13. TROPICAL INDEPENDENCE AND PROOF OF THE GIESEKER-PETRI THEOREM

(for $\rho = 0$)

The goal of this lecture is a tropical proof the Gieseker-Petri theorem, due to Jensen and Payne, in the $\rho = 0$ case.

Let $X$ be a general curve of genus $g$ over $K$ a complete non-archimedean field. The scheme $G_d^r(X)$ parameterizes linear series of degree $d$ and rank $r$ on $X$. In the last lecture we saw the Brill-Noether theorem, which says that $G_d^r(X)$ has pure dimension $\rho = g - (r + 1)(g - d + r)$ if $\rho \geq 0$, and is empty if $\rho < 0$.

**Theorem 13.1** (Gieseker-Petri). The scheme $G_d^r(X)$ is smooth.

It is an established fact about algebraic curves that for a linear series $W \subset L(D_X)$,

$$\dim T_W(G_d^r(X)) = \rho + \dim \ker \mu_W,$$

where $\mu_W : W \otimes L(K_X - D_X) \to L(K_X)$ is the multiplication map. So, to prove the theorem, we must show that $\mu_W$ is injective for all such $D_X$ and $W$. It is enough to show the following result:

**Theorem 13.2.** If $\Sigma(X^\text{an})$ is a generic chain of loops with bridges, then the map $\mu : L(D_X) \otimes L(K_X - D_X) \to L(K_X)$ is injective for all divisors $D_X$.

The proof will combine the tropicalization maps and specialization lemma which we have studied together with combinatorics of divisors on chains of loops. Let $\Gamma$ a metric graph. Recall that $\text{PL}(\Gamma)$ is the set of piecewise linear function on $\Gamma$ with integer slopes, analogous to rational functions. Given a divisor $D$ on $\Gamma$, $|D|$ is the complete linear series of $D$, that is, the set of effective divisors equivalent to $D$. Equivalently,

$$|D| = \{D + \text{div}(\phi) \geq 0 : \phi \in \text{PL}(\Gamma)\}.$$

This object is analogous to the complete linear series $|D_X|$ on $X$.

For a divisor $D$, define $R(D) = \{\phi \in \text{PL} : D + \text{div}(\phi) \geq 0\}$, analogous to the space of rational functions on $X$ with zeros and poles of at most the orders prescribed by a divisor $D_X$ at each point. Observe that we have a map $R(D) \to |D|$ given by $\phi \mapsto D + \text{div}(\phi)$. The set $R(D)$ has the structure of a tropical module, analogous to the vector space structure on $L(D_X)$. This means that for $\phi_1, \ldots, \phi_n \in R(D)$ and $b_1, \ldots, b_n \in \mathbb{R}$, we have $\min_i(\phi_i + b_i) \in R(D)$. In other words, $R(D)$ is closed under “tropical linear combinations”.

Recall the tropicalization maps on divisors and rational functions

$$\text{Trop} : \text{Div}(X) \to \text{Div}(\Gamma).$$
and \(\text{trop} : K(X)^* \to \text{PL}(\Gamma)\),

the latter given by \(f \mapsto (y \mapsto \text{val}_y(f))\). The Poincare-Lelong formula shows that 
\(\text{Trop}(\text{div}(f)) = \text{div}(\text{trop}(f))\). We have also seen the Specialization Lemma, which says that 
\(r(\text{Trop}(D_X)) \geq r(D_X)\), where \(r(\text{Trop}(D_X))\) is the combinatorial rank of a divisor on a metric graph and \(r(D_X)\) is algebraic rank of a divisor on a curve, i.e., 
\(h^0(\mathcal{L}(D_X)) - 1\). This inequality is not an equality, in general. For example, effective divisors on \(X\) stay effective on \(\Gamma\), but non-effective divisors might become effective by cancellation.

We now show how to use the tropical module structure discussed above to prove statements about linear independence of divisors.

**Definition 13.3.** The set of functions \(\{\phi_1, \ldots, \phi_n\} \subseteq \text{PL}(\Gamma)\) is tropically dependent if there exist \(b_1, \ldots, b_n \in \mathbb{R}\) such that \(\min_i (\phi_i + b_i)\) occurs at least twice at every point of \(\Gamma\).

**Example 13.4.** On the segment \([-1, 1]\), the two functions \(x \mapsto 2x\) and \(x \mapsto 2x + 1\) are tropically dependent. Less trivially, the three functions \(x \mapsto |x|\), \(x \mapsto \max\{-x, 2x\} + 37\), and \(x \mapsto \max\{-2x, x\} - 2016\) are tropically dependent.

**Remark 13.5.** This notion of linear dependence does not have matroidal properties. Indeed, one can construct a set of piecewise linear functions with maximal tropically independent subsets of different cardinalities.

Observe that if \(f_1, \ldots, f_r \in K(X)\) are linearly dependent and \(\phi_i = \text{trop}(f_i)\), then \(\phi_1, \ldots, \phi_r\) are tropically dependent. This is clear: write \(\sum_i c_i f_i = 0\), where not all of the \(c_i\) are 0, and take valuations, using the non-archimedean property that \(v(a + b) = \min(v(a), v(b))\) if \(v(a) \neq v(b)\).

Consider the map \(\mu : \mathcal{L}(D_X) \otimes \mathcal{L}(E_X) \to \mathcal{L}(D_X + E_X)\). Choose bases \(\{f_i\}\) and \(\{g_j\}\) for \(\mathcal{L}(D_X)\) and \(\mathcal{L}(E_X)\), respectively. Let \(\phi_i = \text{trop}(f_i)\) and \(\psi_j = \text{trop}(g_j)\). The following lemma follows immediately from the preceding observation:

**Lemma 13.6.** If \(\{\text{trop}(f_i g_j)\} = \{\phi_i + \psi_j\}\) is tropically independent, then \(\mu\) is injective.

We will apply this lemma to piecewise linear functions on the chain of \(g\) loops with generic edge lengths in the special case when \(\rho = 0\). This is the only case we will present, and does not require bridges between the loops; the cases where \(\rho > 0\) are more complicated, requiring additional techniques from Berkovich theory and reductions of rational functions, and the proof when \(\rho > 0\) does require bridges of nonzero length between the loops.

Fix a divisor \(D_X\). We shall apply the lemma to \(\mathcal{L}(D_X)\) and \(\mathcal{L}(K_X - D_X)\) to prove the theorem.

Let \(\Gamma\) be a chain of \(g\) loops with generic edge lengths. The \(i\)th loop has a left endpoint \(v_i\) and a right endpoint \(w_i\), and there are bridges \(b_i\), connecting \(w_i\) to \(v_{i+1}\). We denote the \(i\)th loop by \(\gamma_i\), excluding the right endpoint \(w_i\). The edge lengths \(\ell_i\) and \(m_i\) on the loop \(\gamma_i\) are *generic* if there do not exist integers \(a\) and \(b\) with \(a + b \leq 2g - 2\) and \(\frac{a}{m_i} = \frac{b}{\ell_i}\).

The outline will be:
(1) Show that any divisor in $|K|$ contains no point in one of $\gamma_1, \ldots, \gamma_g$.

(2) Show that, for $D \in \text{Div} \Gamma$, $\psi_0, \ldots, \psi_r \in R(D)$, $\theta = \min_i \psi_i$, and $\Gamma' \subseteq \Gamma$ connected, if $D + \text{div}(\psi_i)$ contains a chip in $\Gamma'$ for each $i$, then so does $D + \text{div} \theta$.

(3) Construct representatives of $|D|$ and $|K - D|$, where $D = \text{trop} D_X$ and use (1) to show that the corresponding functions in $R(K)$ are tropically independent.

Note that, by the lemma above, (3) implies that the map $\mu$ is injective, so once this is established, we will be done.

We first show that (3) follows from (1) and (2). To see this, we shall suppose to the contrary that these functions are tropically dependent and use (2) to construct a divisor in the class of $K$ with a point on each $\gamma_i$, contradicting (1).

Recall the bijection between $v_1$-reduced divisors of degree $d$ and rank $r$ on $\Gamma$ and standard tableaux of shape $(r+1) \times (g-d+r)$ via the bijection of each with lingering lattice paths. It turns out that the involution $D \leftrightarrow K - D$ on divisors corresponds to transposition on tableaux, $P \leftrightarrow P^T$.

Lemma 13.7. Suppose $D$ is a divisor of degree $d$ and rank $r$ on $\Gamma$ and let $P$ be the tableau corresponding to $D$.

(a) There are unique representatives $D_0, \ldots, D_r \in |D|$ such that for each $j$, $D_j - j[v_1] - (r-j)[w_g]$ is effective, and $D_j$ has no chips on $\gamma_i$ if and only if $i$ is in the $j$-th column of $P$.

(b) There are unique representatives $E_0, \ldots, E_{g-d+r-1} \in |K - D|$ such that $E_j - j[v_1] - (g-d+r-1-j)[w_g]$ and $E_j$ has no chips on $\gamma_i$ if and only if $i$ is in the $j$-th row of $P$.

(c) The divisor $D_j + E_k \in |K|$ has no chips on $\gamma_i$ if and only if $i$ is the $(j,k)$ entry of $P$.

Proof. The proof of (a) is a chip-firing argument: start with $j$ chips at $v_0$, fire them left to right, and consider at which loops you pick up a chip. Part (b) follows from (a) by applying the involution. Part (c) follows from (a) and (b). \qed

Proof of (3), assuming (1) and (2). By construction, there exist bases $\{f_j\}$ and $\{g_k\}$ for $\mathcal{L}(D_X)$ and $\mathcal{L}(K_X - D_X)$, respectively, such that $\text{Trop}(D_X + \text{div}(f_j)) = D_j$ and $\text{Trop}(K_X - D_X + \text{div}(g_k)) = E_k$. Let $\phi_j = \text{trop}(f_j) \in R(D)$ and $\psi_k = \text{trop}(g_k) \in R(K - D)$, so $D_j = D + \text{div}(\phi_j)$ and $E_k = E + \text{div}(\psi_k)$.

We claim that the set $\{\phi_j + \psi_k\}$ is tropically independent. Suppose it is not. By modifying these functions by constants, we may assume $\theta = \min_{(j,k) \neq (j_0,k_0)} (\phi_j + \psi_k)$ is achieved at least twice at every point. We show $K + \text{div} \theta$ has a chip on each loop $\gamma_i$.

Fix $i$ and suppose $i$ is the $(j_0,k_0)$ entry of the tableau corresponding to $D$. Because the minimum is achieved twice at every point, $\theta = \min_{(j,k) \neq (j_0,k_0)} (\phi_j + \psi_k)$. Each $K + \text{div}(\phi_j + \psi_k) = D_j + E_k$, where $(j,k) \neq (j_0,k_0)$, has a chip in $\gamma_i$ by part (c) of Lemma 13.7. By (2), so does $K + \text{div} \theta$.

Therefore, $K + \text{div} \theta$ has a chip on each loop of $\Gamma$. But $K + \text{div} \theta \in |K|$, which contradicts (1). \qed

Proof of (1). Fix $D \in |K|$. Write $D = K + \text{div} \psi$ for $\psi \in R(K)$. Let $\Gamma_\psi$ be the closed set where $\psi$ achieves its minimum. The set $\Gamma_\psi$ is a union of edges of $\Gamma$ with no leaves. Indeed, if $\Gamma_\psi$ contains an edge $e$ with leaf $u$, let $v$ be the other end of $e$. Then
\[ K_\Gamma(v) = \deg v - 2, \text{ while } \ord_v(\psi) \leq -(\deg v - 1), \text{ since } \psi \text{ is constant on } e \text{ and has slope at least } 1 \text{ on all other edges incident to } v. \text{ So } D(v) = K_\Gamma(v) + \ord_v(\psi) \leq -1, \text{ a contradiction.} \]

So \( \Gamma_\psi \) is a union of loops, bridges, and halves of loops. Let \( i \) be minimal such that \( \gamma_i \subset \Gamma_\psi \). Then \( \Gamma_\psi \) is disjoint from \( \gamma_1, \ldots, \gamma_{i-1} \), and \( br_{i-1} \). Therefore, \( D \) has no chips on \( \gamma_i \), since

\[
\ord_{\psi_i}(\psi) \geq \begin{cases} 
1 & i \geq 1, \\
0 & i = 1
\end{cases} = K_\Gamma(v_i).
\]

\[ \square \]

**Lemma 13.8.** Suppose \( D \in \Div(\Gamma) \), \( \psi_0, \ldots, \psi_r \in R(D) \), \( \theta = \min_i \psi_i \). Let

\[
\Gamma_i = \{ v \in \Gamma : \theta(v) = \psi_i(v) \}.
\]

Then

\[
\supp(D + \div(\theta)) \cap \Gamma_i = (\supp(D + \div(\psi_i)) \cap \Gamma_i) \cup \partial \Gamma_i.
\]

**Proof of (2).** Fix \( D \in \Div(\Gamma) \) and \( \psi_0, \ldots, \psi_r \in R(D) \), and let \( \theta \) and \( \Gamma_i \) be as defined.

Choose \( i \) such that \( \theta(v) = \psi_i(v) \) for some \( v \in \Gamma' \). Let \( \Gamma'_i = \{ v \in \Gamma' : \theta(v) = \psi_i(v) \} \). We have \( \Gamma'_i \neq \emptyset \) by our choice of \( i \). If \( \Gamma'_i = \Gamma' \), then \( D + \div(\psi_i) = D + \div(\theta) \) on \( \Gamma' \), so \( D + \div(\theta) \) has a chip on \( \Gamma' \) because \( D + \div(\psi_i) \) does. Otherwise, by Lemma 13.8, \( D + \div(\theta) \) has a chip at some point in \( \Gamma' \cap \partial \Gamma'_i \).

**References**