CORRECTION TO "FROBENIUS SPLITTINGS OF TORIC VARIETIES" AND "LATTICE POLYTOPES CUT OUT BY ROOT SYSTEMS AND THE KOSZUL PROPERTY"

SAM PAYNE

Abstract. This note explains an error in Theorem 1.3 of "Frobenius splittings of toric varieties," Algebra Number Theory **3** (2009), no. 1, 107–119 and its effect on the main result of "Lattice polytopes cut out by root systems and the Koszul property," Adv. Math. **220** (2009), no. 3, 926–935.

Let X be a toric variety, and let $\Delta_i = X^{i-1} \times \Delta \times X^{n-i-1}$. Theorem 1.3 in [Pay09a] asserts that if X is diagonally split then X^n is split compatibly with $\Delta_1 \cup \cdots \cup \Delta_{n-1}$. This assertion is false, and the following is a counterexample.

Example 1. Let P be the convex hull of the 3×3 permutation matrices, also known as the 4-dimensional Birkhoff polytope. The normal vectors of the facets of P form a unimodular matrix [Sch86, §19], and it follows that X is diagonally split at all primes [CHP$^+$16, Theorem 1.1]. However, the line bundle corresponding to P gives rise to an embedding of X as a cubic hypersurface in $\mathbb{P}^5$ [HP09]. It follows that $\Delta \times X \cup X \times \Delta$ is not compatibly split in X^3, since every embedding of such a variety is normally presented, and in particular its homoegenous ideal is generated by quadrics [Pay09a, Proposition 2.4].

The error in the proof occurs in the middle of the second paragraph, with the false assertion that a certain explicit splitting π is compatible with $\Delta_1 \cup \cdots \cup \Delta_{n-1}$. Indeed, that splitting is never compatible with $\Delta_1 \cup \cdots \cup \Delta_{n-1}$ when $n \geq 3$. See [CHP$^+$16, Theorem 7.3] for a characterization of splittings of X^n that are compatible with $\Delta_1 \cup \cdots \cup \Delta_{n-1}$.

Note that this error also affects [Pay09a, Theorem 1.4], which is deduced as a corollary to the existence of a compatible splitting of $\Delta_1 \cup \cdots \cup \Delta_{n-1}$, and [Pay09b, Theorems 1.1 and 1.3]. The main arguments in the latter paper show correctly that if $\Sigma(1)$ is contained in a root system of type A, B, C, or D then the toric variety X is diagonally split at q for all odd $q \geq 3$. It follows that any lattice polytope whose facet normals is in one of these root systems is normal, as are Cayley sums of polytopes whose Minkowski sum is such a polytope. However, it does not follow that these polytopes are Koszul. The Koszulness of lattice polytopes whose facet normals are contained in a root system of type A is known, by [BGT97]. The corresponding statement for root systems of type B, C, or D is an open problem.

One additional correction was kindly pointed out to me by C. Eur. In Proposition 2.2, one should assume that the compatibly split subvariety is an equivariantly embedded toric subvariety, as in Proposition 2.1 and Propositions 2.3–2.5.

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References

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Email address: `payne.sam@gmail.com`