NON-ARCHIMEDEAN LIMITS OF DEGENERATING FAMILIES OF COMPLEX CALABI-YAU VARIETIES

MATTIAS JONSSON

Abstract. We discuss the construction of "hybrid" analytic spaces that contain both Archimedean and non-Archimedean information, and their applications to various problems involving degenerations, with particular emphasis on degenerations of Calabi-Yau varieties.

Introduction

1. Berkovich spaces over Banach rings

Among the different approaches to non-Archimedean geometry (see [Con08] for a nice introduction), the one by Berkovich [Berk90] has the interesting feature that the basic definitions make sense over general Banach rings.

Namely, to any Banach ring $(A, \|\cdot\|)$ is associated a space, the *Berkovich spectrum* $M(A)$ consisting of all multiplicative seminorms on A bounded by the given norm on A. It has a natural topology, namely the weakest one in which the evaluation map $M(A) \ni |\cdot| \to |a| \in \mathbf{R}_+$ is continuous for all $a \in A$. Excluding the case of the zero ring, the Berkovich spectrum $M(A)$ is a nonempty compact Hausdorff space.

Now consider a scheme X of finite type over A. To this data, Berkovich in [Berk09] associates a continuous map

$$X^{\mathrm{An}} \to M(A) = (\operatorname{Spec} A)^{\mathrm{An}}$$

of topological spaces[1] satisfying various nice properties. For example, X^{An} is locally compact and locally path connected. It is compact when X is proper.

If X is affine, that is, $X = \operatorname{Spec} B$ for a finitely generated A-algebra B, then X^{An} is the set of multiplicative seminorms on B whose restrictions to A are bounded. In general, X^{An} is obtained by gluing the analytifications of open affine subschemes of X.

We now list several examples of the analytifications procedure above.

1.1. **Complex analytic spaces.** When $A = \mathbf{C}$ is the field of complex numbers, equipped with the usual (Archimedean) norm $|\cdot|_\infty$, then the analytification X^{An} of a scheme of finite type over $\mathbf{C}$ is the complex analytic space X^{hol} associated to X. As a set, X^{hol} equals $X(\mathbf{C})$, the set of $\mathbf{C}$-valued points of X. The functor $X \to X^{\mathrm{hol}}$ has various nice properties known as GAGA [Ser55].

Date: May 25, 2017.

[1] The superscript in X^{An} refers to analytification, but at this point we only consider topological spaces. See [Poi10, Poi13] for some properties of the naturally defined structure sheaf on X^{An}.

1.2. **Berkovich spaces.** Now assume that A is a non-Archimedean field k, i.e. a field equipped with a multiplicative non-Archimedean norm. In this case, X^{An} is a (good, boundaryless) k-analytic space in the sense of [Berk90, Berk93]. Such spaces are usually simply called Berkovich spaces, and one writes X^{an} instead of X^{An}. The functor $X \to X^{\mathrm{an}}$ also satisfies various GAGA properties [Berk90, §3].

There are two non-Archimedean fields of particular relevance for complex degenerations. First, we can equip $\mathbf{C}$ with the *trivial* norm $|\cdot|_0$, in which $|a|_0 = 1$ for all $a \in \mathbf{C}^*$. Second, given $r \in (0,1)$, we can equip the field $\mathbf{C}(\!(t)\!)$ of complex Laurent series with the following norm $|f| = r^{\operatorname{ord}_0(f)}$, where $\operatorname{ord}_0(\sum_i a_i t^i) = \min\{i \mid a_i \neq 0\}$.

1.3. **The hybrid norm on C.** Denote by $\mathbf{C}_{\mathrm{hyb}}$ the Banach ring $(\mathbf{C}, \|\cdot\|_{\mathrm{hyb}})$, where the *hybrid norm* is defined as

$$\|\cdot\|_{\mathrm{hyb}} := \max\{|\cdot|_0, |\cdot|_\infty\},$$

with $|\cdot|_0$ the trivial absolute value and $|\cdot|_\infty$ the usual absolute value.

The elements of the Berkovich spectrum $\mathcal{M}(\mathbf{C}_{\mathrm{hyb}})$ are of the form $|\cdot|_\infty^\rho$ for $\rho \in [0,1]$, interpreted as the trivial absolute value $|\cdot|_0$ for $\rho = 0$. This yields a homeomorphism $\mathcal{M}(\mathbf{C}_{\mathrm{hyb}}) \simeq [0,1]$.

1.4. **Hybrid geometry over C.** If X is a scheme of finite type over $\mathbf{C}$, we denote, as above, by X^{hol} its analytification with respect to the usual absolute value $|\cdot|_\infty$, by X_0^{an} its analytification with respect to the trivial absolute value, and by X^{hyb} its analytification with respect to the hybrid norm $\|\cdot\|_{\mathrm{hyb}}$.

The structure morphism $X \to \operatorname{Spec} \mathbf{C}$ gives rise to a continuous map

$$\lambda\colon X^{\mathrm{hyb}} \to \mathcal{M}(\mathbf{C}_{\mathrm{hyb}}) \simeq [0,1].$$

The fiber $\lambda^{-1}(\rho)$ is equal to the analytification of X with respect to the multiplicative norm $|\cdot|_\infty^\rho$ on $\mathbf{C}$. In particular, we have canonical identifications $\lambda^{-1}(1) \simeq X^{\mathrm{hol}}$ and $\lambda^{-1}(0) \simeq X_0^{\mathrm{an}}$. For $0 < \rho \le 1$, the fiber $\lambda^{-1}(\rho)$ is also homeomorphic to X^{hol}. In fact, we have a a homeomorphism

$$\lambda^{-1}((0,1]) \simeq (0,1] \times X^{\mathrm{hol}},$$

see [Berk09, Lemma 2.1]. See Figure 1 for an illustration of $(\mathbf{P}^1)^{\mathrm{hyb}}$.

1.5. **The hybrid circle.** Now consider the *hybrid circle* of radius $r \in (0,1)$, that is, $C_{\mathrm{hyb}}(r) := \{|t| = r\} \subset \mathbf{A}^{1,\mathrm{hyb}} = (\operatorname{Spec} \mathbf{C}[t])^{\mathrm{hyb}}$. By [Poi10, Prop 2.1.1], this is compact and realized as the Berkovich spectrum of the Banach ring

$$A_r := \left\{ f = \sum_{\alpha \in \mathbf{Z}} c_\alpha t^\alpha \in \mathbf{C}(\!(t)\!) \;\middle|\; \|f\|_{\mathrm{hyb}} := \sum_{\alpha \in \mathbf{Z}} \|c_\alpha\|_{\mathrm{hyb}} r^\alpha < +\infty \right\}.$$

Since $\|c_\alpha\|_{\mathrm{hyb}} \ge |c_\alpha|_\infty$, every $f \in A_r$ defines a continuous function f^{hol} on the punctured closed disc $\overline{\mathbf{D}}_r^*$ that is holomorphic on $\mathbf{D}_r^*$ and meromorphic at 0. In fact, there is a homeomorphism $\overline{\mathbf{D}}_r \xrightarrow{\sim} \mathcal{M}(A_r) \simeq C_{\mathrm{hyb}}(r)$, that maps $z \in \overline{\mathbf{D}}_r \subset \mathbf{C}$ to the seminorm on A_r defined by

$$|f| = \begin{cases} r^{\operatorname{ord}_0(f)} & \text{if } z = 0 \\ r^{\frac{\log |f^{\mathrm{hol}}(z)|_\infty}{\log |z|_\infty}} & \text{otherwise,} \end{cases} \tag{1.1}$$

and via which the map $\lambda\colon C_{\mathrm{hyb}}(r) \to [0,1]$ is given by $\lambda(z) = \frac{\log r}{\log |z|_\infty}$, see [BJ17, Proposition A.4].

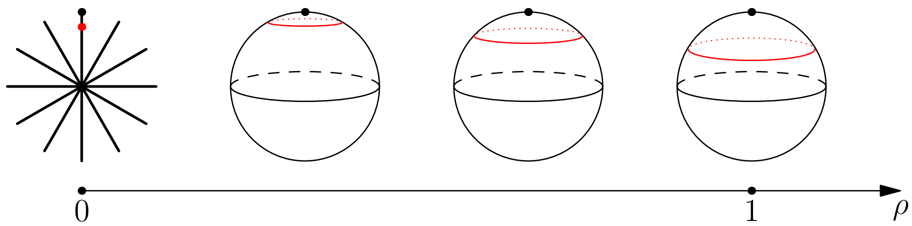

FIGURE 1. The hybrid space $\mathbf{P}^{1,\mathrm{hyb}}$ defined as the analytification of the complex projective line with respect to the hybrid norm $\|\cdot\|_{\mathrm{hyb}}$ on $\mathbf{C}$, together with the canonical map $\lambda\colon \mathbf{P}^{1,\mathrm{hyb}} \to [0,1]$. The fiber $\lambda^{-1}(0)$ is the analytification of $\mathbf{P}^1$ with respect to the trivial norm, and looks like a cone over $\mathbf{P}^1(\mathbf{C})$. All the other fibers are homeomorphic to a sphere. The points on top form a continuous section of λ. The smaller circle in the fiber $\lambda^{-1}(\rho)$ is of radius $e^{-1/\rho}$; these circles converge as $\rho \to 0$ to a unique point in the fiber $\lambda^{-1}(0)$.

1.6. **Geometry over the hybrid circle.** Let now X be a scheme of finite type over A_r for some $r \in (0,1)$. We will associate to X three kinds of analytic spaces.

First, since X is obtained by gluing together finitely many affine schemes cut out by polynomials with coefficients holomorphic on $\mathbf{D}_r^* \subset \mathbf{C}$ and meromorphic at 0, we can associate to X in a functorial way a complex analytic space X^{hol} over $\mathbf{D}_r^*$, which we call its *holomorphic analytification*.

Second, since A_r is contained in $\mathbf{C}((t))$, we may also consider the base change $X_{\mathbf{C}((t))}$ and its *non-Archimedean analytification* $X^{\mathrm{an}}_{\mathbf{C}((t))}$ with respect to the non-Archimedean absolute value r^{ord_0} on $\mathbf{C}((t))$.

Finally, denote by X^{hyb} the analytification of X as a scheme of finite type over the Banach ring A_r, and call it the *hybrid analytification* of X. It comes with a continuous structure map

$$\pi\colon X^{\mathrm{hyb}} \to \overline{\mathbf{D}}_r \simeq M(A_r),$$

and we have canonical homeomorphisms

$$\pi^{-1}(0) \simeq X^{\mathrm{an}}_{\mathbf{C}((t))} \quad \text{and} \quad \pi^{-1}(\mathbf{D}_r^*) \simeq X^{\mathrm{hol}} \tag{1.2}$$

compatible with the projection to $\mathbf{D}_r$.

2. A HYBRID SPACE ARISING FROM DEGENERATIONS

We now explain how the last construction above naturally appears in the context of certain degenerations. Consider a projective family

$$\pi\colon X \to \mathbf{D}^* \tag{2.1}$$

of complex projective varieties. By this we mean that X is given as a closed subspace of the complex manifold $\mathbf{P}^N_{\mathbf{C}} \times \mathbf{D}^*$ for some $N \geq 1$, cut out by homogeneous equations whose coefficients are holomorphic functions on $\mathbf{D}^*$ and meromorphic at $0 \in \mathbf{D}$. Here $\mathbf{D}$ denotes the complex unit disc in $\mathbf{C}$ and $\mathbf{D}^* = \mathbf{D} \setminus \{0\}$.

Now fix any $r \in (0,1)$ and consider the Banach ring A_r above. Every holomorphic function on $\mathbf{D}^*$ that is meromorphic at $0 \in \mathbf{D}$ defines a unique element of A_r, so the base change X_{A_r} is a well-defined projective scheme over A_r. The analytification procedure above therefore gives rise to a map

$$\pi\colon X^{\mathrm{hyb}}_{A_r} \to M(A_r) \simeq \overline{\mathbf{D}}_r. \tag{2.2}$$

Here we are abusing notation, since π already denoted the projection in (2.1). However, one can show that the restrictions of (2.1) and (2.2) to $\mathbf{D}^*_r$ agree. For simplicity, we simply write X^{hyb} instead of $X^{\mathrm{hyb}}_{A_r}$. After rescaling the discs, we have effectively extended (2.1) by inserting the Berkovich space $X^{\mathrm{an}}_{\mathbf{C}((t))}$ as the central fiber.

3. Applications

There are several instances where hybrid spaces can be used to study complex degenerations. For example:

(1) First, we have Berkovich's work on limit mixed Hodge structures [Berk09]. More precisely, Berkovich identified the weight zero part of the above structures as coming from non-Archimedean geometry. We shall not discuss this further here, but rather refer to the original paper.
(2) The hybrid space X^{hyb} was used in [Jon16] to prove that the rescalings of amoebas associated to subvarieties of toric varities, converge to the associated tropical varieties. This was proved much earlier, using different methods, by Mikhalkin [Mik04] and Rullgård [Rul01] in the case of hypersurfaces.
(3) Hybrid spaces can be used in complex dynamics to study the boundary of the parameter spaces of rational maps of the Riemann sphere. This was done at least implicitly by Kiwi and DeMarco-Faber [Kiw06, Kiw15] and explicitly by Favre [Fav16], who moreover studied the situation in higher dimensions.
(4) Degenerations occur naturally when studying the existence of special metrics in Kähler geometry, such as Kähler-Einstein metrics. The construction of hybrid spaces inspired the work in [BHJ16] and [BBJ15].
(5) Finally, hybrid spaces can be used to study degenerating families of volume forms, as in [BJ17]. This will be briefly explained in the next section, in the case of Calabi-Yau varieties.

4. Degenerations of Calabi-Yau varieties

We now explain how degenerations of Calabi-Yau varieties can be studied using hybrid spaces. Consider a projective family

$$\pi\colon X \to \mathbf{D}^* \tag{4.1}$$

of Calabi-Yau varieties, of relative dimension n. Thus we have the same situation as in §2, with the additional information that the relative canonical bundle $K_{X/\mathbf{D}^*}$ is trivial. Applying the procedure above, and rescaling the discs, we extend (4.1) to a hybrid space

$$\pi\colon X^{\mathrm{hyb}} \to \mathbf{D}$$

with the Berkovich space $X^{\mathrm{an}}_{\mathbf{C}((t))}$ as the central fiber.

Since $X_{\mathbf{C}((t))}$ is a Calabi-Yau variety, the analytification $X^{\mathrm{an}}_{\mathbf{C}((t))}$ contains a natural subset $\mathrm{Sk}(X)$, the *Kontsevich-Soibelman skeleton* first introduced in [KS06] and studied in detail in [MN15, NX16]. It is a simplicial complex of dimension $\leq n$.

The skeleton comes equipped with a natural integral affine structure. This allows us to define Lebesgue measure on $\mathrm{Sk}(X)$. More precisely, each simplex σ in $\mathrm{Sk}(X)$ has a uniquely defined (normalized) Lebesgue measure Leb_σ

Of particular interest is the case when the skeleton has dimension n: in this case X is said to be *maximally degenerate*.

Fix a trivializing section $\eta \in H^0(X, K_{X/\mathbf{D}^*})$. This is unique up to multiplication with a holomorphic function on $\mathbf{D}^*$, meromorphic at 0. The restriction $\eta_t := \eta|_{X_t}$ is a non-vanishing holomorphic n-form, and induces a smooth positive measure

$$\nu_t := i^{n^2} \eta_t \wedge \overline{\eta_t}.$$

on X_t. The following result is proved in [BJ17].

Theorem 4.1. *There exist $\kappa \in \mathbf{Q}$ and $c > 0$ such that the following holds:*

(i) $\nu_t(X_t) \sim c|t|^{2\kappa}(\log |t|^{-1})^d$*, where* $d = \dim \mathrm{Sk}(X)$*;*

(ii) *the rescaled measures*

$$\mu_t := |t|^{-2\kappa}(\log |t|^{-1})^{-d}$$

converge, as $t \to 0$, to a Lesbegue type measure μ_0 on $\mathrm{Sk}(X)$ *in the weak topology of measures on X^{hyb}; more precisely, μ_0 is of the form*

$$\mu_0 = \sum_\sigma c_\sigma \,\mathrm{Leb}_\sigma,$$

where σ runs over simplices of $\mathrm{Sk}(X)$ *of maximal dimension* $d = \dim \mathrm{Sk}(X)$*, and $c_\sigma > 0$;*

(iii) *if X is maximally degenerate and admits semistable reduction, then μ_0 is proportional to Lebesgue measure on* $\mathrm{Sk}(X)$*.*

Remark 4.2. Some comments are in order.

(i) After multiplying η by a function of t, we may assume $c = 1$ and $\kappa \geq 0$.

(ii) When X is maximally degenerate, the assumption on semistable reduction is always satisfied after a base change $t \mapsto t^m$ for some $m \geq 1$.

(iii) Convergence results similar to the ones in the theorem were earlier proved in [CLT10], but in a situation where the limit measure lives on the central fiber of an snc model as defined below.

The idea of the proof is as follows. Instead of working on the hybrid space, we fix a holomorphic *model* of X, i.e. manifold $\mathcal{X}$ with a proper flat map $\mathcal{X} \to \mathbf{D}$ such that the preimage above $\mathbf{D}^*$ is isomorphic to X. Assume that $\mathcal{X}$ is an snc model, i.e. the central fiber $\mathcal{X}_0$ is an snc divisor. One can then construct a hybrid space $\mathcal{X}^{\mathrm{hyb}}$, defined as a set as the disjoint union

$$\mathcal{X}^{\mathrm{hyb}} := X \sqcup \Delta_{\mathcal{X}},$$

where $\Delta_{\mathcal{X}}$ is the dual complex of $\mathcal{X}$, and equipped with a natural topology. This construction essentially goes back to Morgan and Shalen [MS84] and even to work by Bergman [Berg71]. The skeleton $\mathrm{Sk}(X)$ is then a subcomplex of $\Delta_{\mathcal{X}}$, and one proves the analogues of (i)–(iii) on the space $\mathcal{X}^{\mathrm{hyb}}$ using rather explicit computations in polar coordinates. This can be shown to imply the convergence result in the theorem, since the hybrid space X^{hyb} is naturally homeomorphic to the inverse limit of the spaces $\mathcal{X}^{\mathrm{hyb}}$, when $\mathcal{X}$ ranges over all snc models. We refer to [BJ17] for details.

5. Towards the Kontsevich-Soibelman conjecture

An influential conjecture by Kontsevich and Soibelman [KS06] (versions of which were stated also by Gross–Wilson and Todorov) concerns the *metric* degeneration of Calabi-Yau varieties. Consider a maximally degenerate family $X \to \mathbf{D}^*$ as above, and fix a relatively ample line bundle L on X. For $t \in \mathbf{D}^*$, let $\omega_t \in c_1(L_t)$ be the unique Ricci-flat metric on X_t in the first Chern class of $L_t := L|_{X_t}$, the existence of which is due to Yau [Yau78]. This turns X_t into a metric space (X_t, d_t), where we further normalize d_t so that the diameter of X_t is one. The Kontsevich-Soibelman conjecture now states that the family of metric spaces (X_t, d_t) converges in the Gromov-Hausdorff topology to the essential skeleton $\mathrm{Sk}(X)$ endowed with a piecewise smooth metric of Monge-Ampère type. The latter means that the metric is locally given as the Hessian of a convex function satisfying a real Monge-Ampère equation.

The theorem above on the convergence of measures on X^{hyb} is at least compatible with the Kontsevich-Soibelman conjecture. Let us be more precise. The Ricci flat metric ω_t on each X_t can be obtained as the curvature form of a metric ϕ_t on L_t satisfying a Monge-Ampère equation involving the measure ν_t. Similarly, on the central fiber it follows from [BFJ15, BG+16] (see also [KT00, Liu11, YZ13]) that there exists a metric ϕ_0 on the line bundle $L^{\mathrm{an}}_{\mathbf{C}((t))}$, unique up to scaling, that solves the *non-Archimedean* Monge-Ampère equation $\mathrm{MA}(\phi_0) = \mu_0$, where μ_0 is normalized Lebesgue measure on $\mathrm{Sk}(X)$.

It is now tempting to approach the Kontsevich–Soibelman conjecture by studying the behavior of ϕ_t as $t \to 0$. More precisely, the metrics ϕ_t (suitably normalized) define a metric ϕ^{hyb} on the hybrid line bundle L^{hyb} over X^{hyb}, and it would be interesting to see if ϕ^{hyb} is a *continuous* metric. Note that this is not obvious since there is no a priori reason why the weak continuity at $t = 0$ of $t \mapsto \mu_t$ would imply continuity of the solutions $t \mapsto \phi_t$. Further, to prove the Kontsevich-Soibelman conjecture, one probably needs to estimate some derivatives of ϕ_t, which may be even harder. Nevertheless, the convergence of measures gives some indication that the Kontsevich-Soibelman conjecture may be true.

References

[Berg71] G.M. Bergman. *The logarithmic limit-set of an algebraic variety.* Trans. Amer. Math. Soc. **157** (1971), 459–469.

[Berk90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields.* Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI (1990).

[Berk93] V. G. Berkovich. *Étale cohomology for non-Archimedean analytic spaces.* Publ. Math. Inst. Hautes Études Sci. **78** (1993), 5–161.

[Berk09] V. G. Berkovich. *A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures.* In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin.* Progr. Math., vol 269, 49–67. Birkhäuser, Boston, MA, 2009.

[BBJ15] R. Berman, S. Boucksom and M. Jonsson. *A variational approach to the Yau-Tian-Donaldson conjecture.* `arXiv:1509.04561`.

[BFJ15] S. Boucksom, C. Favre and M. Jonsson. *Solution to a non-Archimedean Monge-Ampère equation.* J. Amer. Math. Soc. **28** (2015), 617–667.

[BHJ16] S. Boucksom, T. Hisamoto and M. Jonsson. *Uniform K-stability and asymptotics of energy functionals in Kähler geometry.* `arXiv:1603.01026`. To appear in J. Eur. Math. Soc.

[BJ17] S. Boucksom and M. Jonsson. *Tropical and non-Archimedean limits of degenerating families of volume forms.* J. Éc. polytech. Math. **4** (2017), 87–139.

[BG+16] J. I. Burgos Gil, W. Gulber, P. Jell, K. Künnemann and F. Martin. *Differentiability of non-Arcimedean volumes and non-Archimedean Monge-Ampère equations.* `arXiv:1608.01919`.

[CLT10] A. Chambert-Loir and Y. Tschinkel. *Igusa integrals and volume asymptotics in analytic and adelic geometry.* Confluentes Math. **2** (2010), 351–429.

[Con08] B. Conrad. *Several approaches to non-Archimedean geometry.* In *p-adic geometry*, 9–63. Univ. Lecture Ser., 45, Amer. Math. Soc., Providence, RI, 2008.

[DF14] L. DeMarco and X. Faber. *Degenerations of complex dynamical systems.* Forum Math. Sigma **2** (2014), e6, 36 pp.

[DF16] L. DeMarco and X. Faber. *Degenerations of complex dynamical systems II.* Math. Ann. **365** (2014), 1669–1699.

[Fav16] C. Favre. *Degeneration of endomorphisms of the complex projective space in the hybrid space.* `arXiv:1611.08490`.

[Jon16] M. Jonsson. *Degenerations of amoebae and Berkovich spaces.* Math. Ann. **364** (2016), 293–311.

[Kiw06] J. Kiwi. *Puiseux series polynomial dynamics and iteration of complex cubic polynomials.* Ann. Inst. Fourier **56** (2006), 1337–1404.

[Kiw15] J. Kiwi. *Rescaling limits of complex rational maps.* Duke Math. J. **164** (2015), 1437–1470.

[KS06] M. Kontsevich and Y. Soibelman. *Affine structures and non-Archimedean analytic spaces.* The unity of mathematics, 321–385. Progr. Math., 244. Birkhäuser, Boston, 2006.

[KT00] M. Kontsevich and Y. Tschinkel. *Non-Archimedean Kähler geometry.* Unpublished.

[Liu11] Y. Liu. *A non-Archimedean analogue of Calabi-Yau theorem for totally degenerate abelian varieties.* J. Diff. Geom. **89** (2011), 87–110.

[Mik04] G. Mikhalkin. *Decomposition into pairs-of-pants for complex algebraic hypersurfaces.* Topology **43** (2004), 1035–1065.

[MS84] J. W. Morgan, and P. B. Shalen. *Valuations, trees, and degenerations of hyperbolic structures, I* Ann. Math. **120** (1984), 401–476.

[MN15] M. Mustaţă and J. Nicaise. *Weight functions on non-archimedean analytic spaces and the Kontsevich-Soibelman skeleton.* Algebr. Geom. **2** (2015), 365–404.

[NX16] J. Nicaise and C. Xu. *The essential skeleton of a degeneration of algebraic varieties.* Amer. J. Math. **138** (2016), 1645–1667.

[Poi10] J. Poineau. *La droite de Berkovich sur* $\mathbf{Z}$. Astérisque **334** (2010).

[Poi13] J. Poineau. *Espaces de Berkovich sur* $\mathbf{Z}$*: étude locale.* Invent. Math. **194** (2013), 535–590.

[Rul01] H. Rullgård. *Polynomial amoebas and convexity.* Research reports in Mathematics, number 8. Stockholm University, 2001.

[Ser55] J.-P. Serre. *Géométrie algébrique et géométrie analytique.* Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42.

[Yau78] S. T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation.* Comm. Pure Appl. Math. **31** (1978), 339–411.

[YZ13] X. Yuan and S.-W Zhang. *The arithmetic Hodge index theorem for adelic line bundles.* Math. Ann. **367** (2017), 1123–1171.

Dept of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA
E-mail address: `mattiasj@umich.edu`