TOPOLOGY OF TROPICAL MODULI SPACES

MELODY CHAN

These are notes from my talk at the 2017 Simons Symposium on Non-Archimedean and Tropical Geometry, May 14–20, which was an exposition of parts of the preprint [CGP16] with Søren Galatius and Sam Payne.

1. Generalized Δ-complexes

A generalized Δ-complex is a data structure that efficiently captures "simplices with symmetries." To warm up, let us review (or recast, depending on your perspective) the notion of a Δ-*complex*, also known as a semisimplicial set. Let Δ_{inj} be the category whose objects are $[p] = \{0, \ldots, p\}$ for $p \geq 0$ and whose morphisms are injective, order-preserving maps.

Definition 1.1. A Δ-*complex* is a presheaf of sets on Δ_{inj}, that is, a functor

$$X \colon \Delta_{\text{inj}}^{\text{op}} \to \mathbf{Set}.$$

You think of the elements of $X_p = X([p])$ as a list of p-simplices in X. For every $i \in [p]$, write $\delta^i \colon [p\text{–}1] \hookrightarrow [p]$ for the injective map missing i in its image; then $d^i = X\delta^i \colon X_p \to X_{p-1}$ is the data of the i^{th} face map.

Example 1.2. For example, $\mathrm{Hom}_{\Delta_{\text{inj}}}(-, [2])$ is an (oriented) triangle, with one 2-simplex, three edges, and three vertices.

Note that there's a natural geometric realization functor $|\cdot| \colon \mathbf{\Delta\text{-cx}} \to \mathbf{Top}$.

There is a drawback (for us) in this category, which is that it does not admit colimits, e.g. encoding "half a triangle" without resorting to something like barycentric subdivisions. This is rectified as follows. Let I be the category with objects $[p]$ for $p \geq 0$, as before, and whose morphisms are *all* injective maps of sets (not just order-preserving ones). This category is sometimes called FI instead.

Definition 1.3. A *generalized* Δ-*complex* is a presheaf of sets on I, that is, a functor

$$X \colon I^{\text{op}} \to \mathbf{Set}.$$

There's again a natural geometric realization functor $|\cdot| \colon \mathbf{\Delta\text{-cx}} \to \mathbf{Top}$.

Example 1.4. Now $\mathrm{Hom}_I(-, [2])$ is an (unoriented) triangle, with six 2-simplices, six edges, and three vertices.

This category admits small colimits.

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Example 1.5. "Half a triangle": this is obtained as the colimit of a diagram $Y \rightrightarrows Y$ where one arrow is the identity and one arrow is a flip. It has three 2-simplices, 3 edges, and 2 vertices.

Remark 1.6. A similar construction occurs in [HV98], in which the building blocks are cubes $[0,1]^n$ with symmetries. V. Berkovich kindly remarked that he uses a more general construction in [Ber99] with simplices replaced by polysimplicial sets.

Remark 1.7. The category of generalized Δ-complexes is equivalent to the category of smooth, connected generalized cone complexes as defined in [ACP15], [KKMSD73].

One use of these gadgets is that they are a natural encoding of boundary complexes of normal crossings compactifications, as we now explain.

2. Boundary complexes as generalized Δ-complexes

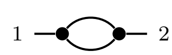
We follow [KKMSD73, Thu07, ACP15]. Suppose first $U \subseteq X$ for X an irreducible, smooth variety, with simple normal crossings boundary $D = X \setminus U$. Then the boundary complex $\Delta(X) = \Delta(U \subseteq X)$ is the dual complex of D: it is has one vertex v_i for every irreducible component of D_i, an edge $v_i v_j$ (for all $i \neq j$) for every irreducible component of $D_i \cap D_j$, and so on.

Now suppose $U \subseteq X$ for an irreducible smooth variety (or separated Deligne Mumford stack), such that $D = X \setminus U$ is now normal crossings. This implies that there is an étale atlas $V \to X$, where V is a scheme, such that the pullback $U_V := U \times_X V \subseteq V$ is a simple normal crossings compactification. Then we define $\Delta(U \subseteq X)$ to be the coequalizer of the diagram

$$\Delta(V \times_X V) \rightrightarrows \Delta(V).$$

This is a generalized Δ-complex.

Example 2.1. Let G be the twice-marked, vertex weighted graph shown here:



It corresponds to a 0-stratum of the Deligne-Mumford compactification $\overline{\mathcal{M}_{1,2}}$. We discuss this example informally as follows. We see two 1-dimensional boundary strata meeting there, corresponding to the two edges of G; except that the boundary strata are indistinguishable from each other. The boundary complex of an appropriate neighborhood looks like a "half interval" (it is a coequalizer of a diagram $X \rightrightarrows X$ where $X = \mathrm{Hom}_I(-,[2])$ is the unordered 1-simplex.)

Example 2.2. This "example" is really a theorem of [ACP15], vastly generalizing and making precise the above discussion. The link of the *tropical moduli space of curves* ([BMV11], see also [Cap13],[Mik06]) is canonically identified with the boundary complex of the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}_{g,n}} \supset \mathcal{M}_{g,n}$. So it has the structure of a generalized Δ-complex.

The topology of boundary complexes is interesting. By the work of [Dan75], [Pay13], [Ste06], [Thu07], it is known that the homotopy type of the boundary complex of a nc compactification $U \subseteq X$ is an invariant of U. Moreover, over $\mathbb{C}$, by Deligne's work [Del75] (see [CGP16, §A] for details in this setup), for any normal crossings compactification of a complex smooth, separated Deligne Mumford stack $U \subseteq X$ of complex dimension d, there is an identification

$$\mathrm{Gr}^W_{2d} H^{2d-i}(U, \mathbb{Q}) \cong \widetilde{H}_{i-1}(\Delta(U \subseteq X), \mathbb{Q})$$

of weight $2d$ ("top-weight") rational cohomology of U with the reduced rational homology of the boundary complex.

Hence the interest in studying the topology of the boundary complex of $\mathcal{M}_{g,n}$, i.e. tropical moduli spaces.

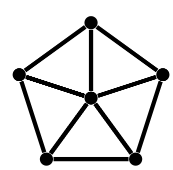
3. Results

Here are some results from [CGP16] on the topology of tropical moduli spaces $\Delta_{g,n}$.

- For $g = 1$, $\Delta_{1,n}$ is homotopy equivalent ot a wedge of $(n-1)!/2$ $(n-1)$-spheres; see also [Get99].
- We have full calculations of $H_*(\Delta_{g,n})$ using a computer, for a range of g, n, including
  - $g = 2$ and $n \leq 8$ (see also [Cha15] for further $g = 2$ results),
  - $g = 3$, $n \leq 4$,
  - $g = 4$, $n \leq 3$,
  - $g = 5$, $n = 0, 1$;
  - $g = 6$, $n = 0$.

In fact, the calculation for $(g, n) = (6, 0)$ implies that $\mathcal{M}_6$ has a unique top weight class, in fact occurring in degree 15. In general, it is known from Euler characteristic considerations [HZ86] that the spaces $\mathcal{M}_g$ will have plenty of odd degree cohomology classes, but it seems that very few explicit ones are known. We note that Tommasi has produced an example of a class of weight 6 in $H^5(\mathcal{M}_4, \mathbb{Q})$ [Tom05].

Our data leads us to conjecture the following infinite family of top-weight cohomology classes in $\mathcal{M}_g$ and $\mathcal{M}_{g,1}$ for g odd. Let W_g be the genus g graph obtained by coning over a g-cycle. Here is a picture of W_5 (from [CGP16]):



Let W'_g be obtained from W_g by marking any vertex (for example, the central one) in W_g.

Then we conjecture that for $g \geq 3$ odd, W_g and W'_g yield nonzero homology classes in the tropical moduli spaces Δ_g and $\Delta_{g,1}$ for all g odd. What we mean by "yield" will be explained a little more in the next section.

Remark 3.1. When $g = 3$ the conjecture is true by work of Looijenga and Bergstrom-Tommasi [Loo93, BT07]. When $g = 5$ and $g = 7$ we have computationally verified the conjecture; the case $g = 5$ can be done by hand, and the case $g = 7$ required extensive computer calculation.

4. Techniques

We briefly highlight some of the combinatorics/combinatorial topology that goes into our results.

First, we have a cellular homology theory for generalized Δ-complexes which is convenient for computation. Given $X\colon I^{\mathrm{op}} \to \mathbf{Set}$ a generalized Δ-complex, define

$$C_p(X) = (\mathbb{Z}X_p \otimes \mathbb{Z}^{\mathrm{sign}})_{S_{p+1}}.$$

There are natural boundary maps which make this into a complex. Similarly, define $C_p(X;\mathbb{Q}) = C_p(X) \otimes \mathbb{Q}$.

Proposition 4.1. There is a natural identifcation

$$H_*(\cdots C_p(X;\mathbb{Q}) \to C_{p-1}(X;\mathbb{Q}) \to \cdots) \cong H_*(|X|;\mathbb{Q}).$$

Example 4.2. This example highlights that the identification holds over $\mathbb{Q}$-coefficients and not necessarily with $\mathbb{Z}$-coefficients. Consider again the half-interval. The relevant complex is

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \to 0,$$

with associated *integral* homology $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ in degrees 1 and 0. On the other hand, the geometric realization of the half-interval is contractible.

Because the definition of C_p involves taking S_{p+1} coinvariants after tensoring with sign, we see that for any dual graph G to a stable curve in $\overline{M}_{g,n}$, if G admits an automorphism that induces a *non-alternating* permutation of $E(G)$, then G drops out in the cellular chain complex associated to the generalized Δ-complex $\Delta_{g,n}$. We derive the following easy criterion to produce cycles in $H_*(\Delta_{g,n})$:

Remark 4.3. If every edge of a dual graph G is contained in a triangle, then G represents a cycle in $H_*(\Delta_{g,n})$.

Indeed, all 1-edge contractions of G are graphs with parallel edges; exchanging parallel edges is a non-alternating automorphism. The criterion in the remark applies to the graphs W_g and W'_g, for instance. (It's much more subtle to verify that these cycles are nonzero in homology, however.)

The second technique I briefly mention is that we have a combinatorial topology criterion, loosely in the spirit of Forman's Discrete Morse theory [For98], for finding sub-generalized-Δ-complexes with contractible geometric realization. In our applications, we find large contractible subcomplexes of $\Delta_{g,n}$. In the case $g = 1$ this already yields the exact homotopy type of $\Delta_{1,n}$. In general, these simplifications extend our computational range.

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Department of Mathematics, Brown University, Box 1917, Providence, RI 02912
E-mail address: melody_chan@brown.edu