UNIFORM BOUNDS ON TORSION POINTS ON CURVES

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This talk reports on joint work with Joseph Rabinoff and David Zureick-Brown.

1. Statement of Results

Theorem 1.1. *Let $C/\mathbf{Q}$ be a nice (smooth, proper, geometrically connected) curve of genus g. Let $r = \operatorname{rank} J_C(\mathbf{Q})$ (with Abel–Jacobi map $\iota\colon C \hookrightarrow J_C$ given by a point $x_0 \in C(\mathbf{Q})$). Then we have*

1. *If $r \leq g-3$, $\#C(\mathbf{Q}) \leq 84g^2 - 98g + 28$,*
2. *Unconditionally, $\#\iota^{-1}(J_{\mathrm{tors}}(\mathbf{Q})) \leq 84g^2 - 98g + 28$*
3. *For p prime, pick $\mathscr{C}/\mathcal{O}_{\mathbf{C}_p}$ stable and suppose $g \geq 2g(C_i) + n_{C_i}$ for all components C_i of the special fiber $\mathscr{C}_0$, where n_{C_i} is the number of nodes on C_i, then $\#\iota^{-1}(J_{\mathrm{tors}}(\overline{\mathbf{Q}})) \leq N(g)$ where $N(g)$ is an explicit, albeit exponential constant.*

The three parts of the theorem refer to rational points, rational torsion, and geometric torsion.

The main idea of the proof is to bound the size of a certain set Z. We form C^{an}, the Berkovich analytification of C. We try to find a collection of basic wide open sets $U_i \subset C^{\mathrm{an}}$ and analytic functions f_i on U_i such that

$$Z \subseteq \bigcup \Big(\{f_i = 0\} \subseteq U_i \Big).$$

The sets U_i need not cover C^{an}. In the two rational cases above, we need only cover arithmetically relevant points, so it suffices that $C(\mathbf{Q}_p)$ be covered by U_i.

2. Chabauty Method

The functions f_i are constructed by the Chabauty–Coleman method. Let Z be the set of rational or torsion points on C. The Jacobian J_C is an Abelian variety and possesses a logarithm,

$$\operatorname{Log}\colon J_C(\mathbf{C}_p) \to \operatorname{Lie} J_C(\mathbf{C}_p) \cong \mathbf{C}_p^g.$$

The relevant points may land in a subspace $V \subset \operatorname{Lie} J_C(\mathbf{Q})$. To find the rational points, we use the p-adic fact that

$$\dim \operatorname{Span}(\operatorname{Log}(\overline{J_C(\mathbf{Q})})) \leq \operatorname{rank} J_C(\mathbf{Q}).$$

Therefore, there exists $V \subseteq \operatorname{Lie} J_C(\mathbf{C}_p)$ such that $\operatorname{Log}(J(\mathbf{Q})) \subseteq V$. For the torsion points, because Lie $J_C(\mathbf{C}_p)$ is torsion-free, $\operatorname{Log}(J_{\mathrm{tors}}(\overline{\mathbf{Q}})) = 0$, so we may set $V = 0$.

Since $\Omega^1(C) = (\operatorname{Lie} J_C)^\vee$, given $\omega \in \Omega^1(C)$, we may define an *Abelian p-adic integral*,

$$f_\omega(x) = {}^{\mathrm{Ab}}\!\!\int_{x_0}^{x} \omega \coloneqq \langle \operatorname{Log}(\iota(x)), \omega \rangle.$$

This integral has no periods and exists for purely p-adic Lie group-theoretic reasons. If $\omega \in V^\perp$, then f_ω vanishes on Z.

The function is $f_\omega(x)$ is only locally analytic. A priori, we do not have much control over its domains of analyticity. This prevents us from bounding its zeroes. A good example of a strange locally analytic function is a branch of logarithm, $\operatorname{Log}\colon \mathbf{G}_m(\mathbf{C}_p) \to \mathbf{C}_p$. It is zero at all roots of unity. This function is analytic on residue discs, but there are infinitely many of them, so we do not get much control over its zeroes.

3. p-adic integration

Let's try to understand p-adic integration more generally to try to get more control over domains of analyticity. Let X be a smooth $\mathbf{C}_p$-analytic space and let

$$\mathscr{P}(X) \coloneqq \{\gamma\colon [0,1] \to X \mid \gamma(0), \gamma(1) \in X(\mathbf{C}_p)\}$$

be the set of paths with endpoints in $X(\mathbf{C}_p)$.

Definition 3.1. An *integration theory* on X is a map

$$\int \colon \mathscr{P}(X) \times Z^1_{\mathrm{dR}}(X) \to \mathbf{C}_p$$

(where Z^1_{dR} are the closed 1-forms) such that

(1) If $U \subset X$ is an open subdomain isomorphic to a polydisc and $\omega|_U = df$ for some analytic function f on U, then

$$\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0)),$$

(2) $\int_\gamma \omega$ only depends on the fixed endpoint homotopy class of γ,
(3) $\int_{\gamma' * \gamma} = \int_\gamma \omega + \int_{\gamma'} \omega$, and
(4) $\omega \mapsto \int_\gamma \omega$ is linear in ω.

Theorem 3.2. *For an Abelian variety A, ${}^{\mathrm{Ab}}\!\int$ is an integration theory on A^{an}.*

The big problem with the Abelian integration theory is that it's not the only integration theory and it's not local. It makes heavy use of the commutative algebraic group properties of A, and its pullback by the Abel–Jacobi map cannot be performed locally on curves.

There's another, more intrinsic integration theory due to Coleman, de Shalit, and Berkovich, which we call Berkovich–Coleman integration, ${}^{\mathrm{BC}}\!\int$. We begin by picking a branch of p-adic logarithm, $\operatorname{Log}\colon \mathbf{G}_m^{\mathrm{an}}(\mathbf{C}_p) \to \mathbf{C}_p$. This integration theory is characterized by the additional two properties:

(1) If $f\colon X \to Y$ is a morphism and $\omega \in Z^1_{\mathrm{dR}}(Y)$, then

$$^{\mathrm{BC}}\!\!\int_\gamma f^*\omega = {}^{\mathrm{BC}}\!\!\int_{f(\gamma)} \omega,$$

(2) If $X = \mathbf{G}_m^{\mathrm{an}}$, then ${}^{\mathrm{BC}}\!\!\int_1^x \frac{\mathrm{d}T}{T} = \mathrm{Log}(x)$.

The first property lets you "analytically continue by Frobenius" which is Coleman's trick for fixing constants of integration across residue discs. Property 2 normalizes the integral on annuli. From these conditions, we can define the integral on curves. By de Jong's theory of alterations, we can extend it to analytic spaces. This integral gives us some control over domains of analyticity because if ω is exact on a subdomain U, we can take a primitive. This makes it possible to bound zeroes of ${}^{\mathrm{BC}}\!\!\int$.

4. Integral comparison theorem

Now, we'll compare ${}^{\mathrm{Ab}}\!\!\int$ and ${}^{\mathrm{BC}}\!\!\int$ on curves. They turn out to be equal on curves of good reduction. Stoll and Besser–Zerbes have also thought about this and have work or work-in-progress. I will make the comparison on the Jacobian J, but everything I say holds for Abelian varieties. We will pass to the Berkovich universal cover of J^{an} which we shall call E^{an}. This is really just Raynaud's uniformization theory in disguise. We have a uniformization cross

$$\begin{array}{ccccc} & & T^{\mathrm{an}} & & \\ & & \downarrow & & \\ M' & \longrightarrow & E^{\mathrm{an}} & \xrightarrow{\pi} & A^{\mathrm{an}} \\ & & \downarrow & & \\ & & B^{\mathrm{an}} & & \end{array}$$

where T^{an} is the analytification of an algebraic torus, B is an Abelian variety with good reduction, and M' is a lattice of rank equal to the dimension of T. For ease of exposition, we will specialize to the maximally degenerate case, so we have $E^{\mathrm{an}} = (\mathbf{G}_m^g)^{\mathrm{an}}$. We pull back the two integration theories to E^{an} to define two logarithms, $\mathrm{Log}_{\mathrm{BC}}, \mathrm{Log}_{\mathrm{Ab}}\colon E^{\mathrm{an}}(\mathbf{C}_p) \to \mathrm{Lie}\, E^{\mathrm{an}}(\mathbf{C}_p) \cong (\mathbf{C}_p)^g$. Again, note $\Omega^1(J) = (\mathrm{Lie}\, J)^\vee = (\mathrm{Lie}\, E^{\mathrm{an}})^\vee$. Define the logarithms by, for $x \in E^{\mathrm{an}}(\mathbf{C}_p)$,

$$\begin{aligned} \langle \mathrm{Log}_{\mathrm{BC}}(x), \omega\rangle &= {}^{\mathrm{BC}}\!\!\int_0^x \omega \\ \langle \mathrm{Log}_{\mathrm{Ab}}(x), \omega\rangle &= {}^{\mathrm{Ab}}\!\!\int_0^x \omega \end{aligned}$$

We have the following observations:

(1) because ${}^{\mathrm{Ab}}\!\!\int$ has no periods, $\mathrm{Log}_{\mathrm{Ab}}$ vanishes on M',
(2) by the normalization property, $\mathrm{Log}_{\mathrm{BC}}\colon (\mathbf{G}_m)^g \to (\mathbf{C}_p)^g$ is just a Cartesian power of the branch of p-adic logarithm that we fixed, and

(3) by the fundamental theorem of calculus for integration theories, $\mathrm{Log}_{\mathrm{BC}} - \mathrm{Log}_{\mathrm{Ab}}$ vanishes on the fiber over the identity of $\mathrm{trop}\colon E(\mathbf{C}_p) \to N_{\mathbf{Q}}$, and so factors as

$$E \xrightarrow{\mathrm{trop}} N_{\mathbf{R}} \xrightarrow{L} \mathrm{Lie}(\mathbf{G}_m)^g$$

for some linear map L.

To compare the integrals on a curve, we set $\Sigma \coloneqq \mathrm{trop}(J) = N_{\mathbf{R}}/\mathrm{trop}(M')$ and let Γ be the skeleton of of C^{an}. By a result of Baker–Rabinoff, the Abel–Jacobi map commutes with tropicalization,

$$\begin{tikzcd} C^{\mathrm{an}} \arrow[r, "\iota"] \arrow[d, "\mathrm{trop}"'] & J^{\mathrm{an}} \arrow[d, "\mathrm{trop}"] \\ \Gamma \arrow[r, "\iota_{\mathrm{trop}}"] & \Sigma \end{tikzcd}$$

The map ι_{trop} is the tropical Abel–Jacobi map which is piecewise-linear on Γ. The consequences are the following:

(1) $\mathrm{Log}_{\mathrm{BC}} - \mathrm{Log}_{\mathrm{Ab}}$ is constant on the inverse image under tropicalization of a vertex in Γ, and
(2) $\mathrm{Log}_{\mathrm{BC}} - \mathrm{Log}_{\mathrm{Ab}}$ is an affine map on the inverse image under tropicalization of an edge in Γ.

It follows that for an annulus in C given by the inverse image under tropicalization of an edge, the condition ${}^{\mathrm{BC}}\!\int \omega = {}^{\mathrm{Ab}}\!\int \omega$ is a codimension 1 condition on ω.

This finally gives us a strategy for bounding rational and torsion points. We will cover Z by open sets U_i for which there exists $\omega \in V^{\perp}$ such that

(1) ${}^{\mathrm{BC}}\!\int \omega = {}^{\mathrm{Ab}}\!\int \omega$ on U_i, and
(2) ω is exact on U_i.

Then the functions f_ω are the desired analytic functions on U_i.

For rational points and rational torsion, we use for our covering

(1) residue discs around rational points not tropicalizing to an edge of Γ, and
(2) annuli tropicalizing to an edge.

For the annuli, we need $\dim V \geq 3$ to find an ω satisfying the integral agreement and exactness conditions. Since the tropicalization of rational points is discrete, we can pick economical coverings that give us enough overconvergence to guarantee that our 1-forms are exact.

For geometric torsion, we use the following for our covering:

(1) Basic wide opens that occur as the inverse image under tropicalization of a flag consisting of a half-open initial segment of an edge, and
(2) annuli that occur as the inverse image under tropicalization of a open sub-interval of an edge.

The tricky genus condition above lets us pick out an exact 1-form on subsets of the first kind.

5. Bounding zeroes

There's another complication. We want to bound the zeroes of f_ω on an annulus. If

$$\omega = \sum_{n \neq 0} a_n T^n \frac{\mathrm{d}\, T}{T},$$

then

$$f_\omega = c + \sum_{n \neq 0} \frac{a_n}{n} T^n.$$

We would like to use a Newton polygon argument but the Newton polygons for ω and f_ω may be quite different even apart from the constant term. The reason is that if $p \mid n$, then $\mathrm{val}(a_n) \neq \mathrm{val}(a_n/n)$. However, if we know how many zeroes ω has on the annulus, this gives us control over the lengths of the segments of the Newton polygon corresponding to zeroes in the annulus. But we need to know more than that! Because of the above divisibility issue, we'd like to know where the segment is located. If it contained a point $(n, \mathrm{val}(a_n))$ where n is divisible by a high power of p, then the Newton polygons of ω and f_ω near that point could be drastically different.

The answer is to consider 1-forms as global objects, in fact, as sections of the canonical bundle, K_C. The canonical bundle has a canonical metric, so we may take

$$F_0 \coloneqq \log \|\omega\| \colon \Gamma \to \mathbf{R}.$$

This function measures the vanishing order of ω on components of the closed fiber of a model of C. Its slopes are bounded by the combinatorics of linear systems on graphs. These give us control of the relevant segments of the Newton polygons. By introducing a small error term, we can bound the zeroes of f_ω on annuli. By using these bounds, together with the Poincaré–Lelong formula, we may bound the zeroes of f_ω on basic wide opens.

References

[1] E. Katz, J. Rabinoff, and D. Zureick-Brown, *Uniform bounds for the number of rational points on curves of small Mordell–Weil rank*, Duke Math. J. **165** (2016), no. 16, 3189–3240.

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