Differential forms and currents on Berkovich spaces

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Abstract. — This is a summary of my lecture at the Simons Symposium on Non-archimedean and tropical geometry, held in St John, April 1–6, 2013. My notes of the talk have been lost with my luggage which got stolen while returning from St John to New York. I thank Matt Baker and Sam Payne for passing their notes to me, as well as Sam for patiently requiring that I put them in form, well beyond the initial deadline.

These notes explain the construction of differential forms and currents on Berkovich spaces, following the joint paper [**4**] with Antoine Ducros. The talk was following a two hour talk by Walter Gubler (see [**7**]) who introduced tropicalizations of algebraic varieties and differential forms on the associated analytic spaces. As in the workshop, I shall assume that the reader is acquainted with this theory. In fact, our paper [**4**] is more general in so that forms and currents can be defined on arbitrary (good) analytic spaces; this is of course essential if one wants that forms/currents give rise to *sheaves*, but I shall try to keep silent about such subtleties here.

1. Calibrations

1.1. Calibrations of a real affine space. — Before we can integrate (n,n)-forms on an analytic space, we need to explain, following Lagerberg [**8**], the definition of the integral of an (n,n)-form on $\mathbf{R}^n$. If

$$\alpha = f\,\mathrm{d}'x_1\wedge \mathrm{d}''x_1\wedge\cdots\wedge \mathrm{d}'x_n\wedge \mathrm{d}''x_n$$

is such a form, where f is an integrable function on $\mathbf{R}^n$, set

$$\int\alpha = \int_{\mathbf{R}^n} f\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Algebraically, and more intrinsically, this construction can be understood as follows: set $V=\mathbf{R}^n$, with its canonical basis $(e_1,\dots,e_n)$, and view $\mathrm{d}''x_1\wedge\cdots\wedge\mathrm{d}''x_n$ as an element of $\bigwedge^n V^*$; what we did is contracting the d''-part of α with the n-vector $\mu = e_1\wedge\cdots\wedge e_n$. In general, one defines a *calibration* of an n-dimensional affine space V as the datum of an orientation of V plus an n-vector $\mu\in\bigwedge^n V$. If α is an (n,n)-form on V, we can then define $\int_V\alpha$ by contracting the d''-part of α with μ.

1.2. Calibrations of polyhedra. — In this paper, a cell is the convex hull of a finite set of points in a real affine space, and a polyhedron is a locally finite union of cells. A cell decomposition of a polyhedron is a locally finite family of cells which cover the polyhedron and whose pairwise intersections are cells which have empty interior in both larger cells.

If P is a polytope of a real affine space, with a cell decomposition, a (n-dimensional) calibration of P is the datum of a calibration of its n-dimensional cells. We identify two calibrations of a polyhedron (associated to two cell decompositions) if they agree up to refining the respective cell decompositions to a common one.

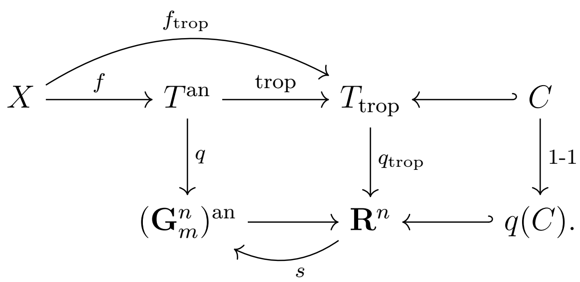
1.3. Canonical calibration of analytic spaces. — We fix a field k which is complete for a non-archimedean absolute value (trivial or not). All analytic spaces that appear below are k-analytic spaces in the sense of Berkovich [**2**].

If X is an analytic space, a moment map is an analytic map $f\colon X \to T^{\mathrm{an}}$ from X to the analytic space associated to a (split) algebraic torus $T \simeq \mathbf{G}_m^d$. Let $N = \mathrm{Hom}(\mathbf{G}_m, T) \simeq \mathbf{Z}^d$ be the lattice of cocharacters of T; the tropicalization of T is the real space $T_{\mathrm{trop}} = N_{\mathbf{R}} \simeq \mathbf{R}^d$. Also recall that the tropicalization map $T^{\mathrm{an}} \to T_{\mathrm{trop}}$ admits a canonical section s whose image is the skeleton of T^{an}. In coordinates, the tropicalization map $\mathrm{trop}\colon T^{\mathrm{an}} \to T_{\mathrm{trop}}$ is the continuous map that sends $x \in (\mathbf{G}_m^d)^{\mathrm{an}}$ to the tuple $(-\log|T_1(x)|, \dots, -\log|T_d(x)|)$, while $s(t_1, \dots, t_d)$ is the point of T^{an} such that for every $\varphi = \sum \varphi_m T^m \in k[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$,

$$|\varphi(s(t_1, \dots, t_d))| = \max_{m \in \mathbf{Z}^d} |\varphi_m| e^{-m_1 t_1 - \dots - m_d t_d}.$$

In particular, $s(0)$ is the Gauss norm on the ring $k[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$ of Laurent series with coefficients in k. (The minus sign and the choice of a basis of logarithms are irrelevant.) We let $f_{\mathrm{trop}} = \mathrm{trop} \circ f\colon X \to T_{\mathrm{trop}}$.

If X is compact and n-dimensional, then it is known that $f_{\mathrm{trop}}(X)$ is a polyhedron of dimension $\leqslant n$. We define in [**4**] a (n-dimensional) calibration of $f_{\mathrm{trop}}(X)$ as follows; we call it the *canonical calibration* of $f_{\mathrm{trop}}(X)$. Consider the diagram



Here, C is an n-dimensional cell of $f_{\mathrm{trop}}(X)$, and $q\colon T \to \mathbf{G}_m^n$ is any morphism of tori chosen in such a way that the induced map from C to $q(C)$ is one-to-one (this property holds generically). The fundamental property of this diagram is that the induced map

$$q \circ f|_{f_{\mathrm{trop}}^{-1}(C)}\colon f_{\mathrm{trop}}^{-1}(C) \to (\mathbf{G}_m^n)^{\mathrm{an}}$$

is *finite and flat* over $s(q(C))$ provided that $q(C)$ and $q \circ f_{\text{trop}}(\partial X)$ do not meet. (It is *finite* because it is zero-dimensional and we avoid the boundary, it is *flat* because the local ring at a point of the skeleton is a field.) Since C is connected (and non-empty), $s(q(C))$ is connected too and we get a well-defined degree $d_{q,C}$.

There is a unique affine map $\sigma_q \colon \mathbf{R}^n \to T_{\text{trop}} \simeq \mathbf{R}^d$ which is a section of q_{trop} and takes $q(C)$ to C. We then define the n-vector

$$\mu_C = d_{q,C} \cdot (\sigma_q)_*(|e_1 \wedge \cdots \wedge e_n|).$$

It is independent of the choice of the auxiliary morphism q. [1]

1.4. Balancing condition. — We keep the same setting as in the previous paragraph. Let F be an $(n-1)$-dimensional face of $f_{\text{trop}}(X)$ and define

$$\partial\mu_F = \sum_{C \supset F} \pm\mu_C,$$

as the sum of all n-vectors μ_C corresponding to n-dimensional cells C of $f_{\text{trop}}(X)$ of which F is an $(n-1)$-dimensional cell. Signs $\pm$ are added to take care of the orientations: we fix an orientation of F, then orient C with the "outgoing normal" rule used in the Stokes's formula, so that μ_C becomes a genuine n-vector.

We have a generalization of the balancing condition in tropical geometry. In our context, it claims that *for any $(n-1)$-dimensional face F of $f_{\text{trop}}(X)$ which is not countained in* $\operatorname{trop}(\partial X)$*, then $\partial\mu_F = 0$.*

For the proof, we consider a generic morphism of tori $q \colon T \to \mathbf{G}_m^n$ such that q_{trop} is one-to-one on each cell C which is adjacent ot F, and such that $q_{\text{trop}}(G) \neq q_{\text{trop}}(F)$ is G is an $(n-1)$-dimensional cell distinct from F. Then

$$q_*(\sum \pm\mu_C) = \sum \pm d_{q,C} \cdot q_*\sigma q, C*(|e_1 \wedge \cdots \wedge e_n|) = \left(\sum \pm d_{q,C}\right) \cdot |e_1 \wedge \cdots \wedge e_n| = 0$$

by invariance of degree on each side of $q(F)$. Since for every generic projection q, $q_*(\partial\mu_F) = 0$, one has $\partial\mu_F = 0$.

1.5. Integral of an (n,n)-form on an analytic space. — In the previous setting, let α be an (n,n)-form on T_{trop}. By definition, its pull-back $f^*\alpha$ is an (n,n)-form on X. We endow $f_{\text{trop}}(X)$ with its canonical calibration $\mu_{X,f}$ and define

$$\int_X f^*\alpha = \int_{f_{\text{trop}}(X)} \alpha\mu_{X,f}.$$

It only depends on the (n,n)-form $f^*\alpha$, and not on the pair (f,α).

1. Beware: The proof of this fact given in the first arXiv version is incorrect!

2. Partitions of unity

Lemma 2.1. — *Let X be an affinoid space and let x, y be distinct points of X. There exists a smooth function u on X such that $u(x) = 1$ and $u(y) = 0$.*

Proof. — Recall that X is the set of multiplicative seminorms on an affinoid algebra $\mathscr{A}$. Be definition, if $x \neq y$, there exists $f \in \mathscr{A}$ such that $|f(x)| \neq |f(y)|$. Assume for the moment that f be invertible; it then gives rise to a morphism from X to $\mathbf{G}_m$, and a smooth function u as requested is obtained as the composition $\varphi \circ f_{\mathrm{trop}} = \varphi(-\log|f|)$ where φ is a $\mathscr{C}^\infty$ compactly supported function ("bump function") on $\mathbf{R}$ such that $\varphi(-\log|f(x)|) = 1$ and $\varphi(-\log|f(y)|) = 0$. In any case, f defines a morphism from X to the affine line $\mathbf{A}^1$. If $f(y) \neq 0$, we take the same function $\varphi(-\log|f|)$ on the locus $f^{-1}(\mathbf{G}_m)$ where f is invertible, and extend it by 0. Then u is smooth on $f^{-1}(\mathbf{G}_m)$ by definition; it is also smooth on a neighborhood of $f^{-1}(0)$ since it vanishes identically there. The case where $f(y) = 0$ is treated similarly, since then $f(x) \neq 0$. □

Given this lemma, the rest of the construction of partitions of unity, etc., is general topology.

Corollary 2.2. — *Let X be a good and Hausdorff analytic space. Then for any $x \in X$ and any neighborhoods U and V of x such that $\overline{V} \subset U$, there is a smooth function u on X such that $u|_V \equiv 1$ and $\operatorname{supp}(u) \subset U$.*

***Corollary 2.3* (Stone-Weierstrass).** — *Let X be a good and Hausdorff analytic space and let U be an open set of X. A continuous function on X with compact support contained in U can be uniformly approximated by smooth functions with compact support contained in U.*

Corollary 2.4. — *If X is good and paracompact, then X admits smooth partitions of unity.*

3. Integrals

3.1. Definition. — Let X be a good and paracompact analytic space, let $n = \dim(X)$. The integral of an (n,n)-form on X is defined in such a way that $\int_X f^*\alpha = \int_{f_{\mathrm{trop}}(X)} \langle \alpha, \mu_f \rangle$ if X is compact, $f \colon X \to T$ is a moment map and α is an (n,n)-form on T_{trop} in the sense of Lagerberg. It satisfies additivity with respect to compact analytic domains: if ω is an (n,n)-form on X and U, V are compact analytic domains of X, then

$$\int_{U\cup V} \omega + \int_{U\cap V} \omega = \int_U \omega + \int_V \omega.$$

There is one subtle points involving partitions of unity. Let ω be an (n,n)-form on X; we want to define $\int_X \omega$. By definition, there exists a family (U_i) of compact analytic domains U_i whose interiors cover X and such that $\omega|_{U_i} = f_i^*(\alpha_i)$ for some moment maps $f_i\colon U_i \to T_i$ and some (n,n)-forms α_i on $T_{i,\mathrm{trop}}$. Take a partition of unity (λ_i) subordinate to the open covering $(\mathring{U}_i)$. Then, one will have

$$\int_X \omega = \int_X (\sum \lambda_i)\omega = \sum_i \int_X \lambda_i \omega = \sum_i \int_{U_i} \lambda_i f_i^*(\alpha_i).$$

The difficulty is that λ_i is smooth, but is not necessarily of the form $f_i^*(u_i)$ for some smooth function u_i on $T_{i,\mathrm{trop}}$, even after refinement of the moment map f_i. Fortunately, one can find a compact analytic domain V_i contained in the interior of U_i and containing $\mathrm{supp}(\omega|_{U_i})$, a moment map $g_i\colon V_i \to T_i'$ and a smooth function u_i on $T'_{i,\mathrm{trop}}$ such that $\lambda_i = g_i^* u_i$ on V_i. One may also assume that $\omega|_{V_i} = g_i^*\alpha_i'$ for some form on $T'_{i,\mathrm{trop}}$.

Then, one can *define*

$$\int_X \omega = \sum_i \int_{V_i} g_i^*(v_i\alpha_i') = \sum_i \int_{g_{i,\mathrm{trop}}(V_i)} \langle v_i\alpha_i', \mu_{g_i}\rangle.$$

3.2. Boundary integral. — One can also defined the integral on the boundary of X of an $(n-1,n)$-form. It is defined in such a way that for a compact analytic space X of dimension n, a moment map $f\colon X \to T$, and an $(n-1,n)$-form α on T_{trop},

$$\int_{\partial X} f^*(\alpha) = \sum_F \int_F \langle \alpha, \partial\mu_f\rangle.$$

(Here, F runs over the $(n-1)$-dimensional cells of a cell decomposition of F, and $\partial\mu_f$ is the boundary calibration of the canonical calibration μ_f.)

The balancing condition implies that if $\mathrm{supp}(\omega)$ is contained in $\mathrm{Int}(X)$, then $\int_{\partial X}\omega = 0$.

3.3. Stokes's formula. — Let ω be an $(n-1,n)$-form on X, compactly supported. Then

$$\int_X \mathrm{d}'\,\omega = \int_{\partial X} \omega.$$

3.4. Green's formula. — Let α be a (p,p)-form, let β be a (q,q)-form, assume that $p+q=n-1$ and that α,β are "symmetric". Then

$$\int_X \alpha\wedge \mathrm{d}'\mathrm{d}''\,\beta - \mathrm{d}'\mathrm{d}''\,\alpha\wedge\beta \int_{\partial X} \alpha\wedge \mathrm{d}''\,\beta - \mathrm{d}''\,\alpha\wedge\beta.$$

The symmetry condition means that when one exchanges d''s and d'''s, the (p,p)-form α gets changed to $(-1)^p\alpha$, and similarly for β. It is an analogue of the reality condition for differential forms on complex spaces.

4. Currents

4.1. Definition. — Let X be an analytic space. We define currents of bidimension (p,q) on X to be continuous linear forms on the space of (p,q)-forms with proper support on X. Two adjective needs an explanation. *Proper* support means that the support of the form is compact and disjoint from the boundary of X. The topology on the space of forms with proper support is naturally defined: a sequence (ω_n) of forms converges to a form ω if there exists a compact set K contained in $\operatorname{Int}(X)$ and containing $\operatorname{supp}(\omega)$ and $\operatorname{supp}(\omega_n)$ for every n, as well as a finite family of compact analytic domains (U_i) whose interiors cover K, moments $f_i\colon U_i \to T_i$ and forms $\alpha_{i,n}$, α_i on $T_{i,\mathrm{trop}}$ such that $\omega_n|_{U_i} = f_i^*\alpha_{i,n}$, $\omega|_{U_i} = f_i^*\alpha_i$, and such that all coefficients of $\omega_{n,i}$ converge uniformly to ω_i when $n \to \infty$, as well as their derivatives.

4.2. Examples. — Integrating (n,n)-forms with proper support on X furnishes a current δ_X of bidimension (n,n).

More generally, let L/k be an ultrametric extension, let Y be a L-analytic space of dimension p and let $\varphi\colon Y \to X_L$ be an analytic map which is topologically proper. The current $\varphi_*\delta_Y$ associates to every (p,p)-form ω with proper support on X the integral $\int_Y \varphi^*(\omega)$. The topological properness of the morphism φ ensures that the support of $\varphi^*(\omega)$ is compact, so that the written integral converges.

In an analogous way, integration on the boundary of Y gives rise to a current $\delta_{\partial Y}$ of bidimension $(p-1,p)$.

Assume that $\dim(X) = n$ and let ω be a form of degree (p,q). It gives rise to a current $[\omega]$ of bidimension $(n-p,n-q)$, defined by the formula $\alpha \mapsto \int_X \alpha \wedge \omega$. Moreover, one can prove that if $[\omega] = 0$, then $\omega|_{\operatorname{Int}(X)} = 0$. If X is boundaryless, or if ∂X has empty interior in X, this gives an embedding of forms into currents.

4.3. Differential calculus. — Let T be a current of bidimension (p,q). We define currents $\mathrm{d}'\,T$ and $\mathrm{d}''\,T$ by duality, via the action of d' and d'' on forms:

$$\mathrm{d}'\,T\colon \alpha \mapsto (-1)^{p+q+1}\langle T, \mathrm{d}'\,\alpha\rangle,$$
$$\mathrm{d}''\,T\colon \alpha \mapsto (-1)^{p+q+1}\langle T, \mathrm{d}''\,\alpha\rangle.$$

These are currents of bidimension $(p-1,q)$ and $(p,q-1)$ respectively. As in differential geometry, some signs must be inserted so that differential calculus on forms and currents match: for a form of bidegree (p,q), one has

$$\mathrm{d}'[\omega] = [\mathrm{d}'\,\omega] \quad \text{and} \quad \mathrm{d}''[\omega] = [\mathrm{d}''\,\omega].$$

Let α be a form of bidegree $(n-1,n)$ with proper support. Then

$$\langle \mathrm{d}'\,\delta_X, \alpha\rangle = \langle \delta_X\, \mathrm{d}'\,\alpha = \int_X \mathrm{d}'\,\alpha = \int_{\partial X} \alpha = 0,$$

by the Stokes's formula and the fact that α has proper support. More generally, let $\varphi\colon Y \to X_L$ be as above, let α be a form of bidegree $(p-1,p)$ on X. By the Stokes's formula on Y, one has

$$\langle \mathrm{d}' \varphi_*(\delta_Y), \alpha\rangle = \langle \varphi_*(\delta_Y), \mathrm{d}' \alpha\rangle = \int_Y \varphi^*(\mathrm{d}' \alpha) = \int_Y \mathrm{d}'(\varphi^*\alpha) = \int_{\partial Y} \varphi^*\alpha = \langle \delta_{\varphi_*(Y)}, \alpha$$

so that

$$\mathrm{d}' \varphi_*(\delta_Y) = \varphi_*(\delta_{\partial Y}).$$

5. The Poincaré–Lelong formula

5.1. On the support of (p,n)-forms. — An important (and slightly unusual) feature of our (p,n)-forms is that their support is very small. In particular it avoids every Zariski closed subset of empty interior. Let indeed ω be any (p,n)-form ω on X, and let x be a point of $\operatorname{supp}(\omega)$. By definition, there is a compact analytic neighborhood U of x and invertible analytic functions on U which gives rise to a moment $f\colon U \to T$ thanks to which ω is defined. Since the (p,n)-form ω is non-zero around x, the tropicalization $f_{\mathrm{trop}}(U)$ must have an n-dimensional cell which contains $f_{\mathrm{trop}}(x)$. This forces the transcendence degree of the residue field of x to be n-dimensional (more precisely, a "graded" version of this). However, if x belonged to a Zariski subset of dimension $d < n$, this transcendence degree would be at most d.

5.2. Statement of the Poincaré–Lelong formula. — Let $f \in \mathscr{M}(X)^*$ be a regular meromorphic function on X; this means that locally, f can be written as the quotient of two holomorphic functions on X, none of which is (locally) a zero divisor. Let $\operatorname{div}(f) = \sum n_i Y_i$ be the divisor of f. It is a formal, locally finite, sum of Zariski closed subsets Y_i of X, with multiplicities n_i given by the order of vanishing (or minus the order of pole) of f along Y_i. We then define a current of bidimension $(n-1, n-1)$ by

$$\delta_{\operatorname{div}(f)} = \sum n_i \delta_{Y_i}.$$

Let U_f be the largest open subset of X on which f is defined as well as its inverse. It is a dense open subset, complementary to the union of the Y_i (those for which $n_i \neq 0$). The function $\log|f|$ is defined and continuous on U_f. For any (n,n)-form with proper support on X, the support of ω is contained in U_f, hence one can consider the integral $\int_{U_f} \log|f|\omega$. This gives rise to a current $[\log|f|]$ on X. Note that in complex geometry, the definition of this current is made more complicated by the logarithmic poles.

The Poincaré Lelong formula is following equality of currents:

$$\mathrm{d}'\mathrm{d}''[\log|f|] = \delta_{\operatorname{div}(f)}.$$

5.3. Another formula. — Let $f\colon X \to \mathbf{A}^1$ be a morphism, let $r > 0$ be a positive real number. Then the function $x \mapsto \log(\max(r, |f(x)|))$ is continuous on X, hence gives rise to a current of bidimension (n,n). Let $\eta_r \in \mathbf{A}^1$ be the point $s(-\log(r))$

of the skeleton of $\mathbf{G}_m$. (Recall that when $r = 1$, it corresponds to the Gauss norm on the ring $k[T]$ of polynomials.) Then one has

$$\mathrm{d}'\mathrm{d}''[\log\max(r, |f(x)|)] = \delta_{f^{-1}(\eta_r)}.$$

5.4. Indications about the proof. — One important part of the proof consists in proving that "tropicalizations are constant" in families. Suppose that X is affinoid and satisfies Serre's S_1 property (for example, that X is reduced). Let $f\colon X \to \mathbf{A}^1$ be an analytic morphism whose fibers are purely $(n-1)$-dimensional, let $g\colon X \to T$ be a moment map. Let $h = (g, f)\colon X \to T \times \mathbf{A}^1$; for any $t \in \mathbf{A}^1$, let $X_t = f^{-1}(t)$; for any $A \subset \mathbf{R}$, let $X_A = \{x \in X\,;\, |f(x)| \in A\}$.

Lemma 5.5. — *There exists a positive real number r, a polyhedron $P \subset T_{\mathrm{trop}}$ of dimension $(n-1)$ and an ($(n-1)$-dimensional) calibration of P such that:*

(1) For every $t \in D(0, r)$, the $(n-1)$-skeleton of $g_{\mathrm{trop}}(X_t)$ is equal to P;

(2) For any compact interval $A \subset (0, r)$ with non-empty interior, the n-skeleton of $h_{\mathrm{trop}}(X_A)$ equals $P \times A$.

Here, by "$(n-1)$-skeleton", we mean the union of the $(n-1)$-dimensional cells of a polyhedron.

We can now describe the proof of the Poincaré-Lelong formula. Let ω be an $(n-1, n-1)$-form on X with proper support. Since X_0 is a Zariski closed subset of X and $\mathrm{d}''\omega$ is a $(n-1, n)$-form, its support is supported on $X_{(s,r)}$ for some $s > 0$. Let W be the subset $\{|f(x)| \geqslant s\}$ of X, so that

$$\langle \mathrm{d}'\mathrm{d}''[\log|f|], \omega\rangle = \int_X \log|f|\,\mathrm{d}'\mathrm{d}''\omega = \int_W \log|f|\,\mathrm{d}'\mathrm{d}''\omega.$$

By Green's formula, this is equal to the sum of three terms

$$\int_W \mathrm{d}'\mathrm{d}''(\log|f|)\omega + \int_{\partial W} \log|f|\,\mathrm{d}''\omega - \int_{\partial W} \mathrm{d}''\log|f|\omega.$$

View f as a moment map on $\mathbf{G}_m$; then $\log|f| = -f^*x$, where x is a real affine function on $\mathbf{R} = \mathbf{G}_{m,\mathrm{trop}}$. Consequently,

$$\mathrm{d}'\mathrm{d}''(x) = \mathrm{d}'\big(\frac{\partial x}{\partial x}\,\mathrm{d}''x\big) = \mathrm{d}'(\mathrm{d}''x) = 0.$$

Consequently, $\mathrm{d}'\mathrm{d}''(\log|f|) = 0$ on W, so that the first term in the previous sum vanishes. As a side remark, this establishes a particular case of the Poincaré-Lelong formula: if f is holomorphic and invertible, then $\mathrm{d}'\mathrm{d}''\log|f| = 0$.

The second term vanishes too: since W is defined by the inequality $|f| \geqslant s$ on X, its boundary ∂W has a part where $\log|f| = s$ (more precisely, $f = \eta_s$) and the remaining part is the intersection $\partial X \cap W$. By assumption, the support of ω vanishes on the latter; by definition of s, the support of $\mathrm{d}''\omega$ vanishes of the former.

Consequently, we have

$$\langle \mathrm{d}'\mathrm{d}''\log|f|, \omega\rangle = -\int_{\partial W} \mathrm{d}''\log|f|\omega.$$

We compute the right hand side tropically, assuming that $\omega = g^*\alpha$. Since ω vanishes on ∂X, it is equal to

$$-\sum_{\substack{\text{cells of the form}\\ F=C\times\{s\}}} \int_F \mathrm{d}'' \log|f| \langle \alpha, \mu_h \rangle = \int_P \omega = \int_{X_0} \omega.$$

The minus sign disappears because of (thanks to) our signs conventions: the positive vector is entering $C \times [s, \infty)$ in the cell $C \times \{s\}$.

6. First Chern form of metrized line bundles

6.1. Metrics. — Let L be a line bundle on X. A continuous metric $\|\cdot\|$ on L is the datum, for any local section s of L of a continuous function $\|s\|$ such that $\|fs\| = |f|\|s\|$ if f is a local analytic function and s a local section and such that $\|s\|$ does not vanish if s is a local frame. Such a metric is smooth if for any local frame s, $\log\|s\|$ is smooth.

Assuming that X is paracompact, partitions of unity easily imply the existence of smooth metrics on every line bundle L.

We write $\overline{L}$ for the datum $(L, \|\cdot\|)$ of a line bundle and of a metric on X.

6.2. Curvature form. — Let $\overline{L}$ be a metrized line bundle on X. There exists a unique $(1,1)$-form $c_1(\overline{L})$ on X which is locally equal to $\mathrm{d}'\mathrm{d}'' \log\|s\|^{-1}$ for any local frame s.

Moreover, let s be a regular meromorphic section of L. Then, $\log\|s\|^{-1}$ is a continuous function on X, with values in $\mathbf{R} \cup \{-\infty, +\infty\}$; however, it is continous on the open subset of X where s is regular and non-vanishing. Consequently, one obtains a well-defined current $[\log\|s\|^{-1}]$ of bidimension (n,n). (As always, $n = \dim(X)$.) The Poincaré Lelong formula then implies the following equality:

$$\mathrm{d}'\mathrm{d}''[\log\|s\|]^{-1} + \delta_{\mathrm{div}(s)} = c_1(\overline{L}). \tag{6.3}$$

This shows that in our setting, Cartier divisors have Green currents.

From this equality, one easily shows by induction that integrating the curvature forms of metrized line bundles recovers intersection numbers.

Proposition 6.4. — *Let X be a proper algebraic variety of dimension n. Let $L_1, \dots, L_n$ be line bundles on X equipped with smooth metrics on X^{an}. Then*

$$\int_{X^{\mathrm{an}}} c_1(\overline{L}_1) \wedge \dots \wedge c_1(\overline{L}_n) = \deg(c_1(L_1) \cdots c_1(L_n) \cap [X]).$$

6.5. Model metrics. — In applications, it is important to consider metrized line bundles given by *models*. Let R be the valuation ring of k. A model of X over R is a R-formal scheme $\mathfrak{X}$, flat and locally topologically finitely generated, together with an isomorphism of its generic fiber $\mathfrak{X}_\eta$ with R. Similarly, a model of a line bundle L is a line bundle $\mathfrak{L}$ on a model $\mathfrak{X}$ together with an isomorphism of its generic fiber $\mathfrak{L}_\eta$ with L.

By a theorem of Raynaud (comparison between rigid analytic geometry and formal geometry), models exist as soon as X is a paracompact strict k-analytic space. In particular, analytic spaces associated to projective algebraic varieties have models. Moreover, those models can be chosen to be projective flat schemes (rather than formal schemes).

A model $(\mathfrak{X}, \mathfrak{L})$ of (X, L) gives rise to a metric on L which is defined as follows. For any formal open subscheme $\mathfrak{U}$ of $\mathfrak{X}$, any section $s \in \Gamma(\mathfrak{U}, \mathfrak{L})$ has a generic fiber s_η which is a section of L on the analytic domain $\mathfrak{U}_\eta$ of X; its norm is $\leqslant 1$ at every point; moreover, if s is a basis, then its norm is 1 at every point. Let us write $\overline{L}$ for the resulting metrized line bundle.

Model metrics are fundamental in applications. For example, these are the only metrics that the classical Arakelov geometry of Gillet–Soulé (see [**6**]) considers, although Zhang proposed in [**9**] an extension to some metrics (which are uniform limits of "semipositive" model metrics). However, model metrics are continuous, but are not smooth, so that line bundles equipped with such metrics do not have a Chern form. Still, they have a Chern *current*, which is given by Equation (6.3): even at points where s does not vanish, the function $\log\|s\|^{-1}$ is only continuous in general, so that its image under $\mathrm{d}'\mathrm{d}''$ is a current. Since products of currents do not exist in general, this forbids a priori to consider analogues of Proposition 6.4.

Nevertheless, one can mimick a theory of complex analysis, due Bedford–Taylor [**1**] and Demailly [**5**], and *define* the product of those Chern currents by approximating the model metric by smooth metrics. This requires an elementary study of positive forms and positive currents, completely analogous to the similar concepts in complex geometry. Then one defines this product for metrics which are uniform limits of smooth semipositive metrics, and conclude by linearity. In particular, if $n = \dim(X)$, the nth power $c_1(\overline{L})^n$ of the Chern current $c_1(\overline{L})$ is defined and is a measure on X. We prove that this measure is a *discrete* measure.

We even give a precise formula for it. For simplicity of exposition, I assume here that the valuation of the field k is discrete, X is a projective normal variety, and $\mathfrak{X}$ is an integrally closed projective scheme over R. (The general case is treated in [**4**].)

Let $(\mathfrak{X}_i)$ be the family of irreducible components of $\mathfrak{X}_s$; for every i, let m_i be the multiplicity of $\mathfrak{X}_i$ in the special fiber $\mathfrak{X}_s$.

Recall that there is a specialization (anticontinuous) map $X^{\mathrm{an}} \to \mathfrak{X}_s$ from the analytic space of X to the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$. By the assumption made here, for every i, the generic point of the irreducible component $\mathfrak{X}_i$ has exactly one preimage, say ξ_i, by the specialization map.

Proposition 6.6. — *One has the following equality of measures*

$$c_1(\overline{L})^n = \sum_i m_i (c_1(\mathfrak{L})^n | \mathfrak{X}_i) \delta_{\xi_i}.$$

In particular, the present formalism of differential forms and currents allows to recover the measures defined in an ad hoc manner in [**3**]. This indicates a strong relation between the present theory and non-archimedean Arakelov geometry which needs to be explorated further.

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