RATIONAL POINTS ON CURVES AND RIEMANN-ROCH FOR GRAPHS

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This is a summary of the 30-minute lecture which I gave at the 2013 Simons Symposium on Tropical and Non-Archimedean Geometry. The goal was to explain some of the ideas behind the proof of the following recent theorem due to Eric Katz and David Zureick-Brown [KZB]:

Theorem 1. *Let $X/\mathbb{Q}$ be an algebraic curve of genus at least 2. Assume that $r := \mathrm{rank} J(\mathbb{Q}) < g$. Let $p > 2r + 2$ be a prime number, and let $\mathfrak{X}$ be a proper regular model for X over $\mathbb{Z}_p$. Then*

$$\#X(\mathbb{Q}) \leq \#\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p) + 2r.$$

Katz and Zureick-Brown give an example in their paper where $r < g - 1$ and this bound is *sharp*.

1. The method of Coleman and Chabauty

Around 1940, Claude Chabauty had the idea that in order to prove that $X(\mathbb{Q})$ is finite (the Mordell conjecture), one could try to show that $X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$ is finite, where $\overline{J(\mathbb{Q})}$ is the p-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$ for some prime number p. Under the assumption that $r < g$, Chabauty proved in [Ch] that this strategy actually works!

In the mid-1980's, Coleman [Co] made Chabauty's finiteness theorem effective in the sense that he was able to give an explicit upper bound for $\#X(\mathbb{Q})$. The theorem of Katz and Zureick-Brown refines Coleman's work, building on intermediate developments due to Lorenzini–Tucker [LT], McCallum–Poonen [MP], and Stoll [St].

One way to summarize the combined ideas of Chabauty and Coleman is as follows (here we follow closely an article of McCallum and Poonen). Let $X/\mathbb{Q}$ be an algebraic curve and let J be its Jacobian. There is a canonical bilinear map

$$J(\mathbb{Q}_p) \times H^0(J_{\mathbb{Q}_p}, \Omega^1) \to \mathbb{Q}_p,$$

denoted $\langle Q, \omega_J \rangle \mapsto \int_0^Q \omega_J$, uniquely characterized by the following two properties:

- For fixed ω_J, the map $\eta_J : J(\mathbb{Q}_p) \to \mathbb{Q}_p$ given by $Q \mapsto \int_0^Q \omega_J$ is a group homomorphism.
- On some open subgroup U of $J(\mathbb{Q}_p)$, one can compute $\int_0^Q \omega_J$ for $Q \in U$ by formally integrating power series in suitable local coordinates.

Date: June 1, 2013.

Now, for $Q, Q' \in X(\mathbb{Q}_p)$ and $\omega_X \in H^0(X_{\mathbb{Q}_p}, \Omega^1)$, we define

$$\int_Q^{Q'} \omega_X := \int_0^{[Q'-Q]} \omega_J,$$

where ω_J is the differential corresponding to ω_X under the canonical isomorphism between $H^0(J_{\mathbb{Q}_p}, \Omega^1)$ and $H^0(X_{\mathbb{Q}_p}, \Omega^1)$.

If $Q, Q' \in X(\mathbb{Q}_p)$ have the same reduction in some proper regular model $\mathfrak{X}$ for X, then $\int_Q^{Q'} \omega_X$ can be calculated by expanding ω_X in a power series with respect to a local parameter on X and formally integrating.

One deduces from this formalism:

Proposition 1. *Let* $r := \mathrm{rank} J(\mathbb{Q})$ *and fix* $P \in X(\mathbb{Q})$. *Let*

$$V_{\mathrm{chab}} = \{\omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1) \mid \int_P^Q \omega = 0 \text{ for all } Q \in X(\mathbb{Q})\}.$$

Then $\dim V_{\mathrm{chab}} \geq g - r$. *In particular, if* $r < g$ *then* $\dim V_{\mathrm{chab}} > 0$.

A Newton polygon argument, together with the assumption that $p > 2r + 2$, yields the fundamental bound

$$\#X(\mathbb{Q}) \leq \sum_{\bar{Q} \in \bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p)} \left(1 + n_{\bar{Q}}\right),$$

where $n_{\bar{Q}} = \min_{\omega \in V_{\mathrm{chab}}} \mathrm{ord}_{\bar{Q}}(\bar{\omega})$.

Thus if we let

$$D_{\mathrm{chab}} = \sum_{\bar{Q} \in \bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p)} n_{\bar{Q}}(\bar{Q}) \in \mathrm{Div}(\bar{\mathfrak{X}})$$

and set $d = \deg(D_{\mathrm{chab}})$, then the theorem boils down to the inequality $d \leq 2r$.

Note that by considering a fixed nonzero $\omega \in V_{\mathrm{chab}}$, one obtains the bound $d \leq 2g - 2$. This gives the Lorenzini–Tucker and McCallum–Poonen bound $\#X(\mathbb{Q}) \leq \#\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p) + 2g - 2$. Coleman's original bound was the special case of this inequality when $p > 2g$ and X has good reduction at p.

2. Stoll's refinement

If X has **good reduction** at p and $r < g - 1$, Stoll improved the bound $d \leq 2g - 2$ to $d \leq 2r$, which implies that $\#X(\mathbb{Q}) \leq \#\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p) + 2r$, by using *Clifford's inequality*. Indeed, in this case it is easy to see that D_{chab} and $K_{\bar{X}} - D_{\mathrm{chab}}$ are both linearly equivalent to effective divisors, so Clifford's inequality implies that

$$r(K_{\bar{X}} - D_{\mathrm{chab}}) = h^0(K_{\bar{X}} - D_{\mathrm{chab}}) - 1 \leq \frac{1}{2}(2g - 2 - d).$$

On the other hand, by the semicontinuity of h^0, we have

$$h^0(K_{\bar{X}} - D_{\mathrm{chab}}) \geq \dim V_{\mathrm{chab}} \geq g - r.$$

Combining these two inequalities gives $g - r - 1 \leq \frac{1}{2}(2g - 2 - d)$ and therefore $d \leq 2r$ as desired.

3. The case of bad reduction and Clifford's inequality for graphs

What to do if X has **bad reduction** at p? Well, first of all, Katz and Zureick-Brown prove a lemma to the effect that one can reduce to the case where $\bar{\mathfrak{X}}$ is semistable; the point is that extending scalars only makes D_{chab} bigger. However, it is well-known (see for example C-LS) that Clifford's inequality fails in general for singular curves, even semistable ones, so another idea is needed. It turns out that one can use *Clifford's inequality for graphs* (or, more generally, *metrized complexes of curves*) to carry out Stoll's idea in the bad reduction case. We now turn to a brief explanation of this.

For simplicity, assume for the moment that $\mathfrak{X}$ is a proper regular semistable model for X which is *totally degenerate*, i.e., every irreducible component of $\bar{\mathfrak{X}}$ is a smooth rational curve. Let G be the dual graph of $\bar{\mathfrak{X}}$, so the vertices of G correspond to the irreducible components and the edges correspond to crossings between these components. There is a divisor $\bar{D}_{\text{chab}}$ on G which records which irreducible component of $\bar{\mathfrak{X}}$ a given point lies on. Clifford's theorem for graphs (which is a consequence of the Riemann-Roch theorem for graphs [BN]) implies that

$$r(K_G - \bar{D}_{\text{chab}}) \leq \frac{1}{2}(2g - 2 - d)$$

and a variant of the so-called *specialization inequality* from [B] (the analogue in this context of semicontinuity) implies that

$$r(\bar{D}_{\text{chab}}) \geq \dim V_{\text{chab}} - 1 \geq g - r - 1.$$

Combining these inequalities yields $d \leq 2r$ as before.

The general case, where $\mathfrak{X}$ is a proper regular semistable model for X which is not assumed to be totally degenerate, follows similarly replacing Clifford's inequality for graphs with *Clifford's inequality for metrized complexes of curves.* The latter can be deduced formally from the Riemann-Roch theorem for metrized complexes of curves due to Amini–Baker [AB], which includes as special cases both Riemann-Roch for graphs and Riemann-Roch for algebraic curves, or from the results of [AC].

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