APPLICATIONS OF THE LI-YAU INEQUALITY IN ARITHMETIC GEOMETRY

OMID AMINI

These are an extended version of the notes of a talk I gave at Simons Symposium on Non-Archimedean and Tropical Geometry. The objective was to give a survey of some recent applications of the spectral gap estimates in graphs and Li-Yau inequality in arithmetic geometry [1].

1. Li-Yau inequality

1.1. **Classical Li-Yau inequality.** We start by recalling the Li-Yau inequality [20]. Let M be a compact surface with a Riemannian metric g. We denote by $d\mu$ the volume form corresponding to its metric, and by $\mu(M)$ the total volume of M. Consider the sphere $\mathbb{S}^2$ with its standard metric g_0, and let $\phi : M \to \mathbb{S}^2$ be a non-degenerate conformal map. The group of conformal diffeomorphisms of $\mathbb{S}^2$, denoted by $\mathrm{Diff}_c(\mathbb{S}^2)$ acts on the set of non-degenerate conformal maps from M to $\mathbb{S}^2$ in a natural way. Define $\mu_c(M, \phi)$ as the supremum volume of M with the respect to the volume forms induced on M from S^2 by the conformal maps in the orbit of ϕ, i.e.,

$$\mu_c(M, \phi) := \sup_{\psi \in \mathrm{Diff}_c(\mathbb{S}^2)} \int_M |\nabla(\psi \circ \phi)|^2 d\mu.$$

The conformal area $\mu_c(M)$ of M (with respect to the conformal structure on M induced by the metric g) is by definition the infimum of $\mu_c(M, \phi)$ over non-degenerate conformal maps $\phi : M \to \mathbb{S}^2$, i.e., $\mu_c(M) := \inf_\phi \mu_c(M, \phi)$.

Theorem 1.1 (Li-Yau [20])**.** *Denote by $\lambda_1 > 0$ the first non-zero eigenvalue of the Laplacian of (M, g). Then $\lambda_1 \mu(M) \leq 2\mu_c(M)$.*

This refines earlier results of Hersch [18] and Szegö [28]. We quickly sketch the proof of the above theorem, which, like the earlier results, uses Hersch lemma.

Lemma 1.2 (Hersch lemma)**.** *Let $\phi : M \to \mathbb{S}^2$ a conformal map. Denote by x_1, x_2, x_3 the coordinate functions on $\mathbb{S}^2$ for the standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$; $x_1^2 + x_2^2 + x_3^2 = 1$. There exists $\psi \in \mathrm{Diff}_c(\mathbb{S}^2)$ such that $\int_M x_i \circ \psi \circ \phi \, d\mu = 0$ for $i = 1, 2, 3$.*

Proof. Let p be a point of $\mathbb{S}^2$ and consider the stereographic projection π_p of $\mathbb{S}^2$ to the hyperplane H_p in $\mathbb{R}^3$ tangent to $\mathbb{S}^2$ at $-p$. For each $t \in (0, 1)$, let $\alpha_{t,p} : H_p \to H$ be the dilation by a factor $1/t$ in H_p, seen as an affine plane with origin at $-p$. Consider the family of conformal maps $\psi_{t,p} = \pi_p^{-1} \circ \alpha_{t,p} \circ \pi_p : \mathbb{S}^2 \to \mathbb{S}^2$. We claim the existence of a t such that for $\psi = \psi_{t,p}$ the conclusion of theorem holds. To see this, consider the map $T : (0, 1) \times \mathbb{S}^2 \to \mathbb{B}_3$,

[1] I would like to thank Simons Foundation and the organizers for the organization of this nice event. Thanks to Antoine Chambert-Loir and David Cohen-Steiner for interesting discussions.

the closed unite ball in $\mathbb{R}^3$, which sends (t,p) to the point with coordinates $\int_M x_i \circ \psi_{t,p} \circ \phi \, d\mu$ for $i = 1,2,3$. The map T can be extended to a map $\overline{T}: [0,1] \times \mathbb{S}^2/\{1\} \times \mathbb{S}^2 \sim \mathbb{B}^3$, so that on the boundary $\{0\} \times \mathbb{S}^2 = \partial\mathbb{B}_3$, $\overline{T}$ restricts to the identity map. Assuming 0 not being in the image of T, one gets a retraction of $\mathbb{B}_3$ to $\partial\mathbb{B}_3$, which leads to a contradiction. $\square$

Proof of Theorem 1.1. Fix an $\epsilon > 0$ and let ϕ be a non-degenerate conformal map $M \to \mathbb{S}^2$ such that $\mu_c(M,\phi) \leq \mu(M) + \epsilon$. By Hersch Lemma, up to replacing ϕ by a conformal map in its orbit for the action of $\mathrm{Diff}_c(\mathbb{S}^2)$, we can assume that $\int_M x_i \circ \phi \, d\mu = 0$ for $i = 0,1,2$, and in addition $\int_M |\nabla\phi|^2 d\mu \leq \mu_c(M,\phi) \leq \mu_c(M) + \epsilon$.

By variational characterization of λ_1, one has

$$\lambda_1 = \inf \frac{\int_M |\nabla f|^2 \, d\mu}{\int_M f^2 \, d\mu},$$

where the infimum is taken over all Lipschitz functions f on M with $\int_M f \, d\mu = 0$. In particular, one has

$$\lambda_1 \int_M (x_i \circ \phi)^2 d\mu \leq \int_M |\nabla x_i \circ \phi|^2 \, d\mu.$$

Summing up over i, gives

$$\begin{aligned}\lambda_1 \, \mu(M) \leq \int_M \sum_i |\nabla x_i \circ \phi|^2 \, d\mu &= \int_M \phi^*(\sum_i |\nabla x_i|^2 d\mu_{\mathbb{S}^2}) \\ &= 2\int_M \phi^*(d\mu_{\mathbb{S}^2}) = 2\int_M |\nabla\phi|^2 \, d\mu \leq 2\mu_c(M) + 2\epsilon.\end{aligned}$$

This holds for any $\epsilon > 0$, from which the theorem follows. $\square$

Let M be a Riemann surface, equipped with a metric of constant curvature in its conformal class, λ_1 and μ the first non-trivial eigenvalue of the Laplacian and the volume of M, respectively. Denote by $\gamma(M)$ the gonality of M, the minimum degree of a (branched) covering $M \to \mathbb{P}^1(\mathbb{C})$.

Corollary 1.3. *For any Riemann surface M,*

$$\lambda_1 \, \mu(M) \leq 8\pi\gamma(M).$$

Proof. It is easy to see that for a conformal map of positive degree d from M to N, one has $\mu_c(M) \leq d\mu_c(N)$. It follows that

$$\lambda_1 \, \mu(M) \leq 2\gamma(M) \, \mu_c(\mathbb{S}^2).$$

One concludes by observing that $\mu_c(\mathbb{S}^2) = 4\pi$. $\square$

1.2. **Combinatorial Li-Yau inequality.** We now discuss a combinatorial version of the above Li-Yau inequality, established in [11].

1.2.1. *Metric graphs, tropical curves, harmonic morphisms and gonality.* We start by recalling some standard definitions and notations related to tropical geometry of curves, see [2] and the references there for a more detailed discussion of the following definitions with several examples.

Given $r \in \mathbb{Z}_{\geq 1}$, we define $S_r \subset \mathbb{C}$ to be a "star with r branches", i.e., a topological space homeomorphic to the convex hull in $\mathbb{R}^2$ of $(0,0)$ and any r points, no two of which lie on a common line through the origin. We also define $S_0 = \{0\}$.

A finite topological graph Λ is the topological realization of a finite graph. So Λ is a compact 1-dimensional topological space such that for any point $p \in \Lambda$, there exists a neighborhood U_p of p in Λ homeomorphic to some S_r; moreover there are only finitely many points p with U_p homeomorphic to S_r with $r \neq 2$.

The unique integer r such that U_p is homeomorphic to S_r is called the valence of p and denoted $\mathrm{val}(p)$. A point of valence different from 2 is called an essential vertex of Λ. The set of tangent directions at p is $T_p(\Lambda) = \varinjlim_{U_p} \pi_0(U_p \setminus \{p\})$, where the limit is taken over all neighborhoods of p isomorphic to a star with r branches. The set $T_p(\Lambda)$ has $\mathrm{val}(p)$ elements.

A metric graph is a finite connected topological graph Λ equipped with a complete inner metric on $\Lambda \setminus V_\infty(\Lambda)$, where $V_\infty(\Lambda) \subsetneq \Lambda$ is some (finite) set of 1-valent vertices of Λ called infinite vertices of Λ. (An inner metric is a metric for which the distance between two points x and y is the minimum of the lengths of all paths between x and y.)

Let Λ be a metric graph. A vertex set $V(\Lambda)$ is a finite subset of the points of Λ containing all essential vertices. An element of a fixed vertex set $V(\Lambda)$ is called a vertex of Λ, and the closure of a connected component of $\Lambda \setminus V(\Lambda)$ is called an edge of Λ. An edge which is homeomorphic to a circle is called a loop edge. An edge adjacent to an infinite vertex is called an infinite edge. We denote by $V_f(\Lambda)$ the set of finite vertices of Λ, and by $E_f(\Lambda)$ the set of finite edges of Λ.

Fix a vertex set $V(\Lambda)$. We denote by $E(\Lambda)$ the set of edges of Λ. Since Λ is a metric graph, we can associate to each edge e of Λ its length $\ell(e) \in \Lambda \cup \{+\infty\}$. Since the metric on $\Lambda \setminus V_\infty(\Lambda)$ is complete, an edge e is infinite if and only if $\ell(e) = +\infty$. The notion of vertices and edges of Λ depends, of course, on the choice of a vertex set; we will fix such a choice without comment whenever there is no danger of confusion.

Fix now vertex sets $V(\Lambda')$ and $V(\Lambda)$ for two metric graphs Λ' and Λ, respectively, and let $\phi : \Lambda' \to \Lambda$ be a continuous map.

- The map ϕ is called a $(V(\Lambda'), V(\Lambda))$-morphism of metric graphs if we have $\phi(V(\Lambda')) \subset V(\Lambda)$, $\phi^{-1}(E(\Lambda)) \subset E(\Lambda')$, and the restriction of ϕ to any edge e' of Λ' is a dilation by some factor $d_{e'}(\phi) \in \mathbb{Z}_{\geq 0}$.
- The map ϕ is called a morphism of metric graphs if there exists a vertex set $V(\Lambda')$ of Λ' and a vertex set $V(\Lambda)$ of Λ such that ϕ is a $(V(\Lambda'), V(\Lambda))$-morphism of metric graphs.
- The map ϕ is said to be finite if $d_{e'}(\phi) > 0$ for any edge $e' \in E(\Lambda')$.

An edge e' of Λ' is mapped to a vertex of Λ if and only if $d_{e'}(\phi) = 0$. Such an edge is said to be contracted by ϕ. A morphism $\phi : \Lambda' \to \Lambda$ is finite if and only if there are no contracted edges, which holds if and only if $\phi^{-1}(p)$ is a finite set for any point $p \in \Lambda$. If ϕ is finite, then

$p' \in V_f(\Lambda')$ if and only if $\phi(p') \in V_f(\Lambda)$. The morphisms of interest are usually assumed to be finite.

The integer $d_{e'}(\phi) \in \mathbb{Z}_{\geq 0}$ in the definition above is called the degree of ϕ along e' (it is also sometimes called the weight of e' or expansion factor along e' in the literature). Since $\ell(\phi(e')) = d_{e'}(\phi) \cdot \ell(e')$, it follows in particular that if $d_{e'}(\phi) \geq 1$ then e' is infinite if and only if $\phi(e')$ is infinite. Let $p' \in V(\Lambda')$, let $v' \in T_{p'}(\Lambda')$, and let $e' \in E(\Lambda')$ be the edge in the direction of v'. The directional derivative of ϕ in the direction v' is by definition the quantity $d_{v'}(\phi) := d_{e'}(\phi)$. If we set $p = \phi(p')$, then ϕ induces a map

$$d\phi(p') \ : \ \left\{v' \in T_{p'}(\Lambda') \ : \ d_{v'}(\phi) \neq 0\right\} \to T_p(\Lambda)$$

in the obvious way.

Let $\phi : \Lambda' \to \Lambda$ be a morphism of metric graphs, let $p' \in \Lambda'$, and let $p = \phi(p')$. The morphism ϕ is harmonic at p' provided that, for every tangent direction $v \in T_p(\Lambda)$, the number

$$d_{p'}(\phi) := \sum_{\substack{v' \in T_{p'}(\Lambda') \\ v' \mapsto v}} d_{v'}(\phi)$$

is independent of v. The number $d_{p'}(\phi)$ is called the degree of ϕ at p'.

We say that ϕ is harmonic if it is surjective and harmonic at all $p' \in \Lambda'$; in this case the number $\deg(\phi) = \sum_{p' \mapsto p} d_{p'}(\phi)$ is independent of $p \in \Lambda$, and is called the degree of ϕ.

1.2.2. *Tropical modifications and tropical curves.* There is an equivalence relation among metric graphs; an equivalence class for this relation will be called a tropical curve.

An elementary tropical modification of a metric graph Λ_0 is a metric graph $\Lambda = [0, +\infty] \cup \Lambda_0$ obtained from Λ_0 by attaching the segment $[0, +\infty]$ to Λ_0 in such a way that $0 \in [0, +\infty]$ gets identified with a finite point $p \in \Lambda_0$.

A metric graph Λ obtained from a metric graph Λ_0 by a finite sequence of elementary tropical modifications is called a tropical modification of Λ_0.

If Λ is a tropical modification of Λ_0, then there is a natural retraction map $\tau : \Lambda \to \Lambda_0$ which is the identity on Λ_0 and contracts each connected component of $\Lambda \setminus \Lambda_0$ to the unique point in Λ_0 lying in the topological closure of that component. The map τ is a (non-finite) harmonic morphism of metric graphs.

Tropical modifications generate an equivalence relation $\sim$ on the set of metric graphs. A tropical curve is an equivalence class of metric graphs (resp. metric graphs) with respect to $\sim$.

By abuse of terminology, we will often refer to a tropical curve in terms of one of its metric graph representatives.

There exists a unique rational tropical curve, which we denote by $\mathbb{TP}^1$. Any rational metric graph whose 1-valent vertices are all infinite is obtained by a sequence of tropical modifications from the metric graph consisting of a unique finite vertex (of genus 0).

Let Λ (resp. Λ') be a representative of a tropical curve C (resp. C'), and assume we are given a harmonic morphism of metric graphs $\phi : \Lambda' \to \Lambda$.

An elementary tropical modification of ϕ is a harmonic morphism $\phi_1 : \Lambda_1' \to \Lambda_1$ of metric graphs, where $\tau : \Lambda_1 \to \Lambda$ is an elementary tropical modification, $\tau' : \Lambda_1' \to \Lambda'$ is a tropical modification, and such that $\phi \circ \tau' = \tau \circ \phi_1$.

A tropical modification of ϕ is a finite sequence of elementary tropical modifications of ϕ.

Two harmonic morphisms ϕ_1 and ϕ_2 of metric graphs are said to be tropically equivalent if there exists a harmonic morphism which is a tropical modification of both ϕ_1 and ϕ_2.

A tropical morphism of tropical curves $\phi : C' \to C$ is a harmonic morphism of metric graphs between some representatives of C' and C, considered up to tropical equivalence, and which has a finite representative.

A tropical curve C is said to have a (non-metric) graph G as its combinatorial type if C admits a representative whose underlying graph is G.

Definition 1.4. A tropical curve C is called d-gonal if there exists a tropical morphism $C \to \mathbb{TP}^1$ of degree d.

The gonality of a tropical curve C is denoted by $\gamma(C)$.

1.2.3. *Morphisms of curves induce morphisms of tropical curves.* Let X and X' be two smooth proper curves over an algebraically closed complete non-Archimedean field K. Consider a morphism $\phi` : X \to X'$, and let $\phi : X^{\mathrm{an}} \to X'^{\mathrm{an}}$ be the induced morphism between the Berkovich analytifications of X and X'^{an}.

Recall (c.f. [5], see also [12, 13, 29]) that a semistable vertex set of the Berkovich analytic curve X^{an} is a finite subset V of type-2 points of X^{an} such that $X^{\mathrm{an}} \setminus V$ is a disjoint union of open balls and (a finite number of) open annuli. Semistable vertex sets are in bijection with semistable models of X over the valuation ring of K. To each semistable vertex sets is associated a skeleton $\Sigma(X, V)$ of the Berkovich curve X^{an}, which is a finite metric graph. These metric graphs are tropically equivalent, and thus varying the semistable vertex sets defines a tropical curve C associated to X.

The proof of the following theorem, as well as more precise statements concerning stronger skeletonized versions of some foundational results of Liu-Lorenzini [22], Coleman [10], and Liu [21] on simultaneous semistable reduction of curves, can be found in [2].

Theorem 1.5. *Let $\phi : X \to X'$ be a fintie morphism of smooth proper curves over K of degree d. Let C and C' be the tropical curves associated to X and X'. Then ϕ induces a tropical morphism $\phi : C \to C'$ of degree d.*

Note that, in particular, the (algebraic) gonality of X over K is bounded below by the (combinatorial) gonality of C. In general the inequality $\gamma(X) \geq \gamma(C)$ can be strict (see [2] for an example of a genus 27 tropical curve C of gonality 4 such that any X over K of genus 27 with associated tropical curve C has gonality at least 5).

In general if the base non-Archimedean field K is not algebraically closed, and $\phi : X \to Y$ is a finite morphism between two smooth proper geometrically connected curves X and Y over K, then one gets a morphism between two tropical curves C and C' by looking at ϕ over the completion of an algebraic closure of K.

1.2.4. *Statement of the combinatorial Li-Yau inequality.* Let C be a tropical curve with combinatorial type a graph G with set of vertices V and set of edges E. Let λ_1 be the first non-trivial eigenvalue of the Laplacian Δ of G. Recall that Δ is the positive semidefinite operator defined on the space of real valued functions on the vertices of G by

$$\Delta(f)(v) = \sum_{u:uv\in E} f(v) - f(u),$$

for any function $f : V \to \mathbb{R}$.

Theorem 1.6 (Cornelissen-Kato-Kool [11])**.** *For any tropical curve C with combinatorial type G, we have*

$$\gamma(C) \geq \lceil \frac{\lambda_1}{\lambda_1 + 4d_{\max}} |G| \rceil,$$

where $d_{\max}$ denotes the maximum valence of vertices of G, and $|G|$ is the number of vertices in G.

Tree-wdith, graph minors, gonality and the spectral bound. We discuss below a generalization of Theorem 1.6, based on the concept of tree-decompositions of graphs and the theory of graph minors, as developed by Robertson-Seymour [25]. We note that Corollary 1.14 and Theorem 1.15 might be slightly weaker than Theorem 1.6, however, they are enough for the arithmetic applications of [11], discussed in Section 2

1.2.5. *Tree-decomposition, minors, and graph minor theorem.* We start by recalling some basic terminology on tree-decompositions of finite graphs.

Let $G = (V, E)$ be a connected graph. A tree-decomposition of G is a pair $(T, \mathcal{X})$ where T is a finite tree on a set of vertices I, and $\mathcal{X} = \{X_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $V = \cup_{i\in I} X_i$,
(2) for any edge e in G, there is a set $X_i \in \mathcal{X}$ which contains both end-points of e,
(3) for any triple i_1, i_2, i_3 of vertices of T, if i_2 is on the path from i_1 to i_3 in T, then $X_{i_1} \cap X_{i_3} \subseteq X_{i_2}$.

Note that the point (3) in the above definition simply means that the subgraph of T induced by all the vertices i which contain a given vertex v of the graph G is connected.

The width of a tree-decomposition $(T, \mathcal{X})$ is defined as $w(T, \mathcal{X}) = \max_{i\in I} |X_i| - 1$. The tree-width of G, denoted by $tw(G)$, is the minimum width of any tree-decomposition of G.

There is a useful duality theorem concerning the tree-width wich allows in practice to bound the tree-width of graphs. The dual notion for tree-width is bramble (as named by B. Reed [24]): a bramble in a finite graph G is a collection of connected subsets of $V(G)$ such that the union of any two of these subsets forms again a connected subset of $V(G)$. (To be more precise, we should say the graph induced on these subsets is connected.) The order of a bramble is the minimum size of a subset of vertices which intersect any set in the bramble. The bramble number of G, denoted by $bn(G)$, is the maximum order of a bramble in G.

Theorem 1.7 (Seymour-Thomas [27])**.** *For any graph G, $tw(G) = bn(G) - 1$.*

(For a more general form of the duality theorem and a conceptual proof see [4].)

Example 1.8. Let H be an $n \times n$ grid. It is easy to see that $bn(H) = n$ by taking brambles formed by crosses in the grid. This shows that grid graphs can have large tree-width. Thus, the tree-width can be unbounded on planar graphs.

The other notion directly related to the concept of the tree-decompositions is the notion of minor in graphs. A graph H is a minor of another graph G, and we write $H \preceq G$, if H can obtained from G by a sequence of operations consisting in
- contracting an edge of G, or
- removing an edge of G.

It is easy to see that tree-width is minor monotone, in the sense that if $H \preceq G$, then $tw(H) \leq tw(G)$. It follows that bounded tree-width graphs cannot have large grid minors.

The main theorem concerning the notion of graph minors is the Robertson-Seymour finiteness theorem which states:

Theorem 1.9 (Robertson-Seymour [25]). *Let $\mathcal{F}$ be a family of graphs which is stable under minors, i.e., if $G \in \mathcal{F}$ and H is a minor of G, then H belongs to $\mathcal{F}$. Then there is a finite number of graphs (possibly empty if $\mathcal{F}$ contains all finite graphs) $H_1, \dots, H_k$ such that G belongs to $\mathcal{F}$ if and only if G does not contain any of H_i as minor.*

In particular, the above theorem is a far reaching generalization of Kuratowski theorem which characterizes planar graphs as the family of graphs which do not contain the complete graph on five vertices K_5, and the complete bipartite graph $K_{3,3}$ on two parts of size three each.

Remark 1.10. Robertson and Seymour prove that tree-width is bounded on the class of graphs with forbidden H-minor if and only if H is planar.

1.2.6. *Gonality and tree-width.* We have the following basic proposition relating the gonality of a tropical curve with combinatorial type G to the tree-width of G.

Proposition 1.11. *For any tropical curve C with combinatorial type $G = (V, E)$, we have $2\gamma(C) \geq tw(C)$.*

Proof. Let $\phi : C \to \mathbb{TP}^1$ be a morphism of degree $\gamma(C)$. Consider the restriction of ϕ to a finite harmonic morphism from a metric graph representative Λ of C with vertex set V and edge set E, and denote by T the image of Λ in $\mathbb{TP}^1$, so T is a finite tree. Let I_1 be a vertex set for T which contains $\phi(V)$, and E_1 be the corresponding set of edges. For each edge e in T_1 take a point in the interior of e, and let I be the new vertex set for T obtained by adding to I_1 all these new vertices.

A tree decomposition $(T, \mathcal{X})$ of G can be defined as follows. For each vertex i in I, consider the preimage $\phi^{-1}(i)$ of i. This set consists of some (possibly empty) vertices $v_1, \dots, v_s$ of G and some (possibly empty) points $x_1, \dots, x_l$ in the interior of some edges $e_1 = u_1w_1, \dots, e_l = u_lw_l$ of G. Define $X_i = \{v_1, \dots, v_s, u_1, w_1, \dots, u_l, w_l\}$. Since ϕ is of degree $\gamma(C)$, $|\phi^{-1}(i)| \leq \gamma(C)$ and thus, X_i has cardinality at most $2\gamma(C)$. It is easy to check that $(T, \mathcal{X} = \{X_i\}_{i\in I})$ is a tree-decomposition of G. This proves the proposition. $\square$

As a corollary, if a graph G is a model of a tropical curve with bounded gonality, then the tree-width of G is bounded, and thus, G cannot contain a large grid as minor.

1.2.7. *Eigenvalue estimates on proper minor-closed family of graphs and generalization of Theorem 1.6.* Let H be a given graph. Consider the family $\mathcal{F}_H$ of all connected graphs G which do not contain H as minor. Note that $\mathcal{F}_H$ is minor closed. For any graph G on n vertices, denote by $\lambda_0(G) = 0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$ all the eigenvalues of the graph Laplacian Δ_G.

Theorem 1.12 ([19])**.** *There is a constant $h = h(H)$ such that for any graph G in $\mathcal{F}_H$ and any $1 \leq k$, we have $\lambda_k(G) \leq \frac{h d_{\max} k}{|G|}$ where $d_{\max}$ is the maximum valence of vertices in G and $|G|$ is the number of vertices in G.*

For graphs which can be embedded in a surface of genus at most g, the following more precise statement holds

Theorem 1.13 ([3])**.** *There is a universal constant c such that for any graph G which can be embedded in a surface of genus at most g, we have*

$$\lambda_k^{nr}(G) \leq c\frac{d_{\max}(g+k)}{m},$$

where λ_k^{nr} are the eigenvalues of the normalized Laplacian of G, and m is the number of edges of G.

(Note that in any graph G, with min- and max-degrees $d_{\min}$ and $d_{\max}$, one has $d_{\min}\lambda_k^{nr}(G) \leq \lambda_k(G) \leq d_{\max}\lambda_k^{nr}(G)$, and similarly, $d_{\min}|G|/2 \leq m \leq d_{\max}|G|/2$.)

We end this subsection with a discussion of the above results in the case of bounded tree-width graphs. A graph of tree-width bounded by some constant N does not contain a grid of size $N \times N$ as minor. It follows that there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for a graph G of tree-width $tw(G)$, one has $\lambda_k(G) \leq f(tw(G)) d_{\max} k/|G|$, where $|G|$ is the number of vertices of G. Combined with Proposition 1.11, one obtains the following corollary.

Corollary 1.14. *For any tropical curve C of combinatorial type G, one has $f(2\gamma(C)) \geq \lambda_k(G).|G|\frac{d_{\max}}{k}$. In particular, if in a family of tropical curves C_i of combinatorial type G_i, $d_{\max}$ is bounded and for some constant k, $\lambda_k(G_i).|G_i|$ tend to infinity, then one has $\gamma(C_i) \to \infty$.*

Note that since $\lambda_1 \leq d_{\max}$, the case of $k = 1$ in this theorem can be regarded as a version of Theorem 1.6 (with some possibly weaker constant). Indeed, the case of λ_1 (i.e., $k = 1$) in the above corollary, is a result of Chandran-Subramanian [8], and is very similar to Theorem 1.6.

Theorem 1.15 ([8])**.** *For any graph $G = (V, E)$, the following holds*

$$tw(G) \geq \lfloor \frac{3|G|}{4} \frac{\lambda_1}{2\lambda_1 + d_{max}} \rfloor - 1.$$

We briefly sketch the proof of the above theorem. First recall the following variational characterization of λ_1:

$$\lambda_1 = \inf_{f:V\to\mathbb{R} \text{ with } \sum_v f(v)=0} \frac{\sum_{uv\in E}\big(f(u)-f(v)\big)^2}{\sum_{v\in V} f(v)^2}.$$

Let Y and Z be two disjoint subsets of V. Applying this to the (test) function f defined by $f(y) = \frac{1}{|y|}$ for $y \in Y$, $f(z) = -\frac{1}{|Y|}$ for $y \in Y$ and $f(w) = 0$ for any $w \in V \setminus (X \cup Y)$, which is a test function for the minimum in the right term of the above equation, we get

$$\lambda_1 \leq \Big(|E| - |E(Y)| - |E(Z)|\Big)\Big(\frac{1}{|Y|} + \frac{1}{|Z|}\Big), \tag{1}$$

where $E(A)$ denotes the set of all edges with both endpoints in A. This is used repeatedly in the proof.

Proof of Theorem 1.15. Denote by n the number of vertices of G. For the sake of a contradiction, assume the inequality does not hold and let $(T, \mathcal{X} = \{X_i\})$ be a tree decomposition of G such that

$$|X_i| \leq \lfloor \frac{3n}{4} \frac{\lambda_1}{2\lambda_1 + d_{max}} \rfloor - 1,$$

for any vertex i of T. Denote by ρ the quantity in the right hand side of the above equation. The following argument shows the existence of a subset X of size at most ρ in G such that each component of $G \setminus X$ has size at most $\frac{n-|X|}{2}$: suppose this is not the case, then we can give orientations to edges of the tree T as follows: consider a vertex i of the tree. The set X_i has size at most ρ. For each neighbor j of i in the tree, let A_j be the union of all the X_k with k in the subtree of $T - \{i\}$ which contains j. Our assumption implies one of the sets $A_j \setminus X_i$ has size strictly larger than $\frac{n-|X_i|}{2}$, for j adjacent to i. Give the orientation $i \to j$ to the edge $e = \{i, j\}$ of the tree. Doing this for any vertex, we given orientation to edges of the tree exactly $|V(T)|$ times. Since T has $|V(T)| - 1$ edges, at least one edge $\{i, j\}$ gets orientated twice, which leads to a contradiction since both $A_i \setminus (X_i \cap X_j)$ and $A_j \setminus (X_i \cap X_j)$ contain strictly larger than $\frac{n-|\mathcal{X}_i \cap \mathcal{X}_j|}{2}$ vertices.

Let X be the set of size at most ρ such that all the connected components $Y_1, \dots, Y_s$ of $G \setminus X$ has size at most $\frac{n-|X|}{2}$. Applying Inequality (1) to the disjoint sets Y_i and $Z_i = V \setminus (X \cup Y_i)$, and noting that the quantity $|E| - |E(Y_i)| - |E(Z_i)|)$ is bounded by $|X| d_{\max}$, show that each Y_i has size at most $\frac{n}{4}$. Consider now the smallest j such that $\overline{Y}_j = Y_1 \cup \cdots \cup Y_j$ has size at least $\frac{n}{4}$. Since each Y_i has size at most $\frac{n}{4}$, $\overline{Y}_j$ has size at most $\frac{n}{2}$. Applying now Inequality (1) to $\overline{Y}_j$ and $\overline{Z}_j = V \setminus (X \cup \overline{Y}_j)$ leads to a contradiction. □

2. Applications

In this section we discuss some recent applications of the above Li-Yau inequalities in arithmetic geometry.

2.1. **Rational points and Galois representations.**

We start by giving an overview of the recent applications of Theorem 1.1 to arithmetic geometry over number fields from [14].

2.1.1. *Rational points.* Let k be a number field. Let X be a smooth geometrically connected curve over k. Consider a family X_i of étale covers of X defined over k. Consider an Archimedean place of k, an embedding to $\mathbb{C}$, and denote by $X_{i,\mathbb{C}}$ and $X_{\mathbb{C}}$ the corresponding Riemann surfaces associated to X_i and X. The fundamental group $\pi_1(X_{i,\mathbb{C}})$ is a subgroup of $\pi_1(X_{\mathbb{C}})$ (we omit the base points), and fixing a symmetric set of generators S for $\pi_1(X_{\mathbb{C}})$ (i.e., $S = S^{-1}$) allows to define the Cayley graph $\mathrm{Cay}(\pi_1(X_{i,\mathbb{C}}) \backslash \pi_1(X_{\mathbb{C}}); S)$ as the quotient of $\mathrm{Cay}(\pi_1(X_{\mathbb{C}}); S)$ by the left action of $\pi_1(X_{i,\mathbb{C}})$ on $\mathrm{Cay}(\pi_1(X_{\mathbb{C}}); S)$. To simplify the notation, we simply write $\mathrm{Cay}(X_i/X; S)$ to denote this finite Cayley graph.

Consider the combinatorial Laplacian of $\mathrm{Cay}(X_i/X;S)$ and let $\lambda_1^{(i)}$ be its first non-trivial eigenvalue.

Theorem 2.1 (Burger [7])**.** *There is a constant $C > 1$ depending only on $X_{\mathbb{C}}$ such that $C^{-1}\lambda_1(X_{i,\mathbb{C}}) \leq \lambda_1^{(i)} \leq C\lambda_1(X_{i,\mathbb{C}})$ for any i.*

Here $X_{i,\mathbb{C}}$ is equipped with a metric of constant curvature.

Proof. By going to the universal cover $\tilde{X}$ and taking a tiling of $\tilde{X}$ obtained by fixing a fundamental domain for the action of $\pi_1(X_{\mathbb{C}})$ on $\tilde{X}$, one can see that each surface $X_{i,\mathbb{C}}$ admits a decomposition into domains isometric to a fixed domain F with piecewise smooth boundary (independent of i) such that the dual complex associated to this tiling is precisely the Cayley graph $\mathrm{Cay}(X_i/X;S)$. The theorem now follows by looking at the discretization functional $\phi : \mathcal{C}^\infty(X_{i,\mathbb{C}}) \to C^0(\mathrm{Cay}(X_i/X;S))$ which sends f to $\phi(f)$ taking a value at a vertex v of $\mathrm{Cay}(X_i/X;S)$ equal to the average of f on the domain corresponding to v in the tiling of $X_{i,\mathbb{C}}$. The inverse of ϕ sends a discrete function defined on vertices of the Cayley graph to a smoothing of the function constant on each domain of the surface $X_{i,\mathbb{C}}$. The ratio between $\lambda_1^{(i)}$ and $\lambda_1(X_{i,\mathbb{C}})$ remains bounded away from zero and infinity, by a non-zero function depending on the first Neumann eigenvalue of the Laplacian operator on F. $\square$

Corollary 2.2. *Assume $\lambda_1^{(i)}|\mathrm{Cay}(X_i/X;S)|$ tends to infinity. Then the gonality of X_i tends to infinity.*

Proof. The volume of $X_{i,\mathbb{C}}$ is $|\mathrm{Cay}(X_i/X;S)|$ times the volume μ of X. By Li-Yau inequality, $\lambda_1(X_{i,\mathbb{C}})|\mathrm{Cay}(X_i/X;S)|\,\mu \leq 8\pi\gamma(X_i,\mathrm{C})$. Since $\lambda_1^{(i)}|\mathrm{Cay}(X_i/X;S)|$ tends to infinity, and $\lambda_1^{(i)}$ is within a constant factor of $\lambda_1(X_{i,\mathbb{C}})$, it follows that $\gamma(X_{i,\mathbb{C}})$ tends to infinity and the result follows. $\square$

Theorem 2.3 ([14])**.** *Let X_i/X be a family of étale covers of X. Assume that $\lambda_1^{(i)}|\mathrm{Cay}(X_i/X;S)| \to \infty$. For any d, the set*

$$\bigcup_{k_1:[k_1:k]\leq d} X_i(k_1)$$

is finite for all but finitely many i.

Proof. Under the hypothesis of the theorem, the gonality $\gamma(X_i)$ of X_i tends to infinity so there is N_d such that for $i \geq N_d$, $\gamma(X_i) > 2d$. By Faltings-Frey theorem [15], the set $\bigcup_{k_1:[k_1:k]\leq d} X_i(k_1)$ is finite for any $i \geq N_d$. $\square$

2.1.2. *Examples of Cayley graphs with large eigenvalues.* The basic example is the example of a family of Cayley graphs of fixed valence which form a family of expanders, i.e., such that the first non-trivial eigenvalue of the Laplacian of graphs in the family is lower bounded by a constant. Consider e.g. a finite index subgroup G of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. Then G satisfies Kazhdan (T) property, and as a consequence, for a fixed symmetric set of generators S for G, the family of Cayley graphs $\mathrm{Cay}(H\backslash G;S)$ where H runs over all finite index subgroup of G form a family of expanders.

Example 2.4. Let X be a smooth curve over a number field k of genus at least two. There exists a family of étale covers $X_i \to X$ such that the Cayley graphs $\mathrm{Cay}(X_i/X;S)$, as defined in the previous section, form a family of expanders with sizes tending to infinity. This is because the topological fundamental group of $X_{\mathbb{C}}$ has a quotient which is isomorphic to $\mathrm{SL}_3(\mathbb{Z})$. By the above results, $\gamma(X_i)$ tends to infinity.

The following recent result of Pyber-Szabó [23] (see also [6]) provides a rich class of examples of Cayley graphs with large eigenvalues. For earlier results of similar type see [16, 17].

Let m be an integer and consider a family of subgroups G_p of $GL_m(\mathbb{F}_p)$ indexed by all but finitely many prime numbers p. Let S_p, $S_p = S_p^{-1}$, be a generating set for G_p of order at most a constant s, for any p. Consider the family of Cayley graphs $\mathrm{Cay}(G_p; S_p)$.

Theorem 2.5 (Pyber-Szabó [23])**.** *If the groups G_p are non-trivial perfect groups generated by their elements of order p, then $\lambda_1(\mathrm{Cay}(G_p; S_p))|\mathrm{Cay}(G_p; S_p)| \to \infty$, when p tends to infinity. More precisely, $\lambda_1(\mathrm{Cay}(G_p; S_p)) >> \frac{1}{\log |G_p|^A}$ for some constant A.*

2.1.3. *Galois representations.*

Theorem 2.6 (Ellenberg-Hall-Kowalski [14])**.** *Let k be a number field and X/k a smooth geometrically connected algebraic curve. Let $\mathcal{A} \to X$ be a principally polarized abelian scheme over X of dimension $g \geq 1$, defined over k, and let*

$$\rho : \pi_1(X_{\mathbb{C}}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$$

be the associated monodromy representation. For any finite extension k_1/k and a rational point $t \in X(k_1)$, let

$$\overline{\rho}_{t,\ell} : \mathrm{Gal}(\overline{k}/k_1) \to \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$$

be the Galois representation associated to the ℓ-torsion points of $\mathcal{A}_t$.

Assume that the image of ρ is Zariski dense in Sp_{2g}. Then the set

$$\bigcup_{k_1 :\ [k_1:k]=d} \{t \in X(k_1) \,|\, \text{the image of } \overline{\rho}_{t,\ell} \text{ does not contain } Sp_{2g}(\mathbb{F}_\ell)\}$$

is finite for any $d \geq 1$ and any but finitely many ℓ (depending on d).

Proof. By assumption the image I of ρ is dense in $\mathrm{Sp}_{2g}(\mathbb{Z})$ which implies that the image I_ℓ of the reduction map $I \to \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is the whole $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ for all but finitely many ℓ. Suppose that for each conjugacy class of a maximal subgroup of $\mathrm{Sp}_{2g}(\ell)$ a fixed representative is designed, and consider all the pairs (ℓ, J) where ℓ is such that $I_\ell = \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ and $J < \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ runs over the representatives of the conjugacy classes of maximal subgroups of $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$. Each such pair (ℓ, J) gives rise to an étale cover $X_{\ell,J} \to X$ with the property that $\mathrm{Cay}(X_{\ell,J}/X; S) = \mathrm{Cay}(J\backslash\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$.

In particular, the set of all k_1-rational points t of X such that I_ℓ is not in the image of $\overline{\rho}_{t,\ell}$ lies in the image of k_1-rational points of a pair (ℓ, J) under the map $\pi_{\ell,J}$. So the theorem follows as soon as it is shown that the number of k_1-rational points of the constructed étale covers $X_{\ell,J}$ of X are finite for any fixed $d \geq 1$ and for extensions $[k_1 : k] = d$. For this, it will be enough to show that the family of étale covers $X_{\ell,J}/X$ verifies the condition of Theorem 2.3.

The group $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is perfect for $\ell \geq 5$ and is generated by its elements of order ℓ. In addition each maximal subgroup J of $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is of index at most $\frac{1}{2}(\ell^g - 1)$. By Theorem 2.5, the Cayley graphs $\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ have $\lambda_1(\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)) >> \frac{1}{\log |\mathrm{Sp}_{2g}(\mathbb{F}_\ell)|^A}$. The Cayley graph $\mathrm{Cay}(J\backslash \mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ is by definition the quotient of $\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ under the left action of J, and thus have the same λ_1. An easy calculation now shows that

$$\lambda_1(\mathrm{Cay}(J\backslash \mathrm{Sp}_{2g}(\mathbb{F}_\ell); S))\, |\mathrm{Cay}(J\backslash \mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)| \to \infty$$

when (ℓ, J) runs over all pairs as above with $\ell \geq 5$, which finishes the proof. □

2.2. **Gonality and rational points of bounded degree of Drinfeld modular curves.** In this section, we discuss arithmetic consequences of the combinatorial Li-Yau inequality from [11]. The main theorem is a linear lower bound in the genus for the gonality of Drinfeld modular curves. This extends the work of Abramovich [1] to positive characteristic case.

2.2.1. *Lower bound on the gonality of X_Γ.* Let K be a function field of genus g over the field of constants $k = \mathbb{F}_q$, of characteristic p. Let ∞ be a fixed place of K of degree δ, and let A be the ring of functions $f \in K$ which have poles at most at ∞.

Let K_∞ be the completion of K at ∞, and denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞. Let $\Omega = \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(K_\infty) = C_\infty \setminus K_\infty$. The group $\mathrm{GL}_2(K)$ acts by fractional linear transformations on Ω.

Consider now Γ an arithmetic subgroup of $\mathrm{GL}_2(K)$: Γ is a congruent subgroup of $\mathrm{GL}(Y) \subseteq \mathrm{GL}_2(K)$ for a rank-two A-lattice Y in K_∞. This means that Γ contains a subgroup of the form $\mathrm{GL}(Y, \mathfrak{n}) := \ker\{\mathrm{GL}(Y) \to \mathrm{GL}(Y/\mathfrak{n}Y)\}$ for an ideal $\mathfrak{n}$ of A.

The group Γ acts on Ω, and the quotient $\Gamma\backslash\Omega$ is a smooth analytic curve which is the analytification of a smooth affine curve Y_Γ defined over a finite (abelian) extension of K_∞. The Drinfeld modular curve X_Γ is the compactification of Y_Γ obtained by adding a finite number of points, called cusps, to Y_Γ.

Theorem 2.7 (Cornelissen-Kato-Kool [11])**.** *Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(K)$. There is a constant $c = c(K, \delta)$, such that the gonality $\gamma(X_\Gamma)$ over $\overline{K}$ satisfies*

$$\gamma(X_\Gamma) \geq c\,.\,(g(X_\Gamma) - 1),$$

where $g(X_\Gamma)$ is the genus of X_Γ.

We briefly discuss the proof of this theorem.

Reduction graph of X_Γ. The group Γ acts by automorphisms on the Bruhat-Tits tree $\mathfrak{T}$ of $\mathrm{PGL}_2(K_\infty)$, and the quotient is a finite graph G with a finite set of infinite rays corresponding to the cusps of X_Γ. The Drinfeld curve X_Γ is a Mumford curve with reduction graph over $\mathbb{F}_{q^\delta}$ isomorphic to G.

Maximum valence of G. The Bruhat-Tits tree $\mathfrak{T}$ is a regular tree of valence $q^\delta + 1$. The graph G being the finite part of a quotient of this tree by a subgroup of the automorphism group, it has maximum valence $d_{\max}$ bounded by $q^\delta + 1$.

First non-trivial eigenvalue of the Laplacian of G for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. In the case $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$, the Laplacian of G can be described in terms of the projection of the Hecke operator on $\mathfrak{T}$ corresponding to the characteristic function of ∞, and a zero-one matrix corresponding to the infinite rays of the quotient of $\mathfrak{T}$ by $\mathrm{GL}(Y, \mathfrak{n})$. Ramanujan-Petersson conjecture for global function fields, proved by Drinfeld, gives an estimate of the form $\lambda_1 \geq q^\delta - 2q^{\delta/2}$ for the first non-trivial eigenvalue of the Laplacian.

Number of vertices of G for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. A direct comparison argument between the two quotient graphs G and G_0 associated to $\mathrm{GL}(Y, \mathfrak{n})$ and $GL(Y)$, respectively, involving the stabilizer of the vertex v_0 of $\mathfrak{T}$ corresponding to the root vertex of $\mathfrak{T}$, leads to a lower bound of the type

$$|G| \geq \frac{1}{q(q^2-1)}[\mathrm{GL}(Y) : \mathrm{GL}(Y, \mathfrak{n})],$$

where $|G|$ is the number of vertices of G.

Gonality of X_Γ for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. Combining the above estimates with the combinatorial Li-Yau inequality gives the existence of a constant c_0, depending only on q and δ, such that for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$,

$$\gamma(X_\Gamma) \geq c_0 \,.\, [\mathrm{GL}(Y) : \Gamma]. \tag{2}$$

The bound on the genus is obtained by applying the Riemann-Hurwitz formula to the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_{\mathrm{GL}(Y)}$, and a careful analysis of the degree of the ramification divisor. Riemann-Hurwitz gives

$$[\mathrm{GL}(Y, \mathfrak{n}) : \mathrm{GL}(Y)] = (g(X_{\mathrm{GL}(Y,\mathfrak{n})}) - 1)\frac{2(q-1)}{2(g(X_{\mathrm{GL}(Y)}) - 1) + R},$$

so it will be essentially enough to give a lower bound on R since $g(X_{\mathrm{GL}(Y)})$ is a constant, depending only on K and δ.

Theorem for general Γ. This follows by looking at the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_\Gamma$. This gives $\gamma(X_\Gamma) \geq \gamma(X_{\mathrm{GL}(Y,\mathfrak{n})})|\Gamma \cap Z|/[\Gamma : \mathrm{GL}(Y, \mathfrak{n})]$, where $Z \simeq \mathbb{F}_q^*$ is the centralizer of $\mathrm{GL}(Y)$. Combining the theorem for $\mathrm{GL}(Y, \mathfrak{n})$ with Riemann-Hurwitz for the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_\Gamma$ gives the result for general Γ.

Note that the inequality (2) holds for more general Γ, for a constant $c_0 = c_0(q, \delta)$.

2.2.2. *Rational points of bounded degree.* It is possible to apply the analogue in positive characteristic of Faltings-Frey theorem [26, 9], along the linear lower bound on the gonality (2) to prove the following theorem.

Suppose that X_Γ is defined over the finite extension L of K.

Theorem 2.8 ([11]). *There is a constant $c_0 = c_0(q, \delta)$ such that the set*

$$\bigcup_{L':\ [L':L] \leq \frac{1}{2}(c_0[\mathrm{GL}(Y):\Gamma]-1)} X_\Gamma(L')$$

is finite.

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves*, Internat. Math. Res. Notices 20 (1996), 1005–1011.

[2] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff, *Lifting harmonic morphisms of tropical curves, metrized complexes, and Berkovich skeleta.* Preprint available at `http://arxiv.org/abs/1303.4812`.

[3] O. Amini and D. Cohen-Steiner, *Improved bounds on higher eigenvalues of graphs*, in preparation.

[4] O. Amini, F. Mazoit, N. Nisse, and S. Thomassé, *Submodular partition functions.* Disc. Math., 309 (2009), 6000-6008.

[5] M. Baker, S. Payne, J. Rabinoff, *Non-Archimedean geometry, tropicalization, and metrics on curves* (2011), Preprint available at http://arxiv.org/abs/1104.0320.

[6] E. Breuillard, B. Green, and T. Tao, *Approximate subgroups of linear groups*, GAFA

[7] M. Burger, *Petites valeurs propres du Laplacien et topologie de Fell*, thèse de doctorat (1986), Econom Druck AG (Basel).

[8] L.S. Chandran and C.R. Subramanian, *A spectral lower bound for the treewidth of a graph and its consequences*. Inf. Proc. Let. 87 (2003), 195–200.

[9] P.L. Clark, *On the Hasse principle for Shimura curves*, Israel J. Math. 171 (2009), 349–365.

[10] R.F. Coleman, *Stable maps of curves*, Documenta Math. Extra volume Kato (2003), 217–225.

[11] G. Cornelissen, F. Kato, and J. Kool, *A combinatorial Li-Yau inequality and rational points on curves*, Preprint available at `http://arxiv.org/abs/1211.2681`.

[12] A. Ducros, *Triangulations et cohomologie étale sur une courbe analytique p-adique*, J. Algebraic Geom. 17 (2008), 503–575.

[13] A. Ducros, *La structure des courbes analytiques*, book available at http://www.math.jussieu.fr/ ducros/.

[14] J. Ellenberg, C. Hall, and E. Kowalski, *Expander graphs, gonality and variation of Galois representations*, Duke Math. J. 161 (2012), 1233–1275.

[15] G. Frey, *Curves with infinitely many points of fixed degree*, Israel J. Math. 85 (1994), 79–83.

[16] H. Helfgott, *Growth and generation in* $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, Ann. of Math. 167 (2008), 601–623.

[17] N. Gill and H. Helfgott, *Growth of small generating sets in* $SL_n(\mathbb{Z}/p\mathbb{Z})$, International Math. Res. Notices (2011) 4226–4251.

[18] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris 270 (1970), 1645–1648.

[19] J.A. Kelner, J.R. Lee, G.N. Price, and S.H. Teng, *Metric uniformization and spectral bounds for graphs*. Geometric and Functional Analysis 21 (2011), 1117–1143.

[20] P. Li and S. T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. math. 69 (1982), 269–291.

[21] Q. Liu, *Stable reduction of finite covers of curves*, Compositio Math. 142 (2006), 101–118.

[22] Q. Liu and D. Lorenzini, *Models of curves and finite covers*, Compositio Math. 118 (1999), 61–102.

[23] L. Pyber and E. Szabó, *Growth in finite simple groups of Lie type of bounded rank*, Preprint available at `http://arxiv.org/abs/1005.1858`.

[24] B.A. Reed, *Tree width and tangles: A new connectivity measure and some applications*, Surveys in Combinatorics 241 (1997), 87–162.

[25] N. Robertson and P. Seymour, *Graph Minors. I - XXIII*, J. Comb. Theory Series B. 1983–2010.

[26] A. Schweizer, *On the uniform boundedness conjecture for Drinfeld modules*, Math. Z. 244 (2003), 601–614.

[27] P.D. Seymour and R. Thomas, *Graph searching and a min-max theorem for tree-width*. J. Comb. Theory, Series B, 58 (1993), 22-33.

[28] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal 3 (1954), 343–356.

[29] M. Temkin, *Introduction to Berkovich analytic spaces*, Preprint available at http://arxiv.org/abs/1010. 2235v1.

CNRS - DMA, École Normale Supérieure, Paris

E-mail address: `oamini@math.ens.fr`