A MOTIVIC FUBINI THEOREM FOR THE TROPICALIZATION MAP

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ABSTRACT. This is a write-up of a lecture delivered at the 2017 Simons Symposium on non-archimedean and tropical geometry. The lecture was based on a joint paper with Sam Payne, "A tropical motivic Fubini theorem with applications to Donaldson-Thomas theory" (arXiv:1703.10228).

Notation. Let k be an algebraically closed field of characteristic zero. We set

$$R_0 = k[\![t]\!],\ K_0 = k(\!(t)\!),\ R = \bigcup_{n>0} k[\![t^{1/n}]\!],\ K = \bigcup_{n>0} k(\!(t^{1/n})\!).$$

We denote by

$$\mathrm{val} : K^* \to \mathbb{Q}$$

the t-adic valuation on K, and we set $\mathrm{val}(0) = \infty$. Finally, we set

$$\widehat{\mu} = \varprojlim \mu_n(k) = \mathrm{Gal}(K/K_0).$$

0. MOTIVATION

Let V be a smooth k-variety and $f : V \to \mathbb{A}^1_k$ a regular function on V. Denef and Loeser have defined an object ψ_f^{mot}, called the *motivic nearby fiber* of f, that lies in the Grothendieck ring $K_0^{\widehat{\mu}}(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ of k-varieties with $\widehat{\mu}$-action, localized with respect to the class $\mathbb{L}$ of the affine line. This object should be viewed as a motivic incarnation of the nearby cycles complex of f. It has important applications in birational geometry and singularity theory. It also plays a central role in the theory of refined curve counting, more specifically in the motivic Donaldson-Thomas theory of Kontsevich and Soibelman, where it appears as a motivic upgrade of the Behrend function and the virtual Euler characteristic. In order to get a reasonable theory, several mathematicians were led to conjecture various identities involving the motivic nearby fiber. The standard way to compute the motivic nearby fiber is Denef and Loeser's formula in terms of a log resolution for f; however, computing a log resolution can be quite involved, and even impossible if one only has some qualitative information about the general shape of f. The aim of this talk is threefold:

(1) explain a new interpretation of the motivic nearby fiber, based on Hrushovki and Kazhdan's theory of motivic integration;
(2) use this interpretation to devise a new computational tool: a motivic Fubini theorem for the tropicalization map;
(3) deduce a proof of the Davison-Meinhardt conjecture on the motivic nearby fibers of weighted homogeneous polynomials (the same method also yields a short proof of Kontsevich and Soibelman's integral identity conjecture, first proven by Lê Quy Thuong).

1. A new interpretation of the motivic nearby fiber

1.1. **Semi-algebraic sets.** Let X be a K-scheme of finite type. A semi-algebraic subset of X is a finite Boolean combination of subsets of $X(K)$ of the form

$$\{x \in U(K) \mid \mathrm{val}(f(x)) \geq \mathrm{val}(g(x))\} \subset X(K) \tag{1.1.1}$$

where U is an affine open subvariety of X and f, g are regular functions on U. If X is of the form $X_0 \times_{K_0} K$, for some K_0-scheme of finite type X_0, then we say that S is defined over K_0 if we can write it as a finite Boolean combination of sets of the form (1.1.1) such that U, f and g are defined over K_0. Note that this property depends on the choice of X_0; if we want to make this choice explicit, we will also call S a semi-algebraic subset of X_0 (even though it is not an actual subset of X_0).

The basic examples of semi-algebraic sets are provided by the following constructions.

(1) If X is a K-scheme of finite type, then every constructible subset S of $X(K)$ is semi-algebraic, because we can write the condition $f(x) = 0$ for some algebraic function f in the equivalent form $\mathrm{val}(f(x)) \geq \mathrm{val}(0)$. If X and the constructible subset S are defined over K_0, then S is also defined over K_0 as a a semi-algebraic set.

(2) For every integer $n \geq 0$ we consider the *tropicalization map*

$$\mathrm{trop} : \mathbb{G}^n_{m,K_0}(K) \to \mathbb{Q}^n : (x_1, \ldots, x_n) \mapsto (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)).$$

A *polyhedron* in $\mathbb{Q}^n$ is a finite intersection of half-spaces of the form

$$\{u \in \mathbb{Q}^n \mid \ell(u) \geq c\}$$

with $\ell \in (\mathbb{Q}^n)^\vee$ and $c \in \mathbb{Q}$. A *constructible subset* of $\mathbb{Q}^n$ is a finite Boolean combination of polyhedra. If Γ is a constructible subset of $\mathbb{Q}^n$, then $\mathrm{trop}^{-1}(\Gamma)$ is a constructible subset of $\mathbb{G}^n_{m,K_0}$.

(3) If $\mathscr{X}$ is an R_0-scheme of finite type, we can consider the *specialization map*

$$\mathrm{sp}_{\mathscr{X}} : \mathscr{X}(R) \to \mathscr{X}(k)$$

defined by reducing the coordinates of an R-point modulo the maximal ideal in R. For every constructible subset C of $\mathscr{X}(k)$, the set $\mathrm{sp}_{\mathscr{X}}^{-1}(C)$ is a semi-algebraic subset of $\mathscr{X}_{K_0}$.

If X and X' are K-schemes of finite type and S and S' are semi-algebraic subsets of X and X', respectively, then a morphism of semi-algebraic sets $f : S \to S'$ is a map whose graph is semi-algebraic in $X \times_K X'$. If $X = X_0 \times_{K_0} K$ and $X' = X'_0 \times_{K_0} K$ and S and S' are defined over K_0, then we say that f is defined over K_0 if its graph has this property. It follows from Robinson's quantifier elimination for algebraically closed valued fields that the image of a morphism of semi-algebraic sets is again a semi-algebraic set. If the morphism is defined over K_0, then the same holds for its image. Every semi-algebraic set defined over K_0 carries a natural action of the Galois group $\widehat{\mu} = \mathrm{Gal}(K/K_0)$, and every semi-algebraic morphism defined over K_0 is $\widehat{\mu}$-equivariant.

1.2. **The motivic volume.** The theory of motivic integration of Hrushovski and Kazhdan assigns to every semi-algebraic set S over K_0 a *motivic volume* $\mathrm{Vol}(S)$ in the Grothendieck ring $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)$ of k-varieties endowed with an action of $\widehat{\mu}$. This motivic volume then yields further geometric invariants of semi-algebraic sets (for instance, Euler characteristics and Hodge-Deligne polynomials) *via* the various motivic realization morphisms. Hrushovski and Kazhdan's theory is based on the model theory of algebraically closed valued fields and formulated in the language of first order logic. We can translate the part that we will need in geometric terms in the following way.

Theorem 1.2.1 (Hrushovski-Kazhdan)**.** *There is a unique way to attach to every semi-algebraic set S over K_0 a motivic volume $\mathrm{Vol}(S)$ in $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)$ such that the following properties are satisfied.*

1. *The motivic volume is additive with respect to disjoint unions, multiplicative with respect to Cartesian products, and invariant under semi-algebraic bijections.*
2. *If $S = \mathrm{trop}^{-1}(\Gamma)$ for some constructible subset Γ of $\mathbb{Q}^n$, then*
$$\mathrm{Vol}(S) = \chi'(\Gamma) \cdot [\mathbb{G}^n_{m,k}]$$
where $\widehat{\mu}$ acts trivially on $\mathbb{G}^n_{m,k}$. Here χ' denotes the bounded Euler characteristic*, the unique additive invariant on constructible subsets of $\mathbb{Q}^n$ such that $\chi'(\Gamma) = 1$ for every non-empty polyhedron Γ.*
3. *Let X_0 be a smooth K_0-scheme of finite type and let $\mathscr{X}$ be a smooth R-model of $X = X_0 \times_{K_0} K$ such that the $\widehat{\mu}$-action on X extends to $\mathscr{X}$. Let C be a constructible subset of $\mathscr{X}(k)$ that is stable under the action of $\widehat{\mu}$, and let $S = \mathrm{sp}^{-1}_{\mathscr{X}}(C)$. Then S is a semi-algebraic subset of X_0, and $\mathrm{Vol}(S) = [C]$ in $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)$. We can think of S as a semi-algebraic set over K_0 with "potential good reduction."*

The motivic nearby fiber has a natural interpretation as the motivic volume of a semi-algebraic set.

Proposition 1.2.2 (N.-Payne)**.** *Let V be a smooth k-variety and let*
$$f : V \to \mathbb{A}^1_k = \mathrm{Spec}\, k[t]$$
be a regular function on V. Set $\mathscr{X} = V \times_{k[t]} k[\![t]\!]$. Then ψ^{mot}_f is the image of $\mathrm{Vol}(\mathscr{X}(R))$ *under the localization morphism*
$$\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k) \to \mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)[\mathbb{L}^{-1}].$$

Note that $\mathscr{X}(R)$ can be identified with the set of R-valued points v on V such that $f(v) = t$. The proof of the proposition is based on an explicit computation of both sides of the equality on a log resolution for f. The expression of ψ^{mot}_f as a motivic volume has two advantages:

1. it shows that ψ^{mot}_f is well-defined without inverting $\mathbb{L}$ (this is non-trivial, because Borisov has shown that $\mathbb{L}$ is a zero-divisor in the Grothendieck ring);
2. the invariance of the motivic volume under semi-algebraic bijections provides extra flexibility in the calculation of the motivic nearby fiber; in particular, we can generate interesting semi-algebraic decompositions by means of *tropical geometry*.

2. A motivic Fubini theorem

2.1. **Constructible functions.** Let A be an abelian group and let n be a non-negative integer. We say that a function
$$\varphi : \mathbb{Q}^n \to A$$

is *constructible* if there exists a partition of $\mathbb{Q}^n$ into finitely many constructible subsets $\sigma_1, \ldots, \sigma_r$ such that φ takes a constant value $a_i \in A$ on σ_i for each i in $\{1, \ldots, r\}$. In that case, we define the integral of φ with respect to the bounded Euler characteristic χ' by means of the formula

$$\int_{\mathbb{Q}^n} \varphi \, d\chi' = \sum_{i=1}^{r} a_i \chi'(\sigma_i) \quad \in A.$$

If Γ is a constructible subset of V, then we also write

$$\int_{\Gamma} \varphi \, d\chi' = \int_{\mathbb{Q}^n} (\varphi \cdot \mathbf{1}_\Gamma) \, d\chi'$$

where $\mathbf{1}_\Gamma$ is the characteristic function of Γ.

2.2. **The Fubini theorem.** Our main result is a motivic Fubini theorem for the tropicalization map, which states that we can compute the motivic volume of a semi-algebraic subset S of $\mathbb{G}^n_{m,K_0}$ by computing the motivic volumes of the fibers of the tropicalization map

$$\mathrm{trop} : \mathbb{G}^n_{m,K_0}(K) \to \mathbb{Q}^n$$

restricted to S and then integrating the resulting function on $\mathbb{Q}^n$ with respect to the bounded Euler characteristic χ'.

Theorem 2.2.1 (Motivic Fubini theorem; N.-Payne)**.** *Let n be a positive integer and let S be a semi-algebraic subset of $\mathbb{G}^n_{m,K_0}$. Then the function*

$$\mathrm{trop}_* \mathbf{1}_S : \mathbb{Q}^n \to \mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k) : w \mapsto \mathrm{Vol}(S \cap \mathrm{trop}^{-1}(w))$$

is constructible, and

$$\mathrm{Vol}(S) = \int_{\mathbb{Q}^n} \mathrm{trop}_* \mathbf{1}_S \, d\chi'$$

in $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)$.

The proof consists of a reduction to the case where S is of the form $\mathrm{trop}^{-1}(\Gamma) \cap X_0(K)$ for some constructible subset Γ of $\mathbb{Q}^n$ and some schön closed subvariety X_0 of $\mathbb{G}^n_{m,K_0}$. In that case, one can make explicit computations of the motivic volumes by means of well-chosen semi-algebraic decompositions induced by the theory of tropical compactifications.

In many interesting situations, one can show that the function $\mathrm{trop}_* \mathbf{1}_S$ is constant on certain constructible subsets Γ of $\mathbb{Q}^n$, *without knowing anything about the value* of $\mathrm{trop}_* \mathbf{1}_S$ on Γ. If, moreover, $\chi'(\Gamma) = 0$, the Fubini theorem implies that $\mathrm{Vol}(S \cap \mathrm{trop}^{-1}(\Gamma)) = 0$ so that we can discard this piece of S from the calculation of the motivic volume. This idea has led to short and natural proofs of the integral identity conjecture of Kontsevich and Soibelman, and the following conjecture of Davison and Meinhardt.

3. Application: the Davison-Meinhardt conjecture

3.1. **Statement of the conjecture.** The conjecture of Davison and Meinhardt states that the motivic nearby fiber of a weighted homogeneous polynomial is given by the most natural candidate, namely, the class of the so-called *affine Milnor fiber* $f^{-1}(1)$.

Conjecture 3.1.1 (Davison–Meinhardt, 2011)**.** *Let $f : \mathbb{A}^n_k \to \mathbb{A}^1_k$ be a regular function defined by a weighted homogenous polynomial of degree d with weights $w_1, \ldots, w_n > 0$. Then $\psi_f^{\mathrm{mot}} = [f^{-1}]$ in $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)[\mathbb{L}^{-1}]$, where the $\widehat{\mu}$-action on $f^{-1}(1)$ factors through the $\mu_d(k)$-action with weights $w_1, \ldots, w_n$.*

Davison and Meinhardt proved their conjecture in the homogeneous case (all weights w_i equal to 1) by means of a computation on a log resolution for f. We will now explain how one can deduce the general case from our motivic Fubini theorem.

3.2. **Proof of the conjecture.** In fact, we will prove a stronger equality, without inverting $\mathbb{L}$: setting $\mathscr{X} = \mathbb{A}^n_k \times_{k[t]} k[\![t]\!]$, we will show that

$$\mathrm{Vol}(\mathscr{X}(R)) = [f^{-1}(1)]$$

in $\mathbf{K}^{\widehat{\mu}}(\mathrm{Var}_k)$. The conjecture then follows at once from our interpretation of the motivic nearby fiber explained above.

Set $\mathscr{Y} = f^{-1}(1) \times_k R$, endowed with the diagonal $\widehat{\mu}$-action. Then $\mathscr{Y}(R)$ is a semi-algebraic set defined over K_0 with "potential good reduction" and $\mathrm{Vol}(\mathscr{Y}(R)) = [f^{-1}(1)]$ by the definition of the motivic volume. Since f is weighted homogeneous, we have a map

$$\mathscr{X}(K) \to \mathscr{Y}(R) : (x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n) = (t^{-w_1/d} x_1, \ldots, t^{-w_n/d} x_n).$$

This is a semi-algebraic bijection defined over K_0 (since it is $\widehat{\mu}$-equivariant). The problem is that it does not identify $\mathscr{X}(R)$ with $\mathscr{Y}(R)$, but with a larger subset S' of $\mathscr{Y}(K)$ defined by the conditions $\mathrm{val}(y_i) \geq -w_i/d$ for $i = 1, \ldots, n$. Since the motivic volume is additive and invariant under semi-algebraic bijections, it suffices to show that $\mathrm{Vol}(S' \setminus \mathscr{Y}(R)) = 0$. By additivity and induction on the dimension n, it is enough to prove that $\mathrm{Vol}(S) = 0$, where

$$S = (S' \setminus \mathscr{Y}(R)) \cap \mathbb{G}^n_{m,K_0}(K).$$

The points y of $S' \cap \mathbb{G}^n_{m,K_0}$ that lie in $\mathscr{Y}(R)$ are exactly the ones satisfying $\mathrm{val}(y_i) \geq 0$ for all i. From the fact that $\mathscr{Y}$ is obtained by base change to R from a scheme defined over k, one can deduce that $\mathrm{trop}_* \mathbf{1}_S$ is constant on the relative interiors of the cones of a finite polyhedral fan in $\mathbb{Q}^n$. Now it suffices to observe that the intersection of a relatively open cone with the strip

$$\{u \in \mathbb{Q}^n \mid u_i \geq -w_i/d \text{ for all } i,\ u_i < 0 \text{ for some } i\}$$

has bounded Euler characteristic zero, so that $\mathrm{Vol}(S) = 0$ by the Fubini theorem.

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