TROPICAL BRILL-NOETHER THEORY

6. Riemann–Hurwitz theory for finite graphs

In the series of lectures that preceded today, we discussed Riemann-Roch theory for finite graphs, including divisors and their ranks, Jacobians, and the Riemann-Roch theorem itself. In the theory of Riemann surfaces, the divisor group and Jacobian constructions can be extended to both covariant and contravariant functors, for holomorphic maps.

The goal of this lecture is to introduce and discuss the "right" notion of morphism, or at the very least, one with good properties, in the category of finite graphs. The need for a restricted notion of morphism is clear. An arbitrary morphism $G \to G'$ of graphs, namely one mapping vertices to vertices and preserving incidence relations (see Definition 6.1) behaves quite differently than a morphism of Riemann surfaces. For instance, there exist many such morphisms with $g(G) < g(G')$, while there are no non-constant morphisms from a Riemann surface to one of higher genus. This dissonance, and many others, are removed by imposing the *harmonicity condition* for morphisms. The notion seems to have been first introduced into graph theory by Urakawa, in a study of discrete analogues of Green kernels (2000). It was adapted to divisor theory on graphs by Baker and Norine, and it is their paper (2009) that serves as the main reference for today. The metric theory was studied by M. Chan (2013). The metrized complex analogue, hybridizing holomorphism maps of algebraic curves and harmonic morphisms of metric graphs was systematically studied by Amini, Baker, Brugallé, and Rabinoff (2015).

Let G and G' be two finite graphs. Note that in what follows, graphs will be taken to be connected, with no loop edges.

Definition 6.1. *A* morphism of graphs $\phi : G \to G'$ *is a map* $\phi : V(G) \cup E(G) \to V(G') \cup E(G')$ *such that* $\phi(V(G)) \subset \phi(V(G'))$ *and for every* $x \in V(G)$ *and every* $e \in E(G)$ *with* $x \in e$ *one of the following holds:*

- $\phi(e) \in E(G')$ *and* $\phi(x) \in \phi(e)$.
- $\phi(e) = \phi(x)$ *(i.e. the edge* e *is contracted).*

Throughout the remainder of this section $\phi : G \to G'$ will be a morphism of graphs.

Definition 6.2. *Let* $x \in V(G)$ *be a vertex and* $\phi(x) = y \in G'$ *its image. Consider the induced map* $\overline{\phi} : \mathrm{Star}(x) \to \mathrm{Star}(y)$. *The morphism* ϕ *is said to be* locally of pure degree at x *if the number of preimages of an edge* $e' \in \mathrm{Star}(y)$ *is independent of* e'.

Definition 6.3. *A morphism* $\phi : G \to G'$ *is said to be* harmonic *if for all vertices* $x \in G$, ϕ *is locally of pure degree at* x. *In other words, "*ϕ *is everywhere locally of pure degree".*

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We begin with a non-example.

Example 6.4. Let G be the star graph with 1 central vertex x attached of 6 edges and G' the graph with 1 central vertex y attached to 2 edges. The morphism $G \to G'$ sending x to y, and 3 edges of G to each edge in G' is harmonic. Conversely, the morphism sending 2 edges to one edge of G' and 4 edges to the other edge is not harmonic. See Figure 1 below.

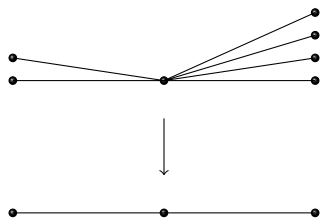

FIGURE 1. An example of a non-harmonic morphism of finite graphs.

Example 6.5. In Figure 2 is depicted a fairly complicated example of a harmonic morphism of graphs. The reader is encouraged to check the local pure degree condition for several of the vertices of the source and study the example to gain a feeling for the global structure harmonic morphisms. This figure is taken directly from Baker and Norine's 2009 paper.

Example 6.6. An important class of morphisms of graphs and tropical curves are contractions of bridge edges. For instance, these play an interesting role in the study of the Abel–Jacobi map in tropical geometry. While harmonic morphisms closely resemble holomorphic maps of Riemann surfaces, there are interesting morphisms that do not fall into the harmonic category. Bridge contractions are among these as illustrated by the morphism depicted in Figure 3.

There are two natural local numerical invariants associated to harmonic morphisms: the vertical and horizontal multiplicity.

Definition 6.7. *The* vertical multiplicity *of $\phi : G \to G'$ at a vertex $x \in V(G)$ is*

$$v_\phi(x) = \#\{e \in E(G) | x \in e, \phi(e) = \phi(x)\}.$$

Definition 6.8. *The* horizontal multiplicity *of ϕ at $x \in V(G)$ is*

$$m_\phi(x) = \#\{e \in E(G) | x \in e, \phi(e) = e'\}$$

for any fixed choice of $e' \in E(G')$ incident to $\phi(x)$. Note that this is well-defined by harmonicity.

A harmonic morphism is said to be *non-degenerate* if its horizontal multiplicity is everywhere positive.

An unwinding of definitions yields a simple but important relationship between the valency of a vertex, its image, and the multiplicities defined above:

$$(\star) \qquad \mathrm{val}(x) = \mathrm{val}(\phi(x)) \cdot m_\phi(x) + v_\phi(x).$$

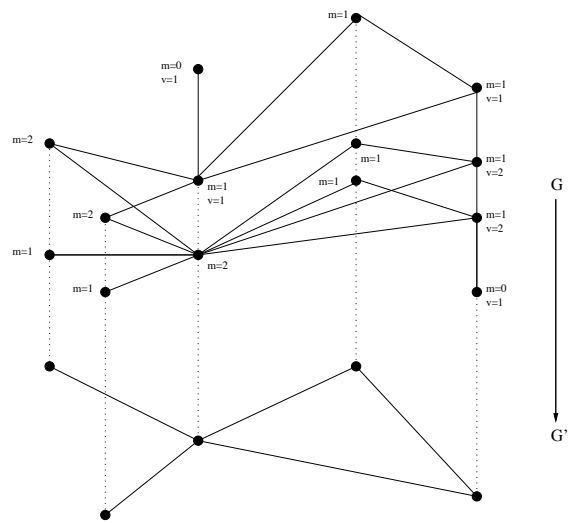

FIGURE 2. An example of a harmonic morphism of finite graphs. The labelings m and v indicate the vertical and horizontal multiplicity, as introduced in Definitions 6.7 and 6.8

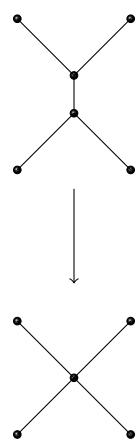

FIGURE 3. A non-harmonic bridge contraction.

The following result shows that there is a well-defined notion of *degree* for a harmonic morphism of finite graphs.

Lemma 6.9. *Fix an edge e' of G'. The quantity* $\deg(\phi) = \#\{e \in E(G) | \phi(e) = e'\}$ *is independent of $e' \in E(G')$.*

Proof. Let x' be a vertex of G' and e'_1 and e'_2 be two edges incident to x'. We first examine the local behavior of ϕ near x'. Since ϕ is harmonic, for each $x \in V(G)$ such that $\phi(x) = x'$ we have

$$\{e \in E(G) | x \in e, \phi(e) = e'_1\} = \#\{\tilde{e} \in E(G) | x \in \tilde{e}, \phi(\tilde{e}) = e'_2\}. \tag{1}$$

Now observe that

$$\#\{e \in E(G) | \phi(e) = e'_1\} = \sum_{x \in \phi^{-1}(x') \text{ vertices}} \#\{e \in E(G) | x \in e, \phi(e) = e'_1\}.$$

Reindexing using (1), the quantity above can in turn be expressed as

$$\#\{\tilde{e} \in E(G) | \phi(\tilde{e}) = e'_2\}.$$

This proves that the lemma holds for any two edges that are incident. Since the graph is connected, any two edges are contained in a path, and the result follows by iteratively applying the argument above to consecutive edges in this path. □

In algebraic geometry, the degree of a morphism is generally defined by counting preimages, with an appropriate correction for ramification on the source. The same holds for harmonic morphisms. The proof is a formal manipulation of the statement above, and we leave it to the interested reader.

Lemma 6.10. *For any $y \in V(G')$,*

$$\deg(\phi) = \sum_{\substack{x \in V(G) \\ \phi(x) = y}} m_\phi(x).$$

Another unwinding of definitions yields the following proposition.

Proposition 6.11. *Let $\phi : G \to G'$ be a harmonic morphism, and assume with $|V(G)| \geq 2$. Then*

- $\deg(\phi) = 0$ *iff ϕ is constant.*
- $\deg(\phi) \geq 1$ *iff ϕ is surjective.*

In the theory of Riemann surfaces, the divisor and Jacobian groups form bivariant functors, as one can pull back and push forward divisors, in a manner that respects linear equivalence. We now define the analogous constructions for harmonic morphisms.

Definition 6.12. *Let $\phi : G \to G'$ be a harmonic morphism.*

- *The pushforward morphism $\phi_* : \mathrm{Div}(G) \to \mathrm{Div}(G')$ is defined as*

$$\phi_*(D) = \sum_{x \in V(G)} D(x)\phi(x).$$

- *The pullback morphism $\phi^* : \mathrm{Div}(G') \to \mathrm{Div}(G)$ is defined as*

$$\phi^*(D') = \sum_{y \in V(G')} \sum_{\substack{x \in V(G) \\ \phi(x) = y}} m_\phi(x) D'(y) \cdot [x].$$

We have the following consequence for the pullback of a divisor.

Lemma 6.13. *If $\phi : G \to G'$ is harmonic and $D' \in \mathrm{Div}(G')$, then*

$$\deg(\phi^* D') = \deg(\phi)\deg(D').$$

Proof. This follows directly from Lemma 6.10. □

We now come to the central result of this lecture.

Theorem 6.14. *(Riemann-Hurwitz)*

(1) $K_G = \phi^* K_{G'} + R_G$ where the ramification divisor R_G is defined as

$$R_G = 2 \sum_{x \in V(G)} (m_\phi(x) - 1)(x) + \sum_{x \in V(G)} v_\phi(x)(x).$$

(2) If $g = g(G)$ and $g' = g(G')$, then

$$2g - 2 = \deg(\phi)(2g' - 2) + \sum_{x \in V(G)} 2(m_\phi(x) - 1) + v_\phi(x).$$

(3) If ϕ is non-constant then $2g - 2 \geq \deg(\phi)(2g' - 2)$ and $g \geq g'$.

Proof. To begin, observe that

$$\phi^* K_{G'}(x) = m_\phi(x)(\mathrm{val}(\phi(x)) - 2).$$

By the valency expression ($\star$) above we see that

$$\begin{aligned} K_G(x) &= \mathrm{val}(x) - 2 \\ &= \mathrm{val}(\phi(x)) m_\phi(x) + v_\phi(x) - 2 \\ &= \phi^* K_{G'}(x) + 2m_\phi(x) + v_\phi(x) - 2 \\ &= (\phi^* K_{G'} + R_G)(x). \end{aligned}$$

The claim (1) follows. Calculating the degrees on both sides and applying Lemma 6.13 yields (2). For (3) it suffices to show that $\deg(R_G) \geq 0$. This is clear unless G has vertical leaves, i.e. 1-valent vertices x such that $m_\phi(x) = 0$, for if this doesn't happen then either $m_\phi(x) \geq 1$ or $v_\phi(x) \geq 2$.

We now proceed by induction on the number of vertical leaf edges. If e is such an edge, then we can contract it to form a new graph $\overline{G}$. The resulting map $\overline{\phi} : \overline{G} \to G'$ is still harmonic, and $\deg(R_{\overline{G}}) = \deg(R_G)$. □

In order to state further functoriality properties of harmonic morphisms, we make the following definition.

Definition 6.15. *Let $f : V(G) \to \mathbb{Z}$ and $f' : V(G') \to \mathbb{Z}$ be functions on G and G' respectively, and ϕ a harmonic morphism as before.*

- *The* pushforward *$\phi_* f : V(G') \to \mathbb{Z}$ is defined by the formula*

$$\phi_* f(y) = \sum_{\substack{x \in V(G) \\ y = \phi(x)}} m_\phi(x) f(x).$$

- *The* pullback *$\phi^* f : V(G) \to \mathbb{Z}$ is defined by composition.*

The pushforward and pullback morphisms for integer valued functions on the vertices are compatible with taking principal divisors. That is, $\phi^* \operatorname{div}(f') = \operatorname{div}(\phi^* f)$ and $\phi_* \operatorname{div}(f) = \operatorname{div}(\phi_* f)$. The proofs of these results are straightforward after careful book keeping. We refer the reader to Baker and Norine's article for the proofs. As a consequence, we obtain two functors from finite graphs to abelian groups, extending the assignment $G \mapsto \operatorname{Jac}(G)$.The *Picard functor* is a contravariant functor sending a morphism ϕ to ϕ^* and the *Albanese* functor is a covariant functor sending a morphism ϕ to ϕ_*.

In algebraic geometry, the pushfoward morphism on Jacobians associated to a ramified cover of Riemann surfaces is always surjective. The same is true of finite graphs.

Proposition 6.16. *Let $\phi : G \to G'$ be a non-constant harmonic morphism. Then $\phi_* : \operatorname{Jac}(G) \to \operatorname{Jac}(G')$ is surjective.*

Proof. From Proposition 6.11 and the linearity of ϕ_*, it follows that ϕ_* is surjective at the level of divisor groups, which implies surjectivity for Jacobians. □

The pullback morphism on Jacobians for a ramified cover of Riemann surfaces $\pi : X \to X'$ is injective precisely when π has a nontrivial unramified abelian subcover, and this is a subtle condition to check, in general. For finite graphs however, the situation is much simpler. The following short proof was communicated to us by Nikolay Malkin, a seminar participant.

Theorem 6.17. *Let $\phi : G \to G'$ be a non-constant harmonic morphism. Then $\phi^* : \operatorname{Jac}(G') \to \operatorname{Jac}(G)$ is injective.*

Proof. The divisor group $\operatorname{Div}(G)$ is isomorphic to the subgroup of functions $f \in \mathbb{Q}^{V(G)}$ such that $\Delta f \in \mathbb{Z}^{V(G)}$, modulo the constant functions. The integer valued functions $f \in \mathbb{Z}^{V(G)}$ yield the principal divisors by mapping f to $\operatorname{div}(f)$. Thus, we see that the Jacobian is the group of $\mathbb{Q}/\mathbb{Z}$-valued harmonic functions on G, up to constant functions. That is,

$$\operatorname{Jac}(G) \cong \{f \in (\mathbb{Q}/\mathbb{Z})^{V(G)} | \Delta f = 0\}/(\text{ConstantFunctions}).$$

The map $\phi : G \to G'$ induces the map

$$\frac{\{f' \in (\mathbb{Q}/\mathbb{Z})^{V(G')} : \Delta f' = 0\}}{(\text{ConstantFunctions})} \cong \operatorname{Jac}(G') \to \operatorname{Jac}(G) \cong \frac{\{f \in (\mathbb{Q}/\mathbb{Z})^{V(G)} : \Delta f = 0\}}{(\text{ConstantFunctions})}$$

by $f \mapsto (v \mapsto f(\phi(v)))$. Injectivity is now clear. □

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