TROPICAL CURVES, GRAPH COMPLEXES, AND TOP WEIGHT COHOMOLOGY OF $\mathcal{M}_g$

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Abstract. We study the topology of a space $\Delta_g$ parametrizing stable tropical curves of genus $g$ with volume 1, showing that its reduced rational homology is canonically identified with both the top weight cohomology of $\mathcal{M}_g$ and also with the genus $g$ part of the homology of Kontsevich’s graph complex. Using a theorem of Willwacher relating this graph complex to the Grothendieck–Teichmüller Lie algebra, we deduce that $H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$ is nonzero for $g = 3, g = 5,$ and $g \geq 7,$ and in fact its dimension grows at least exponentially in $g$. This disproves a recent conjecture of Church, Farb, and Putman as well as an older, more general conjecture of Kontsevich. We also give an independent proof of another theorem of Willwacher, that homology of the graph complex vanishes in negative degrees.

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1. Introduction

Fix an integer $g \geq 2$. In this paper, we study the topology of a space $\Delta_g$ that parametrizes isomorphism classes of genus $g$ tropical curves of volume 1. Tropical curves are certain weighted, marked metric graphs; see §2.1 for the precise definition.

Interest in the space $\Delta_g$ is not limited to tropical geometry. Indeed, $\Delta_g$ may be identified homeomorphically with the following spaces:

(1) the link of the vertex in the tropical moduli space $M_g^{\text{trop}}$ [ACP15, BMV11];
(2) the dual complex of the boundary divisor in $\overline{\mathcal{M}}_g$, the algebraic moduli space of stable curves of genus $g$ (Corollary 6.7);
(3) the quotient of the simplicial completion of Culler–Vogtmann outer space by the action of the outer automorphism group $\text{Out}(F_g)$ [CV03 §5.2], [Vog15 §2.2];
(4) the topological quotient of Harvey’s complex of curves on a surface of genus \(g\) by
the action of the mapping class group \[\text{Har81}\]; and
(5) the topological quotient of Hatcher’s complex of sphere systems in certain 3-
manifolds \[\text{Hat95}\].

Our primary focus will be on the interpretations (1) and especially (2) from tropical and
algebraic geometry: we apply combinatorial topological calculations on \(\Delta_g\) to compute
previously unknown invariants of the complex algebraic moduli space \(M_g\). One such
application gives a lower bound on the size of \(H^{4g-6}(\mathcal{M}_g, \mathbb{Q})\), as follows.

**Theorem 1.1.** The cohomology \(H^{4g-6}(\mathcal{M}_g; \mathbb{Q})\) is nonzero for \(g = 3, 5, \) and \(g \geq 7\). Moreover, \(\dim H^{4g-6}(\mathcal{M}_g; \mathbb{Q})\) grows at least exponentially. More precisely,
\[
\dim H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) > \beta^g + \text{constant}
\]
for any \(\beta < \beta_0\), where \(\beta_0 \approx 1.3247 \ldots\) is the real root of \(t^3 - t - 1 = 0\).

The nonvanishing for \(g = 3\) was known previously; Looijenga famously showed that the
unstable part of \(H^6(\mathcal{M}_3; \mathbb{Q})\) has rank 1 and weight 12 \[\text{Loo93}\].

To put Theorem 1.1 in context, recall that the virtual cohomological dimension of \(M_g\) is \(4g - 5\) \[\text{Har86}\]. Church, Farb, and Putman conjectured that, for each fixed \(k > 0\),
\(H^{4g-4-k}(\mathcal{M}_g; \mathbb{Q})\) vanishes for all but finitely many \(g\) \[\text{CFP14, Conjecture 9}\]. While this is true for \(k = 1\) \[\text{CFP12, MSS13}\], Theorem 1.1 shows that it is false for \(k = 2\). Furthermore, as observed by Morita, Sakasai, and Suzuki \[\text{MSS15, Remark 7.5}\], the Church-Farb-
Putman conjecture is implied by a more general statement conjectured by Kontsevich two
decades earlier \[\text{Kon93, Conjecture 7C}\], which we now recall. In the same paper where
he introduced the graph complex, Kontsevich studied three infinite dimensional Lie alge-
bras, whose homologies are free graded commutative algebras generated by subspaces of
primitive elements. Each contains the primitive homology of the Lie algebra \(\text{sp}(2\infty)\) as a
direct summand. For one of these Lie algebras, denoted \(a_\infty\), the complementary primitive homology is
\[
PH_k(a_\infty)/PH_k(\text{sp}(2\infty)) \cong \bigoplus_{m>0,2g-2+m>0} H^{4g-4+2m-k}(\mathcal{M}_{g,m}/S_m; \mathbb{Q}),
\]
where \(S_m\) denotes the symmetric group acting on the moduli space \(\mathcal{M}_{g,m}\) of curves with
\(m\) marked points by permuting the markings. See \[\text{Kon93, Theorem 1.1(2)}\].

Kontsevich conjectured that the homology of each of these Lie algebras should be
finite dimensional in each degree. In particular, for each \(k\), the cohomology group
\(H^{4g-2-k}(\mathcal{M}_{g,1}; \mathbb{Q})\) should vanish for all but finitely many \(g\). Note that the composition
\[
H^*(\mathcal{M}_g; \mathbb{Q}) \rightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q}) \rightarrow H^{*+2}(\mathcal{M}_{g,1}; \mathbb{Q}),
\]
where the second map is cup product with the Euler class, is injective. This is because
further composing with Gysin pushforward to \(H^*(\mathcal{M}_g; \mathbb{Q})\) is multiplication by \(2 - 2g\).
Therefore, Theorem 1.1 shows that \(PH_2(a_\infty)\) is infinite dimensional, disproving Kontsevich’s conjecture and giving a negative answer to \[\text{MSS15, Problem 7.4}\].

Theorem 1.1 and further applications discussed in Section 7 will be established via
combinatorial topological calculations on the space \(\Delta_g\), which may be identified with the
dual complex of the Deligne-Mumford stable curve compactification $\overline{M}_g$ of $M_g$. Here and throughout the paper, we work with varieties and Deligne-Mumford stacks over $\mathbb{C}$. Recall that Deligne has defined a natural weight filtration on the rational singular cohomology of any complex algebraic variety which gives, together with the Hodge filtration on singular cohomology with complex coefficients, a mixed Hodge structure [Del71, Del74]. When the variety is the complement of a normal crossings divisor in a smooth and proper variety, one graded piece of this filtration can be calculated as the reduced homology of the dual complex of the divisor, a topological space that records the combinatorics of intersections and self-intersections of irreducible components of the divisor. We review the details of this construction in the slightly more general setting of Deligne-Mumford stacks in Section 6.

It is worth noting that we allow arbitrary normal crossings divisors here, not just simple normal crossings. This added generality allows us to consider the Deligne-Mumford stable curve compactification $\overline{M}_g$ of $M_g$. However, the combinatorial topology of the resulting dual complexes is also more general. While the dual complex of a simple normal crossings divisor is a $\Delta$-complex in the standard sense of [RS71] (and used now in textbooks such as [Hat02]), the analogous construction of the dual complex of a normal crossings divisor produces a symmetric $\Delta$-complex, as defined and studied in Section 3.

The graded pieces of the weight filtration on the cohomology of a $d$-dimensional variety are supported in degrees between 0 and $2d$, and we refer to the $2d$-graded piece, denoted $\text{Gr}_{2d}^W$, as the top weight cohomology. As a result of the general discussion described above, the interpretation of $\Delta_g$ as the dual complex of the boundary divisor in the Deligne–Mumford compactification of $M_g$ gives an identification of its reduced rational homology with the top graded piece of the weight filtration on the cohomology of $M_g$.

**Theorem 1.2.** There is an isomorphism

$$\text{Gr}_{6g-6}^W H^*_{6g-6-k}(M_g; \mathbb{Q}) \xrightarrow{\cong} \tilde{H}_{k-1}(\Delta_g; \mathbb{Q}),$$

identifying the reduced rational homology of $\Delta_g$ with the top graded piece of the weight filtration on the cohomology of $M_g$.

Our proof of Theorem 1.2 produces a specific isomorphism. After composing with the surjection from $H^*(M_g; \mathbb{Q})$ to its top weight quotient, this may be rewritten as a degree-preserving surjection in relative homology

$$H_*(\overline{M}_g, \partial \overline{M}_g; \mathbb{Q}) \rightarrow H_*(M_g^{\text{trop}}, \Delta_g; \mathbb{Q})$$

using Poincaré-Lefschetz duality in the domain and contractibility of $M_g^{\text{trop}}$ in the codomain. It can be seen (cf. Remark 6.9 below) that this surjection is in fact induced by a map of pairs of topological spaces.

We study $\Delta_g$ mainly from a combinatorial point of view. In Section 3, we develop some basic notions for a category of symmetric $\Delta$-complexes (§3.2). This is a modification of the usual category of $\Delta$-complexes in which simplices can be glued to each other, and to themselves, along maps that do not necessarily preserve the orderings of the vertices. The topological space $\Delta_g$ will be identified with the geometric realization of such a symmetric
\(\Delta\)-complex. In §3.3, we develop a theory of cellular chains and cochains for symmetric \(\Delta\)-complexes, whose rational homology and cohomology coincide with the rational singular homology and cohomology of the geometric realization. In the case of \(\Delta_g\) it gives a relatively small chain complex calculating its rational homology.

The cellular chain complex of \(\Delta_g\) is used to prove Theorem 1.3 below, which relates the homology of \(\Delta_g\) to the homology of the graph complex \(G(g)\) introduced by Kontsevich [Kon93, Kon94]. Recall that \(G(g)\) is a chain complex of rational vector spaces, with one generator \([\Gamma, \omega]\) for each pair \((\Gamma, \omega)\) of a connected abstract graph \(\Gamma\) of genus \(g\) without loops in which every vertex has valence at least 3, together with a total order \(\omega\) on its set of edges. These generators are subject to the relations \([\Gamma, \omega] = \text{sgn}(\sigma)[\Gamma', \omega']\) if there is an isomorphism of graphs \(\Gamma \sim \Gamma'\) under which the total orderings are related by the permutation \(\sigma\). In particular, \([\Gamma, \omega] = 0\) when \(\Gamma\) admits an automorphism inducing an odd permutation on its set of edges. A genus \(g\) graph \(\Gamma\) with \(v\) vertices and \(e\) edges is in homological degree \(v - (g + 1) = e - 2g\). (This convention agrees with [Wil15] but is shifted by \(g + 1\) compared to [Kon93].) The boundary \(\partial([\Gamma, \omega])\) is the alternating sum of the graphs obtained by collapsing a single edge of \(\Gamma\), where the sign in the alternating sum is according to the total ordering \(\omega\). (If \([\Gamma, \omega] \neq 0\) then \(\Gamma\) has no parallel edges, so collapsing an edge will not create any loops.) The graph complex \(G(g)\) has been studied intensively, including in the past few years. See, e.g., [CV03, CGV05, DRW15, Wil15].

We will show that the cellular chain complex computing the reduced rational homology of \(\Delta_g\) contains a degree-shifted copy of \(G(g)\) as a direct summand, and that the complementary summand is acyclic. Passing to homology gives the following:

**Theorem 1.3.** For \(g \geq 2\), there is an isomorphism

\[
H_k(G(g)) \cong H_{2g+k-1}(\Delta_g; \mathbb{Q}).
\]

Combining Theorems 1.2 and 1.3 then gives a surjection

\[
H^{4g-6-k}(\mathcal{M}_g, \mathbb{Q}) \twoheadrightarrow H_k(G(g)).
\]

In particular, nonvanishing graph homology groups yield nonvanishing results for cohomology of \(\mathcal{M}_g\).

The full structure of the homology of the graph complex remains mysterious, but several interesting substructures and many nontrivial classes are known and understood. In particular, the linear dual of \(\bigoplus_g H_0(G(g))\) carries a natural Lie bracket, and is isomorphic to the Grothendieck-Teichmüller Lie algebra \(\mathfrak{grt}_1\) by the main result of [Wil15]. The Lie algebra \(\mathfrak{grt}_1\) is known to contain a free Lie subalgebra with a generator in each odd degree \(g \geq 3\) ([Bro12]). These results let us deduce Theorem 1.1.

To the best of our knowledge, the only previously known nonvanishing top weight cohomology group on \(\mathcal{M}_g\) is \(\text{Gr}_{12}^W H^6(\mathcal{M}_3, \mathbb{Q})\), which has rank 1 by the work of Looijenga mentioned above [Loo93]. Once the general setup of the paper is in place, the result of Looijenga’s computation of this top weight cohomology group can be recovered immediately. It corresponds to the 1-dimensional subspace of graph homology spanned by the complete graph on four vertices. Note in general that the top weight cohomology of \(\mathcal{M}_g\) is non-tautological and unstable, since stable and tautological classes are of weight
equal to their cohomological degree. The method presented here probes one piece of the cohomology of $\mathcal{M}_g$ that is especially suited to combinatorial study.

The identification of top weight cohomology of $\mathcal{M}_g$ with graph homology, provided by Theorems 1.2 and Theorem 1.3, also yields interesting nonvanishing results in degrees other than $4g - 6$. For instance, the nontrivial class in $H_3(G^{(6)})$ discovered by Bar-Natan and McKay [BNM] proves nonvanishing of $H^{13}(\mathcal{M}_6; \mathbb{Q})$. This seems to be the second known example of a nonzero odd-degree cohomology group of $\mathcal{M}_g$: the first example of known odd-degree cohomology of $\mathcal{M}_g$ is $H^5(\mathcal{M}_4; \mathbb{Q})$ which has rank 1 (and weight 6) by [Tom05]. The interest and difficulty in exhibiting odd cohomology classes on $\mathcal{M}_g$ was highlighted by Harer and Zagier over three decades ago. They observed that no such classes were known at the time of their writing, and standard methods could produce classes only in even degree, while their Euler characteristic computations showed that such classes are abundant when $g \gg 0$ is even: $(-1)^{g+1} \chi(\mathcal{M}_g)$ grows like $g^{2g}$. See [HZ86, p. 458] and [Har88, p. 210].

Finally, we may also use the connection between cohomology of $\mathcal{M}_g$ and graph homology to give an application in the other direction, namely from $\mathcal{M}_g$ to graph complexes. Using Harer’s computation of the virtual cohomological dimension of $\mathcal{M}_g$ [Har86] and the vanishing of $H^{4g-5}(\mathcal{M}_g; \mathbb{Q})$ [CFP12, MSS13], we give an independent proof of the following recent result of Willwacher [Wil15, Theorem 1.1].

**Theorem 1.4.** The graph homology groups $H_k(G^{(g)})$ vanish for $k < 0$.

Relations between graph (co)homology and (co)homology of moduli spaces of curves were also considered by Kontsevich, but the relationships he studied are conceptually quite different. For example, he relates genus $g$ curves to genus $2g$ graph homology where we relate genus $g$ curves to genus $g$ graph homology. The three different Lie algebras mentioned above correspond to three different types of decorations on graphs, and each comes with a corresponding graph complex that computes homology (or cohomology) of an appropriate moduli space of decorated graphs. The Lie algebra $a_\infty$ corresponds to graphs decorated with ribbon structure, and moduli spaces of ribbon graphs are homotopy equivalent to moduli spaces of curves with marked points. This is related to the fact that a punctured Riemann surface deformation retracts to a graph, which remembers a ribbon structure from the deformation. The cohomology of $\mathcal{M}_g$ injects into the cohomology of $\mathcal{M}_{g,1}$, via pullback to the universal curve, and $\mathcal{M}_{g,1}$ is homotopy equivalent to a moduli space of ribbon graphs of first Betti number $2g$ that bound exactly 1 open disk. Forgetting the ribbon structure gives a proper map from this moduli space of ribbon graphs to a moduli space of undecorated graphs. The rational homology of the latter space is computed by the graph complex $G^{(2g)}$ [Kon93, Section 3].

Here, however, we relate the cohomology of $\mathcal{M}_g$ to the graph complex $G^{(g)}$, which computes the rational homology of a space of graphs of first Betti number $g$, not $2g$. The graphs appear not as deformation retracts of punctured curves, but rather as dual graphs of stable degenerations.

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2. Graphs, tropical curves, and moduli

In this section, we recall in more detail the construction of the topological space $\Delta_g$ as a moduli space for tropical curves, which are marked weighted graphs with a length assigned to each edge.

2.1. Weighted graphs and tropical curves. Let $G$ be a finite graph, possibly with loops and parallel edges. All graphs in this paper will be connected. Write $V(G)$ and $E(G)$ for the vertex set and edge set, respectively, of $G$. A weighted graph is a connected graph $G$ together with a function $w: V(G) \to \mathbb{Z}_{\geq 0}$, called the weight function. The genus of $(G,w)$ is

$$g(G,w) = b_1(G) + \sum_{v \in V(G)} w(v),$$

where $b_1(G) = |E(G)| - |V(G)| + 1$ is the first Betti number of $G$.

The valence of a vertex $v$ in a weighted graph, denoted val$(v)$, is the number of half-edges of $G$ incident to $v$. In other words, a loop edge based at $v$ counts twice towards val$(v)$, once for each end, and an ordinary edge counts once. We say that $(G,w)$ is stable if for every $v \in V(G)$,

$$2w(v) - 2 + \text{val}(v) > 0.$$

For $g \geq 2$, this is equivalent to the condition that every vertex of weight 0 has valence at least 3.

2.2. The category $J_g$. The connected stable graphs of genus $g$ form the objects of a category which we denote $J_g$. The morphisms in this category are compositions of contractions of edges $G \to G/e$ and isomorphisms $G \to G'$. For the sake of removing any ambiguity about what that might mean, let us give a formal and precise definition of $J_g$.

Formally, then, a graph $G$ is a finite set $X(G) = V(G) \sqcup H(G)$ (of “vertices” and “half-edges”), together with two functions $s_G, r_G : X(G) \to X(G)$ satisfying $s_G^2 = \text{id}$ and $r_G^2 = r_G$ and that

$$\{ x \in X(G) \mid r_G(x) = x \} = \{ x \in X(G) \mid s_G(x) = x \} = V(G).$$

Informally: $s_G$ sends a half-edge to its other half, while $r_G$ sends a half-edge to its incident vertex. We let $E(G) = H(G)/(x \sim s_G(x))$ be the set of edges. The definition of weights, genus, and stability is as before.

The objects of the category $J_g$ are all connected stable graphs of genus $g$. For an object $G = (G,w)$ we shall write $V(G)$ for $V(G)$ and similarly for $H(G)$, $E(G)$, $X(G)$, $s_G$ and $r_G$. Then a morphism $G \to G'$ is a function $f : X(G) \to X(G')$ with the property that

$$f \circ r_G = r_G' \circ f \quad \text{and} \quad f \circ s_G = s_G' \circ f,$$
and subject to the following three requirements:

- Each \( e \in H(G') \) determines the subset \( f^{-1}(e) \subset X(G) \) and we require that it consists of precisely one element (which will then automatically be in \( H(G) \)).
- Each \( v \in V(G') \) determines a subset \( S_v = f^{-1}(v) \subset X(G) \) and \( S_v = (S_v, r|S_v, s|S_v) \) is a graph; we require that it be connected and have \( g(S_v, w|S_v) = w(v) \).

Composition of morphisms \( G \to G' \to G'' \) in \( J_g \) is given by the corresponding composition \( X(G) \to X(G') \to X(G'') \) in the category of sets.

Our definition of graphs and the morphisms between them is standard in the study of moduli spaces of curves and agrees, in essence, with the definitions in [ACG11, X.2] and [ACP15, §3.2], as well as those in [KM94] and [GK98].

**Remark 2.1.** We also note that any morphism \( G \to G' \) can be alternatively described as an isomorphism following a finite sequence of *edge collapses*: if \( e \in E(G) \) there is a morphism \( G \to G/e \) where \( G/e \) is the marked weighted graph obtained from \( G \) by collapsing \( e \) together with its two endpoints to a single vertex \([e] \in G/e\). If \( e \) is not a loop, the weight of \([e]\) is the sum of the weights of the endpoints of \( e \) and if \( e \) is a loop the weight of \([e]\) is one more than the old weight of the end-point of \( e \). If \( S = \{e_1, \ldots, e_k\} \subset E(G) \) there are iterated edge collapses \( G \to G/e_1 \to (G/e_1)/e_2 \to \cdots \) and any morphism \( G \to G' \) can be written as such an iteration followed by an isomorphism from the resulting quotient of \( G \) to \( G' \).

We shall say that \( G \) and \( G' \) have the same *combinatorial type* if they are isomorphic in \( J_g \). In fact there are only finitely many isomorphism classes of objects in \( J_g \), since any object has at most \( 6g - 6 \) half-edges and \( 2g - 2 \) vertices; and for each possible set of vertices and half-edges there are finitely many ways of gluing them to a graph, and finitely many possibilities for the weight function. In order to get a small category \( J_g \) we shall tacitly pick one object in each isomorphism class and pass to the full subcategory on those objects. Hence \( J_g \) is a *skeletal* category. (Although we shall usually try to use language compatible with any choice of small equivalent subcategory \( J_g \).) It is clear that all \( \text{Hom} \) sets in \( J_g \) are finite, so \( J_g \) is in fact a finite category.

Replacing \( J_g \) by some choice of skeleton has the effect that if \( G \) is an object of \( J_g \) and \( e \in E(G) \) is an edge, then the marked weighted graph \( G/e \) is likely not equal to an object of \( J_g \). Given \( G \) and \( e \), there is a *morphism* \( q: G \to G' \) in \( J_g \) factoring through an isomorphism \( G/e \to G' \). The pair \((G', q)\) is unique up to unique isomorphism (but of course the map \( q \) or the isomorphism \( G/e \to G' \) on their own need not be unique). By an abuse of notation, we shall henceforward write \( G/e \in J_g \) for the codomain of this unique morphism, and similarly \( G/e \) for its underlying graph.

**Definition 2.2.** Let us define functors

\[
H, E: J_g^{\text{op}} \to (\text{Finite sets, injections})
\]

as follows. On objects, \( H(G) = H(G) \) is the set of half-edges of \( G = (G, w) \) as defined above. A morphism \( f: G \to G' \) determines an injective function \( H(f): H(G') \to H(G) \), sending \( e' \in H(G') \) to the unique element \( e \in H(G) \) with \( f(e) = e' \). We shall write \( f^{-1} = H(f): H(G') \to H(G) \) for this map. This clearly preserves composition and
identities, and hence defines a functor. Similarly for $E(G) = H(G)/(x \sim s_G(x))$ and $E(f)$.

2.3. Moduli space of tropical curves. We now recall the construction of moduli spaces of stable tropical curves, as the colimit of a diagram of cones parametrizing possible lengths of edges for each fixed combinatorial type. The construction follows [BMV11, Cap13].

Fix an integer $g \geq 2$. A length function on $G = (G, w) \in J_g$ is an element $\ell \in \mathbb{R}_{>0}^{E(G)}$, and we shall think geometrically of $\ell(e)$ as the length of the edge $e \in E(G)$. A genus $g$ stable tropical curve is then a pair $\Gamma = (G, \ell)$ with $G \in J_g$ and $\ell \in \mathbb{R}_{>0}^{E(G)}$, and we shall say that $(G, \ell)$ is isometric to $(G', \ell')$ if there exists an isomorphism $\phi: G \to G'$ in $J_g$ such that $\ell' = \ell \circ \phi^{-1}: E(G') \to \mathbb{R}_{>0}$. The volume of $(G, \ell)$ is $\sum_{e \in E(G)} \ell(e) \in \mathbb{R}_{>0}$.

We can now describe the underlying set of the topological space $\Delta_g$, which is the main object of study in this paper. It is the set of isometry classes of genus $g$ stable tropical curves of volume 1. We proceed to describe its topology and further structure as a closed subspace of the moduli space of tropical curves.

**Definition 2.3.** Fix $g \geq 2$. For each object $G \in J_g$ define the topological space

$$\sigma(G) = \mathbb{R}_{\geq 0}^{E(G)} = \{ \ell: E(G) \to \mathbb{R}_{\geq 0} \}.$$

For a morphism $f: G \to G'$ define the continuous map $\sigma f: \sigma(G') \to \sigma(G)$ by

$$(\sigma f)(\ell') = \ell: E(G) \to \mathbb{R}_{\geq 0},$$

where $\ell$ is given by

$$\ell(e) = \begin{cases} \ell'(e') & \text{if } f \text{ sends } e \text{ to } e' \in E(G'), \\ 0 & \text{if } f \text{ collapses } e \text{ to a vertex}. \end{cases}$$

This defines a functor $\sigma: J_g^{\text{op}} \to \text{Spaces}$ and the topological space $M_g^{\text{trop}}$ is defined to be the colimit of this functor.

In other words, the topological space $M_g^{\text{trop}}$ is obtained as follows. For each morphism $f: G \to G'$, consider the map $L_f: \sigma(G') \to \sigma(G)$ that sends $\ell': E(G') \to \mathbb{R}_{>0}$ to the length function $\ell: E(G) \to \mathbb{R}_{>0}$ obtained from $\ell'$ by extending it to be 0 on all edges of $G$ that are collapsed by $f$. So $L_f$ linearly identifies $\sigma(G')$ with some face of $\sigma(G)$, possibly $\sigma(G)$ itself. Then

$$M_g^{\text{trop}} = \left( \coprod \sigma(G) \right) \big/ \{ \ell' \sim L_f(\ell') \},$$

where the equivalence runs over all morphisms $f: G \to G'$ and all $\ell' \in \sigma(G')$.

As we shall explain in more detail in Section 3, $M_g^{\text{trop}}$ naturally comes with more structure than a plain topological space; it is an example of a generalized cone complex, as defined in [ACP15, §2]. This formalizes the observation that $M_g^{\text{trop}}$ is glued out of the cones $\sigma(G)$.

The volume defines a function $v: \sigma(G) \to \mathbb{R}_{\geq 0}$, given explicitly as $v(\ell) = \sum_{e \in E(G)} \ell(e)$, and for any morphism $G \to G'$ in $J_g$ the induced map $\sigma(G') \to \sigma(G)$ preserves volume. Hence there is an induced map $v: M_g^{\text{trop}} \to \mathbb{R}_{\geq 0}$, and there is a unique element in $M_g^{\text{trop}}$.
with volume 0 which we shall denote $\bullet_g$. The underlying graph of $\bullet_g$ consists of a single vertex with weight $g$.

**Definition 2.4.** We let $\Delta_g$ be the subspace of $M^\text{trop}_g$ parametrizing curves of volume 1, i.e., the inverse image of 1 $\in \mathbb{R}$ under $v : M^\text{trop}_g \to \mathbb{R}_{\geq 0}$.

Thus $\Delta_g$ is homeomorphic to the link of $M^\text{trop}_g$ at the cone point $\bullet_g$. Moreover, it inherits the structure of a symmetric $\Delta$-complex, as we shall define in Section 3 from the generalized cone complex structure on $M^\text{trop}_g$. See \S 4.2-4.3.

2.4. Kontsevich’s graph complex. Let us briefly recall the definition of the graph complex, first defined by Kontsevich in [Kon93]. This chain complex comes in two versions, differing by some important signs. Kontsevich’s original paper is mostly focused on what he calls the “even” version of the graph complex; it is related to invariants of odd-dimensional manifolds and by Willwacher’s results to deformations of the operad $e_n$ for odd $n$. This is the same version as considered by e.g. [CV03] and [CGV05]. The other version, called “odd” in [Kon93], is related to invariants of even-dimensional manifolds, deformations of the operad $e_n$ for even $n$, and by the main theorem of [Wil15] to the Grothendieck–Teichmüller Lie algebra. It is the latter version which is relevant to our paper and shall be recalled here. Both are considered in [BNM] where they are called the “fundamental example” and the “basic example” of graph homology, respectively (their assertion that the basic example does not occur in nature is of course no longer true).

As already recalled in the introduction, the graph complex is defined by letting $G^{(g)}$ be the rational vector space generated by $[\Gamma, \omega]$ where $\Gamma$ is a connected graph of genus $g$ (Euler characteristic $1 - g$) without loops, all of whose vertices have valence at least 3. The “orientation” $\omega$ is a total ordering on the set of edges (not half-edges) of $\Gamma$, and this notation is subject to the relation $[\Gamma, \omega] = \text{sgn}(\sigma)[\Gamma', \omega']$ if there exists an isomorphism of graphs $\Gamma \cong \Gamma'$ under which the total orderings are related by a permutation $\sigma$. It follows from this relation that $[\Gamma, \omega] = 0$ if $\Gamma$ has at least two parallel edges, since then there is an automorphism of $\Gamma$ inducing an odd permutation of its edge set. The boundary map in this chain complex is induced by

$$\partial[\Gamma, \omega] = \sum_{i=0}^{n} (-1)^i [\Gamma/e_i, \omega|_{E(\Gamma/e_i)}],$$

where $\omega = (e_0 < e_1 < \cdots < e_n)$ is the total ordering on the set $E(\Gamma)$ of edges of $\Gamma$, the graph $\Gamma/e_i$ is the result of collapsing $e_i \subset \Gamma$ to a point, and $\omega|_{E(\Gamma/e_i)}$ is the induced ordering on the subset $E(\Gamma/e_i) = E(\Gamma) \setminus \{e_i\} \subset E(\Gamma)$.

**Example 2.5.** For $g \geq 3$, let $W_g \in G^{(g)}$ be the “wheel graph” with $g$ trivalent vertices, one $g$-valent vertex, and $2g$ edges arranged in a wheel shape with $g$ spokes, and with some chosen ordering of its edge set. The graph underlying $W_5$ is depicted in Figure 1. Then $\partial W_g = 0$. This gives a non-zero cycle for odd $g$, which we also denote $W_g$.

Indeed, any contraction of a single edge $e$, spoke or non-spoke, leads to a graph $W_g/e$ with two parallel edges, which then represents the zero element in the graph complex.
The automorphism group of $W_g$ is isomorphic to $S_4$ when $g = 3$ and is isomorphic to the dihedral group $D_{2g}$ when $g > 3$, and it is easy to verify that it acts by even permutations on $E(W_g)$ when $g$ is odd. Hence $W_g \neq 0 \in G^{(g)}$ for odd $g$. (Notice that so far we are only making the elementary claim that it is non-zero on the chain level, although it in fact turns out to represent a non-zero homology class.) On the other hand, the involutions in the dihedral group act by odd permutations on $E(W_g)$ for even $g$, and hence $W_g = 0$ in this case.

Grading conventions differ from author to author. In Kontsevich’s original paper, the grading of this chain complex is by number of vertices $|V(\Gamma)|$. We shall instead use conventions better suited for comparison with [Wil15], in which the degree of $\Gamma$ is $|V(\Gamma)| - (g + 1)$. In this grading the wheel graph has degree 0. As we shall see later, it also has the effect of making $G^{(g)}$ a connective chain complex, i.e., its homology vanishes in negative degrees. Willwacher’s paper [Wil15] considers the linearly dual cochain complex which he denotes $GC$ or $GC_2$, so that

$$GC = \prod_{g=2}^{\infty} \text{Hom}(G^{(g)}, \mathbb{Q}),$$

where $\text{Hom}(\cdot, \mathbb{Q})$ denotes the graded dual. Elements in this cochain complex are functions assigning $f(\Gamma, \omega) \in \mathbb{Q}$, for example there is a cochain $W_g^\vee$ given by sending the wheel graph $W_g \mapsto \pm 1$ (depending on orientations) and any other graph to 0. In other words it is a dual basis element in the basis for $G^{(g)}$ given by graphs without automorphisms inducing odd permutations of their edge set. In Willwacher’s grading convention, the differential on $GC$ raises the degree by 1, the cohomological degree of (the dual basis element corresponding to) $[\Gamma, \omega]$ being $|V(\Gamma)| - (g + 1) = |E(\Gamma)| - 2g$.

The differential on $GC$ is then given by precomposing with (2.4.1). Because sign conventions are important, we shall pause here to double-check that the differential on $GC$ used by Willwacher is in fact obtained in this way. In [Wil15] the differential is inherited from one on a larger vector space called $fGC$, which has the structure of a differential graded Lie algebra by its definition as a deformation complex of a map of operads. As explained e.g. in [KWZ17] §2 or [DR12 Proposition 8.3], $fGC$ also has a graphical description, and its differential can be defined in terms of its Lie bracket, whose simple description in terms of graphs we now recall.

To define $fGC$, let us first write $fG^{(g)}$ for the chain complex that is defined analogously to $G^{(g)}$, except that we no longer require that the vertices in graphs have valence at least 3. As a vector space, $fG^{(g)}$ has a basis consisting of connected, loopless graphs $\Gamma$ of genus $g$ whose automorphism group induces only alternating permutations of the edge
set, one for each isomorphism class. Each such graph is equipped with an arbitrary fixed orientation \( \omega \), and has degree \(|V(\Gamma)| - (g + 1)\). Then we let

\[
fGC = \prod_{g \geq 2} \text{Hom}(fG^{(g)}, \mathbb{Q}),
\]

where \text{Hom} again denotes graded dual. Each \( fG^{(g)} \) is now an infinite-dimensional \( \mathbb{Q} \)-vector space when \( g > 2 \), but \( fGC \) arises as a completion of the bigraded vector space

\[
\bigoplus_{g \geq 2} \bigoplus_{k \geq -g} \text{Hom}(fG^{(g)}_{k}, \mathbb{Q}).
\]

where each summand is finite-dimensional, with \([\Gamma, \omega]^\vee\) having bidegree \((g, k = |V(\Gamma)| - (g + 1))\). Thus we may describe the Lie bracket on \( fGC \), which will be of bidegree \((0, 0)\), by defining it on dual basis elements \([\Gamma, \omega]\). Let \([\Gamma_1, \omega_1]\) and \([\Gamma_2, \omega_2]\) be connected, loopless, oriented graphs. Write \( \Gamma_i = [\Gamma_i, \omega_i] \) and \( \Gamma_i^\vee = [\Gamma_i, \omega_i]^\vee \). Define a pre-Lie algebra structure by letting \( \Gamma_1^\vee \cdot \Gamma_2^\vee \) be the sum of all (duals to) graphs obtained by inserting \( \Gamma_2 \) in place of one vertex \( v \) of \( \Gamma_1 \), summing over all ways of distributing the half-edges of \( \Gamma_1 \) at \( v \) over the vertices of \( \Gamma_2 \). The order of edges in each new graph is given by taking the edges of \( \Gamma_1 \) first, in order \( \omega_1 \), then the edges of \( \Gamma_2 \) in order \( \omega_2 \). Define the Lie bracket by

\[
[\Gamma_1^\vee, \Gamma_2^\vee] = \Gamma_1^\vee \cdot \Gamma_2^\vee - (-1)^{|\Gamma_1^\vee||\Gamma_2^\vee|}\Gamma_1^\vee \cdot \Gamma_2^\vee
\]

where \(|\Gamma_i^\vee| = |V(\Gamma_i)| - (g + 1)\) denotes the cohomological degree in \( fGC \). The Lie bracket \([\cdot, \cdot] \) on \( fGC \) restricts to \( GC \), since it does not produce vertices of lower valence. Finally, the differential on \( fGC \) is

\[
\partial \Gamma^\vee = \frac{1}{2} [\bullet \cdots \vee, \Gamma^\vee].
\]

Explicitly, for \([\Gamma, \omega] \in fG^{(g)}\), we have

\[
(\text{2.4.2}) \quad \partial [\Gamma, \omega]^\vee = (-1)^{|E(\Gamma)|} \sum_{v \in V(\Gamma)} \sum_{\{H, H'\}} [\Gamma_{H,H'}, \omega_{H,H'}]^\vee
\]

where \( \{H, H'\} \) runs over nontrivial unordered partitions of the set of half-edges at \( v \), and \( \Gamma_{H,H'} \) denotes the graph obtained by replacing \( v \) by two vertices connected by a new edge, reconnecting \( H \) to one vertex and \( H' \) to the other. The orientation \( \omega_{H,H'} \) is obtained from \( \omega \) by placing the new edge last.

As explained in ([Wil15, Proposition 3.4]), the differential on \( fGC \) also restricts to \( GC \); while it is possible that \( \Gamma_{H,H'} \) has a vertex of valence 2, each such graph arises in two ways with cancelling signs. Thus, if every vertex of \( \Gamma \) has valence at least 3, then in the formula \( (\text{2.4.2}) \) one may equivalently restrict to unordered partitions \( \{H, H'\} \) where \(|H|, |H'| \geq 2\).

Now a short sign computation shows that the differential in \( (\text{2.4.2}) \) is the dual to \( (\text{2.4.1}) \). The sign computation amounts to the following observation. Suppose \( \Gamma' \) is a graph with \(|E(\Gamma)| + 1 \) edges, and \( \omega' = e_0 < \cdots < e_{|E(\Gamma)|} \) is an ordering of \( E(\Gamma') \). Then for each \( i = 0, \ldots, |E(\Gamma)| \), changing \( \omega' \) by moving \( e_i \) to be last in order is achieved by multiplying \( \omega' \) by a cycle of length \(|E(\Gamma)| - i + 1\), which has sign \((-1)^{|E(\Gamma)| - i} \). Multiplying by this sign rectifies \( (\text{2.4.1}) \) and \( (\text{2.4.2}) \).
The main result of Willwacher’s paper gives an isomorphism between the Grothendieck–Teichmüller Lie algebra and graph cohomology in degree 0
\[ H^0(GC) \cong \mathfrak{grt}_1. \]
A connected genus \( g \) graph gives a degree-0 cochain if it has precisely \( 2g \) edges (and hence \( g + 1 \) vertices). Any element of \( H^0(GC) \) may be evaluated on the cycle \( W_g \). The dual element \( W_g^\vee \) has cohomological degree 0 in \( GC \), but is likely not a cocycle. By definition, the Lie algebra \( \mathfrak{grt}_1 \) consists of elements \( \phi \) of the completed free Lie algebra on two elements, satisfying certain explicit equations which we shall not recall (see [Wil15 §6.1]). An important consequence of this isomorphism is the following.

**Theorem 2.6 ([Wil15]).** For any odd \( g \geq 3 \) there exists an element \( \sigma_g \in H^0(GC) \) with \( \langle \sigma_g, W_g \rangle \neq 0. \) Hence \( [W_g] \neq 0 \in H_0(G^{(g)}) \), i.e., the wheel cycle \( W_g \) is not a boundary.

**Proof.** Starting from a suitable Drinfeld associator, Willwacher in [Wil15, Section 9] translates the corresponding element \( \sigma_g \in \mathfrak{grt}_1 \) into \( GC \) and proves that the resulting cocycle in \( GC \) has non-zero coefficient of \( W_g^\vee \) and hence pairs non-trivially with the chain \( W_g \). See also [RW14] for a more direct construction of cocycles representing \( \sigma_g \). \( \square \)

**Theorem 2.7.** The cohomology \( H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) \) is nonzero for \( g = 3 \), \( g = 5 \), and \( g \geq 7 \). Moreover, \( \dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) \) grows at least exponentially. More precisely,
\[ \dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) > \beta^g + \text{constant} \]
for any \( \beta < \beta_0 \), where \( \beta_0 \approx 1.3247 \ldots \) is the real root of \( t^3 - t - 1 = 0 \).

**Proof.** Let \( V \) denote the graded \( \mathbb{Q} \)-vector space generated by symbols \( \sigma_{2i+1} \) in degree \( 2i+1 \) for each \( i \geq 1 \), and let \( \text{Lie}(V) \) be the free Lie algebra on \( V \). As explained in [Wil15], the result of [Bro12] implies that the classes \( \sigma_{2i+1} \in \mathfrak{grt}_1 \) together with the Lie algebra structure on \( \mathfrak{grt}_1 \) gives rise to an injection
\[ \text{Lie}(V) \hookrightarrow \mathfrak{grt}_1 \cong H^0(GC) \cong \left( \bigoplus_{g \geq 2} H_0(G^{(g)}) \right)^\vee. \]
Thus \( \sigma_3, \sigma_5 \neq 0 \in \mathfrak{grt}_1 \), and since any even number \( g \geq 8 \) may be written as \( 3 + (g-3) \) with \( g - 3 > 3 \), we also have \( [\sigma_3, \sigma_{g-3}] \neq 0 \in \mathfrak{grt}_1 \), which gives rise to a non-zero homomorphism \( H_0(G^{(g)}) \to \mathbb{Q} \). More specifically, for \( g \geq 8 \) even, \( H_0(G^{(g)}) \) contains a non-zero homology class whose Lie cobracket contains a term \( W_3 \otimes W_{g-3} \).

For the asymptotic statement, we shall compute the Poincaré series (i.e., the generating function for dimension of graded pieces) of \( \text{Lie}(V) \), using a variant of Witt’s formula for the dimension of the graded pieces of a free, finitely generated Lie algebra that is generated in degree 1, and then appeal to (2.4.3). The Poincaré series of \( V \) is \( f(t) = t^3/(1 - t^2) \). The universal enveloping algebra \( U(\text{Lie}(V)) \) is isomorphic to the free associative algebra \( \bigoplus_{n \geq 0} V^\otimes n \), so has Poincaré series \( 1/(1 - f) \). Now let \( S(\text{Lie}(V)) \) denote the free commutative \( \mathbb{Q} \)-algebra on \( \text{Lie}(V) \); it has Poincaré series
\[ \prod_{d \geq 0} \frac{1}{(1 - t^d)^{A_d}}, \]
Therefore \( p \) is the sought-after coefficients of the Poincaré series for \( \text{Lie}(V) \). The Poincaré-Birkhoff-Witt theorem implies that \( U(\text{Lie}(V)) \cong S(\text{Lie}(V)) \) as graded vector spaces, so \( 1/(1 - f) = \prod_{n \geq 0} 1/(1 - t^n)^{A_n} \). Applying \( t^n \frac{d}{dt} \log(\cdot) \) to both sides yields

\[
(2.4.4) \quad p(t) := \frac{t^3(3 - t^2)}{(1 - t^2)(1 - t^2 - t^3)} = \sum_{d \geq 0} dA_d \frac{t^d}{1 - t^d}.
\]

Write \( p(t) = \sum_{n \geq 0} a_n t^n \). To analyze the \( a_n \), notice that \( p(t) \) has five simple poles, at the roots of \( (1 - t^2)(1 - t^2 - t^3) = 0 \). There is a unique root \( \alpha \approx 0.75488 \ldots \) having smallest magnitude, and \( \text{Res}_\alpha p(t) = -\alpha \) (the exact value of the residue is not important). Therefore \( p(t) = -\alpha/(t - \alpha) + \sum_{n \geq 0} b_n t^n = \sum_{n \geq 0} (\frac{1}{\alpha^n} + b_n) t^n \), where \( \sum_{n \geq 0} b_n t^n \) converges on a disc centered at 0 of radius \( \alpha \). Therefore \( b_n \alpha^n \to 0 \) and \( a_n \alpha^n = (\frac{1}{\alpha^n} + b_n) \alpha^n \to 1 \). Setting \( \beta_0 = 1/\alpha \approx 1.32472 \ldots \), then \( a_n \to \beta_0^n \).

Now equating coefficients in (2.4.4) yields \( a_n = \sum_{d|n} dA_d \), so

\[
A_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d
\]

by Möbius inversion. Since \( a_n \) grows exponentially, the summand when \( d = n \), namely \( a_n/n \), eventually dominates the other terms in the sum, and \( A_n \) grows faster than \( \beta_0^n \) for any \( \beta < \beta_0 \). \( \square \)

In comparison, asymptotic size of the tautological ring of \( \mathcal{M}_g \) is bounded above by \( C^{\sqrt{g}} \) for a constant \( C \). Indeed, its Poincaré series is dominated coefficient-wise by that of the polynomial ring

\[
\mathbb{Q}[\kappa_1, \kappa_2, \ldots], \quad \deg \kappa_i = 2i.
\]

where \( \kappa_i \) has degree \( 2i \), and \( \mathcal{M}_g \) has virtual cohomological dimension \( 4g - 5 \) \cite{Har86}. A rough bound may be obtained by calculating \( \dim \mathbb{Q}[\kappa_1, \kappa_2, \ldots]_{2n} = p(n) \) where \( p(n) \) is the number of partitions of \( n \), which is well-known to grow as \( A \cdot B^{\sqrt{n}}/n \) for constants \( A \) and \( B \). Therefore the dimension of the tautological ring is bounded by \( \sum_{n=1}^{2g-3} p(2n) < 2g \cdot p(2g) \sim C \cdot D^{\sqrt{g}} \) for constants \( C \) and \( D \).

On the other hand, the Euler characteristic estimates by Harer–Zagier mentioned earlier imply that the size of the top weight part of \( H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) \) as \( g \to \infty \) is still negligible in comparison to the entire \( H^*(\mathcal{M}_g; \mathbb{Q}) \) (and hence in comparison to the largest single Hodge number of \( \mathcal{M}_g \)).

3. Symmetric semi-simplicial objects

**Definition 3.1.** For \( p \geq -1 \) an integer, we set

\[
[p] = \{0, \ldots, p\}.
\]

This notation includes \([-1] = \emptyset \) by convention.

Recall that \( \Delta^p \subset \mathbb{R}^{p+1} \) is the convex hull of the standard basis vectors \( e_0, \ldots, e_p \); its points are \( t = (t_0, \ldots, t_p) = \sum t_i e_i \) with \( t_i \geq 0 \) and \( \sum t_i = 1 \). Associating the standard simplex \( \Delta^p \) to the number \( p \) may be promoted to a functor from finite sets to topological
spaces; for a finite set $S$ define $\Delta^S = \{ a: S \to [0, \infty) \mid \sum a(s) = 1 \}$ in the Euclidean topology and for any map of finite sets $\theta: S \to T$, define $\theta_*: \Delta^S \to \Delta^T$ by
\[
(\theta_*a)(t) = \sum_{\theta(s) = t} a(s).
\]
The usual $p$-simplex is recovered as $\Delta^p = \Delta^{|[p]|}$ with $[p] = \{0, \ldots, p\}$.

3.1. Recollections on $\Delta$-complexes. Let us write $e_i \in \Delta^p$ for the $i$th vertex, $0 \leq i \leq p$. We order the set of vertices in $\Delta^p$ as $e_0 < \cdots < e_p$. Recall that a $\Delta$-complex $X$ is a topological space obtained by gluing simplices $\Delta^p$ together along injective face maps $\Delta^q \to \Delta^p$, where the gluing maps are affine and induce an order preserving injective map on vertices.

An equivalent, but more combinatorial definition instead encodes the set of $p$-simplices for all $p$, together with the gluing data of which $(p-1)$ simplex is glued to each face of each $p$-simplex. Let $\Delta_{\text{inj}}$ be the category with one object $[p] = \{0, \ldots, p\}$ for each integer $p \geq 0$, in which the morphisms $[p] \to [q]$ are the order preserving injective maps.

We shall take the following as the official definition of a $\Delta$-complex (sometimes known as “semi-simplicial set” in the more recent literature, especially when emphasizing this functorial point of view).

**Definition 3.2.** A $\Delta$-complex is a functor $X: \Delta_{\text{inj}}^{\text{op}} \to \text{Sets}$.

Translating from the combinatorial/functorial description to the geometric one uses the geometric realization
\[
|X| = \left( \prod_{p=0}^{\infty} X([p]) \times \Delta^p \right) / \sim,
\]
where $\sim$ is the equivalence relation generated by $(x, \theta, a) \sim (\theta^* x, a)$ for $x \in X([q]), \theta: [p] \to [q]$ in $\Delta_{\text{inj}}$, and $a \in \Delta^p$. Each element $x \in X([p])$ determines a map of topological spaces $x: \Delta^p \to |X|$, and the functor $X: \Delta_{\text{inj}}^{\text{op}} \to \text{Sets}$ may be recovered from the topological space $|X|$ together with this set of maps from simplices. Many textbook sources (e.g., [Hat02]), take the official definition of $\Delta$-complex to be a topological space equipped with a set of maps from simplices satisfying certain axioms. The intuition is that $X([p])$ has one element for each $p$-simplex and the functoriality determines how the simplices are glued together. In any case, these two different approaches produce equivalent categories. The combinatorial/functorial terminology is less tied to the category of sets, and one speaks also about semi-simplicial groups, semi-simplicial spaces, etc; these are functors from $\Delta_{\text{inj}}^{\text{op}}$ into the appropriate category.

As is customary, we shall usually write $X_p = X([p])$. Let us also write $\delta^i: [p-1] \to [p]$ for the unique order preserving injective map whose image does not contain $i$, and $d_i: X_p \to X_{p-1}$ for the induced map. The $\Delta$-complex $X$ is then determined by the sets $X_p$ for $p \geq 0$ and the maps $d_i: X_p \to X_{p-1}$ for $i = 0, \ldots, p$. These satisfy $d_id_j = d_{j-1}d_i$ for $i < j$, and any sequence of sets $X_p$ and maps $d_i$ satisfying this axiom uniquely specifies a $\Delta$-complex.
An augmented $\Delta$-complex is a functor $(\Delta_{\text{inj}} \cup \{-1\})^{\text{op}} \to \text{Sets}$, where $[-1] = \emptyset$ is added to $\Delta_{\text{inj}}$ as initial object. The geometric realization $|X|$ then comes with a continuous map $\epsilon: |X| \to X_{-1}$. We shall usually identify the category of (non-augmented) $\Delta$-complexes with a full subcategory of the augmented ones, by setting $X_{-1}$ to be a singleton.

3.2. Symmetric $\Delta$-complexes. We now generalize the notion of $\Delta$-complexes to allow gluing also along maps $\Delta^p \to \Delta^p$ that do not preserve the ordering of the vertices. This includes gluing along maps from $\Delta^p$ to itself induced by permuting the vertices. We begin with a combinatorial description.

**Definition 3.3.** Let $I$ be the category with the same objects as $\Delta_{\text{inj}} \cup \{-1\}$, but whose morphisms $[p] \to [q]$ are all injective maps $\{0, \ldots, p\} \to \{0, \ldots, q\}$. A symmetric $\Delta$-complex (or symmetric semi-simplicial set) is a functor $X: I^{\text{op}} \to \text{Sets}$.

Such a functor is given by a set $X_p$ for each $p \geq -1$, actions of the symmetric group $S_{p+1}$ on $X_p$ for all $p$, and face maps $d_i: X_p \to X_{p-1}$ for $0 \leq i \leq p$. The face maps satisfy the usual simplicial identities as well as a compatibility with the symmetric group action. We have chosen the name in analogy with the “symmetric simplicial sets” in the literature (e.g., [Gra01]), which is a similar notion also including degeneracy maps. The geometric realization of $X$ is given by formula (3.1.1), where the equivalence relation now uses all morphisms $\theta$ in $I$.

Symmetric $\Delta$-complexes also come with a set $X_{-1} = X(\emptyset)$ and there is an augmentation map $|X| \to X_{-1}$. (So strictly speaking “augmented symmetric $\Delta$-complexes” would be a more accurate name, but we use “symmetric $\Delta$-complexes” for brevity. We again identify the non-augmented version with the full subcategory in which $X_{-1}$ is a singleton.)

The standard orthant $\mathbb{R}_{\geq 0}^p = \prod_{i=0}^p [0, \infty)$ is functorial in $[p] \in I$ by letting $\theta \in I([p],[q])$ act as $\theta_*(t_0, \ldots, t_p) = \sum t_i e_{\theta(i)}$, where $e_i \in \mathbb{R}^p$ denotes the $i$th standard basis vector. Replacing $\Delta^p$ by the standard orthant in the definition of $|X|$ we arrive at the cone over $X$:

$$CX = \left( \bigoplus_{p=-1}^{\infty} X_p \times \mathbb{R}_{\geq 0}^p \right)/\sim,$$

where $\sim$ is the equivalence relation generated by $(x, \theta_* a) \sim (\theta^* x, a)$ for $p, q \geq -1, x \in X_q, a \in \mathbb{R}_{\geq 0}^p$, and $\theta \in I([p],[q])$. The maps $\ell: \mathbb{R}_{\geq 0}^p \to \mathbb{R}$ given by $(t_0, \ldots, t_p) \mapsto \sum t_i$ are compatible with this gluing, and induce a canonical map

$$\ell_X: CX \to \mathbb{R}_{\geq 0},$$

so that $X \mapsto CX$ naturally takes values in the category of spaces over $\mathbb{R}_{\geq 0}$. We have canonical homeomorphisms $\ell_X^{-1}(1) = |X|$ and $\ell_X^{-1}(0) = X_{-1}$, and from $\ell_X^{-1}([0,1])$ to the mapping cone of the augmentation $|X| \to X_{-1}$. The inclusions $X_{-1} \subset \ell_X^{-1}([0,1]) \subset CX$ are both deformation retractions. It follows that the quotient $CX/|X|$ deformation retracts to the mapping cone of $|X| \to X_{-1}$. When $X_{-1} = \{\ast\}$ the space $CX$ deformation retracts to the cone over $|X|$ (hence the name) and $CX/|X|$ deformation retracts to the unreduced suspension $S|X|$.
Example 3.4. The representable functor $I(-,[p]): I^{op} \to \text{Sets}$ has geometric realization $|I(-,[p])| \cong \Delta^p$.

Example 3.5. An (abstract) simplicial complex $K$ with vertex set $V$ determines a symmetric $\Delta$-complex $X_K: I^{op} \to \text{Sets}$, sending $[p]$ to the set of injective maps $f: [p] \to V$ whose image spans a simplex of $K$. The realizations of $K$ as a simplicial complex and $X_K$ as a symmetric $\Delta$-complex are canonically homeomorphic.

Example 3.6. A typical example of a symmetric $\Delta$-complex in which the symmetric groups do not act freely is the half interval given as a coequalizer of the two distinct morphisms $X = \text{colim}(I(-,[1]) \rightrightarrows I(-,[1]))$, where $X_{-1}$, $X_0$, and $X_1$ are one-element sets and $X_p = \emptyset$ for $p \geq 2$. The unique element in $X_1$ gives a map $\Delta^1 \to |X|$ which is not injective; it identifies $|X|$ with the topological quotient of $\Delta^1$ by the action of $\mathbb{Z}/2\mathbb{Z}$ that reverses the orientation of the interval.

3.3. Cellular chains. We introduce a chain complex calculating the rational singular homology of $|X|$ and the relative homology of $(CX, |X|)$.

Definition 3.7. Let $R$ be a commutative ring and write $RX_p$ for the free $R$-module spanned by the set $X_p$. The group of cellular $p$-chains $C_p(X;R)$ is

$$C_p(X;R) = (R^\text{sgn} \otimes_{R_{S^{p+1}}} RX_p)$$

where $R^\text{sgn}$ denotes the action of $S_{p+1}$ on $R$ via the sign.

The boundary map $\partial: C_p(X) \to C_{p-1}(X)$ is the unique map that makes the following diagram commute:

$$\begin{array}{ccc}
RX_p & \xrightarrow{\sum(-1)^i(d_i)_*} & RX_{p-1} \\
\downarrow & & \downarrow \\
C_p(X;R) & \xrightarrow{\partial} & C_{p-1}(X;R).
\end{array}$$

To see that such a map exists, let us write $\tau_j: X_p \to X_p$ for the map induced by the bijection $(j, j-1)$ and calculate

$$d_i \circ \tau_j = \begin{cases} 
\tau_j \circ d_i & \text{for } i > j, \\
d_{i-1} & \text{for } i = j, \\
d_{i+1} & \text{for } i = j-1, \\
\tau_{j-1} \circ d_i & \text{for } i < j-1.
\end{cases}$$

Therefore, $\pi \circ (\sum(-1)^i(d_i)_*) \circ \tau_j = -\pi \circ (\sum(-1)^i(d_i)_*)$, as required.

Lemma 3.8. The homomorphism defined by this formula satisfies $\partial^2 = 0$. □

Similarly, we define cochains

$$C^p(X;R) = \text{Hom}_Z(C_p(X;Z), R) = \text{Hom}_{R_{S^{p+1}}}(RX_p, R^\text{sgn}),$$
with coboundary $\delta = (-1)^{p+1} \partial^V : C^p(X; R) \to C^{p+1}(X; R)$. In other words, $C^p(X; R)$ is the $R$-module consisting of all set maps $\phi : X_p \to R$ which satisfy $\phi(\sigma x) = \text{sgn}(\sigma) \phi(x)$ for all $x \in X_p$ and all $\sigma \in S_{p+1}$.

To compare our cellular theory to singular homology, write $t_p \in C_p^{\text{sing}}(\Delta^p)$ for the chain given by the identity map of $\Delta^p$, and $t'_p \in C_p^{\text{sing}}(\Delta^p)$ for its barycentric subdivision (in the sense of e.g. [Hat02 p. 122] or [Bres83 §IV.17]). This chain represents a generator of $H_p^{\text{sing}}(\Delta^p, \partial \Delta^p)$, satisfies $\sigma_x(t'_p) = \text{sgn}(\sigma)t'_p$ on the chain level, and $\partial t'_p = \sum(-1)^i (d_i)_{*}(t'_{p-1})$. Now, any element $x \in X_p$ gives a map $x : \Delta^p \to |X|$, and we define a natural transformation

$$C^*_p(X; Z) \to C^*_p(|X|; Z)$$

$$x \mapsto x_*(t'_p),$$

(3.3.1)

The properties of $t'_p$ ensure that this is well defined for each $p \geq 0$, and that it defines a chain homomorphism $C_*(X; Z)_{\geq 0} \to C_*(|X|; Z)$ where we write $C_*(X; Z)_{\geq 0}$ for the quotient by $ZX_{-1}$. Define natural transformations of homology and cohomology with coefficients in $R$ by applying $R \otimes_Z (-)$ or $\text{Hom}_Z(-, R)$ to (3.3.1).

**Proposition 3.9.** The homomorphisms

$$H_p(C_*(X; R)_{\geq 0}, \partial) \to H^p(\Delta^p; |X|; R)$$

$$H^p(C^*(X; R)_{\geq 0}, \delta) \leftarrow H^p_{\text{sing}}(\Delta^p; |X|; R)$$

induced by the (co)chain homomorphisms defined above are isomorphisms, provided the orders of stabilizers of $S_{p+1}$ on $X_p$ are invertible in $R$ (e.g., if the actions are all free or if $\mathbb{Q} \subset R$). Under this assumption, there are induced isomorphisms

$$H_p(C_*(X; R), \partial) \cong H^p_{\text{sing}}(CX, |X|; R)$$

$$H^p(C^*(X; R), \delta) \cong H^{p+1}_{\text{sing}}(CX, |X|; R).$$

When $X_{-1}$ is a singleton the right hand sides here are reduced homology and cohomology $\tilde{H}^p_{\text{sing}}(|X|; S_{p+1})$ and $\tilde{H}^p_{\text{sing}}(|X|; R)$.

**Proof.** The symmetric $\Delta$-complex $X$ is filtered by subcomplexes $X^{(p)} \subset X$ defined by setting $X_q^{(p)} = X_q$ for $q \leq p$ and $X_q^{(p)} = \emptyset$ for $q > p$. The quotient space $|X^{(p)}|/|X^{(p-1)}|$ may be identified with the orbit space

$$|X^{(p)}|/|X^{(p-1)}| \cong \left( \frac{X_p \times \Delta^p}{X_p \times \partial \Delta^p} \right)/S_{p+1},$$

and the induced map

$$R^\text{sgn} \otimes_{R S_{p+1}} RX \to H^p_{\text{sing}}((X_p \times \Delta^p)/S_{p+1}, (X_p \times \partial \Delta^p)/S_{p+1}; R)$$

is an isomorphism under the assumption. Now proceed by induction on skeleta, using the five-lemma and the long exact sequences associated to the pairs $(X^{(p)}, X^{(p-1)})$, exactly as in the proof of [Hat02 Theorem 2.2.27].

For the augmented statement use $|X| \to X_{-1}$ to add one more term to the singular chain complex

$$\cdots \to C^1_{\text{sing}}(|X|; R) \to C^0_{\text{sing}}(|X|; R) \to RX_{-1} \to 0$$
This complex calculates $H^\text{sing}_{n+1}(CX, |X|; R)$, since the inclusion $X_{-1} \subset CX$ is a deformation retraction. The claim is now easily deduced from the absolute case, and cohomology is similar. \hfill \Box

Henceforth we shall use the same notation $H_*(-; R)$ and $H^*(-; R)$ for the singular and cellular theories.

**Definition 3.10.** For a symmetric $\Delta$-complex $X$ define

\[
H_p(X; R) = H_p(C_*(X; R), \partial)
\]

\[
H^p(X; R) = H^p(C^*(X; R), \delta).
\]

When $X_{-1}$ is a singleton these agree with $\tilde{H}_p(|X|; R)$ and $\tilde{H}^p(|X|; R)$ respectively, provided orders of stabilizers of $S_{p+1}$ on $X_p$ are invertible in $R$.

### 3.4. Symmetric semi-simplicial spaces.

**Definition 3.11.** A symmetric semi-simplicial space is a functor $X : I^\text{op} \to \text{Top}$. The cone $CX$ and the map $\ell_X : CX \to \mathbb{R}_{\geq 0}$ is defined by the same formula (3.2.1) as above, giving each $X_p \times \mathbb{R}_{\geq 0}^{[p]}$ the product topology. The geometric realization $|X|$ is then defined as $\ell_X^{-1}(1)$, or equivalently as a quotient of $\bigsqcup X_p \times \Delta^p$.

In the following example, we will be especially interested in the case when $B$ is a smooth projective complex variety (or DM stack) and $A \to B$ is the normalization of a normal crossings divisor.

**Example 3.12.** Let $f : A \to B$ be any continuous map of spaces. For an object $[p] \in I$ we may consider the subspace

\[
A_p = \{((a_0, \ldots, a_p), b) \in A^{p+1} \times B \mid f(a_0) = \cdots = f(a_p) = b, a_i \neq a_j \text{ for } i \neq j\}.
\]

Permuting and forgetting the $a_i$ coordinates makes this into a functor $A_\bullet : [p] \mapsto A_p$ from $I^\text{op}$ to $\text{Top}$.

Obviously $A_p$ is an open subspace of the $(p+1)$-fold fiber product $A \times_B \cdots \times_B A$, and if $f$ is locally injective and $A$ and $B$ are Hausdorff it is also a closed subspace. Let us assume this is the case. The augmentation gives a map $|A_\bullet| \to B$ whose image equals the image of $f$. The resulting map

\[
|A_\bullet| \to f(A)
\]

is proper, and is a weak equivalence under fairly mild assumptions on $f$, as we now explain. The inverse image in $|A_\bullet|$ of any point $x \in f(A)$ is a simplex with vertex set $f^{-1}(x)$ and hence contractible. If $B$ is a locally compact metric space, then one may show that $|A_\bullet|$ is metrizable, and it follows that $|A_\bullet| \to f(A)$ is a weak equivalence by Smale’s homotopical version of the Vietoris-Begle theorem (\cite{Sma57}). Since $CA_\bullet \simeq A_{-1} = B$ we then get an induced isomorphism on homology

\[
H_k(CA_\bullet, |A_\bullet|) \xrightarrow{\cong} H_k(B, f(A)).
\]

Furthermore, in this situation we may define a symmetric $\Delta$-complex as $X_p = \pi_0(A_p)$. The resulting map $A_p \to X_p$ of symmetric semi-simplicial spaces induces a map in homology
which combined with the isomorphism above may be viewed as a homomorphism

\[(3.4.1) \quad H_k(B, f(A)) \to H_k(CX, |X|).\]

In the applications we have in mind, \(f: A \to B\) will be a map of orbifolds, and it is better to let \(A_p\) be the coarse space of the orbifold fiber products \(A \times_B \cdots \times_B A\). (If we don’t pass to coarse space we should work with pseudofunctors from \(I^{op}\) to the 2-category of orbifolds.) The realization \(|A_*|\) then maps to the image of the coarse space of \(A\) in that of \(B\). The fibers of this map over a point coming from \(x \in B\) will be the quotient of the simplex with vertex set \(f^{-1}(x)\) by the action of the isotropy group of \(x\). In particular it is still contractible, so the same arguments apply.

In the main case of interest, where \(f: A \to B\) is the normalization of a normal crossings divisor in a complex projective variety (or DM stack), the symmetric \(\Delta\)-complex \(X\) in the above example will be our definition of the boundary complex. The pair \((CA_*, |A_*|)\) is a homotopical model for \((B, f(A))\) and comes with a map to \((CX, |X|)\), cf. Remark 6.9. See also Section 8 for another example of a symmetric semi-simplicial space of interest for translating from graph cohomology classes to classes in \(H_*(M_g)\).

4. Symmetric \(\Delta\)-complexes

We return to the case of symmetric semi-simplicial sets, in slightly more detail. We have described a chain complex for calculating the homology of \(|X|\) for a symmetric \(\Delta\)-complex \(X\). In practice it will be studied using the following observation.

**Lemma 4.1.** Let \(X\) be a symmetric \(\Delta\)-complex, and let \(R\) be a commutative ring. Suppose we are given subsets \(T_p \subset X_p\) for some \(p\), and suppose that either

- the induced map \(S_{p+1} \times T_p \to X_p\) is a bijection for all \(p\), or
- \(Q \subset R\), the composition \(T_p \to X_p \to X_p/S_{p+1}\) is injective, the stabilizer of any \(x \in T_p\) is contained in \(A_{p+1}\), and any point \(x \in X_p\) whose stabilizer is contained in \(A_{p+1}\) is in the \(S_{p+1}\)-orbit of some \(x' \in T_p\).

Then the map

\[RT_p \to C_p(X; R)\]

is an isomorphism of \(R\)-modules. \(\square\)

This means that \(H_*(|X|; \mathbb{Q})\) may often be calculated by a rather small chain complex: it has one generator for each element in a set of representatives for orbits of elements with alternating stabilizers.

**Remark 4.2.** A similar construction was used in [HV98] to find a small model for the rational chains of a certain space, except that instead of our \(\Delta^p/H\) for \(H < S_{p+1}\) their basic building blocks are of the form \([0, 1]^n/H\) for certain subgroups \(H\) of the symmetry group of a cube. There is also a general construction used by [Ber99], in which simplices are replaced by polysimplicial sets.
4.1. Colimit presentations and subdivision. A particular role is played by the representable functors \( I(-, [p]): I^{\text{op}} \rightarrow \text{Sets} \). As we have already seen, the geometric realization of \( I(-, [p]) \) is canonically homeomorphic to the simplex \( \Delta^p \). Moreover, an arbitrary symmetric \( \Delta \)-complex is isomorphic to the colimit of a diagram consisting of representable functors and morphisms between them, encoding the idea that a symmetric \( \Delta \)-complex is “glued out of simplices.” This is a special case of a general fact about presheaves of sets on a small category, cf. [ML98, §III.7], but let us spell out explicitly how it works in our case.

Given \( X: I^{\text{op}} \rightarrow \text{Sets} \), define a category \( J_X \) whose objects are pairs \( ([p], x) \) with \( x \in X([p]) \) and whose morphisms \( ([p], x) \to ([p'], x') \) are the \( \theta \in I([p'], [p]) \) with \( X(\theta)(x) = x' \). For later use we point out that \( \theta \) is an isomorphism in \( J_X \) if and only if \( p = p' \). Let \( j_X: J_X^{\text{op}} \rightarrow I \) be the functor given on objects by \( j_X([p], x) = [p] \), and note that there is a canonical morphism of symmetric \( \Delta \)-complexes

\[
\text{colim}_{([p], x) \in J_X} I(-, [p]) \rightarrow X,
\]

assembled from the morphisms \( x: I(-, [p]) \to X \). Using that colimits in the category of symmetric \( \Delta \)-complexes are calculated object-wise, it is easy to verify that this morphism is in fact always an isomorphism.

We will sometimes use this to reduce a statement about all symmetric \( \Delta \)-complexes to a statement about representable ones, assuming of course that the statement is preserved by taking colimits.

**Lemma 4.3.** The functor \( X \mapsto |X| \) from symmetric \( \Delta \)-complexes to topological spaces preserves all small colimits.

**Proof.** Recall that any functor which admits a right adjoint will automatically preserve all small colimits [ML98, V.5].

A right adjoint to geometric realization may be defined as follows. Let \( Z \) be a topological space, and let \( \text{Sing}(Z) \) be the symmetric \( \Delta \)-complex which sends \( [p] \) to the set of all continuous maps \( \Delta^p \rightarrow Z \). The resulting functor \( \text{Sing} \) is right adjoint to geometric realization. Indeed, given \( X: I^{\text{op}} \rightarrow \text{Set} \) and given a topological space \( Z \), a natural transformation from \( X \) to \( \text{Sing}(Z) \) amounts to a choice of a continuous map \( \Delta^p \rightarrow Z \) for every element of \( X([p]) \), such that the choices are compatible with the gluing data \( X(\theta) \) for all \( \theta \in \text{Mor}(I) \). These precisely give the data of a continuous map \( |X| \rightarrow Z \). Moreover, the association is natural with respect both to maps \( X \to X' \) and to maps \( Z \to Z' \). \( \square \)

The homeomorphisms \( |I(-, [p])| = \Delta^p \), natural in \( [p] \in I \), together with Lemma 4.3 and the fact that an arbitrary symmetric \( \Delta \)-complex is canonically isomorphic to a colimit of representable functors, characterizes the geometric realization functor \( X \mapsto |X| \) up to natural homeomorphism. Similar facts hold for \( X \mapsto CX \).

Let us briefly discuss how to construct the barycentric subdivision of a symmetric \( \Delta \)-complex. Barycentric subdivision will be a functor from symmetric \( \Delta \)-complexes to (augmented) \( \Delta \)-complexes. The main point is that barycentric subdivision should preserve all small colimits, so it suffices to explain how to barycentrically subdivide the symmetric \( \Delta \)-complex \( I(-, [p]) \), functorially in \( [p] \). Explicitly, define the subdivision \( \text{sd}(I(-, [p])) \) by sending \( [q] \in \Delta_{\text{inj}} \cup \{-1\} \) to the set of all flags \( \emptyset \subset A_0 \subset \cdots \subset A_q \subseteq [p] \).
subdivision \( \text{sd}(X) \) of a general \( X : I^{\text{op}} \to \text{Sets} \) is then defined as the colimit in augmented \( \Delta \)-complexes

\[
\text{sd}(X) = \text{colim}_{([p], x) \in J_X} \text{sd}(I(-, [p])).
\]

Explicitly, this spells out to the formula

\[
\text{sd}(X)([q]) = (\coprod_p X_p \times \text{sd}(I(-, [p]), [q]) / \sim,
\]

where \( \sim \) is the equivalence relation generated by \((x, \theta^*b) \sim (X(\theta)(x), b)\) whenever \(x \in X_p, b \in \text{sd}(I(\cdot, [p'])), \) and \(\theta \in I([p'], [p])\). The formula incidentally also makes sense when \(X\) is a symmetric semi-simplicial space. Alternatively, \(\text{sd}(X)([p])\) may be explicitly described as the set of equivalence classes of functors \(\sigma : (0 < \cdots < p) \to J_{\text{op}}\) sending all non-identity morphisms to non-isomorphisms, up to natural isomorphism of functors. In any case, let us emphasize that the subdivision of a symmetric \(\Delta\)-complex is an (augmented) ordinary \(\Delta\)-complex, not a symmetric one.

**Lemma 4.4.** *The geometric realizations of a symmetric \(\Delta\)-complex \(X\) and the \(\Delta\)-complex \(\text{sd}(X)\) are canonically homeomorphic.*

**Proof.** Since both geometric realization and barycentric subdivision preserve all small colimits, it suffices to construct a natural homeomorphism

\[
|I(-, [p])| \cong |\text{sd}(I(-, [p]))|,
\]

which is done in the usual way: the left hand side is \(\Delta^p\), a non-empty subset \(A \subset [p]\) determines a face of \(\Delta^p\), and the corresponding vertex on the right hand side is sent to the barycenter of that face; extend to an affine map on each simplex. \(\square\)

**Remark 4.5.** Colimit presentations may be used to make many other definitions, or illuminate old ones. For example, the *join* \(X * Y\) of two symmetric \(\Delta\)-complexes \(X\) and \(Y\) may be defined by requiring \((I(-, [p])) * (I(-, [q])) = I(-, [p] \amalg [q])\) and requiring \(X * Y\) to preserve colimits in \(X\) and \(Y\) separately. The chains functor \(X \mapsto C_*(X; R)\) that we defined above also preserve colimits, so it suffices to define it on representables. The shifted chains functor, sending \(X\) to \(C_*(X; R)\) shifted so that \(RX_{-1}\) is in degree 0, is characterized up to natural isomorphism by its value on the point \(I(-, [0])\) together with the properties that it sends join of symmetric \(\Delta\)-complexes to tensor product of chain complexes, and preserves all colimits.

### 4.2. The tropical moduli space as a symmetric \(\Delta\)-complex

Let us return to the tropical moduli space \(\Delta_g\), which we defined in Section 2. To illustrate how the definitions of this section work for \(\Delta_g\), we will give two descriptions that exhibit \(\Delta_g\) as the geometric realization of a symmetric \(\Delta\)-complex. The first description presents \(\Delta_g\) as a colimit of a diagram of symmetric \(\Delta\)-complexes; the second is an explicit description as a functor \(X : I^{\text{op}} \to \text{Sets}\).

The category \(J_g\) from §2.2 has a unique final object: a single vertex, of weight \(g\). For the first description of \(\Delta_g\) as a colimit of a diagram of \(\Delta\)-complexes, choose for each object \(G \in J_g\) a bijection \(\tau = \tau_G : E(G) \to [p]\) for the appropriate \(p \geq -1\). This chosen bijection will be called the *edge-labeling* of \(G\). The terminal object has \(p = -1\), and all
non-terminal objects have $p \geq 0$. We require no compatibility between the edge-labelings for different $G$, but a morphism $\phi: G \to G'$ determines an injection

$$[p'] \xrightarrow{\tau_{G'}^{-1}} E(G') \xrightarrow{\phi^{-1}} E(G) \xrightarrow{\tau_G} [p],$$

where the middle arrow is the induced bijection from the edges of $G'$ to the non-collapsed edges of $G$ as in Definition 2.2. This gives a functor $F: (J_g)^{op} \to I$ sending $G$ to the codomain $[p]$ of $\tau_G$, and hence induces a functor from $(J_g)^{op}$ to symmetric $\Delta$-complexes, given as $G \mapsto I(-, F(G))$, whose colimit $X$ has geometric realization $|X| = \Delta_g$ and cone $CX = M_g^{top}$.

Indeed, colimit commutes with geometric realization by Lemma 4.3, so the geometric realization of $X$ is the colimit of the functor $G \mapsto |I(-, F(G))|$ from $J_g^{op}$ to $Top$. But we have a homeomorphism $|I(-, F(G))| = \Delta^p$, where $[p] = F(G)$. Furthermore, if $G' \in J_g$ is an object with $p' + 1$ edges, then the injection of label sets $F(\phi): [p'] \to [p]$ determined as above by a morphism $\phi: G \to G'$, induces a gluing of the simplex $|I(-, F(G'))|$ = $\Delta^p$ to a face of $|I(-, F(G))| = \Delta^p$. This agrees with the gluing obtained from the gluing of $\sigma(G')$ to a face of $\sigma(G)$ in Definition 2.3 by restricting to the length-one subspaces $\Delta^p \subset \sigma(G) = \mathbb{R}^{E(G)}_{\geq 0}$ and $\Delta^{p'} \subset \sigma(G) = \mathbb{R}^{E(G)}_{\geq 0}$.

For the second description of $\Delta_g$ as the geometric realization of a symmetric $\Delta$-complex, we explicitly describe a functor $X: I^{op} \to Sets$ as follows. The elements of $X_p$ are equivalence classes of pairs $(G, \tau)$ where $G \in J_g$ and $\tau: E(G) \to [p]$ is an edge labeling; two edge-labelings are considered equivalent if they are related by an isomorphism $G \cong G'$ (including of course automorphisms). Here $G$ ranges over all objects in $J_g$ with exactly $p + 1$ edges. (Using that in §2.2 we tacitly picked one element in each isomorphism class in $J_g$, the equivalence relation is generated by actions of the groups $Aut(G)$.) This defines $X: I^{op} \to Sets$ on objects.

Next, for each injective map $\iota: [p'] \to [p]$, define the following map $X(\iota): X_p \to X_{p'}$; given an element of $X_p$ represented by $(G, \tau: E(G) \to [p])$, contract the edges of $G$ whose labels are not in $\iota([p']) \subset [p]$, then relabel the remaining edges with labels $[p']$ as prescribed by the map $\iota$. The result is a $[p']$-edge-labeling of some new object $G'$, and we set $X(\iota)(G)$ to be the element of $X_{p'}$ corresponding to it.

Hereafter, we will use $\Delta_g$ to refer to this symmetric $\Delta$-complex and write $|\Delta_g|$ for the topological space. To avoid double subscripts, we write $\Delta_g([p])$ for the set of $p$-simplices of $\Delta_g$. Then $H_k(\Delta_g) = \tilde{H}_k(|\Delta_g|)$ since $\Delta_g([-1]) = \{\ast\}$. In fact, the interpretation of $|\Delta_g|$ as a moduli space of stable tropical curves of genus $g$ and volume 1 described in §2.3 is, strictly speaking, not logically necessary for the main results of this paper. Nevertheless, we find this modular interpretation of $|\Delta_g|$ to be a useful point of view.

4.3. Generalized cone complexes and symmetric $\Delta$-complexes. We now briefly discuss the generalized cone complexes of [ACP13] §2 and their relationship to the symmetric $\Delta$-complexes described here. We will see that the category of symmetric $\Delta$-complexes is equivalent to the category of smooth generalized cone complexes, by which we mean the category whose objects are generalized cone complexes built out of copies of standard orthants in $\mathbb{R}^n$, and whose arrows are face morphisms. In the next two paragraphs we recall the precise definition.
Recall that there is a category of cones $\sigma$ and face morphisms $\sigma \to \sigma'$. A cone is a topological space $\sigma$ together with an “integral structure”, i.e., a finitely generated subgroup of the group of continuous functions $\sigma \to \mathbb{R}$ satisfying a certain condition. A face morphism $\sigma \to \sigma'$ is a continuous function satisfying another condition. We do not recall the details, since we will not need the full category of all cones. The special case we need is the standard orthant $\mathbb{R}^p_{\geq 0} = \prod_{i=0}^p \mathbb{R}_{\geq 0}$ together with the abelian group $\mathbb{M}$ generated by the $p+1$ projections $\mathbb{R}^p_{\geq 0} \to \mathbb{R}$ onto the axes, for $p \geq -1$. The face morphisms $\mathbb{R}^p_{\geq 0} \to \mathbb{R}^q_{\geq 0}$ are precisely those induced by $\theta \in I([p], [q])$. In other words, if we write $\Sigma([p]) = \mathbb{R}^p_{\geq 0}$ with this integral structure, we have defined a functor $I \to \text{Cones, face morphisms}$

$$ [p] \mapsto \Sigma([p]), $$

which is full and faithful. Let us say that a cone is smooth if it is isomorphic to $\Sigma([p])$ for some $p \geq -1$; then the category of smooth cones and face morphisms between them is equivalent to $I$.

In [ACP15] §2.6, a generalized cone complex is a topological space $X$ together with a presentation as $\text{colim}(r \circ F)$, where $F: J \to \text{Cones, face morphisms}$ is a functor from a small category $J$, and $r$ denotes the forgetful functor from cones to topological spaces. We say that a generalized cone complex is smooth if it is isomorphic to a colimit of smooth cones.

Let us now describe the smooth generalized cone complex $\Sigma_X$ associated to a symmetric $\Delta$-complex $X: I^{op} \to \text{Sets}$. This is essentially the same as the space denoted $CX$ in §3.2, where we defined it as the colimit of the composition

$$ J_X \xrightarrow{u} I \to \text{Top}, $$

where the last functor sends $[p] \mapsto C([p]) = \mathbb{R}^p_{\geq 0}$. To get the generalized cone complex we regard this last functor as taking values in generalized cone complexes instead. Following [ACP15] we write $\Sigma_X$ for the resulting generalized cone complex: i.e., $\Sigma_X$ denotes the space $CX$ together with its presentation as a colimit of cones.

Next we describe the correspondence in the other direction: how to associate a symmetric $\Delta$-complex to a smooth generalized cone complex. We first extend the notion of “face morphism” between cones to morphisms between generalized cone complexes. If $\Sigma$ is a smooth generalized cone complex and $\sigma$ is a smooth cone, let us say that a morphism $\sigma \to \Sigma$ is a face morphism if it admits a factorization as $\sigma \to \sigma' \to \Sigma$, where the second map $\sigma' \to \Sigma$ is one of the cones in the colimit presentation of $\Sigma$ and the first map is a face morphism of cones. If $\Sigma$ and $\Sigma'$ are generalized cone complexes, a morphism $\Sigma \to \Sigma'$ is a face morphism if the composition $\sigma \to \Sigma \to \Sigma'$ is a face morphism for all cones $\sigma \to \Sigma$ in the colimit presentation of $\Sigma$. We may then define a functor $X_\Sigma: I^{op} \to \text{Sets}$ whose value $X_\Sigma([p])$ is the set of face morphisms $\Sigma([p]) \to \Sigma$.

These processes are inverse and give an equivalence of categories between symmetric $\Delta$-complexes and the category whose objects are smooth generalized cone complexes and whose morphisms are face morphisms between such. Geometrically, if $X: I^{op} \to \text{Sets}$ is a symmetric $\Delta$-complex, the geometric realization $|X| \to X_{-1}$ is the link of the cone points.
in the corresponding generalized cone complex $\Sigma_X$. The cone points themselves become the elements of the set $X_{-1}$.

**Remark 4.6.** The notion of *morphism* between (smooth) generalized cone complexes used in [ACP15, §2.6] contains many other morphisms, in addition to face morphisms. For instance, the map $\Sigma([1]) = (\mathbb{R}_{\geq 0})^2 \rightarrow \mathbb{R}_{\geq 0} = \Sigma([0])$ given in coordinates as $(x_0, x_1) \mapsto x_0 + x_1$ is a morphism of cones and hence generalized cone complexes, but is not a face morphism. These additional morphisms are necessary to make the construction of skeletons of toroidal varieties (and DM stacks) functorial with respect to arbitrary toroidal morphisms, but are not needed for the purposes of this paper.

## 5. Graph complexes and cellular chains on $\Delta_g$

In this section we prove Theorem 1.3.

**Definition 5.1.** Let $C^{(g)}$ be the rational chain complex with one generator $[G, \omega]$ of degree $p$ for each object $G \in J_g$ and each bijection $\omega$ from $E(G)$ to an object $[p] \in \Delta_{\text{inv}}$. These generators are subject to the relations $[G, \omega] = \text{sgn}(\sigma)[G', \omega']$ if there exists an isomorphism $G \rightarrow G'$ in $J_g$ inducing the permutation $\sigma$ of the set $[p] = \{0, \ldots, p\}$.

**Lemma 5.2.** There is an isomorphism $\tilde{H}_k(|\Delta_g|; \mathbb{Q}) \cong H_k(C^{(g)})$.

**Proof.** The cellular chain complex from §3.3 applied to the particular $X: I^{\text{op}} \rightarrow \text{Sets}$ described in §4.2, more precisely the “second description” in that section, is isomorphic to $C^{(g)}$. \hfill \square

**Definition 5.3.** Let $A^{(g)} \subset C^{(g)}$ be the subcomplex spanned by those $[G, \omega]$ for which $G$ has all vertex weights zero and has no loops. Let $B^{(g)} \subset C^{(g)}$ be the subcomplex spanned by those $[G, \omega]$ which either has a loop or a vertex with positive weight.

**Lemma 5.4.** These are in fact subcomplexes, and hence the natural map $A^{(g)} \oplus B^{(g)} \rightarrow C^{(g)}$

is an isomorphism of chain complexes.

**Proof.** It is clear that the boundary map sends $B^{(g)}$ into itself and that the indicated map is an isomorphism of graded vector spaces.

It may seem like $A^{(g)}$ is not a subcomplex, since collapsing an edge in a graph without loops may result in a graph with a loop. However, if $G/e$ has more loops than $G$ does, then in fact the edge $e$ must have had a parallel edge, and hence $[G, \omega] = 0$ and hence of course $\partial[G, \omega] = 0$ is in $A^{(g)}$. Similarly, for the sum of the vertex weights in $G/e$ to be strictly higher than that of $G$, the edge $e$ must have been a loop so this does not occur in $A^{(g)}$. \hfill \square

**Lemma 5.5.** The chain complex $A^{(g)}$ is isomorphic to a shift of Kontsevich’s graph complex $G^{(g)}$. In our grading conventions, the isomorphism is

$$G_k^{(g)} \rightarrow A_{k+2g-1}^{(g)}, [G, \omega] \mapsto [G, \omega].$$
Proof. The map described sends generators to generators and matches the relations under isomorphisms $G \cong G'$, and matches the boundary maps in the two chain complexes. To convince oneself that the degrees are as indicated, recall that the wheel graph, with $2g$ edges, is a 0-cycle in the domain but corresponds to the $(2g - 1)$-dimensional simplex mapping to $\Delta_g$ obtained by varying edge lengths in the wheel graph. In both complexes degrees go down by one if an edge is collapsed. □

The following proposition implies that the resulting split injection of $H_0(G^{(g)})$ into $\tilde{H}_{2g-1}(|\Delta_g|; \mathbb{Q})$ is an isomorphism. It is quite similar to the acyclicity result established in [CGV05, Theorem 2.2].

Proposition 5.6. The chain complex $B^{(g)}$ has vanishing homology in all degrees.

The complex $B^{(g)}$ calculates the reduced homology of the subspace of $\Delta_g$ consisting of graphs containing a loop or a vertex of positive weight. In a follow-up paper we shall show that this space is in fact contractible.

Proof. For any $G$ and any $e \in E(G)$, say $e$ is a stem if $G/(E(G) \setminus e)$ is isomorphic to $1 - \bullet - \bullet_{g-1}$.

Then $G$ has a loop or positive weight if and only if it admits a morphism from some $G'$ having a stem. Moreover, for every $G$ having a loop or positive weight, we assert two graph-theoretic statements:

1. There exists a $\tilde{G}$ with a stem and a morphism $\phi: \tilde{G} \to G$ presenting $G$ as the quotient of $\tilde{G}$ by contracting zero or more stems, such that for any other such morphisms $\phi': G' \to G$ there exists a (not necessarily unique) map $\psi: \tilde{G} \to G'$ with $\phi = \psi \circ \phi'$.

2. With $\tilde{G}$ as above, for any two morphisms $\theta_1, \theta_2: \tilde{G} \to G$, there exists an isomorphism $\psi: G \to G$ with $\theta_2 = \psi \theta_1$.

The idea is that if $G$ has any loops not separated from the rest of $G$ by a stem then such a stem may be introduced by uncontraction, and similarly if $G$ has any vertices of weight $w > 1$, or any vertices of weight 1 not separated from the rest of $G$ by a stem, then one may uncontract this vertex into $w$-many stems separating a weight-1 vertex from the rest of $G$.

Now for $i \geq 0$, let $B^{(g),i}$ denote the subcomplex of $B^{(g)}$ spanned by those graphs $G$ with at most $i$ edges that are not stems. Then the subcomplexes $B^{(g),i}$, for $i = 0, \ldots, 3g - 3$, filter $B^{(g)}$.

Next, for each $i > 0$, we claim vanishing of relative homology of the pair $(B^{(g),i}, B^{(g),i-1})$. The chain complex associated to this pair is spanned by those $[G, \omega]$ having a loop or positive weight, satisfying in addition that $G$ has exactly $i$ non-stem edges. Furthermore, the boundary of $[G, \omega]$ is a signed sum of 1-edge-contractions by stems. By assertion (1), this chain complex is a direct sum of the following subcomplexes $B^{(g),i}(G)$, one for each $G$ with $i$ non-stems that is maximal with respect to the contraction of stems. Here, $B^{(g),i}(G)$ is the subcomplex spanned by $G$ and all of its contractions by stems.
Now we claim that $B^{(g),i}(G)$ is acyclic. Indeed, assertion (2) shall imply that it is the rational cellular chain complex associated to a pair in which the first space retracts onto the second. The pair in question is

$$(\Delta^{[\text{E}(G)]-1}/\text{Aut}(G), Z/\text{Aut}(G))$$

where $Z$ is the union of the $i$ facets of $\Delta^{[\text{E}(G)]-1}$ that contain all vertices of $\Delta^{[\text{E}(G)]-1}$ corresponding to stems of $G$. Since $0 < i < |\text{E}(G)|$, there is a natural deformation retraction of $\Delta^{[\text{E}(G)]-1}$ onto $Z$, and this retraction is $\text{Aut}(G)$-equivariant.

It remains to see that $B^{(g),0}$ is acyclic; in fact it is two-dimensional with rank one boundary map. Generators are graphs in which every single edge is a stem. In particular the graphs have no loops, so every stem must separate a weight-1 vertex from the rest of the graph. It is not hard to classify such $G$: there is one isomorphism class for each $h \in \{0,\ldots,g\}$, given by a graph with a single central vertex of weight $g - h$, to which are attached $h$ edges ending in a weight-1 vertex. Hence the underlying graph is a wedge of $h$ intervals. These graphs all admit odd automorphisms, so the corresponding generator for the graph complex vanishes, except when $h = 0$ and $h = 1$. The boundary of the $h = 1$ graph is the $h = 0$ graph. □

6. Boundary complexes

The theory of dual complexes for simple normal crossings divisors is well-known. They may be constructed as $\Delta$-complexes, with the $\Delta$-complex structure depending on a choice of total ordering on the irreducible components of the divisor. Many applications involve the fact that the homotopy types (and even simple homotopy types) of boundary complexes, the dual complexes of boundary divisors in simple normal crossings compactifications, are independent of the choice of compactification. The same is also true for Deligne-Mumford (DM) stacks [Har17]. Boundary complexes were introduced and studied by Danilov in the 1970s [Dan75], and have become an important focus of research activity in the past few years, with new connections to Berkovich spaces, singularity theory, geometric representation theory, and the minimal model program. See, for instance, [Ste08, ABW13, Pay13, KX16, Sim16, dFKX17].

In order to apply combinatorial topological properties of $\Delta_g$ to study the moduli space of curves $\mathcal{M}_g$ using the compactification by stable curves, we must account for the facts that $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are stacks, not varieties, and that the boundary divisor in $\overline{\mathcal{M}}_g$ has normal crossings, but not simple normal crossings. The latter of those two complications is the more serious one; when the irreducible components of the strata have self-intersections, the fundamental groups of strata may act nontrivially by monodromy on the analytic branches of the boundary and this needs to be accounted for. Once that is properly understood, passing from varieties to stacks is relatively straightforward.

In this section we explain how dual complexes of normal crossings divisors are naturally interpreted as symmetric $\Delta$-complexes and, in particular, the dual complex of the boundary divisor in the stable curves compactification of $\mathcal{M}_g$ is naturally identified with $\Delta_g$.

6.1. Dual complexes of simple normal crossings divisors. We begin by recalling the notion of dual complexes of simple normal crossings divisors, using the language of
(regular) symmetric Δ-complexes introduced in Section 3. In §6.2 we will explain how to interpret dual complexes of normal crossings divisors in smooth Deligne–Mumford (DM) stacks as symmetric Δ-complexes of §3.3. Here and throughout, all of the varieties and stacks that we consider are over the complex numbers, and all stacks are separated and DM.

Let $X$ be a $d$-dimensional smooth variety. Recall (cf. [Sta17, Tag 0BI9]) that a simple normal crossings divisor is an effective Cartier divisor $D \subset X$ which is (Zariski) locally cut out by $x_1 \cdots x_d$ for a regular system of parameters $x_1, \ldots, x_d$ in the local ring at any $p \in D$. It is a normal crossings divisor if it becomes a simple normal crossing divisor after pulling back along a surjective étale map. In that case it is a simple normal crossings divisor if all irreducible components are smooth. Recall that the strata of $D$ may be defined inductively as follows. The $(d - 1)$-dimensional strata of $D$ are the irreducible components of the smooth locus of $D$; and for each $i < d - 1$, the $i$-dimensional strata are the irreducible components of the regular locus of $D \setminus (D_{d-1} \cup \cdots \cup D_{i+1})$, where $D_j$ temporarily denotes the union of the $j$-dimensional strata of $D$.

If $D \subset X$ has simple normal crossings, then the dual complex $\Delta(D)$ is naturally understood as a regular symmetric Δ-complex whose geometric realization has one vertex for each irreducible component of $D$, one edge for each irreducible component of a pairwise intersection, and so on. The inclusions of faces correspond to containments of strata. It is augmented, with $(-1)$-simplices the set of irreducible components (equivalently, connected components) of $X$. Equivalently, using our characterization of symmetric Δ-complexes in terms of presheaves on the category $I$ given in §3.2, $\Delta(D)$ is the presheaf whose value on $[p]$ is the set of pairs $(Y, \phi)$, where $Y \subset X$ is a stratum of codimension $p + 1$, i.e., codimension $p$ in $D$ for $p \geq 0$, and $\phi$ is an ordering of the components of $D$ that contain $Y$, with maps induced by containments of strata. Dual complexes can also be defined in exactly the same way for simple normal crossings divisors in DM stacks.

Remark 6.1. In the literature, it is common to fix an ordering of the irreducible components of the simple normal crossings divisor $D$. The corresponding ordering of the vertices induces a Δ-complex structure on $\Delta(D)$. Working with dual complexes as symmetric Δ-complexes may be slightly more natural, in that it avoids this choice of an ordering, and certainly it generalizes better to the construction of dual complexes for divisors with (not necessarily simple) normal crossings as symmetric Δ-complexes, given in §6.2.

In the literature it is also commonly assumed that $X$ is irreducible, and hence there is no need for keeping track of $(-1)$-simplices and augmentations. This is sufficient for studying a single irreducible variety at a time, but comes with some technical inconveniences. In particular, certain auxiliary constructions such as the étale covers and fiber products appearing later in this section, do not preserve irreducibility. It is convenient to set up the language in a way that applies without assuming irreducibility.

6.2. Dual complexes of normal crossings divisors. We now discuss the generalization to normal crossings divisors $D$ in a smooth DM stack $X$ which are not necessarily simple normal crossings, i.e., the irreducible components of $D$ are not necessarily smooth and may have self-intersections. This situation is more subtle, even for varieties, due
to monodromy. In the stack case, when the boundary strata have stabilizers, the monodromy may be nontrivial even for zero-dimensional strata. This phenomenon appears already at the zero-dimensional strata of $\overline{M}_g$ given by stable curves having nontrivial automorphisms, i.e., the strata corresponding to (unweighted) trivalent graphs of first Betti number $g$ with nontrivial automorphisms.

Let $X$ be a smooth variety or DM stack, not necessarily irreducible. Recall that a divisor $D \subset X$ has normal crossings if and only if there is an étale cover by a smooth variety $X_0 \to X$ in which the preimage of $D$ is a divisor with simple normal crossings.

Note that this étale local characterization of normal crossings divisors is the same for varieties and DM stacks.

Following [ACP15], the dual complex may be defined étale locally, in the following way.

Choose a surjective étale map $X_0 \to X$ for which the divisor $D \times_X X_0 \subset X_0$ has simple normal crossings, set $X_1 = X_0 \times_X X_0$ and observe that the divisor $D \times_X X_1 \subset X_1$ also has simple normal crossings. The two projections $D \times_X X_1 \to D \times_X X_0$ give rise to two maps of symmetric $\Delta$-complexes

\[(6.2.1) \Delta(D \times_X X_1) \to \Delta(D \times_X X_0),\]

and we define $\Delta(D)$ to be the coequalizer of those two maps, in the category of symmetric $\Delta$-complexes. It is shown in [ACP15] (in the equivalent language of (smooth) generalized cone complexes) that, up to isomorphism, the resulting symmetric $\Delta$-complex does not depend on the choice of $X_0 \to X$ (cf. also Lemma 6.4 below). That recipe makes sense also in the more general case where $X$ is a smooth DM stack and $D \subset X$ is a normal crossings divisor, using a sufficiently fine étale atlas $X_0 \to X$, and $\Delta(D)$ is independent of the choice of $X_0 \to X$ in this generality as well.

We now give an equivalent and more direct description of $\Delta(D)$ as a functor $I^{\text{op}} \to \text{Sets}$. Let $\widetilde{D} \to X$ denote the normalization of $D \subset X$, and for $[p] \in I$ write

$$\widetilde{D}_p = (\widetilde{D} \times_X \cdots \times_X \widetilde{D}) \setminus \{(z_0, \ldots, z_p) \mid z_i = z_j \text{ for some } i \neq j\}.$$ 

This construction is completely analogous to the one in Example 3.12, except applied to the map $\widetilde{D} \to X$ in varieties (or DM stacks) over $\mathbb{C}$ instead of the map $A \to B$ in the category of spaces. We have $\widetilde{D}_0 = \widetilde{D}$ and $\widetilde{D}_{-1} = X$. Then $\widetilde{D}_p \to X$ is a local complete intersection morphism whose conormal sheaf is a vector bundle of rank $(p + 1)$ ([Sta17, Tag 0CBR]). In particular $\widetilde{D}_p$ is smooth over $\mathbb{C}$ of dimension $d - p$ if $X$ is smooth over $\mathbb{C}$ of dimension $d + 1$.

**Definition 6.2.** Let $X$ be a smooth variety or DM stack, let $D \subset X$ be a normal crossings divisor, and write $\widetilde{D}_p \to X$ for the construction defined for all $[p] \in I^{\text{op}}$ above. In this situation, define the symmetric $\Delta$-complex $\Delta(D)$ by letting $\Delta(D)_p$ be the set of irreducible components (= connected components) of $\widetilde{D}_p$.

We point out that in the case of stacks, the association $[p] \mapsto \widetilde{D}_p$ will only be a pseudofunctor, but the set of irreducible components will be functorial in $[p] \in I$.

In the case that $D$ has simple normal crossings, we now have two definitions of $\Delta(D)$, one in Definition 6.2 and one in [6.1]. Let us explain how to reconcile the two definitions.
If \( D \) has smooth components \( Z_0, \ldots, Z_r \) then \( \tilde{D}_p \) may be calculated by distributing fiber product over disjoint union: it is

\[
\tilde{D}_p = \prod_{\theta: [p] \to [r]} Z_{\theta(0)} \times_X \cdots \times_X Z_{\theta(r)},
\]

where the disjoint union is over all injective maps of sets \( \theta: [p] \to [r] \). Each such fiber product maps isomorphically to a subset \( Z_{\theta(0)} \times_X \cdots \times_X Z_{\theta(r)} \cong Z_{\theta(0)} \cap \cdots \cap Z_{\theta(r)} \subset X \), which is smooth over \( \mathbb{C} \) and has codimension \( p + 1 \) in \( X \), but need not be connected. Each connected component is the closure in \( X \) of precisely one stratum of codimension \( p + 1 \). The function \( \theta \) gives an ordering on the \( p + 1 \) components of \( D \) which contain this stratum. Hence we have produced a bijection between the components of \( \tilde{D}_p \) and the set of \( p \)-simplices of the symmetric \( \Delta \)-complex described in \([6.1]\) it is easy to see that this bijection is natural with respect to maps \([p] \to [p] \) in \( I \).

We note that a closed point of \( \tilde{D}_p \) corresponds precisely to a closed point \( x \) in a codimension \( p \) stratum of \( D \), together with an ordering \( \sigma \) of the \( p + 1 \) local analytic branches of \( D \). Hence \( \Delta(D)_p \) may be described more transcendentally as the set of equivalence classes of pairs \((x, \sigma)\), where \((x, \sigma)\) is equivalent to \((x', \sigma')\) if there is a path (continuous in the analytic topology) within the stratum taking \( x \) to \( x' \), and that following the ordering of the branches along this path takes \( \sigma \) to \( \sigma' \).

**Remark 6.3.** Recall that a \( \Delta \)-complex \( X \) is regular if the maps \( \Delta^p \to |X| \) associated to \( \sigma \in X_p \) for all \( p \geq 0 \) are all injective. This definition makes sense equally well for symmetric \( \Delta \)-complexes \( X \) and is equivalent to the condition that every edge of \( X \) has two distinct endpoints, i.e., for any \( e \in X_1 \), we have \( d_0(e) \neq d_1(e) \).

The dual complex of a normal crossings divisor will be a regular symmetric \( \Delta \)-complex exactly when \( D \) has simple normal crossings, meaning that every irreducible component of \( D \) is smooth. Indeed, the irreducible components of \( D \) are smooth if and only if at every codimension 1 stratum of \( D \), the two analytic branches belong to distinct irreducible components. This is equivalent to the condition that \( d_0(e) \neq d_1(e) \) for \( e \in \Delta(D)_1 \).

**Lemma 6.4.** The association \( D \mapsto \Delta(D) \) satisfies étale descent in the sense that if \( X_0 \to X \) is an étale cover and \( X_1 = X_0 \times_X X_0 \), then

\[
\Delta(D \times_X X_1) \to \Delta(D \times_X X_0) \to \Delta(D)
\]

is a coequalizer diagram.

**Proof.** Let us apply the same construction of normalization and iterated fiber product to the divisors \( D \times_X X_0 \subset X_0 \) and \( D \times_X X_1 \subset X_1 \). Since normalization commutes with étale base change (\([\text{Sta17 Tag 07TD}]\)), and since \( \tilde{D}_p \subset \tilde{D} \times_X \cdots \times_X \tilde{D} \) is defined by a property which is checked in fibers over \( X \), we get three fiber squares

\[
\begin{array}{ccc}
\tilde{D}_p \times_X X_1 & \longrightarrow & \tilde{D}_p \times_X X_0 \\
\downarrow & & \downarrow \\
X_1 & \\ \longrightarrow & \longrightarrow & X,
\end{array}
\]
in which \( \Delta(D \times X X_1)_p \) is the set of components of \( \tilde{D}_p \times X X_1 \) and \( \Delta(D \times X X_0)_p \) is the set of components of \( \tilde{D}_p \times X X_0 \).

Since \( X_1 = X_0 \times X X_0 \) is also a fiber product, this diagram may be unfolded to a 3-dimensional cubical diagram (the bottom square is the fiber product defining \( X_1 \)), all of whose square faces are known to be pullback, except possibly the top. It follows that the top of this cube, i.e., the commutative square

\[
\begin{array}{ccc}
\tilde{D}_p \times X X_1 & \longrightarrow & \tilde{D}_p \times X X_0 \\
\downarrow & & \downarrow \\
\tilde{D}_p \times X X_0 & \longrightarrow & \tilde{D}_p \\
\end{array}
\]

(6.2.2)

is also pullback. It then induces a pullback of sets of \( K \)-valued points for any \( K \), and in particular when \( K \) is an algebraic closure of the function field of an irreducible component of \( \tilde{D}_p \). The two maps \( \tilde{D}_p \times X X_0 \to \tilde{D}_p \) are étale and surjective, so they induce surjections for such \( K \). We conclude that for any pair of irreducible components of \( \tilde{D}_p \times X X_0 \) mapping to the same component of \( \tilde{D}_p \) there exists a point in \( \tilde{D}_p \times X X_1 \) mapping to those two components, i.e., that the map of sets \( \Delta(D \times X X_1)_p \to (\Delta(D \times X X_0)_p) \times_{\Delta(D)_p} (\Delta(D \times X X_0)_p) \) induced by taking sets of irreducible components in (6.2.2) is surjective.

This surjectivity implies that the coequalizer of \( \Delta(D \times X X_1)_p \rightrightarrows \Delta(D \times X X_0)_p \) induces a pullback of sets of \( \tilde{D}_p \), so this finishes the proof.

The following example illustrates how étale descent can be used to compute the dual complex of a normal crossings divisor, with monodromy, as a coequalizer.

**Example 6.5.** Consider the Whitney umbrella \( D = \{x^2y = z^2\} \) in \( X = \mathbb{A}^3 \setminus \{y = 0\} \), as in [ACP15, Example 6.1.7]. Then the dual complex \( \Delta(D) \) is the half interval of Example 3.6.

We will explain this calculation two different ways in order to demonstrate the equivalent constructions of the boundary complex.

Let \( X_0 \cong \mathbb{A}^2 \times \mathbb{G}_m \to X \) be the degree 2 étale cover given by a base change \( y = u^2 \). Then \( D' = D \times X X_0 = \{x^2u^2 - z^2 = 0\} \) is simple normal crossings, and \( D'' = D \times X X_1 = D' \times X X 2 \cong D_0 \times \mathbb{Z}/2\mathbb{Z} \), since \( D' \) is degree 2 over \( D \). Explicitly, one component of \( D'' \) parametrizes pairs \((p, p)\) of points in \( D' \), and the other parametrizes pairs \((p, q)\) with \( p \neq q \) lying over the same point of \( D \). So \( \Delta(D') \cong I(-, [1]) \) is an unordered 1-simplex or an “interval,” and \( \Delta(D'') \) is two intervals, and the two maps \( \Delta(D'') \rightrightarrows \Delta(D'') \) differ by one flip, making the coequalizer a half interval.

Second, we have the normalization map \( E = \mathbb{A}^2_{x,u} - \{u = 0\} \to D \) sending \((x, u)\) to \((x, u^2, xu)\). Then \( \tilde{D}_0 \), while \( \tilde{D}_1 \) is isomorphic to the 1-dimensional stratum \( Y = \{x = z = 0\} \cong \mathbb{G}_m \) in \( X \); it has closed points \((0, u), (0, -u)\) \( \in E \times X E \). So \( \tilde{D}_0 \) and \( \tilde{D}_1 \) each have a single irreducible component, which completely determines \( \Delta(D) \).

As an immediate consequence of Lemma 6.4, we deduce the following compatibility with the construction in [ACP15].
Corollary 6.6. Let $X$ be a smooth variety or DM stack with the toroidal structure induced by a normal crossings divisor $D \subset X$. Then the dual complex $\Delta(D)$ is the symmetric $\Delta$-complex associated to the smooth generalized cone complex $\Sigma(X)$.

Proof. Indeed, [ACP15] defines $\Sigma(X)$ as the coequalizer in generalized cone complexes of $\Sigma(X_1) \rightrightarrows \Sigma(X_0)$. The identification of symmetric $\Delta$-complexes with smooth generalized cone complexes preserves all colimits. □

Most important for our purposes is the special case where $X = \overline{M}_g$ is the Deligne–Mumford stable curves compactification of $M_g$ and $D = \overline{M}_g \setminus M_g$ is the boundary divisor.

Corollary 6.7. The dual complex of the boundary divisor in the moduli space of stable curves with marked points $\Delta(\overline{M}_g \setminus M_g)$ is $\Delta_g$.

Proof. Modulo the translation from symmetric $\Delta$-complexes to (smooth) generalized cone complexes given in Corollary 6.6, this is one of the main results of [ACP15] and we refer there for a thorough treatment. Let us outline the argument in the notation and setup of the present paper, where both objects are symmetric $\Delta$-complexes, i.e., functors $I^\text{op} \to \text{Sets}$. The main ingredient is the irreducibility of moduli spaces from (DM69).

The complement $\mathcal{D} = \overline{M}_g \setminus M_g$ is a normal crossings divisor in $\overline{M}_g$ and hence its normalization $\widetilde{\mathcal{D}} \to \mathcal{D}$ is smooth over $\mathbb{C}$. The open stratum of $\mathcal{D}$ has a locally closed embedding into the universal 1-nodal curve $\partial \overline{M}_{g,1} \to \overline{M}_g$ as the universal node, and by a flatness argument its closure in $\partial \overline{M}_{g,1}$ may be identified with $\overline{M}_g$. In other words, $\overline{M}_g$ is the moduli stack of nodal curves with one marked node. By normalizing the marked node, we get an equivalence of stacks

$$\overline{M}_g \cong (\overline{M}_{g-1,2} \amalg \prod_{i=1}^{g-1} \overline{M}_{i,1} \times \overline{M}_{g-i,1})/S_2.$$  

(6.2.3)

Now, $\Delta_g([0])$ is the set of isomorphism classes of pairs of an object $G \in J_g$ together with a bijection $E(G) \to [0]$, so the underlying graph of $G$ has precisely one edge. If this edge is a loop, the vertex must weight $g - 1$; if not, the graph must be an edge between two distinct vertices whose weights have sum $g$. Hence there is a continuous map

$$\overline{\mathcal{D}} \to \Delta_g([0])$$

induced by sending $\overline{M}_{g-1,2}$ to the loop with vertex weight $g - 1$, and $\overline{M}_{i,1} \times \overline{M}_{g-i,1}$ to the 1-edge graph with two vertices of weights $i$ and $g - i$. Since the stacks $\overline{M}_{g-1,2}$ and $\overline{M}_{i,1} \times \overline{M}_{g-i,1}$ are irreducible, we deduce that the above map induces a bijection between the set of irreducible components of $\overline{\mathcal{D}}$ and $\Delta_g([0])$.

For $[p] = \{0, \ldots, p\}$ the stack $\overline{\mathcal{D}}_p$ is similarly the moduli stack of nodal curves with $p + 1$ marked distinct nodes, labeled by the finite set $[p]$. If desired, $\overline{\mathcal{D}}_p$ may be realized inside the $(p + 1)$-fold fiber product of $\mathcal{C}_g$ over $\overline{M}_g$. In analogy with the case $p = 0$ we may define a map

$$\overline{\mathcal{D}}_p \to \Delta_g([p]),$$

(6.2.4)
by sending a nodal curve with $p+1$ marked labeled distinct nodes to its dual graph, with all edges corresponding to unmarked nodes contracted, which then has edge set labeled by $[p]$. This map is continuous when the codomain is given the discrete topology, and the inverse image of the element of $\Delta_g([p])$ given by $G \in \mathcal{J}_g$ and a bijection $\tilde{E}(G) \to [p]$, admits a dominant map from the stack

$$\prod_{v \in V(G)} \mathcal{M}_{w(v), r^{-1}(v)},$$

where $r^{-1}(v) \subset H(G)$ is the set of half-edges emanating from $v$, and $\mathcal{M}_{w(v), r^{-1}(v)}$ is the moduli stack of nodal curves of genus $w(v)$ and distinct marked points labeled by $r^{-1}(v)$. Since each of these moduli stacks is again irreducible, so are the products, and hence we have constructed a bijection from the set of irreducible components of $\mathcal{D}_p$, which is our definition of the set $\Delta(D)_p$, to $\Delta_g([p])$. This bijection is easily seen to be an isomorphism of functors: forgetting an $i \in [p]$ corresponds to forgetting a marked node, which has the effect of collapsing the corresponding edge in the dual graph.

6.3. **Top weight cohomology.** Let $\mathcal{X}$ be a smooth variety or DM stack of dimension $d$ over $\mathbb{C}$. The rational singular cohomology of $\mathcal{X}$, like the rational cohomology of a smooth variety, carries a canonical mixed Hodge structure, in which the weights on $\mathcal{H}^k$ are between $k$ and $\min\{2k, 2d\}$. Since the graded pieces $\text{Gr}^W_j H^*(\mathcal{X}; \mathbb{Q})$ vanish for $j > 2d$, we refer to $\text{Gr}^W_{2d} H^*(\mathcal{X}; \mathbb{Q})$ as the top weight cohomology of $\mathcal{X}$. The standard identification of the top weight cohomology of a smooth variety with the reduced homology of its boundary complex carries through essentially without change for DM stacks. For completeness, we include the details.

**Theorem 6.8.** Let $\mathcal{X}$ be a smooth and separated DM stack of dimension $d$ with a normal crossing compactification $\overline{\mathcal{X}}$ and let $\mathcal{D} = \overline{\mathcal{X}} \smallsetminus \mathcal{X}$. Then there is a natural isomorphism

$$\text{Gr}^W_{2d} H^{2d-k}(\mathcal{X}; \mathbb{Q}) \cong H_{k-1}(\Delta(\mathcal{D}); \mathbb{Q}),$$

whose codomain is $\overline{H}_{k-1}(\Delta(\mathcal{D}); \mathbb{Q})$ when $\mathcal{X}$ is irreducible.

**Proof.** First, we reduce to the case where $\mathcal{D}$ has simple normal crossings, by a finite sequence of blowups, as follows. Let $\overline{\mathcal{X}} \to \overline{\mathcal{X}}$ be the morphism obtained by first blowing up the zero-dimensional strata of $\mathcal{D}$, and then the strict transforms of the 1-dimensional strata, and so on. We claim that the divisor $\mathcal{D}' = \overline{\mathcal{X}} \smallsetminus \overline{\mathcal{X}}$ has simple normal crossings and that $\Delta(\mathcal{D}')$ is the barycentric subdivision of $\Delta(\mathcal{D})$, as defined in §4.1.1.

To verify this claim, choose an étale cover $X_0 \to \overline{\mathcal{X}}$ such that $D_0 = \mathcal{D} \times_\overline{\mathcal{X}} X_0$ has simple normal crossings, and let $D_1 = D_0 \times_D D_0$. Then $\Delta(D_1) \rightarrow \Delta(D_0) \to \Delta(\mathcal{D})$ is a coequalizer, as is $\Delta(D'_1) \rightarrow \Delta(D'_0) \to \Delta(\mathcal{D}')$, where $D'_1 = \mathcal{D}' \times_{\mathcal{D}} D_i$. Now, standard computations in local toric coordinates show that the induced sequence blowups of the strata of $D_0$ and $D_1$ produces stellar subdivision along the corresponding faces of the dual complex, and the end result of this particular sequence of stellar subdivisions is the barycentric subdivision of $\Delta(D_i)$ for $i = 0, 1$ (see [CLS11, Definition 3.3.17 and Exercise 11.1.10]). Hence $\Delta(D'_i)$ is the barycentric subdivision of $\Delta(D_i)$, for $i = 0, 1$. Barycentric subdivision of symmetric $\Delta$-complexes commutes with coequalizers, by the construction in §4.1.1 so we conclude
that $\Delta(\mathcal{D}')$ is the barycentric subdivision of $\Delta(\mathcal{D})$. Furthermore, $\mathcal{D}'$ has simple normal crossings by Remark 6.3 since the dual complex $\Delta(\mathcal{D}')$ is a barycentric subdivision and hence a regular $\Delta$-complex.

We may therefore assume that $\mathcal{D}$ has simple normal crossings. For the remainder of the argument, we closely follow the proof for simple normal crossings divisors in algebraic varieties given in [Pay13, Sections 2 and 4]. The one additional fact needed is that the cohomology of a smooth DM stack $\mathcal{Y}$ with projective coarse moduli space $Y$ is pure, meaning that $H^k$ has pure weight $k$, for all $k$. To see this, note that the natural map $\mathcal{Y} \to Y$ induces an isomorphism $H^*(Y; \mathbb{Q}) \to H^*(\mathcal{Y}; \mathbb{Q})$ (see [Beh04] or [Edi13, Theorem 4.40]) and, since $Y$ is a compact Kähler $V$-manifold, its cohomology is pure [PS08, Theorem 2.43].

Let $\mathcal{D}_1, \ldots, \mathcal{D}_r$ be the irreducible components of $\mathcal{D}$, each of which is smooth with projective coarse moduli space. The weight filtration on the cohomology of $\mathcal{D}$ is determined by the cohomology of the components, their intersections, and the maps between them; indeed, as explained in various sources such as [EZ83, p. 78], [KK98, Chapter 4, §2], and [Bak10], there is a complex of $\mathbb{Q}$-vector spaces

$$0 \to \bigoplus_{i=1}^r H^j(\mathcal{D}_i; \mathbb{Q}) \xrightarrow{\delta_0} \bigoplus_{i_0<i_1} H^j(\mathcal{D}_{i_0} \times \overline{\mathcal{D}}_{i_1}; \mathbb{Q}) \xrightarrow{\delta_1} \bigoplus_{i_0<i_1<i_2} H^j(\mathcal{D}_{i_0} \times \overline{\mathcal{D}}_{i_1} \times \overline{\mathcal{D}}_{i_2}; \mathbb{Q}) \xrightarrow{\delta_2} \cdots,$$

with differentials given by signed sums of restriction maps, and the cohomology of this complex gives the $j$-graded pieces of the weight filtrations on the cohomology groups of $\mathcal{D}$. More precisely, there are natural isomorphisms

$$\text{Gr}_i^W H^{i+j}(\mathcal{D}; \mathbb{Q}) \cong \frac{\ker \delta_i}{\text{im} \delta_{i-1}},$$

for all $i$. In the special case when $j$ is zero, the complex above computes the cellular cohomology of the dual complex $\Delta(\mathcal{D})$, so we obtain natural isomorphisms

$$(6.3.1) \quad W_0 H^j(\mathcal{D}; \mathbb{Q}) \cong H^j(\Delta(\mathcal{D}); \mathbb{Q}),$$

for all $j$.

Then the long exact sequence of the pair $(\overline{\mathcal{X}}, \mathcal{D})$

$$\cdots \to H^{k-1}(\overline{\mathcal{X}}; \mathbb{Q}) \to H^{k-1}(\mathcal{D}; \mathbb{Q}) \to H^k_c(\mathcal{X}; \mathbb{Q}) \to H^k(\overline{\mathcal{X}}; \mathbb{Q}) \to \cdots$$

induces long exact sequences of graded pieces

$$\cdots \to \text{Gr}_i^W H^{k-1}(\overline{\mathcal{X}}; \mathbb{Q}) \to \text{Gr}_i^W H^{k-1}(\mathcal{D}; \mathbb{Q}) \to \text{Gr}_i^W H^k_c(\mathcal{X}; \mathbb{Q}) \to \text{Gr}_i^W H^k(\overline{\mathcal{X}}; \mathbb{Q}) \to \cdots$$

for $0 \leq j \leq 2d$ [PS08, Proposition 5.54].

Taking $j = 0$, and using (6.3.1) together with the fact that $H^k(\overline{\mathcal{X}}; \mathbb{Q})$ is pure of weight $k$, then gives

$$(6.3.2) \quad H^{k-1}(\Delta(\mathcal{D}); \mathbb{Q}) \cong W_0 H^k_c(\mathcal{X}; \mathbb{Q}).$$

Finally, note that the Poincaré duality pairing $H^k_c(\mathcal{X}; \mathbb{Q}) \times H^{2d-k}(\mathcal{X}; \mathbb{Q}) \to \mathbb{Q}$ induces perfect pairings on graded pieces

$$\text{Gr}_j^W H^k_c(\mathcal{X}) \times \text{Gr}_{2d-j}^W H^{2d-k}(\mathcal{X}) \to \mathbb{Q},$$
for $0 \leq j \leq 2k$. Dualizing (6.3.2) therefore gives

$$H_{k-1}(\Delta(D); \mathbb{Q}) \cong \text{Gr}_{2d}^W H^{2d-k}(\mathcal{X}; \mathbb{Q}),$$

as required. \hfill \square

Remark 6.9. The resulting surjection

$$H^{2d-k}(\mathcal{X}; \mathbb{Q}) \to H_{k-1}(\Delta(D); \mathbb{Q})$$

may be rewritten, using Poincaré-Lefschetz duality in the domain and the definition of reduced homology of augmented symmetric $\Delta$-complexes in the codomain, as a surjection

(6.3.3)

$$H_k(\overline{\mathcal{X}}, \partial \overline{\mathcal{X}}; \mathbb{Q}) \to H_k(C(\Delta(D)), |\Delta(D)|; \mathbb{Q}).$$

Written this way, it can be seen that the homomorphism is an instance of the homomorphism (3.4.1) in Example 3.12, applied to the normalization $\tilde{D} \to \overline{\mathcal{X}}$ of the boundary divisor $D \subset \mathcal{X}$.

7. Applications

We now proceed to use the identification of top weight cohomology of $\mathcal{M}_g$ with reduced homology of the symmetric $\Delta$-complex $\Delta_g$ developed in the preceding sections, in combination with known nonvanishing and vanishing results for graph homology and cohomology of $\mathcal{M}_g$, to prove the applications stated in the introduction.

Theorem 1.2. There is an isomorphism

$$\text{Gr}_{6g-6} W H^{6g-6-k}(\mathcal{M}_g; \mathbb{Q}) \cong \tilde{H}_{k-1}(|\Delta_g|; \mathbb{Q}),$$

identifying the reduced rational homology of $\Delta_g$ with the top graded piece of the weight filtration on the cohomology of $\mathcal{M}_g$.

Proof. Let $\mathcal{D} = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. Then $\Delta_g$ is naturally identified with the dual complex $\Delta(\mathcal{D})$, by Corollary 6.7. The theorem is therefore the special case of Theorem 6.8 where $\mathcal{X} = \mathcal{M}_g$ and $\overline{\mathcal{X}} = \overline{\mathcal{M}}_g$. \hfill \square

We now prove our nonvanishing result for $H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$.

Theorem 1.1. The cohomology $H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$ is nonzero for $g = 3$, $g = 5$, and $g \geq 7$. In fact, $\dim H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$ grows at least exponentially; precisely,

$$\dim H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) > \beta^g + \text{constant}$$

for any $\beta < \beta_0$, where $\beta_0 \approx 1.3247 \ldots$ is the real root of $t^3 - t - 1 = 0$.

Proof. By Theorems 1.3 and 1.2 we have a natural surjection $H^{4g-6}(\mathcal{M}_g) \to H_0(G(g))$. Therefore the result follows from Theorem 2.7. \hfill \square

Note that the nonvanishing unstable cohomology group $\text{Gr}_{12}^W H^6(\mathcal{M}_3; \mathbb{Q})$ found by Looijenga [Loo93] is identified with the span of $[W_3]$ in $H_0(G(3))$. Hence, the nonvanishing, unstable, top weight cohomology that we describe, especially those corresponding to the spans of $[W_g]$ for odd $g \geq 5$, may be naturally seen as direct generalizations.

We also record the following nonvanishing result of an odd-degree cohomology group, as discussed in the introduction:
Corollary 7.1. The cohomology group $H^{15}(\mathcal{M}_6; \mathbb{Q})$ is nonzero.

Proof. By Theorems 1.2 and 1.3, the nontrivial class in $H_3(G^{(6)})$ discovered computationally in [BNM] implies nonvanishing of $H^{15}(\mathcal{M}_6; \mathbb{Q})$. □

We conclude with an application in the other direction, using known vanishing results for $\mathcal{M}_g$ to reprove a recent vanishing result of Willwacher for graph homology.

Theorem 1.4. The graph homology groups $H_k(G^{(g)})$ vanish for $k < 0$.

Proof. The virtual cohomological dimension of $\mathcal{M}_g$ is $4g - 5$ [Har86]. Furthermore, $H^{4g-5}(\mathcal{M}_g; \mathbb{Q})$ vanishes [CFP12, MSS13]. Therefore $H^{4g-5-k}(\mathcal{M}_g; \mathbb{Q})$ vanishes for $k < 0$. The theorem follows, since $H^{4g-5-k}(\mathcal{M}_g; \mathbb{Q})$ surjects onto $H_k(G^{(g)})$. □

8. Generalizations of abelian cycles

The injection $H_k(G^{(g)})^\vee \to H_{4g-5-k}(\mathcal{M}_g; \mathbb{Q})$ allows us, in particular, to produce non-zero homology classes in the mapping class group from classes in $\text{grt}_1 \cong H^0(\text{GC}) \cong \prod_g H_0(G^{(g)})^\vee$. It is natural to ask for a more explicit description of the resulting homology classes. In this section we shall outline how to transport a class represented by a cocycle $\alpha: G^{(g)} \to \mathbb{Q}$ through these isomorphisms. More details (and proofs) will appear in a sequel.

Let $\mathcal{M}_g^{\text{thick}} \subset \mathcal{M}_g$ denote the subspace given in the hyperbolic model for $\mathcal{M}_g$ as those hyperbolic surfaces in which no non-trivial geodesic has length less than $\epsilon$, for a suitably small $\epsilon > 0$. Then $\mathcal{M}_g^{\text{thick}} \subset \mathcal{M}_g$ is a deformation retract [HZ86, p. 476]. Its boundary consists of hyperbolic surfaces with at least one geodesic of length $\epsilon$, but it is better regarded as an orbifold with corners. Let us not spell out explicitly what this means, but mention that it comes with a cover by orbifold charts

$$\mathbb{R}^S_\geq 0 \times \mathbb{R}^T \to \mathcal{M}_g^{\text{thick}}$$

for finite sets $S$ and $T$ (varying from chart to chart).

**Definition 8.1.** Let us write $B = \mathcal{M}_g^{\text{thick}}$ and let $A$ consist of pairs of a hyperbolic surface in $B$ together with a choice of closed geodesic of length $\epsilon$. There is a map of orbifolds $A \to B$, locally modeled on the projections

$$\{(s, x, y) \in S \times \mathbb{R}^S_\geq 0 \times \mathbb{R}^T \mid x_s = 0\} \to \mathbb{R}^S_\geq 0 \times \mathbb{R}^T.$$

From this map $A \to B$, define a symmetric semi-simplicial space $[p] \mapsto A_p$ as in Example 3.13.

In other words, $A_p$ is the space of isometry classes of pairs consisting of a hyperbolic genus $g$ surface in $B$ together with an ordered $(p + 1)$-tuple of distinct geodesics of length $\epsilon$, considered up to isometry preserving the ordered tuple. Then $S_{p+1} = I([p], [p])$ acts by permuting the geodesics, and $d_i: A_{p+1} \to A_p$ is induced by forgetting the $i$th geodesic.

Finally, let $\partial^p \mathcal{M}_g^{\text{thick}}$ denote the image of the map $A_p \to \mathcal{M}_g^{\text{thick}}$ induced by $\emptyset \subset [p]$. 
The symmetric $\Delta$-complex defined as $[p] \mapsto \pi_0(A_p)$ is isomorphic to $\Delta_g$. This may be seen by identifying the orbifold underlying $A_p$ with an $(S^1)^{p+1}$-bundle over the complex analytic orbifold underlying $\tilde{D}_p \setminus d_0(\tilde{D}_{p+1})$, up to homotopy, or, more directly, by sending a hyperbolic surface with $(p + 1)$ ordered labeled geodesics to the dual graph of the nodal $2$-manifold obtained by collapsing the geodesics.

A cochain $C^{(q)}_k \to \mathbb{Q}$ is naturally identified (by extending to zero on graphs with loops and weights) with a cochain $\alpha \in C^p(\Delta_g; \mathbb{Q})$. By definition, such a cochain is a function $\alpha : \Delta_g([p]) = \pi_0(A_p) \to \mathbb{Q}$ which is alternating under the action of $S_{p+1}$ on $A_p$. Hence we may regard such a cochain as an element $\alpha \in H^0(A_p; \mathbb{Q})$ on which a permutation $\sigma \in S_{p+1}$ acts as $\text{sgn}(\sigma)$. Such a cochain is a cocycle exactly when it is in the kernel of

$$(H^0(A_p; \mathbb{Q}) \otimes \mathbb{Q}^{\text{sgn}})_{S_{p+1}} \xrightarrow{\sum(-1)^i(d_i)^*} (H^0(A_{p+1}; \mathbb{Q}) \otimes \mathbb{Q}^{\text{sgn}})_{S_{p+2}}.$$

Now, each $A_p$ is a rational homology manifold with boundary, and comes with a canonical orientation $[A_p] \in H_{d-p}(A_p, \partial A_p; \mathbb{Q})$, where $d = 6g - 7$; this comes from identifying the orbifold underlying $A_p$ with an $(S^1)^{p+1}$ bundle over the complex analytic orbifold underlying $\tilde{D}_p$, and combining the orientation on $\tilde{D}_p$ coming from its complex structure with the orientation on the fibers of the bundle induced by the ordering of the geodesics. These orientations are compatible: this means that $\sigma \in S_{p+1}$ acts on $[A_p]$ as $\text{sgn}(\sigma)$, and that the homomorphism $(d_i)_* : H_{d-p-1}(A_{p+1}, \partial A_{p+1}) \to H_{d-p-1}(\partial A_p, d_i(\partial A_{p+1}))$ sends $[A_{p+1}]$ to the image of $[A_p]$ under the connecting homomorphism for the triple

$$d_i(\partial A_{p+1}) \subset \partial A_p \subset A_p.$$

Poincaré duality, i.e., cap product with these fundamental classes, now identifies the above homomorphism with a homomorphism

$$(8.1) \quad H_{d-p}(A_p, \partial A_p; \mathbb{Q})_{S_{p+1}} \xrightarrow{\sum(d_i)^*} H_{d-p-1}(A_{p+1}, \partial A_{p+1}; \mathbb{Q})_{S_{p+2}},$$

where the signs in both the $S_{p+1}$ action and the boundary homomorphism have canceled with those in the fundamental classes. A cocycle $\alpha \in C^p(\Delta_g; \mathbb{Q})$ gives a Poincaré dual $\text{PD}([\alpha]) \in H_{d-p}(A_p, \partial A_p; \mathbb{Q})_{S_{p+1}}$ in the kernel of $(8.1)$. Mapping into $A_{-1}$ sends all spaces into $\partial^pM_g^{\text{thick}}$ and the map $(8.1)$ fits into a commutative square

$$\begin{array}{ccc}
H_{d-p}(A_p, \partial A_p; \mathbb{Q})_{S_{p+1}} & \xrightarrow{1/(p+1)!} & H_{d-p-1}(A_{p+1}, \partial A_{p+1}; \mathbb{Q})_{S_{p+2}} \\
\downarrow & & \downarrow \frac{1}{(p+2)!} \\
H_{d-p}(\partial^pM_g^{\text{thick}}, \partial^{p+1}M_g^{\text{thick}}; \mathbb{Q}) & \xrightarrow{1/(p+2)!} & H_{d-p-1}(\partial^{p+1}M_g^{\text{thick}}, \partial^{p+2}M_g^{\text{thick}}; \mathbb{Q}),
\end{array}$$

where the bottom row is the connecting homomorphism for the triple. The class $\text{PD}([\alpha])$ in the upper left corner therefore maps to a class in $H_{d-p}(\partial^pM_g^{\text{thick}}, \partial^{p+1}M_g^{\text{thick}})$, which admits a lift to homology relative to $\partial^{p+2}M_g^{\text{thick}}$. Since that space has no homology above degree $(d - p - 2)$, another long exact sequence shows that this class lifts uniquely to $H_{d-p}(\partial^pM_g^{\text{thick}}, \mathbb{Q})$. By a similar argument, one checks that the image of this class in $H_{d-p}(\partial^{p-1}M_g^{\text{thick}}; \mathbb{Q})$ is unchanged by adding a coboundary to $\alpha$, and hence one gets a well defined class in $H_{d-p}(M_g^{\text{thick}}; \mathbb{Q})$ depending only on the cohomology class $[\alpha] \in H^p(\Delta_g; \mathbb{Q})$. 

In the special case where \( p = 3g - 4 \), generators of \( C^p(\Delta_g; \mathbb{Q}) \) are trivalent graphs and automatically cocycles since \( C^{p+1}(\Delta_g; \mathbb{Q}) = 0 \), and the resulting classes in

\[
H_{3g-3}(\mathcal{M}_g; \mathbb{Q}) = H_{3g-3}(\text{Mod}_g; \mathbb{Q})
\]

are exactly the abelian cycles associated to maximal collections of commuting Dehn twists. In this way, homology classes on \( \mathcal{M}_g \) associated to graph cohomology classes in \( H^*(\mathcal{G}^{(g)}) \) may be seen as generalizations of abelian cycle classes for the mapping class group.

**References**


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