About Hrushovski and Loeser's work on the homotopy type of Berkovich spaces

Antoine Ducros
Institut de mathématiques de Jussieu (Université Pierre-et-Marie Curie)
Institut universitaire de France

Introduction

At the end of the eighties, V. Berkovich suggested a new approach to non-archimedean analytic geometry ([2], [3]). One of the main advantages of his theory is that it provides spaces enjoying very nice topological properties: although they are defined over fields that are totally disconnected and often not locally compact, Berkovich spaces are locally compact and locally *pathwise* connected. Moreover, they have turned out to be "tame" objects – in the informal sense of Grothendieck's *Esquisse d'un programme*; let us illustrate this rather vague assertion by several examples.

i) The generic fiber of any polystable formal scheme admits a strong deformation retraction to a closed subset homeomorphic to a finite polyhedron (Berkovich, [4]).

ii) Smooth analytic spaces are locally contractible; this is also proved in [4] by Berkovich, by reduction to i) through de Jong's alterations.

iii) If X is an algebraic variety over a non-archimedean, complete field k, then every semi-algebraic subset of X^{an} has finitely many connected components, each of which is semi-algebraic; this was proved by the author in [5].

iv) Let X be a compact analytic space and let f be an analytic function on X; for every $\varepsilon \geqslant 0$, denote by X_ε the set of $x \in X$ such that $|f(x)| \geqslant \varepsilon$. There exists a finite partition $\mathscr{P}$ of $\mathbb{R}_+$ in intervals such that for every $I \in \mathscr{P}$ and every $(\varepsilon, \varepsilon') \in I^2$ with $\varepsilon \leqslant \varepsilon'$, the natural map $\pi_0(X_{\varepsilon'}) \to \pi_0(X_\varepsilon)$ is bijective. This has been established by Poineau in [10] (it had already been proved in the particular case where f is invertible by Abbes and Saito in [1]).

In their recent work [9], Hrushovski and Loeser vastly improve i), ii), iii), and iv), and provide more generally a new and very fruitful framework, based upon advanced tools of model theory, for studying tameness phenomena in Berkovich geometry.

Let us now make precise the extent to which [9] generalizes i), ii), iii) and iv). Let k be a non-archimedean field, let X be a *quasi-projective* algebraic variety over k, and let V be a semi-algebraic subset of X^{an}. Hrushovski and Loeser prove the following.

a) There exists a strong deformation retraction from V to one of its closed subsets homeomorphic to a finite polyhedron.

b) The topological space V is locally contractible.

c) If $\varphi : X \to Y$ is a morphism of algebraic varieties over k, the set of homotopy types of fibers of the map $\varphi^{\mathrm{an}}|_V : V \to Y^{\mathrm{an}}$ is finite.

d) Let f be a function belonging to $\mathscr{O}_X(X)$. For every $\varepsilon \geqslant 0$ let us denote by V_ε the set of $x \in V$ such that $|f(x)| \geqslant \varepsilon$. There exists a finite partition $\mathscr{P}$ of $\mathbb{R}_+$ in intervals such that for every $I \in \mathscr{P}$ and every $(\varepsilon, \varepsilon') \in I^2$ with $\varepsilon \leqslant \varepsilon'$, the embedding $V_{\varepsilon'} \hookrightarrow V_\varepsilon$ is a homotopy equivalence.

The purpose of this note is to make a quick survey[1] of [9], and especially to give a rough description of the aforementioned new framework which Hrushovski and Loeser have developed.

1 Model theory of valued fields

1.1 Definable functors

Let k be a field endowed with an abstract Krull valuation (we do not require it to have height one); we fix an algebraic closure $\bar{k}$ of k, and an extension of the valuation of k to $\bar{k}$. Let M be the category of algebraically closed valued extensions of $\bar{k}$ whose valuation is non-trivial (morphisms are isometric $\bar{k}$-embeddings). Any k-scheme of finite type X gives rise to a functor from M to Sets which will be again denoted by X.

A sub-functor V of X will be said to be *k-definable* if it can be defined, locally for the Zariski-topology of X, by a boolean combination of inequalities of the form $|f| \bowtie \lambda|g|$ where f and g are regular functions, where $\lambda \in |k|$, and where $\bowtie \in \{\leqslant, \geqslant, <, >\}$. A natural transformation from a k-definable sub-functor V of X to a k-definable sub-functor W of Y will be said to be k-definable if its graph is a k-definable sub-functor of $X \times Y$.

A k-definable sub-functor of an algebraic k-variety is entirely determined by its value on *any* field $F \in \mathsf{M}$.

There is also a notion of an *abstract* (*i.e.*, non-embedded) k-definable functor: this is a functor from M to Sets which is isomorphic to a k-definable sub-functor of some algebraic variety X over k (we will not care here about the canonicity of this sub-functor; for some comments on the precise way to overcome this issue, see [7], §1). One can define k-definable transformations between abstract k-definable functors in a straightforward way, and we get thus a category.

This category is not yet big enough to work with easily. One needs to enlarge it furthermore by also calling "k-definable" the quotient of a k-definable functor by a k-definable equivalence relation. *From now on, we use "k-definable" in this new sense.* This will not cause any conflict: if we start with a k-algebraic variety X, a sub-functor of X is k-definable in this new sense if and only if it is k-definable in the original sense.

Here are some new examples of k-definable functors. We leave to the reader the proof of their k-definability (which amounts to write them as nice quotients).

- The functors $\Gamma : F \mapsto |F^\times|$, and $\Gamma_0 : F \mapsto |F|$.
- The functor $F \mapsto \widetilde{F}$ (we denote by $\widetilde{F}$ the residue field of F).
- The functor that sends F to the set of closed balls of F.

[1]The reader may also refer to the more detailed survey [7].

It is easily seen that a sub-functor of Γ_0^n is k-definable if and only if it can be defined by a boolean combination of monomial inequalities with coefficients in $|k|$; we will simply call such a functor a k-definable *polyhedron*.

If a and b are two elements of $|k|$ with $a \leqslant b$, the interval $[a; b]$ is the functor that sends F to $\{c \in |F|, a \leqslant c \leqslant b\}$. It is k-definable. One can concatenate k-definable intervals (one makes the quotient of their disjoint union by the identification of successive endpoints), and one thus gets k-definable *generalized* intervals[2].

1.2 The functor $\widehat{V}$

Let $X = \text{Spec } A$ be an affine algebraic variety over k. The key construction of Hrushovski and Loeser is the following one, which more or less mimics Berkovich's construction, but with a model-theoretic and definable flavour.

Let $F \in \mathsf{M}$. We denote by $\widehat{X}(F)$ be the set of multiplicative semi-norms $\varphi : A \otimes_k F \to |F|$ satisfying the following property.

$(*)$ *For every finite dimensional k-vector space E of A, there exists a k-definable natural transformation $\Phi_E : E \to \Gamma_0$ such that $\varphi|_{E \otimes_k F} = \Phi_E(F)$.*

Let us make some comments. Such a transformation Φ_E is unique (because it is determined by its values on any field belonging to M, hence in particular by its value on F which we have prescribed). Now for every valued extension F' of F belonging to M, one can consider $\Phi_E(F')$, which is a map from $E \otimes_k F'$ to $|F'|$. Those maps glue well when E goes through the set of all finite dimensional subspaces of k, and we get that way a canonical extension of φ to a multiplicative semi-norm $A \otimes_k F' \to |F'|$ which also satisfies $(*)$, with the same Φ_E 's; roughly speaking, this extension is defined by the same formulas as φ.

We thus get a natural embedding $\widehat{X}(F) \hookrightarrow \widehat{X}(F')$ which makes $\widehat{X}$ a functor $\mathsf{M} \to \mathsf{Set}$.

This construction extends to a slightly more general situation.

- First of all, if V is a sub-functor of X defined by a boolean combination of inequalities of the form $|f| \bowtie \lambda |g|$ (with f and g in A and $\lambda \in |k|$) one defines $\widehat{V}$ as the sub-functor of $\widehat{X}$ that consists of the semi-norms that satisfy the same combination of inequalities.

- Now if V is a k-definable sub-functor of any algebraic variety over k, one defines $\widehat{V}$ by performing the above constructions locally and glueing them.

The topology on $\widehat{V}(F)$. Let V be a k-definable sub-functor of an algebraic variety over k and let $F \in \mathsf{M}$. We are going to define a topology on $\widehat{V}(F)$. It is sufficient to describe it for V an affine algebraic variety, say $V = \text{Spec } A$; the general case is obtained by restricting and glueing. Now any element a of $A \otimes_k F$ defines by evaluation of semi-norms a natural map $\widehat{V}(F) \to |F|$, and we endow $\widehat{V}(F)$ with the coarsest topology making those maps continuous (for the order topology of $|F|$).

[2]Exercise: prove that a k-definable generalized interval is not in general k-definably isomorphic to a single k-definable interval, but that it is so if the concatenated intervals only have non-zero endpoints.

Remark. If $F \subset F'$, then the topology on $\widehat{V}(F)$ is in general strictly coarser than the topology induced from that on $\widehat{V}(F')$; indeed, if $|F'|$ has infinitely small elements with respect to $|F|$, then $\widehat{V}(F)$ is discrete in $\widehat{V}(F')$.

Remark. If D is a polyhedron or a generalized interval, then $D(F)$ also inherits a topology for every F, induced by the order topology on $|F|$; again, this topology is not in general compatible with isometric embeddings of valued fields.

Example. Let $F \in \mathsf{M}$. For every $a \in F$ and $r \in |F|$, the map $\eta_{a,r}$: $\sum a_i(T-a)^i \mapsto \max |a_i| \cdot r^i$ is a multiplicative semi-norm on $F[T]$ which belongs to $\widehat{\mathbb{A}^1_k}(F)$. Two semi-norms $\eta_{a,r}$ and $\eta_{b,s}$ are equal if and only if the closed balls $B(a,r)$ and $B(b,s)$ of F are equal. We get that way a functorial bijection between $\widehat{\mathbb{A}^1_k}(F)$ and the set of closed balls of F; therefore $\widehat{\mathbb{A}^1_k}$ is k-definable.

In general, Hrushovski and Loeser prove the following: *let V be a k-definable sub-functor of an algebraic k-variety X. The functor $\widehat{V}$ is pro-k-definable, and k-definable if* $\dim X \leqslant 1$.

Remark. The pro-definability of $\widehat{V}$ comes from general arguments of model-theory, which hold in a very general context, and not only in the theory of valued fields. But the definability of $\widehat{V}$ in dimension 1 is far more specific to the situation; it ultimately relates on Riemann-Roch's theorem for curves.

2 Homotopy type of $\widehat{V}$ and links with Berkovich spaces

We are now going to explain very roughly Hrushovski and Loeser's strategy for proving assertion a) from the end of introduction. For the sake of simplicity, we will deal only with *strict* semi-algebraic subsets of Berkovich spaces, that is, those which can be defined involving only real numbers belonging to $|k|$.

2.1 The link with Berkovich theory

For this paragraph, we assume that the valuation $|.|$ of k takes real values, and that k is complete.

Let X be an algebraic variety over k, and let V be a k-definable sub-functor of X. The inequalities that define V also define without any ambiguity a strict semi-algebraic subset V^{an} of X^{an} (and any strict semi-algebraic subset of X^{an} is of that kind).

Let $F \in \mathsf{M}$ be such that:
- $|F| = \mathbb{R}_+$;
- $\widetilde{F}$ is an algebraic closure of $\widetilde{k}$;
- F is maximal for those properties.

(Using Zorn's lemma, one immediately sees that such a field F does exist). Any point of $\widehat{V}(F)$ can be interpreted in a suitable affine chart as a multiplicative semi-norm with values in $|F| = \mathbb{R}_+$; it thus induces a point of V^{an}.

We thus have built a map $\pi : \widehat{V}(F) \to V^{\mathrm{an}}$. One proves that π is continuous, surjective, and topologically proper; and that it is a homeomorphism if $F = k$.

Most of Hrushovski and Loeser's work is actually devoted to the study of the homotopy type of $\widehat{V}$; at the end of their paper, they use the map π to transfer their results to the Berkovich setting. We will thus now only focus on what happens inside the "hat" world.

2.2 The 'hat-world' avatar of assertion a)

We don't assume anymore that the valuation $|.|$ of k takes real values. In what follows, a k-definable map f between k-definable functors carrying a topology over every $F \in \mathsf{M}$ will be said to be continuous if $f(F)$ is continuous for every $F \in \mathsf{M}$.

Proposition. *Let X be a* quasi-projective *k-variety, and let V be a k-definable sub-functor of X. There exists:*

- *a k-definable generalized interval I, with endpoints o and e;*
- *a k-definable sub-functor S of $\widehat{V}$;*
- *a polyhedron P;*
- *a k'-definable homeomorphism $S \simeq P$, for a suitable finite extension k' of k inside $\overline{k}$;*
- *a continous k-definable map $h : I \times \widehat{V} \to S$ satisfying the following properties for every $F \in \mathsf{M}$, every $x \in \widehat{V}(F)$ and every $t \in I(F)$:*
  - ◇ *$h(o, x) = x$, and $h(e, x) \in S(F)$;*
  - ◇ *$h(t, x) = x$ if $x \in S(F)$;*
  - ◇ *$h(e, h(t, x)) = h(e, x)$.*

*Comment.*We will refer to the existence of k', of P, and of the k'-definable homeomorphism $S \simeq P$ by saying that S is a *twisted polyhedron.* The finite extension k' of k cannot be avoided; indeed, it reflects the fact that the Galois action on the homotopy type of $\widehat{V}$ is non necessarily trivial. In the Berkovich language, think of the $\mathbb{Q}_3$-elliptic curve $E : y^2 = x(x-1)(x-3)$. The analytic curve $E^{\mathrm{an}}_{\mathbb{Q}_3(i)}$ admits a Galois-equivariant deformation retraction to a circle, on which the conjugation exchanges two half-circles; it descends to a deformation retraction of E^{an} to a compact interval.

Comment. The quasi-projectivity assumption can likely be removed, but it is currently needed in the proof for technical reasons.

2.3 Rough sketch of the proof

The steps of the proof are the following.

1) *The case of $\widehat{\mathbb{P}^1_k}$.* Using the explicit description of $\widehat{\mathbb{A}^1_k}$ (as the space of closed balls) Hrushovski and Loeser explain how to build a deformation retraction from $\widehat{\mathbb{P}^1_k}$ to a prescribed "finite tree".

2) *The case of algebraic curves.* Let X be a projective curve over k, and let $f : X \to \mathbb{P}^1_k$ be a finite, flat morphism. Hrushovski and Loeser explain how to choose a finite tree in $\widehat{\mathbb{P}^1_k}$ so that the corresponding retraction (step 1) lifts to a retraction of $\widehat{X}$ to a twisted finite graph; one can also require it so satisfying some auxiliary conditions which will be useful in the proof. The key point is to control the behaviour of the cardinality of fibers of the fibers $\widehat{f}$ (as a function from $\widehat{\mathbb{P}^1_k}$ to $\mathbb{N}$). The definability of $\widehat{X}$ and $\widehat{\mathbb{P}^1_k}$ plays a crucial role for that purpose.

3) *The general case.* After having reduced to the case where X is of pure dimension n for some n, one proceeds by induction on n, the case $n = 0$ being obvious. One can assume (by enlarging it) that X is projective. The goal is to build a k-definable deformation retraction from the whole $\widehat{X}$ to a twisted polyhedron S which preserves the characteristic function of $\widehat{V}$; this will be sufficient, since the image of $\widehat{V}$ will be a k-definable subset of S, hence also a twisted polyhedron.

   3a) *The preparation.* One can blow up X *along a finite set of closed points* so that the resulting variety X' admits a morphism $X' \to \mathbb{P}^{n-1}_k$ whose generic fiber is sa curve; here we use the fact that X is projective. Now let E be the union of the exceptional divisors of $X' \to X$. If one builds a deformation retraction from $\widehat{X'}$ to a twisted polyhedron S which preserves the characteristic function of $\widehat{E}$, it will descend to $\widehat{X}$ because every connected component of $\widehat{E}$ collapses to a point. Hence we reduce to the case where X itself admits a morphism to $\mathbb{P}^{n-1}_k$ with the generic fiber of dimension 1.

   3b) *Mixing a homotopy on the base and a homotopy inside the fibers.* One applies the relative version of step 2) to a dense open subset of the fibration $X \to \mathbb{P}^{n-1}$ to build a suitable fiberwise homotopy, and the induction hypothesis which allows to build a deformation retraction from $\widehat{\mathbb{P}^{n-1}_k}$ to a twisted polyhedron satisfying some additional constraints ensuring that it can be composed in some sense with the latter fiberwise homotopy; we get that way a homotopy h_1 deforming $\widehat{X} \setminus \widehat{D}$ to a twisted polyhedron S_0, where D is a Zariski-open subset of some divisor on X.

   3c) *Fleeing away from $\widehat{D}$: the inflation homotopy.* One builds a "inflation homotopy" $h_{\text{inf}} : J \times \widehat{X} \to \widehat{X}$ such that $h_{\text{inf}}(t, x) \notin \widehat{D}$ as soon as t is not the origin of J. The concatenation of h_1 and h_{inf} defines now a deformation from the whole $\widehat{X}$ to a twisted polyhedron $S_1 \subset S_0$, the only remaining problem being that S_1 has no reason to be pointwise fixed at every time.

   3d) *Fixing this last problem: the polytopal homotopy.* Nevertheless, one can perform the preceding construction so that there exists a twisted polyhedron $S \subset S_1$ which is pointwise fixed at every time (simply because it is "too big to move", in some sense). The last step then consists in building a polytopal homotopy h_{pol} from S_1 to S; now the concatenation of h_{pol}, h_1 and h_{inf} has the required property.

3 An application of the definability of $\widehat{C}$ for C a curve

We again assume that the valuation of k takes real values, and that k is complete. For every n, denote by S_n the "skeleton" of $\mathbb{G}_m^{n,\mathrm{an}}$. This is the set of semi-norms of the form $\eta_{r_1,\ldots,r_n} := \sum a_I \underline{T}^I \mapsto \max |a_I| \underline{r}^I$. It is naturally homeomorphic to $(\mathbb{R}_+^\times)^n$, hence to $\mathbb{R}^n$ through a logarithm map. This provides it with a rational piecewise-linear structure.

Now let X be a k-analytic space of dimension n, and let $\varphi : X \to \mathbb{G}_m^{n,\mathrm{an}}$ be a morphism. In [8], the author has proven that $\varphi^{-1}(S_n)$ inherits a canonical rational piecewise-linear structure, with respect to which the restriction of φ is a piecewise immersion. This generalizes his preceding work [6], in which some additional assumptions on X were needed, and in which the canonicity of the PL structure (answering a question by Temkin) had not been addressed.

Moreover, the proof given in [6] used de Jong's alterations, while that given in [8] replaces it by the definability of $\widehat{C}$ for C an algebraic curve, which we have mentioned above.

Let us quickly explain where this definability is involved. To prove our result, we first algebraize the situation by standard arguments; that is, we reduce to the case where $X = \mathscr{X}^{\mathrm{an}}$ with $\mathscr{X}$ irreducible, and where φ is induced by a dominant, generically finite algebraic map $\mathscr{X} \to \mathbb{G}_m^n$. Now the key point is the following: *there exist finitely many rational functions on $\mathscr{X}$ whose norms separate the pre-images of x for every $x \in S_n$.*

In fact, we establish the more general, purely valuation-theoretic following theorem. Let k be an arbitrary valued field, let $n \in \mathbb{N}$ and let L be a finite extension of $k(T_1, \ldots, T_n)$. There exists a finite subset E of L such that the following hold: *for every ordered abelian group G containing $|k|$ and any n-uple $\underline{r}$ of elements of G, the elements of E separate the extensions of the valuation $\eta_{\underline{r}}$ of $k(T_1, \ldots T_n)$ to L.*

The proofs goes by induction on n. The crucial step is of course the one that consists in going from $n-1$ to n; it should be thought of as a finiteness result of *algebraic geometry of relative dimension* 1. And the crucial fact that allows to establish this finiteness result is precisely the definability of $\widehat{C}$ for C a curve; the link between our valuation-theoretic problem and the "hat" world is the fact that if F is any field belonging to M with $G \subset |F|$ then for every $r \in G$ one can see η_r as a point of $\widehat{\mathbb{A}_k^1}(F)$.

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