A local approach to the non-Archimedean Monge-Ampère equation

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References:

[BFJ] Boucksom, S.; Favre, C.; Jonsson, M.; *The Non-Archimedean Monge-Ampère Equation* arXiv:1502.05724.

[BGJKM] Burgos Gil, J. I.; Gubler, W.; Jell, P.; Kuennemann, K.; Martin, F.; *Differentiability of non-Archimedean volumes and non-Archimedean Monge-Ampère equations (with an appendix by Robert Lazarsfeld)* arXiv:1608.01919.

1 Complex Monge-Ampère equation

Assume there are given

- X a compact Kähler manifold of dimension d;
- ω a Kähler form on X with Kähler class $[\omega]$;
- Ω a volume form on X such that

$$\int_X \Omega = \int_X \omega^{\wedge d}.$$

Then there exists a unique $(1,1)$ form η on X such that

- $\eta \in [\omega] \in H^{1,1}(X, \mathbb{C})$;
- $\eta^{\wedge d} = \Omega$.

This result was conjectured by Calabi (1954) who proved the uniqueness and was proved by Yau (1978).
A geometric version of this result reads as follows.

- X a projective smooth complex variety of dimension d;
- L an ample line bundle on X;
- Ω a volume form on X such that

$$\int_X \Omega = \int_X c_1(L)^{\wedge d}.$$

Then there exists a smooth metric $\|\cdot\|$ on L, unique up to scaling, such that

$$c_1(L, \|\cdot\|)^{\wedge d} = \Omega.$$

Here $c_1(L)$ denotes the first Chern class of L, while $c_1(L, \|\cdot\|)$ denotes the first Chern form of $(L, \|\cdot\|)$.

2 Non-Archimedean setup

For the remainder of the notes we fix the following setup

- K a complete field with respect to a discrete valuation, $|\cdot|$ the absolute value;
- K° the corresponding DVR, $S = \mathrm{Spec}(K^\circ)$;
- $\tilde{K}$ the residue field;
- X a normal projective variety over K, X^{an} the associated analytic Berkovich space;
- L a line bundle on X.

Definition. A model of X is a projective flat scheme $\mathscr{X}$ over S such that the generic fibre of $\mathscr{X}$ is X. A model of L is a line bundle $\mathscr{L}$ on $\mathscr{X}$ such that $\mathscr{L}\mid_X = L$.

A model $\mathscr{L}$ of L defines a continuous metric $\|\cdot\|_{\mathscr{L}}$ on L^{an} as follows.
Given the model $\mathscr{X}$, there is a reduction map $\mathrm{red}\colon X^{\mathrm{an}} \to \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fibre of the model. Let $p \in X^{\mathrm{an}}$ and let $\mathcal{U}$ be a neighborhood of $\mathrm{red}(p)$ with a trivialization $\varphi\colon \mathscr{L}\mid_{\mathcal{U}} \xrightarrow{\simeq} \mathcal{O}_{\mathcal{U}}$. Then, for $s \in L(U \cap X)$,

$$\|s(p)\|_{\mathscr{L}} := |\varphi(s)(p)|.$$

Definition.
- A metric $\|\cdot\|$ on L^{an} is called a *model* metric if there exists an integer $k \geq 1$ and a model $\mathscr{L}$ of $L^{\otimes k}$ such that $\|\cdot\|^{\otimes k} = \|\cdot\|_{\mathscr{L}}$.
- Such model metric is called *semipositive* if $\mathscr{L}$ is nef.
- A *continuous semipositive metric* is a uniform limit of semipositive model metrics.

Let $(L_1, \|\cdot\|_1), \dots, (L_d, \|\cdot\|_d)$ be line bundles on X with semipositive continuous metrics. Then there is a unique Radon measure μ on X^{an}, denoted

$$c_1(L_1, \|\cdot\|_1) \wedge \cdots \wedge c_1(L_d, \|\cdot\|_d)$$

and called the *Chambert-Loir* measure, such that

1. It is linear in each $(L_i, \|\cdot\|_i)$;
2. If, for each $i = 1, \dots, d$, $\|\cdot\|_i = \|\cdot\|_{\mathscr{L}_i}$ is a model metric defined on a common normal model $\mathscr{X}$, then

$$\mu = \sum_Y \mathrm{mult}_{\mathscr{X}_s}(Y) \cdot \deg_{\mathscr{L}_1,\dots,\mathscr{L}_d}(Y) \cdot \delta_{\xi_Y},$$

   where Y runs through the set of irreducible components of the special fibre $\mathscr{X}_s$ and ξ_Y is the unique point of X^{an} whose reduction is the generic point of Y.

3. μ is continuous in the weak topology with respect to uniform convergence of metrics.

- The Chambert-Loir measure appears in equidistribution theory.
- The total mass of the Chambert-Loir measure is given by

$$\mu(X^{\mathrm{an}}) = \deg_{L_1,\dots,L_d}(X).$$

3 Non-Archimedean Monge-Ampère equation

We are now in position to state the non-Archimedean analogue of the Monge-Ampère equation.
Assume that L is ample, and let μ be a Radon measure on X^{an} such that

$$\mu(X^{\mathrm{an}}) = \deg_L(X).$$

Question: Does there exist a continuous semipositive metric $\|\cdot\|$ on L^{an} such that

$$c_1(L, \|\cdot\|)^{\wedge d} = \mu?$$

- Yuan and Zhang (2011) have proven the unicity up to scaling.

The existence of solutions to the Non-Archimedean Monge-Ampère equation was established by Boucksom, Favre and Jonsson under some hypothesis.

Theorem 1 ([BFJ]). *Assume*

1) $\mathrm{char}(\tilde{K}) = 0$*;*

2) *X is smooth with an SNC model $\mathscr{X}$. Let $\Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$ be the skeleton of $\mathscr{X}$;*

3) *μ is a positive Radon measure with* $\mathrm{supp}(\mu) \subset \Delta_{\mathscr{X}}$*;*

†) *X is defined over the function field of a curve C over $\tilde{K}$ and K is the completion of $\tilde{K}(C)$ at a closed point of C.*

Then there exists a continuous semipositive metric $\|\cdot\|$ on L^{an} such that $c_1(L, \|\cdot\|)^{\wedge d} = \mu$.

- The algebraicity hypothesis **†** allows the use of global methods on a model of X over the curve C.
- The aim of this work is to remove the algebraizability hypothesis **†** giving a local proof of this theorem.
- The main tools for this local approach are the *non-Archimedean volumes* and the *holomorphic Morse inequalities*.

4 Algebraic volumes, cohomological functions and holomorphic Morse inequalities

Let k be a field, Y a projective variety over k of dimension d, and D a Cartier divisor on Y. The *volume* of D measures the asymptotic growth of the space of global sections of $\mathcal{O}(mD)$.

$$\mathrm{vol}(D) = \limsup_{m\to\infty} \frac{h^0(Y, \mathcal{O}(mD))}{m^d/d!} \in [0, \infty[.$$

Küronya has introduced the *asymptotic cohomological functions* that measure the asymptotic growth of the higher cohomology groups of $\mathcal{O}(mD)$.

$$\widehat{h}^q(D) = \limsup_{m\to\infty} \frac{h^q(Y, \mathcal{O}(mD))}{m^d/d!} \in [0, \infty[$$

- In the definition of the volume the limsup is actually a limit. This is not known for the asymptotic cohomological functions.
- Clearly $\widehat{h}^0(D) = \mathrm{vol}(D)$.

The *holomorphic Morse inequalities* give us a bound of the asymptotic cohomological functions.

Theorem (Holomorphic Morse inequalities)**.** *Let D and E be nef Cartier divisors on Y. Then*

$$\widehat{h}^q(D-E) \leq \binom{n}{q} D^{n-q}E^q.$$

- Proved by Demailly (1985) in the case $k = \mathbb{C}$ with analytic methods.
- Proved by Angelini (1996) when $\mathrm{char}(k) = 0$ with algebraic methods.
- Lazarsfeld (2016) in an Appendix to [BGJKM] gives a proof valid for any projective scheme over any field.

5 Non-Archimedean volumes

The local non-Archimedean analogue of the algebraic volumes are the non-Archimedean volumes.
We go back to the non-Archimedean setup and let $\|\cdot\|$ be a metric on L^{an}. The *space of small sections* of L is

$$\widehat{H}^0(X, L, \|\cdot\|) = \{s \in \Gamma(X, L) \mid \|s\|_{\sup} \leq 1\}.$$

It is a finite K° module.

Definition. Given two continuous metrics $\|\cdot\|_1$ and $\|\cdot\|_2$, the *non-Archimedean volume* is defined as

$$\mathrm{vol}(X, L, \|\cdot\|_1, \|\cdot\|_2) = \limsup_{m\to\infty} \frac{d!}{m^{d+1}} \ell_{K^\circ}\left(\frac{\widehat{H}^0(X, L^{\otimes m}, \|\cdot\|_1^{\otimes m})}{\widehat{H}^0(X, L^{\otimes m}, \|\cdot\|_2^{\otimes m})}\right) \in \mathbb{R}$$

where the length of a virtual module is defined as

$$\ell_{K^\circ}(M_1/M_2) = \ell_{K^\circ}(M_1/(M_1 \cap M_2)) - \ell_{K^\circ}(M_2/(M_1 \cap M_2)).$$

Theorem 2 (BGJKM). *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are semipositive metrics on L, then*

$$\mathrm{vol}(X, L, \|\cdot\|_1, \|\cdot\|_2) = \frac{-1}{d+1} \sum_{j=0}^{d} \int_{X^{\mathrm{an}}} \log \frac{\|\cdot\|_1}{\|\cdot\|_2} c_1(L, \|\cdot\|_1)^{\wedge(d-j)} c_1(L, \|\cdot\|_2)^{\wedge j}.$$

- Boucksom and Erikson have an independent proof.
- Boucksom and Bergman have proven an Archimedean analogue.

6 Differentiability

Theorem 3 (BGJKM). *Let $\|\cdot\|$ be a continuous semipositive metric on L^{an} and $f\colon X^{\mathrm{an}} \to \mathbb{R}$ a continuous function. Then*

$$\mathrm{vol}(L, e^{-\varepsilon f}\|\cdot\|, \|\cdot\|) = \varepsilon \int_{X^{\mathrm{an}}} f c_1(L, \|\cdot\|)^d + o(\varepsilon)$$

when $\varepsilon \to 0$.

- $\tilde{K}$ of arbitrary characteristic.
- No global assumption as the condition † in Theorem 1.
- This result was proposed by Kontsevich-Tschinkel.
- Differentiability for arithmetic volumes was proved by Chen and Yuan using global methods.
- Inspired by the strategy of Abbes-Bouche and Yuan.
- In the global case it is enough to prove a one sided bound. In the local case one needs a two sided bound. This is achieved through the holomorphic Morse inequalities and the asymptotic cohomological functions, that allow us to have a better control on the first cohomology groups.

7 Orthogonality

Now we assume

- $\mathrm{char}(\tilde{K}) = 0$.
- X smooth projective variety over K.
- L ample line bundle on X.
- $\|\cdot\|$ a continuous metric on L^{an}.

Definition. The semipositive envelope of $\|\cdot\|$ is the metric

$$P(\|\cdot\|) = \inf\{\|\cdot\|' \mid \|\cdot\|' \geq \|\cdot\| \text{ continuous and semipositive}\}$$

where the infimum is taken pointwise.

Theorem 4 ([BFJ]). *$P(\|\cdot\|)$ is a continuous semipositive metric.*

Lemma 5 ([BGJKM]). $\mathrm{vol}(L, \|\cdot\|, P(\|\cdot\|)) = 0$.

Theorem 6 (Orthogonality property [BGJKM]).

$$\int_{X^{\mathrm{an}}} \log\left(\frac{P(\|\cdot\|)}{\|\cdot\|}\right) c_1(L, P(\|\cdot\|))^{\wedge d} = 0.$$

Proof. Write

$$\varphi := \log\left(\frac{P(\|\cdot\|)}{\|\cdot\|}\right) \geq 0.$$

Then φ is continuous by Theorem 4. For $\varepsilon \in [0, 1]$,

$$\|\cdot\| \leq e^{-\varepsilon\varphi} P(\|\cdot\|) \leq P(\|\cdot\|).$$

Therefore $P(e^{-\varepsilon\varphi} P(\|\cdot\|)) = P(\|\cdot\|)$. By Lemma 5 and Theorem 3

$$0 = \mathrm{vol}(L, e^{-\varepsilon\varphi} P(\|\cdot\|), P(\|\cdot\|)) = \varepsilon \int_{X^{\mathrm{an}}} \varphi c_1(L, P(\|\cdot\|))^{\wedge d} + o(\varepsilon).$$

$\square$

- Boucksom Favre and Jonsson prove the same result assuming the hypothesis **†** to reduce to the complex case and to use global arguments on a model over the curve.

Theorem 7. *Theorem 1 holds without the hypothesis* **†**.

Proof. Same proof as in [BFJ] once the orthogonality property is established without the hypothesis **†**. $\square$