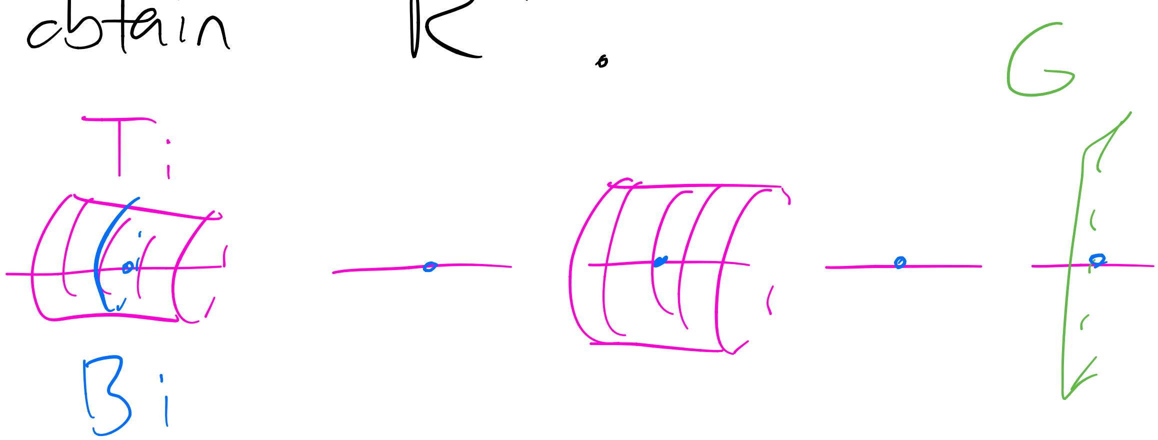
uncrossed

crossed

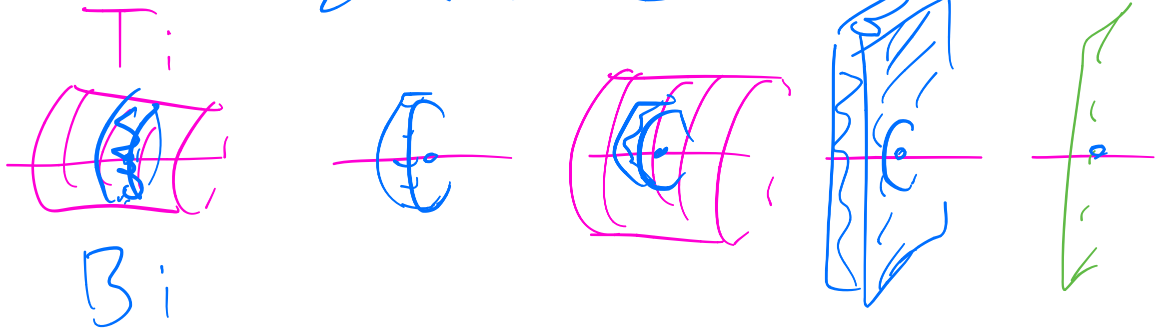




Take B_i still centered about point b_i in R. Choose arcs from b_i to z and compress T_i along disks parallel to tube around arcs + G to obtain R''.



2-handle

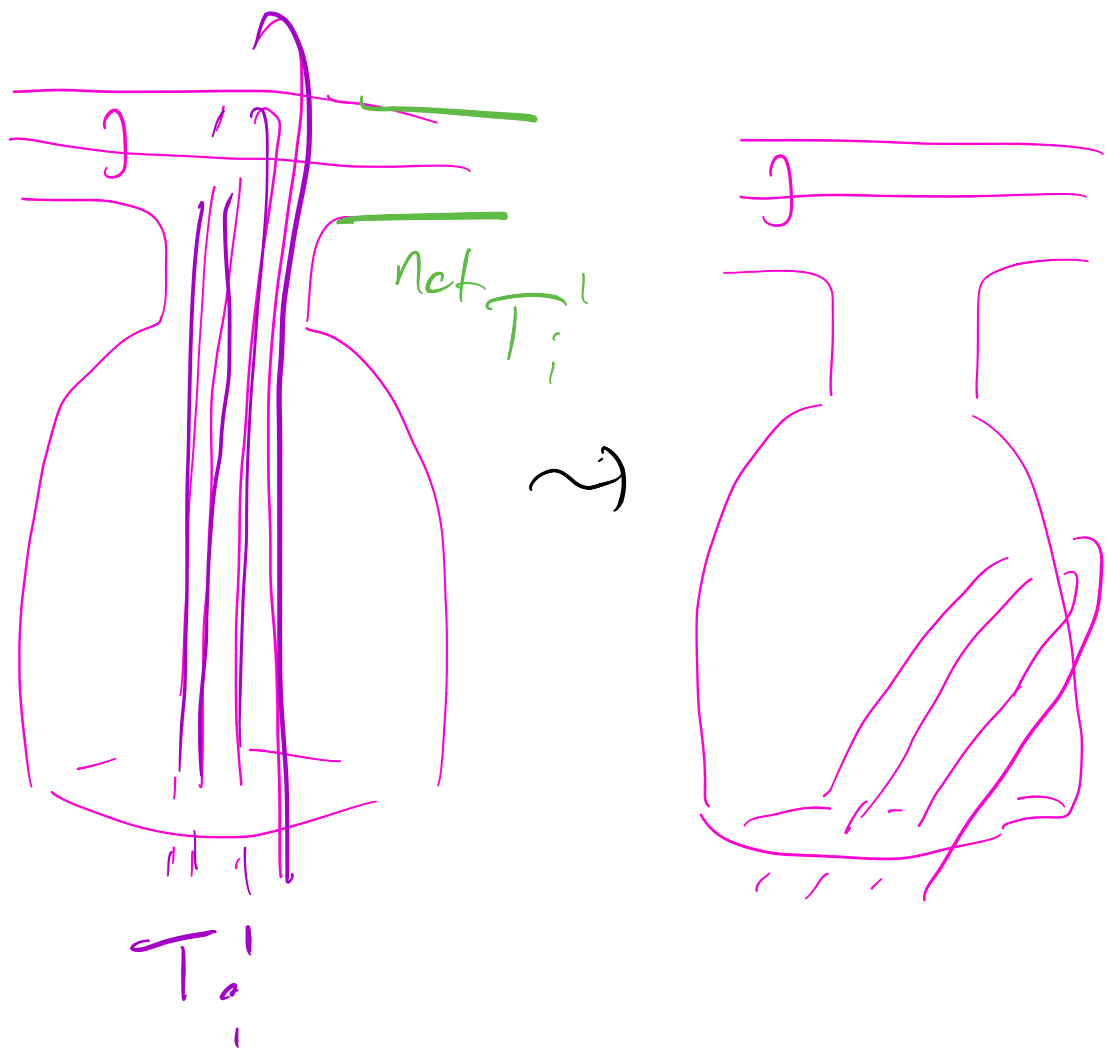


$\rightsquigarrow$ cobordism $M_2 \subset X^4 \times I$
from S_+ to R''

$M_2 = S_+ \times I \cup n$ 2-handles

$N = M_1 \cup M_2$
= concordance from
R' to R''

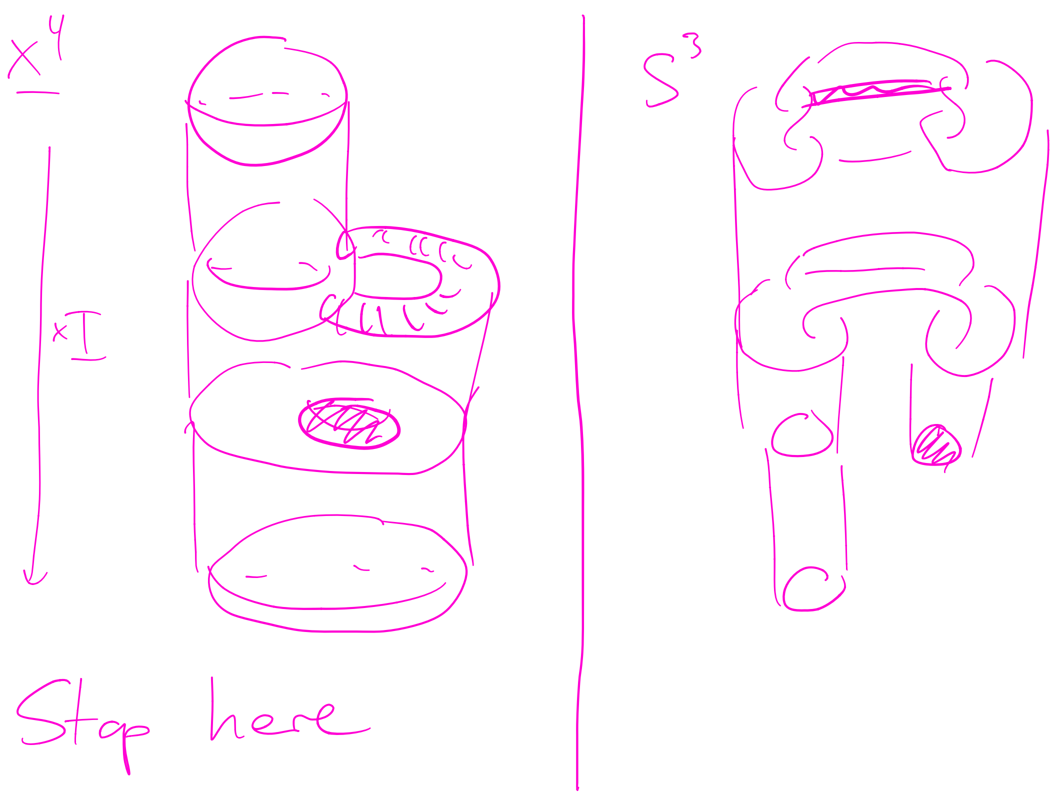
• R'' is a (realization of a) tubed surface on R!



uncrossed $w_i \rightsquigarrow$ single tubes

crossed $w_i \rightsquigarrow$ double tubes

$\therefore$ If $\pi_1(X^4)$ no 2-torsion, then R'' isotopic to $R \rightsquigarrow$ done.



Stop here

Thm (4D LBT Gabai)

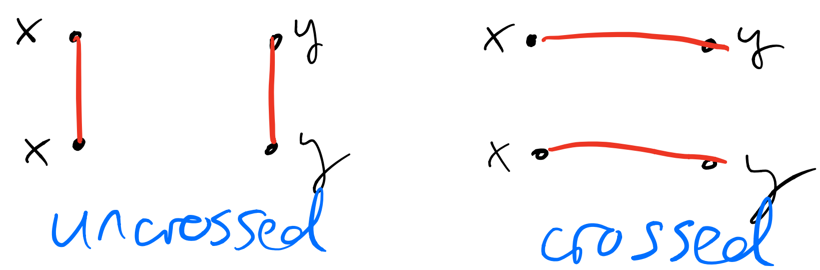
Let R, R' be htpc 2-spheres embedded in X^4 with mutual transverse sphere G.

Say finger moves $f_1, \ldots, f_n$ followed by Whitney moves $w_1, \ldots, w_n$ take R' to R.

Let $S = R'$ after $f_1, \ldots, f_n$.

Let $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ be preimages of self-intersections of S (made $2n$ choices)

Say w_i Whitney move on disk W_i, banding W_i connects



Let η_i be the arc in X^4 with banding on R defining inverse finger move to w_i.

List $\{ [\eta_i] \in \pi_1(X^4, z) \mid w_i \text{ crossed} \}$

If every element of 2-torsion appears even # of times, then R and R' isotopic.

Thm
Same as above
except only R transverse to G
$\Rightarrow$ conclude R and R'
are concordant.

The remainder of the talk is
joint work with Michael
Klug.

Teichner - Schneiderman

Let $R, R' \hookrightarrow X^4$ 2-spheres w/ common dual G. Assume R, R' htpc. Then R, R' isotopic iff $fq(R, R') = 0$. (automatically if $\pi_1 X^4$ no 2-torsion, then $fq(R,R') = 0$)

fq = Freedman-Quinn invariant

Defined for homotopic pair of 2-spheres $A, B \hookrightarrow X^4$

$fq(A,B) \subset \mathbb{F}_2 T_M / \mu_3(\pi_3 M)$

where $T_M \subset \pi_1 X^4$ is the 2-torsion subset ($T_M = \{\text{2-torsion elements of } \pi_1 X\}$) and $\mathbb{F}_2 T_M$ is a vector space.

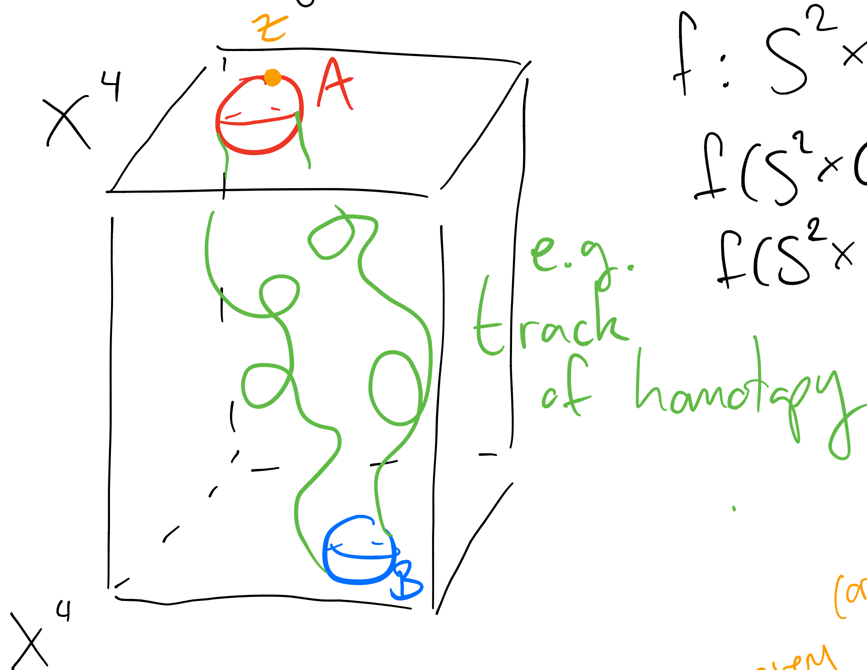
Def (fq)

Have immersion

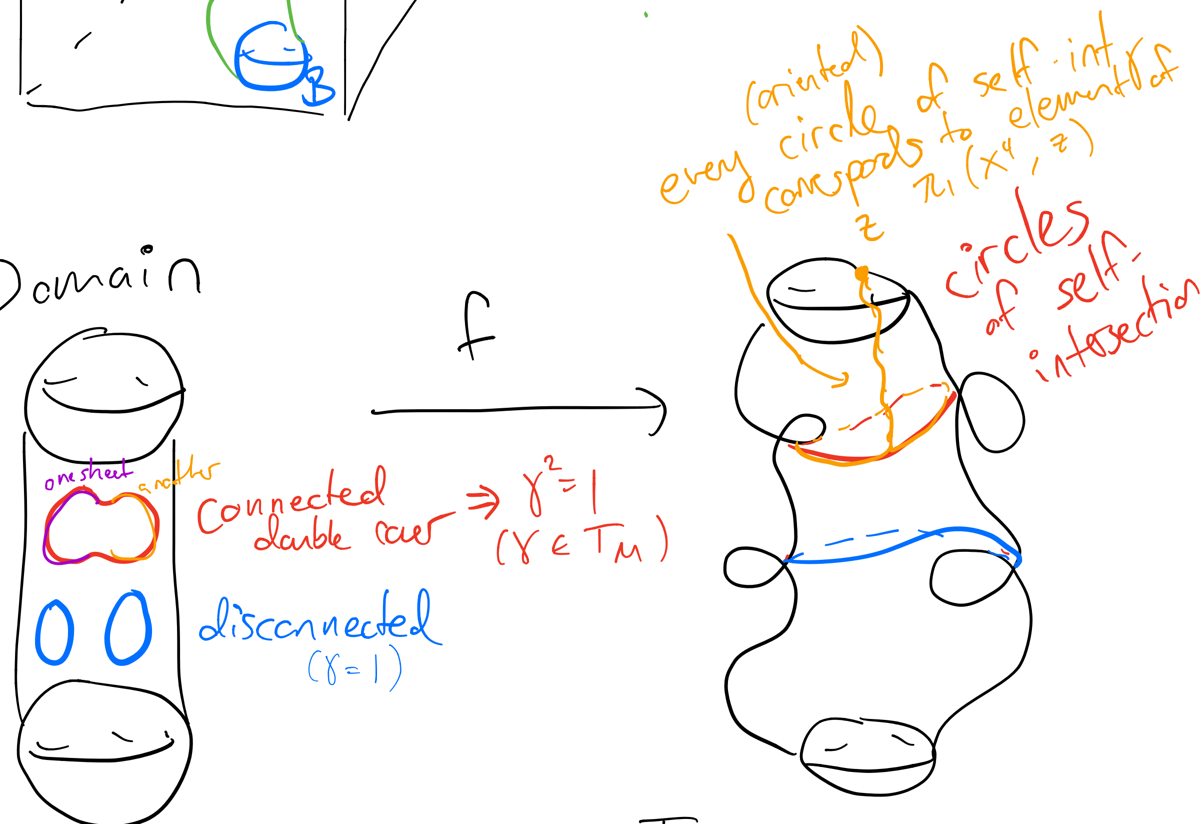
$$f: S^2 \times I \looparrowright X^4 \times I$$

$$f(S^2 \times 0) = A \times 0$$

$$f(S^2 \times 1) = B \times 1$$



Domain



$$fq(A,B) = \sum a_i \gamma_i \in \mathbb{F}_2 T_M / \mu_3 \pi_3 M$$

$\gamma_i \in T_M$, a_i = #times γ_i a self-int of f with connected double cover (mod 2)

Turns out this does not depend on choice of f. (Up to $\mu_3 \pi_3 M$)

Rmk If A, B concordant, then $fq(A, B) = 0$,

If $\pi_1 X^4$ no 2-torsion, then $fq(A, B) = 0$.

Sunukjian (2013)

If $R, T \hookrightarrow X^4$ are 2-spheres

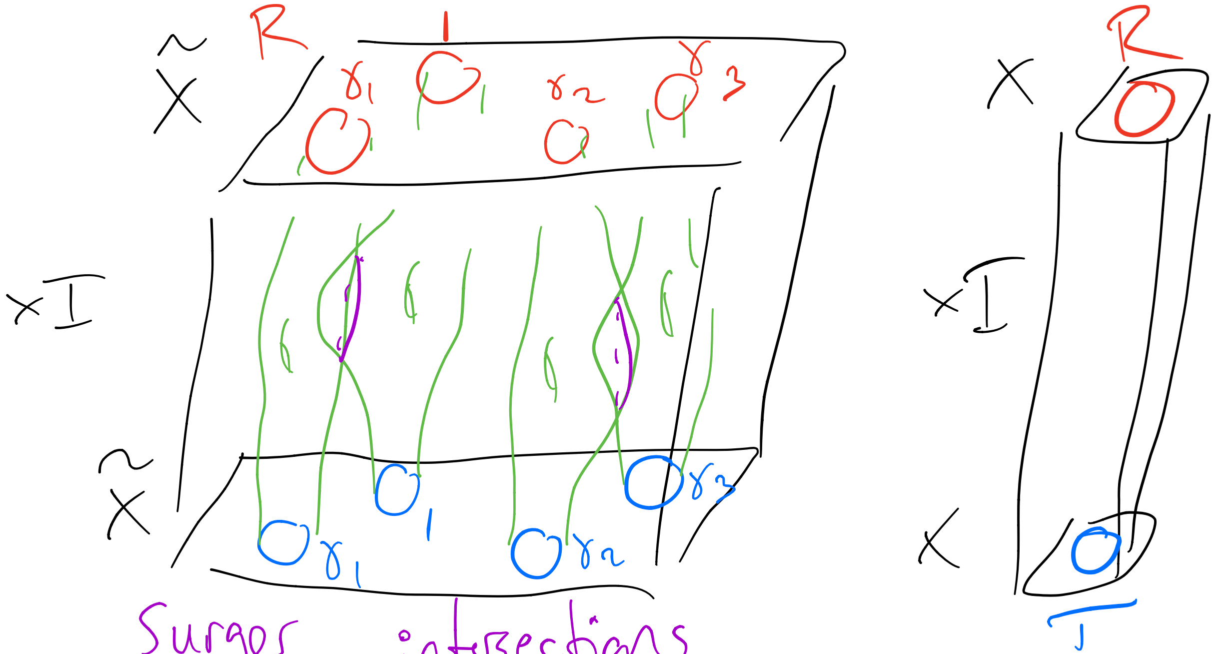
- R, T homotopic
- $i : \pi_1(X \setminus R) \to \pi_1(X)$ isomorphism (i.e. meridian of R nullhtpc in $X \setminus R$)
- $\pi_1(X)$ no 2-torsion

Then R and T are concordant.

(Didn't mention this before Because originally no 2-torsion hypothesis)

Idea of proof

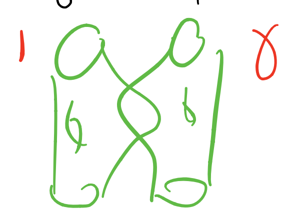
- First prove for $\pi_1 X = 0$
- Now for other π_1, lift to universal cover



Surger intersections away equivariantly

Problem: what if $\gamma \in \pi_1 X$ fixes some circle setwise?

$\gamma^2 = 1$



Michael Klug grad student @ Berkeley/Max Planck

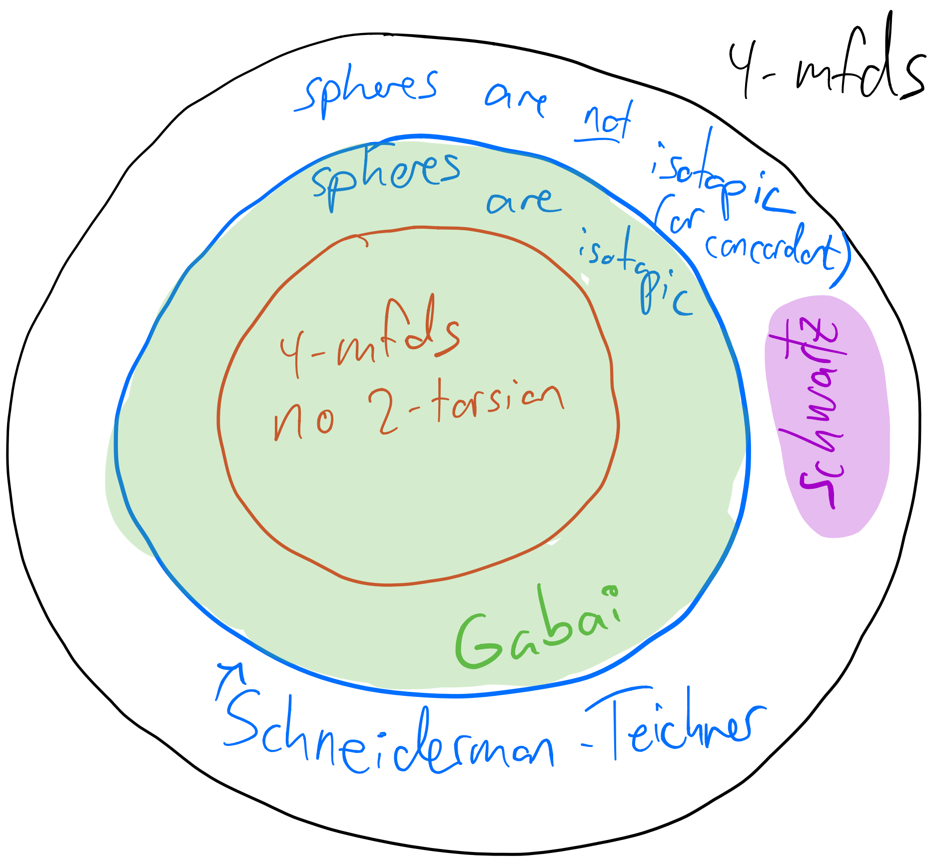
Thm (Klug - M)

Let $R, T \hookrightarrow X^4$ 2-spheres

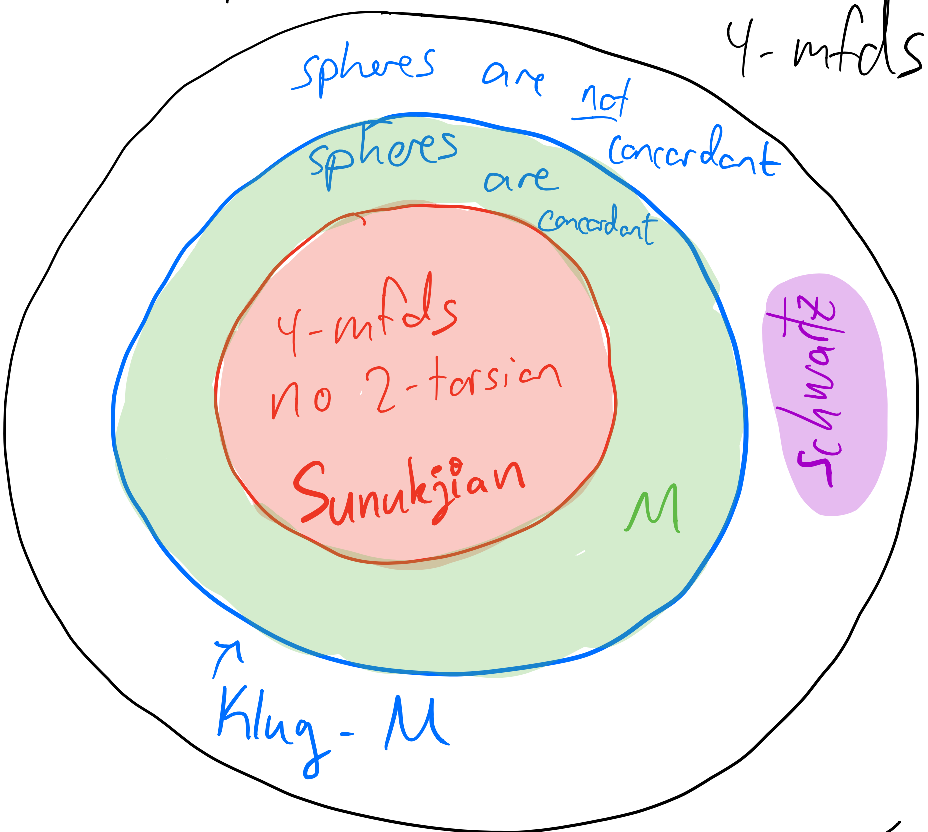
- R, T homotopic
- $i: \pi_1(X \setminus R) \to \pi_1(X)$ isomorphism

Then R, T concordant
iff $fq_\sigma(R, T) = 0$

Lightbulbs (htpc spheres w/ common dual)



Htpc spheres, <u>one</u> has dud



Zeeman:

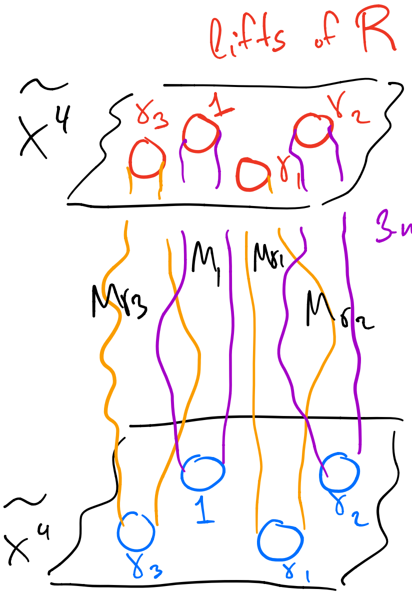
5-twist spun trefoil K has group $A_5 \rtimes \overset{\langle \mu \rangle}{\mathbb{Z}}$

surger μ to get $S^2 \times S^2$, K htpc to $S^2 \times pt$

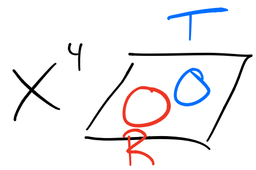
$\pi_1(S^2 \times S^2 \setminus K) \cong A_5$ so K not isotopic to $S^2 \times pt$

Pf

Redefine f_q in terms of covering spaces



lift to universal cover

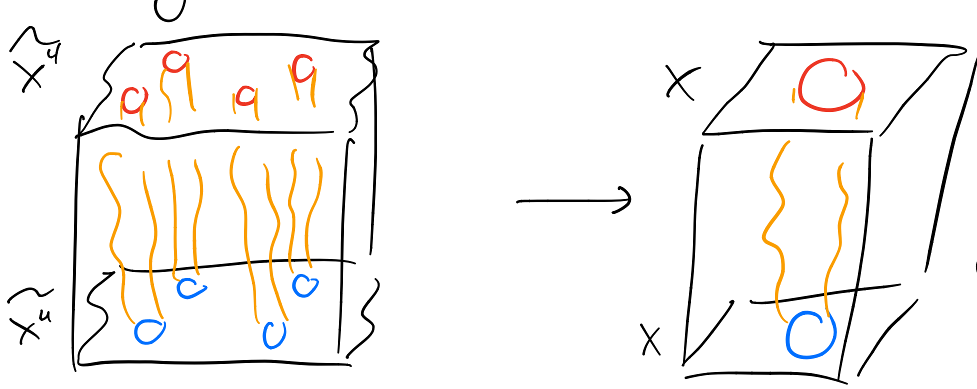


3mflds $\hookrightarrow \tilde{X}^4 \times I$

equivariant wrt $\pi_1 X^4 \circlearrowright \tilde{X} \times I$

Now $\sqcup S^2 \times I$ has circle self-ints, which are acted on by $\pi_1 X^4$

If no circle of self-intersection is fixed by some nontrivial $\gamma \subset \pi_1 X^4$, then Sunukjian: Ambiently equivariantly surger the self-intersections away



R, T concordant

Say $C \subset M_1 \cap M_2$ a circle of self-int fixed setwise by γ

So $\gamma \cdot 1 = 2 \quad \gamma \cdot 2 = 1$

$\Rightarrow \gamma^2 = 1$

<u>Def</u> $fq(R,T) = \sum a_i \gamma_i$

over $\gamma_i \subset T_2$ (2-torsion in $\pi_1 X$)

$a_i = \# M_1 \cap M_{\gamma_i}$ fixed by γ_i (mod 2)

$$\left(\text{in } \mathbb{F}_2 T_2 / \mu_3 \pi_3 M \right)$$

<u>Thm</u> (KM) Equivalent to Schneiderman-Teichner definition

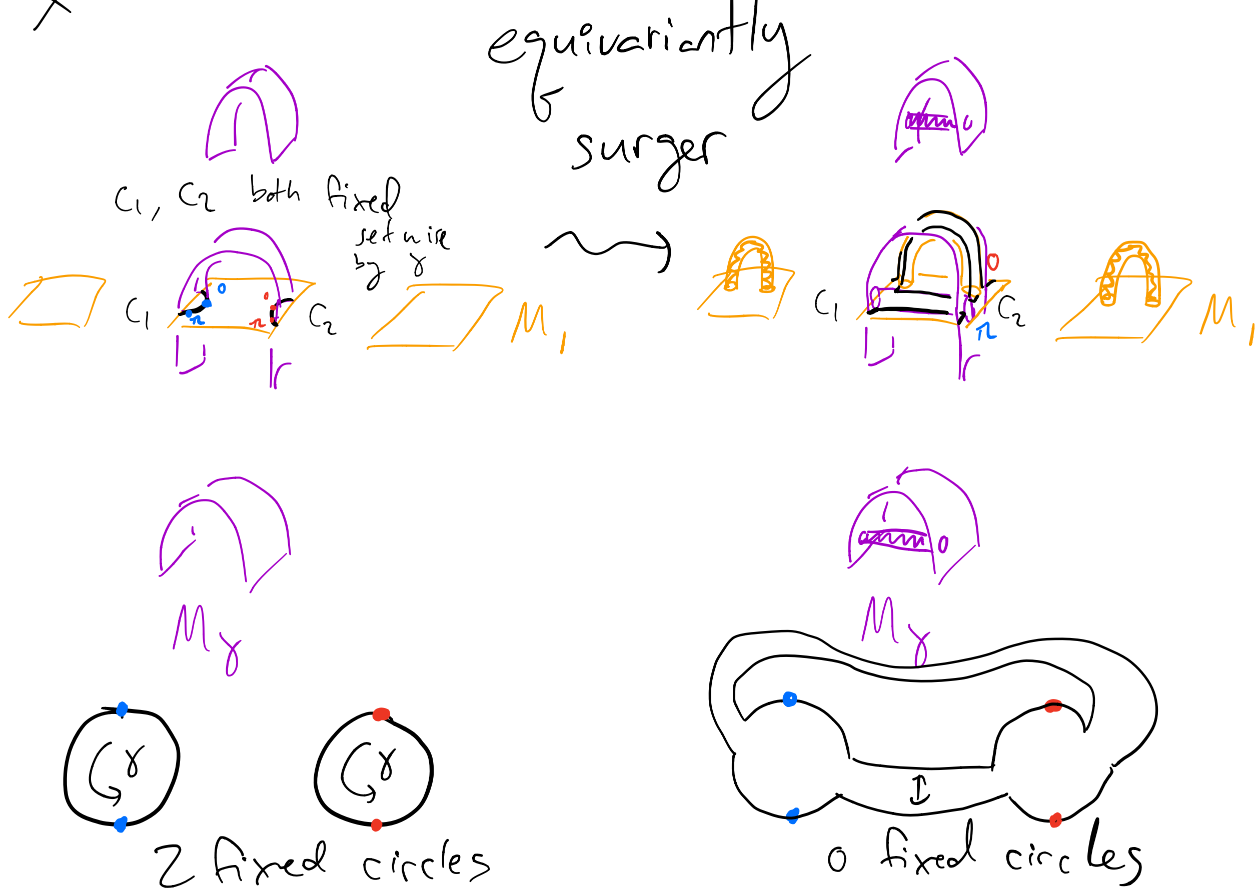
$\Rightarrow$ If nonzero, then R, T not concordant

And if $= 0$

(ignore $\mu_3 \pi_3 M$ part)

Then have even # circles in $M_1 \cap M_\gamma$ fixed by γ

$\tilde{X}^4 \times I$



Repeat until no fixed circles,
then apply Sunukjian's construction

$\Rightarrow R, T$ are concordant.