SIMONS SYMPOSIUM: NOTES FROM A LECTURE BY FRANÇOIS LOESER ON MOTIVIC INTEGRATION, APRIL 4, 2013

1. p-adic Integration

Igusa introduced p-adic integration to solve the following question of Borevich and Shafarevich.

Question 1.1 ([BS66]). *Let $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ and consider the sequence of integers*

$$N_m = \#\{x \in (\mathbb{Z}/p^{m+1}\mathbb{Z})^n \ : \ f(x) = 0 \mod p^{m+1}\}.$$

Is the generating function

$$P(T) = \sum N_m T^m$$

a rational function of T?

In his proof of an affirmative answer [Igu75], Igusa showed that $P(T)$ is related to the p-adic integral

$$I(s) = \int_{\mathbb{Z}_p^n} |f|^s dx$$

by a change of variables formula taking T to p^{-s}. If f is a monomial, then this p-adic integral can be computed by hand and shown to be a rational function in p^{-s}. If f is not necessarily a monomial then, roughly speaking, one uses resolution of singularities over $\mathbb{Q}_p$ to monomialize f, and then proceeds by a similar direct computation.

Serre considered a variant of this generating function, in which one counts only those solutions to f mod p^{m+1} that lift to solutions over $\mathbb{Z}_p$, and asked the analogous rationality question.

Question 1.2 ([Ser81]). *Let*

$$\widetilde{N}_m = \#\{x \in (\mathbb{Z}/p^{m+1}\mathbb{Z})^n \ : \exists y \in \mathbb{Z}_p^n \text{ with } f(y) = 0 \text{ and } y \equiv x \mod p^{m+1}\}.$$

Is the generating function

$$\widetilde{P}(t) = \sum \widetilde{N}_m T^m.$$

rational?

In [Den84] Denef gave an affirmative solution, by a proof which again involves a change of variables relating $\widetilde{P}(T)$ to a p-adic integral. In this case, the integral that appears is

$$\widetilde{I}(s) = \int_{\mathbb{Z}_p^n} d(x, V)^s dx,$$

where $d(x, V)$ denotes the p-adic distance from x to the hypersurface V where $f = 0$. The main new difficulty is that $d(x, V)$ is no longer the norm of a polynomial and is not monomialized by a resolution of singularities.

Over the real numbers, the function $d(x, V)$ computing distance to a hypersurface is semialgebraic. This is also true over $\mathbb{Q}_p$, with suitable definitions. More precisely, let K be a field and consider the smallest family $\mathcal{S}(K^n)$, $n \in \mathbb{N}$, of Boolean algebras of subsets of K^n, containing zero loci of polynomials with coefficients in K and such that, if $X \in \mathcal{S}(K^n)$ and $p_i : K^n \to K^{n-1}$ is the projection omitting the i's coordinate, $1 \leq i \leq n$, $p_i(X) \in \mathcal{S}(K^{n-1})$. A semialgebraic set is an element of some $\mathcal{S}(K^n)$. Semialgebraic sets are exactly sets definable in the first-order language of rings with coefficients in k. One says a function is semialgebraic if its graph is a semialgebraic set. Tarski showed that a subset of $\mathbb{R}^n$ is definable if and only if it is a finite Boolean combination of closed semialgebraic sets, each defined by finitely many polynomials conditions of the form $f = 0$ or $f \geq 0$ [Tar48]. Over $\mathbb{Q}_p$, thanks to a theorem of Macintyre [Mac76], the same statement holds if the closed semialgebraic sets are instead defined by conditions of the form $f = 0$ or $f = y^r$, for $r = 2, 3, \ldots$, with f a polynomial.

Remark 1.3. With this definition, it is a nice exercise to check that sets defined by valuative conditions of the form $\mathrm{val}(f) \geq \mathrm{val}(g)$ with f anf g polynomials are also semialgebraic over $\mathbb{Q}_p$.

Macintyre's theorem is the main ingredient of Denef's proof of the rationality of Serre's generating function $\widetilde{P}(T)$.

Dependence on parameters. In his study of p-adic integrals, Denef obtained a good understanding of how p-adic integrals vary with parameters. More precisely, let X be a definable subset of $\mathbb{Q}_p^n$. Consider the $\mathbb{Q}$-algebra $\mathcal{C}_{\mathbb{Q}_p}(X)$ generated by $\mathrm{val}(h(x))$ and $p^{-\mathrm{val}\, g(x)}$, for definable functions g, h from X to $\mathbb{Q}_p^*$.

Theorem 1.4 (Denef [Den85])**.** *If $\varphi \in \mathcal{C}_{\mathbb{Q}_p}(X \times \mathbb{Q}_p^r)$ then the function taking $\lambda \in \mathbb{Q}_p^r$ to*

$$\int \varphi(x, \lambda) dx$$

belongs to $\mathcal{C}_{\mathbb{Q}_p}(\mathbb{Q}_p^r)$.

Here, the integral should be defined to be zero if the function is not integrable.

Remark 1.5. Note that the integrals in this theory take real values. This is an important distinction from the p-adic valued integrals appearing in the work of Coleman and Berkovich, discussed in earlier lectures at the Symposium

Remark 1.6. The analogues of both rationality questions are open for polynomials over $\mathbb{F}_p((t))$. Rationality for P would follow from resolution of singularities, but for $\widetilde{P}$ even this is not known.

2. Motivic Integration

The theory of motivic integration, as introduced by Kontsevich is his 1995 lecture at Orsay, replaces $\mathbb{Q}_p$ with the field of formal Laurent series $\mathbb{C}((t))$, and the integrals no longer take values in $\mathbb{R}$, but rather in the localization

$$\mathcal{M} = K(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}],$$

of the Grothendieck ring of complex varietiesor in a completion $\widehat{\mathcal{M}}$ a completion of $\mathcal{M}$. Here $[\mathbb{L}] = [\mathbb{A}^1_{\mathbb{C}}]$ is the class of the affine complex line.

Kontsevich proved a suitable change of variables formula in this motivic integration theory, and used it to prove birational invariance of Hodge numbers for Calabi-Yau varieties. This strengthened a theorem of Batyrev, that birational Calabi-Yau varieties have the same Betti numbers, which had been proved earlier using p-adic integration [Bat99].

A very brief sketch of the main idea in the proof is that birational Calabi-Yau varieties X and Y carry nowhere vanishing gauge forms ω_X and ω_Y, respectively. Then, since X and Y are birational, their $\mathbb{C}[[t]]$ points and gauge forms agree outside a locus of infinite codimension (over $\mathbb{C}$). By an appropriate limiting procedure, it follows that

$$\int_{X(\mathbb{C}[[t]])} \omega_X = \int_{Y(\mathbb{C}[[t]])} \omega_Y.$$

and this integration theory is defined such that $\int_{X(\mathbb{C}[[t]])} \omega_X$ and $\int_{Y(\mathbb{C}[[t]])} \omega_Y$ are equal to $[X]$ and $[Y]$, respectively, in $\widehat{\mathcal{M}}$.

3. Joint work with Cluckers [CL08]

This work is a motivic generalization of the work of Denef that is described above. One considers definable subsets of $\mathbb{C}((t))^n$ are cut out by conditions such as $\mathrm{val}(f) \geq \mathrm{val}(g)$. For X definable over $\mathbb{C}((t))$, we define a ring $\mathcal{C}(X)$, analogous to the ring $\mathcal{C}_{\mathbb{Q}_p}$ defined above, for constructible subsets of $\mathbb{Q}_p^n$. In this construction,

$$\mathcal{C}(\mathrm{pt}) = \mathcal{M}\left[\left(\frac{1}{1-\mathbb{L}^{-i}}\right)_{i\geq 1}\right],$$

and there is a well-behaved notion of integration, such that $\mathcal{C}$ is stable with respect to integration with parameters.

If X is defined over $\mathbb{Q}$ or $\mathbb{Q}((t))$, then there are natural specializations to $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$. Following a method initiated by Ax and Kochen [AK66], this allows to compare results over pairs of fields such as these, with isomorphic residue fields and value groups. This applies for instance to the Fundamental Lemma in Langlands theory [CHL11] or to the local integrability of Harish-Chandra characters [CGH].

4. Connections with Nonarchimedean Geometry

Now we consider a geometric analogue of the numbers N_m from the beginning of the talk. Let X be a smooth variety of dimension d over $\mathbb{C}$, with

$$f : X \to \mathbb{A}^1_{\mathbb{C}}.$$

The analogue of N_m that we study is

$$\mathfrak{X}_m = \{\varphi \in X(\mathbb{C}[t]/t^{m+1}) \ : \ f(\varphi) = t^m \mod t^{m+1})$$

Theorem 4.1 (Denef-Loeser [DL02])**.** *For any $m \geq 1$,*

$$\chi_c(\mathfrak{X}_m) = \mathrm{tr}(M^n, H^*(F)),$$

where M denotes the monodromy operator on the cohomology of the Milnor fiber F.

The proof is via motivic integration and resolution of singularities. Nicaise and Sebag proved the following more general result based on nonarchimedean geometry.

Theorem 4.2 ([NS07]). *If X is smooth and proper over $\mathbb{C}((t))$ then*

$$\mathrm{tr}(\varphi^m, H^*(X \otimes \overline{\mathbb{C}((t))}, \mathbb{Q}_\ell) = \chi_c(S(X \otimes \mathbb{C}((t^{1/m}))))$$

in $\mathcal{M}/(\mathbb{L}-1)$, where S denotes the motivic Serre invariant, and φ is a topological generator for $\mathrm{Gal}(\overline{\mathbb{C}((t))}|\mathbb{C}((t)))$.

This formula looks very much like a Lefschetz trace formula for the action of φ, but the proof of Nicaise and Sebag uses resolution of singularities and direct computation of motivic integrals.

In recent joint work with Hrushovski [HL11], we give a new proof which is more conceptual and does not use resolution of singularities. Instead, it involves a Lefschetz trace formula and étale cohomology of analytic domains in Berkovich spaces.

One may introduce the generating function

$$P_{\mathrm{mot}}(T) = \sum_{m \geq 1} [\mathfrak{X}_m] \, \mathbb{L}^{-md} \, T^m$$

in $\mathcal{M}[[T]]$, which is a motivic analogue of the series P considered at the beginning. One can prove it is rational, and the *monodromy conjecture* predicts that its poles are related to eigenvalues of the monodromy. We refer to the survey [Loe09] for a precise statement.

References

[AK66] J. Ax and S. Kochen, *Diophantine problems over local fields. III. Decidable fields*, Ann. of Math. (2) **83** (1966), 437–456.

[Bat99] V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge, 1999, pp. 1–11.

[BS66] A. Borevich and I. Shafarevich, *Number theory*, Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20, Academic Press, New York, 1966.

[CGH] R. Cluckers, and J. Gordon and I. Halupczok, *Local integrability results in harmonic analysis on reductive groups in large positive characteristic*, preprint, arXiv:1111.7057.

[CHL11] R. Cluckers, T. Hales and F. Loeser, *Transfer Principle for the Fundamental Lemma*, Chapter in *On the Stabilization of the Trace Formula*, edited by L. Clozel, M. Harris, J.-P. Labesse, B.-C. Ngô, International Press of Boston, ISBN: 978-1-57146-227-5, (2011).

[CL08] R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration*, Invent. Math., **173** (2008), 23–121.

[Den84] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. **77** (1984), no. 1, 1–23.

[Den85] J. Denef, *On the evaluation of certain p-adic integrals*, in Séminaire de théorie des nombres, Paris 1983–84, 25–47, Progr. Math., **59**, Birkhäuser Boston, Boston, MA, 1985.

[DL02] J. Denef and F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), no. 5, 1031–1040.

[HL11] E. Hrushovski and F. Loeser, *Monodromy and the Lefschetz fixed point formula*, preprint, arXiv:1111.1954, 2011.

[Igu75] J.-I. Igusa, *Complex powers and asymptotic expansions. II. Asymptotic expansions*, J. Reine Angew. Math. **278/279** (1975), 307–321.

[Loe09] F. Loeser, *Seattle lectures on motivic integration.* Algebraic geometrySeattle 2005. Part 2, 745–784, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[Mac76] A. Macintyre, *On definable subsets of p-adic fields*, J. Symbolic Logic **41** (1976), no. 3, 605–610.

[NS07] J. Nicaise and J. Sebag, *Motivic Serre invariants, ramification, and the analytic Milnor fiber*, Invent. Math. **168** (2007), no. 1, 133–173.

[Ser81] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 54, 323–401.

[Tar48] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, RAND Corporation, Santa Monica, Calif., 1948.