EHUD HRUSHOVSKI, SIMONS SYMPOSIUM TALK, APRIL 4, 2013

1. Geometric Integration over valued fields

I was asked to talk about work with Kazhdan on integration over valued fields.[1] The integration theory of [HK06], [HK08] differs from the logical motivic integration of Denef, Loeser, Cluckers, discussed in the preceding talk by Loeser, mainly in that we work with an arbitrary value group Γ rather than a discrete value group $\mathbb{Z}$, and take into account definable sets in Γ^n. Notably, this includes the case $\Gamma = \mathbb{R}$. Such definable sets are piecewise linear and closely connected to tropical geometry.

We consider the first order theory of valued fields VF. The basic structure includes addition and multiplication on a field F, and a valuation map val into an ordered Abelian group Γ.[2] In particular the residue ring $\mathcal{O} = \{x : \mathrm{val}(x) \geq 0\}$ and the maximal ideal $\mathcal{M} = \{x : \mathrm{val}(x) > 0\}$ can be defined. The residue map from the valuation ring $\mathcal{O}$ into the residue field k can then be interpreted, or can be taken itself as basic. Geometric motivic integration essentially involves both of these maps.

Integration was first studied from a logical viewpoint by Denef, in order to shed light on the existing theory of integration for $\mathbb{Q}_p$. Here we concentrate on fields such as the field of Puiseux series $K_{Puiseux} = \bigcup_n \mathbb{C}((t^{1/n}))$, with value group $\mathbb{Q}$, where the classical theory - based on local compactness - does not apply at all. The residue field is assumed to have characteristic zero. (the results apply in large enough characteristic, for a problem of given 'degree'.)

By a theorem of Ax and Kochen ([AK66]), the truth or falsity of a first order sentence in the Henselization of F is determined by the truth value of certain other sentences of the residue field and value group of F. The integration theory we describe depends on an extension of this result from *sentences*, or formulas in 0 variables, that can only be true or false, to *formulas in n variables*, that define a subset of n-space over the value field. We show that a definable set can be transformed into one arising in a simple manner from definable sets over the residue field and value group; it follows that appropriate integration theories on the residue field and value group yield an integration theory on the valued field.

Quantifier elimination. The basic framework is ACVF_F, the first order theory of algebraically closed valued fields extending the valued field F. Elimination of quantifiers in this theory is due to A. Robinson [Rob56]. There is a parallel analytic version of this theory, where similar results have been proved by Lipshitz and Z. Robinson [Li93], [LR98].

[1]This text is based on notes taken by Sam Payne during the talk and very kindly provided to me.

[2]One may allow analytic functions if desired; the results reported here remain valid. But it is also valuable to know that local geometry can be carried out purely semi-algebraically, analogously to Nash geometry over the reals.

A definable subset $D \subset K^n$ is a finite Boolean combination of sets cut out by conditions of the form

$$\mathrm{val}(f) \leq \mathrm{val}(g) \quad \text{or} \quad h = 0$$

for polynomials f, g, and h with coefficients in F. These are F-semialgebraic sets.

Residue conditions are also definable in this theory. For instance, the condition $\mathrm{res}(x) = \mathrm{res}(y)$ is equivalent to $\mathrm{val}(x) = \mathrm{val}(y)$ and $\mathrm{val}(x) < \mathrm{val}(x - y)$.

Such sets should really be called *quantifier-free definable*, as the general notion of definability allows quantifiers $(\exists x)(\forall y) \cdots$. The term *definable sets* is justified in this setting by Robinson's quantifier elimination theorem, stating that *this class of sets is closed under coordinate projections* (or equivalently, existential and universal quantifiers.)

Consequences of quantifier elimination. On the value group, the only basic relations are the addition map, and the linear ordering. One corollary of quantifier-elimination in this language is that any subset of Γ^n definable using the valued field structure - for instance, the image under the tropicalization map of an algebraic set - is in fact cut out by $\mathbb{Q}$-linear inequalities. This statement, or similar ones in more concrete form, is encountered at the foundations of tropical geometry (as we heard in Diane Maclagan's talk.)

Similarly for the residue field: in the ACVF setting, a definable subset of k^n is by definition one whose pullback to $\mathcal{O}^n$ is definable, possibly using quantifiers over the valued field. But Robinson's theorem implies that the definable subsets of k^n are precisely the constructible sets in the sense of algebraic geometry (finite Boolean combinations of Zariski closed sets.)

This also holds jointly: definable subsets of $k^n \times \Gamma^n$ are generated by rectangles, products of a definable subset of each. In this sense the residue field and the value group are orthogonal parts the first order theory of the valued field.

Another corollary is that all algebraic closed valued extensions $K|F$ have the same first order properties. In particular, there is a Nullstellensatz, which says that a semialgebraic set with points over some valued extension of F has points over every algebraically closed valued extension.

An interesting model of $ACVF_{\mathbb{C}}$ was introduced by A. Robinson.

Example 1.1. Fix an ultrafilter u on $\mathbb{N}$ (viewed as an element of the Stone-Cech compactification of $\mathbb{N}$). Consider sequences $a_n \in \mathbb{C}$ satisfying the growth condition

$$|a_n| \leq n^k$$

for some k. After factoring out null sequences, this gives a valued field, with valuation given by

$$\mathrm{val}(a) = \lim_{n \to u} \frac{-\log |a_n|}{\log n}$$

The valuation ring $\mathcal{O}_{Robinson}$ consists of elements represented by bounded sequences.

Strictly speaking $K_{Robinson}$ depends on the choice of u; however the first order theory does not depend on the choice of u, as follows from Robinson's theorem.

See [KT] for an application: Fix a left-invariant metric on a real Lie group such as $SL_n(\mathbb{R})$. Looking at this metric space from a distance approaching infinity, it resembles more and more closely the building $SL_n(K_{Robinson})/SL_n(\mathcal{O}_{Robinson})$, in a very precise sense. In particular the limit is "essentially" independent of choices, as conjectured by Gromov.

Another application is related to Grigory Mikhalkin's talk in this conference. Consider for instance a rational polytope $P \subset \mathbb{R}^n$. It can be viewed as a definable set $X \subset \mathbb{R}^n$ and we take $\Gamma = \mathbb{R}$. By Robinson's theorem, P is the image under tropicalization of a semi-algebraic (or an algebraic) subset of $K_{Puiseux}$ if and only if it is the image of such a subset of $K_{Robinson}$. Now realizability over $K_{Robinson}$ is equivalent to being the limit of the amoebas of a sequence of complex varieties. Therefore, a consequence of elimination of quantifiers for ACVF is that a tropical variety is realizable as a limit of amoebas if and only if it is the tropicalization of a variety over Puiseux series.

Proofs of quantifier elimination. Model theory often offers conceptual but ineffective proofs of quantifier elimination, based on compactness arguments. Let me illustrate this by showing that pushforward by a finite morphism preserves the class of quantifier-free definable sets. We can reduce to the case of a finite Galois cover $f : X \to Y$, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, for F-algebras A and B; let G be the Galois group. Define $\mathcal{V}(X)$ to be the space of all valuations on A that extend the given valuation on F. It is a compact but totally disconnected space, equipped with a topology in which $\{x : \nu_x(f) \geq \nu_x(g)\}$ is both open and closed; the clopen sets correspond precisely to their Boolean combinations, i.e. the quantifier-free definable sets. Thus quantifier elimination for finite maps amounts to showing that the restriction map $\mathcal{V}(X) \to \mathcal{V}(Y)$ is closed and open. But indeed the induced map $\mathcal{V}(X)/G \to \mathcal{V}(Y)$ is continuous and bijective, hence a homeomorphism.

The general case - pushforward under an arbitrary morphism - is still based on compactness but a model-theoretic form is required, namely the compactness theorem of first order logic.

Regardless of how it is proved quantifier elimination is always effective ipso facto, in the sense of Gödel's general recursive functions: there is an algorithm to find a quantifier-free definable set equivalent to a given formula, guaranteed to terminate, but with no estimate provided on the runtime. In various cases of valued fields, Weispfennig has given primitive-recursive procedures, [We84]. In discussions with Bernd Sturmfels and Diane Maclagan in this meeting, we noted existence of algorithms with multiply exponential running times.

The Grothendieck (semi)rings of definable sets. Geometric motivic integration works with a slight generalization of the residue field k, namely Temkin's graded residue field (cf. [Tem04], rediscovered as RES in [HK06].) It is just the union of all k^*-torsors of the form $\{x : \mathrm{val}(x) = \alpha\}/1 + \mathcal{M}$, with the class of definable sets induced from ACVF_F; where $\alpha = \mathrm{val}(f)/m$ for some $f \in F, m \in \mathbb{N}$. Over an an algebraically closed base field F, one can identify RES with (a redundant presentation of) the residue field k. We will not expand on this point in the present talk.

We study the semiring of definable sets up to definable bijections

$$K_+(VF) = \{[X] : X \subset K^n \text{ definable over } F\}$$

where $[X] = [Y]$ if and only if there is a definable bijection $f : X \xrightarrow{Y}$. This is the Grothendieck semi-ring of the theory; disjoint union, direct product being the sum and product operations.

For many applications it is helpful to pass from this semiring to the associated ring, i.e. formally adjoing additive inverses. However, additive inverses lead to considerable loss of information. In some cases the associated ring may even be zero; for instance the bijection $n \mapsto n+1$ in the theory of the $\mathbb{N}$ yields $[\{0\}] = [\mathbb{N}] - [\mathbb{N}^{>0}] = 0$. The semiring setting is in addition smoothly compatible with operations such as passing to rational points over a Henselian subfield, restricting to bounded setes, etc. We thus develop the foundations in the semiring setting.

Volumes. The semiring $K_+(VF)$ has the properties of a universal Euler characteristic. For motivic integration, it is also natural to consider $K_+ \operatorname{vol}(VF)$, the semiring of definable sets up to definable measure preserving bijections. Here, a bijection f is measure preserving if

$$\operatorname{val}(\det(\partial f_i / \partial x_i)) = 0$$

away from a proper closed subvariety. (Any definable map is almost everywhere differentiable; this can be shown abstractly, or else as a corollary of quantifier-elimination; in ACVF_F, any definable map is piecewise algebraic.) Any possible integration theory on definable sets will factor through $K_+ \operatorname{vol}(VF)$. We will in fact be able to classify them completely.

The results for K_+, $K_+\mathrm{vol}$ are entirely parallel; moreover it turns out that K_+^{vol} can easily be described in terms of K_+.

Canonical lifts; embeddings of $K_+\Gamma, K_+ \mathrm{RES}$ into $K_+(VF)$. The semirings of both Γ and RES, reflecting the tropical and algebro-geometric worlds, each admit natural embeddings into $K_+(VF)$. Let us begin with the tropical analog, the scissors-congruence semigroup of definable subsets of Γ^n, with respect to the group $GL_n(\mathbb{Z}) \ltimes \Gamma(F)^n$. In other words we form We form a Grothendieck semiring $K_+(\Gamma[n])$ as above; an element is an equivalence class of definable subsets of Γ^n, where X, Y are equivalent if there exists a definable bijection $f : X \to Y$ that is piecewise in $GL_n(\mathbb{Z}) \ltimes \Gamma(F)^n$. In other words $X = \bigcup_{i=1}^m X_i$ with X_i definable, and there exist matrices $M_i \in GL_n(\mathbb{Z})$ and an element $\gamma_i \in \Gamma(F)^n$, such that for $x \in X_i$, $f(x) = M_i x + \gamma_i$. The *volume preserving* transformations are those that, in addition, preserve the function $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^m x_i$, $\Gamma^n \to \Gamma$.

It is easy to see that if X, Y are equivalent, via f, then f lifts to a definable bijection between their pullbacks to K^n. Thus there exists a natural homomorphism

$$K_+(\Gamma) \to K_+(VF)$$

taking a definable subset $Z \subset \Gamma^n$ to its "tropical pullback"

$$\operatorname{Trop}^{-1}(Z) = \{x \in (K^*)^n \ : \ \operatorname{val}(x) \in Z\}.$$

Similarly, in each dimension n, there is a natural pullback map

$$K_+(\mathrm{RES})[n] \to K_+(VF).$$

For a Zariski open subset $W \subset k^n$, the map simply takes $[W]$ to the class of the inverse image in the valuation ring, $\mathrm{res}^{-1}(W) \subset \mathcal{O}^n$. But here we must also consider étale maps $h : W \to k^n$; in an appropriate sense $[W]$ is mapped to $[W \times_h \mathcal{O}^n]$. In particular if $W \cong V(k)$ for a smooth scheme V over $\mathcal{O}$, then $[W] \mapsto [V(\mathcal{O})]$.

We write $K_+(\cdot)[*]$ for the direct sum of the $K_+(\cdot)[n]$ in each dimension. Here is the basic statement.

Theorem 1.2. *The natural map*

$$K_+(\Gamma)[*] \otimes K_+(\mathrm{RES})[*] \to K_+(VF)$$

is surjective. The same is true for the semirings K_+^{vol}.

The statement is effective: given a definable $X \subset K^n$, one has to find a partition $X = \bigcup_{i=1}^r X_i$, and a definable transformation $g_i : X_i \to Y_i$ where Y_i is a 'rectangle', product of a canonical lift from RES^l with a canonical lift from Γ^k, $k + l = n$. In fact the transformations will be generalized transvections, compositions of functions that translate one variable by an additive quantity depending on the rest of the variables (which remain fixed); thus when it comes to volumes, they are automatically measures-preserving.

Proof of surjectivity. The proof of the theorem is easy, and we sketch it. We first put together Γ and RES geometrically, into a single structure RV; it is defined to be $K/(1 + \mathcal{M})$, with the induced structure. The multiplicative structure on RV fits into an exact sequence $1 \to k^* \to RV \to \Gamma \to 0$. (It may not seem very different from RES, but RES is the pullback of $\Gamma(F^{alg})$ whereas RV is a definable set, whose set of points, and their image in Γ, depends functorially on $K \geq F^{alg}$.) The map of the theorem factors as

$$K_+(\Gamma)[*] \otimes K_+(\mathrm{RES})[*] \to K_+(RV)[*] \to K_+(VF)$$

and one has to show surjectivity of both arrows. For the left arrow, this is essentially the orthogonality of RES and Γ referred to above; a definable subset of $\mathrm{RES} \times \Gamma$ is a finite union of rectangles, and while the exact sequence of RES is not split, still a definable subset can be moved to a finite union of rectangles.

For the second one, one needs to consider a slight generalization of the theorem, applying to subsets of $K^n \times RV$ (admitting projections to K^n with finite fibers.) This allows a straightforward reduction to relative dimension 1 and then, by a devissage / compactness argument, to dimension one over an extension field of F (which however may be relabeled as F now). We are thus reduced to the case of $X \subset K$. Here quantifier-elimination presents X very as a Boolean combination of a finite set of balls and a finite set of points, and the argument becomes hands-on and elementary. For instance if X is a closed ball, defined it suffices to move it so that it contains 0, and then it is a pullback of a subset of Γ. The delicate point is to make sure of Galois invariance, essential for the devissage argument. X may be for instance a closed ball minus a finite set of maximal open sub-balls, without centers in F. In this case it can be moved to the pullback of a Zariski open subset of the affine line over k (whose complement is a finite set without $k(F)$-rational points.) The general case is a combination of these.

This ends the sketch of the proof. Most classical applications of motivic integration follow from this surjectivity. For instance, we can recover the results of Denef and Loeser [DL01] on expressing uniformly in p the values of p-adic integrals. Here is a version for volumes of definable sets that follows immediately from the volume version of Theorem 1.2.

Corollary 1.3. *Given a polynomial f with coefficients in $\mathbb{Q}$, there are $r \in \mathbb{N}$, affine schemes $W_i \subset \mathbb{A}^{m_i}_{\mathbb{Z}}$, and rational polytopes $Y_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, r$, such that for large enough primes p,*

$$\mathrm{vol}\{x \in \mathbb{Q}_p^n : \mathrm{val}\, x_i \geq 0, \mathrm{val}(f(x) - p) = 1\} = \sum_{i=1}^{r} p^{-m_i} |W_i(k)| \cdot \mathrm{vol}\{x \in \mathbb{Q}_p^n : \mathrm{val}(x) \in Y_i(\mathbb{Z})\}$$

The left hand side is a kind of p-adic Milnor fiber, given by way of example; it could be replaced by an arbitrary bounded definable set. On the right we see the finite field contribution $W_i(k)$; each point has weight p^{-m_i} corresponding to the volume it contributes; multiplied by the Γ-contribution, the volume of the pullback of $Y_i(\mathbb{Z}) := Y_i \bigcap \mathbb{Z}^n$ (since here $\Gamma = \mathbb{Z}$.) The latter, combinatorial expression can be written explicitly as $\sum_N a_N p^{-N}$, where $a_N = \{x \in Y_i \bigcap \mathbb{Z}^{n_i} : \sum_{j=1}^{n_i} x_j = N\}|$, and reduced to a finite expression by various methods (including 'motivically'; see below.)

The kernel. Understanding the *kernels* the maps of Theorem 1.2 is harder. The final answer is however simple. The tensor product should be understood to be over the semiring of finite subsets of Γ, since for instance the canonical lift of the point $0 \in \Gamma$ is the same as the lift of $G_m(k) \subset RES$; namely $\mathcal{O} \setminus \mathcal{M}$. One additional relation generates the kernel:

$$[\Gamma^{>0}]_1 \otimes 1 + 1 \otimes [0]_0 = 1 \otimes [1]_1$$

On the left side, one sees the positive ray of Γ, whose canonical lift is $\mathcal{M} \setminus \{0\}$; adding a point in dimension 0, one has a class whose lift is $\mathcal{M}$. On the right hand side, one has the lift of the element 1 of the residue field; this is the ball $1 + \mathcal{M}$. Translation by 1 shows that these classes coincide in $K_+(VF)$. It turns out that this relation generates the entire kernel, in both the case of K_+ and $K_+ \mathrm{vol}$.

The Grothendieck semiring of the value group. What can be said about the two component semirings individually? Let us pass to the associated rings. In order not to lose too much information, on the Γ side, we restrict to definable sets that are bounded below.

The Grothendieck ring of RES is a version of the Grothendieck ring of varieties over a field F, deeply related to motivic algebraic geometry. Sometimes motivic integration can yield information regarding it, see the results on complements below.

The Grothendieck ring of Γ is related to tropical geometry and is much more susceptible, in principle, to a complete analysis. Over an algebraically closed base (a restriction that can probably be removed), and with rational coefficients, this is carried out in [HK08]; the method is integration by parts, with attention to arithmetic issues. (We must give formulas for the content of polytopes that works both for volumes over $\mathbb{R}$ and for number of points over $\mathbb{Z}$, as well as over non-archimedean situations.)

As a corollary we can determine the Grothendieck rings with rational coefficients; here is the result for K_{vol}. The associated dimension-free ring $K_{vol,df}$ is basically the ring of quotients of two volumes of the same dimension (see [HK08], Theorem 3.24 for details.).

Theorem 1.4. *Let F be algebraically closed, $A = \Gamma(F)$. The homomorphism of Theorem 1.2 induces an isomorphism*

$$K_{vol,df}(VF)' \cong K^{df}_{\mathbb{Q}}(\mathrm{Var}_F)[t^A, q^A]'$$

On the right we have polynomial rings, with exponents from A, and two kinds of variables: q^α denotes the normalized volume of a ball of radius α; t^α is a logarithmic quantity, the normalized volume of the hyperbolic region $\{(x,y) : 0 \leq \mathrm{val}(x) = -\mathrm{val}(y) \leq \alpha\}$. The ring $K^{df}_{\mathbb{Q}}(\mathrm{Var}_F)$ is the Grothendieck ring of varieties over F with rational coefficients, and with $[G_m]$ inverted, but where we retain only classes $\frac{a}{[G_m]^n}$ with $\dim(a) \leq n$. We also invert $L^n - 1$ where $L = 1 + \frac{[pt]}{[G_m]}$ is the ratio of class the affine line to a point; this localization and the corresponding one on the left are denoted by adding $'$ following the notation for the ring. This reduces integration theories on valued-field definable sets over F to quantities formed algebraically from varieties over the residue field.

Application to varieties with isomorphic complements. The second component the Grothendieck semiring of varieties, admits no such simple description. When passing to the ring, an additional difficulty arises, in that equality of classes $[X] = [Y]$ is not defined via any direct relation between two classes X, Y, but rather between their complements in some possibly higher dimensional variety Z into which both embed. The geometric meaning of this for X, Y is not transparent. Assuming X, Y are curves over $\mathbb{Q}$, by counting points of X, Y, Z over finite fields in and subtracting, we see that X, Y have the same number points over large finite fields; when one of them has genus ≥ 2, it follows by Faltings that they are birational. It was this observation that led Kontsevich and Gromov to ask if X, Y must always be birational. This (inter alia) was proved by Larsen-Luntz [LL04] using a strong form of resolution of singularities. It turns out that motivic integration can also, and very naturally, return information about this question.

We have an constructible isomorphism $f : Z(\mathbb{C})\backslash X \to Z(\mathbb{C})\backslash Y$. It extends to any bigger algebraically closed field; in particular taking $K = K_{Puiseux}$, we have $f : Z(K) \setminus X \to Z(K) \setminus Y$. View Z as a scheme over $\mathcal{O}_K$ (descending to k); and let $\rho : Z(\mathcal{O}_K) \to Z(k)$ be the residue map. Then f restricts to a constructible bijection $\rho_V^{-1}(X) \setminus X \to \rho_V^{-1}(Y) \setminus Y$. Thus - as lower dimensional sets can be ignored - $\rho_V^{-1}(X), \rho_V^{-1}(Y)$ have the same motivic volume. Computing this volume we find that $[X \times \mathbb{A}^{n-d}] = [Y \times \mathbb{A}^{n-d}]$ in the Grothendieck group of n-dimensional varieties. We obtain:

Theorem 1.5. *Let X, Y be two smooth, proper d-dimensional subvarieties of of a smooth n-dimensional variety Z, and assume $Z \setminus X, Z \setminus Y$ are isomorphic. Then X, Y are stably birational, i.e. $X \times \mathbb{A}^{n-d}, Y \times \mathbb{A}^{n-d}$ are birationally equivalent. If X, Y contain no rational curves, then X, Y are birationally equivalent.*

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